Relation / Function

Uniqueness, Existence

For Relation $R : X \to Y$

Uniqueness $!$

- $\forall x1, x2.(R(x1) = R(x2)) \to x1 = x2$

Existence $\exists$

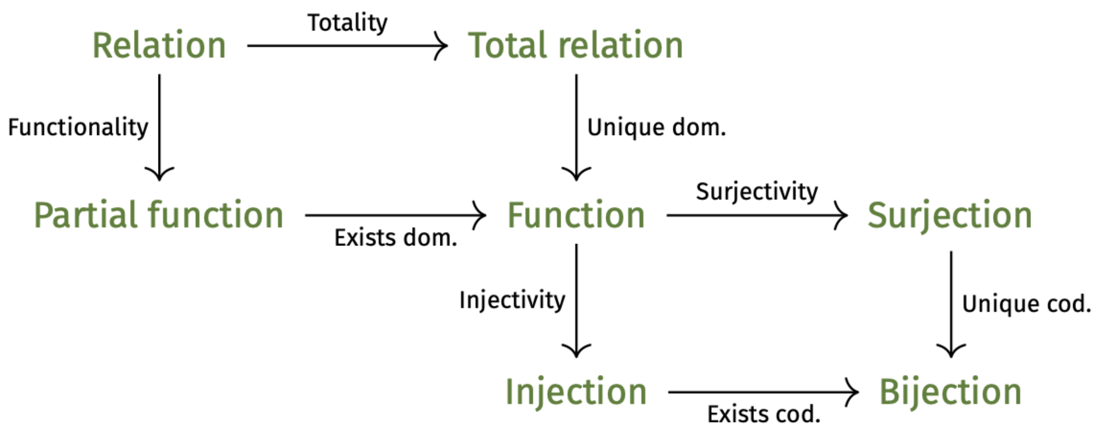
- $\forall y.\exists x.y = R(x)$

Existence of x $\Leftrightarrow$ whole coverage of y

Injective-Functionality, Surjective-Totality

All concepts derived from $!$ uniqueness and $\exists$ existence.

Relationship:



If $f : A \to B$ is injective function, its inverse $f^{-1} : B \to A$ has functionality. [both need uniqueness]

If $f : A \to B$ is surjective function, its inverse $f^{-1} : B \to A$ has totality. [both need existence]

[See more at the below definition]

Note:

Not the other way around total relation (one property only, totality) `$\implies$`surjective function (surjective + functionality + totality). *[See more below how to fix it.]*

Definition and Property

$R : A \to B$

Let the **domain** of the function to be D_f, the **codomain / image** be Im_f.

1. (Partial) Functionality of relation

Functionality requires ! uniqueness of b.

$$\forall b_1, b_2 \in B . aRb_1 \wedge aRb_2 \implies b_1 = b_2 .$$

2. Totality of relation

Totality requires ∃ existence of b.

$$\forall a \in A . \exists b \in B . aRb .$$

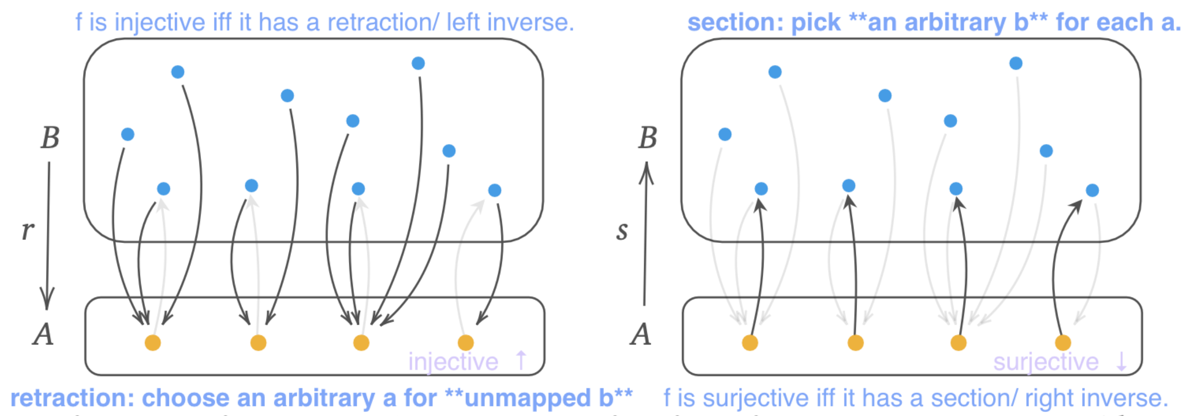
$D_f = A$. (Existence of b ⇔ whole coverage of a)

3. Injective of function

Injection requires ! uniqueness of a.

$$\forall a_1, a_2 \in A . (f(a1) = f(a2)) \implies a_1 = a_2 .$$

- 1: For (finite) set A and B,
  - If there exists an injection $f : A \to B \Leftrightarrow \#A \leq \#B$.
  - Disprove a surjection:$\#A > \#B$.
- 2: injective is equivalent to left-cancellable:
  - $f : B \to C$ is an injection ⇔ for all sets A and functions $g, h : A \to B$, if $f \circ g = f \circ h$ then $g = h$
- 3: f is injective iff it has a left inverse / retraction.



4. Surjective of function

Surjection requires ∃ existence of a.

$$\forall b \in B . \exists a \in A . f(a)=b.$$

$Im_f = B$. (Existence of a ⇔ whole coverage of b)

- 1: For (finite) set A and B,
  - If there exists a surjection $f : A \to B \Leftrightarrow \#A \geq \#B$.
  - Disprove a surjection: $\#A < \#B$.
  - (Note: for infinite set also holds, proved in third year math course)
    - Proof : Schröder–Bernstein / SCB theorem
- 2: surjective is equivalent to right-cancellable:
  - $f : A \to B$ is a surjection ⇔ for all sets C and functions $g, h : B \to C$, $g \circ f = h \circ f$ implies $g = h$
- 3: f is surjective iff it has a right inverse / section.

5. Retraction r

the action of choosing something from a topological space back in subspace.

$B \to A$ maps an element $b \in B$ (in a cluster) to its group label in $r(b) \in A$.

Examples are

- Cities grouped by countries,
- Students grouped by subject/college,
- Products grouped by brands,
- Employees grouped by department,
- and so on (more in the Databases).

Equip B with an implicit partitioning.

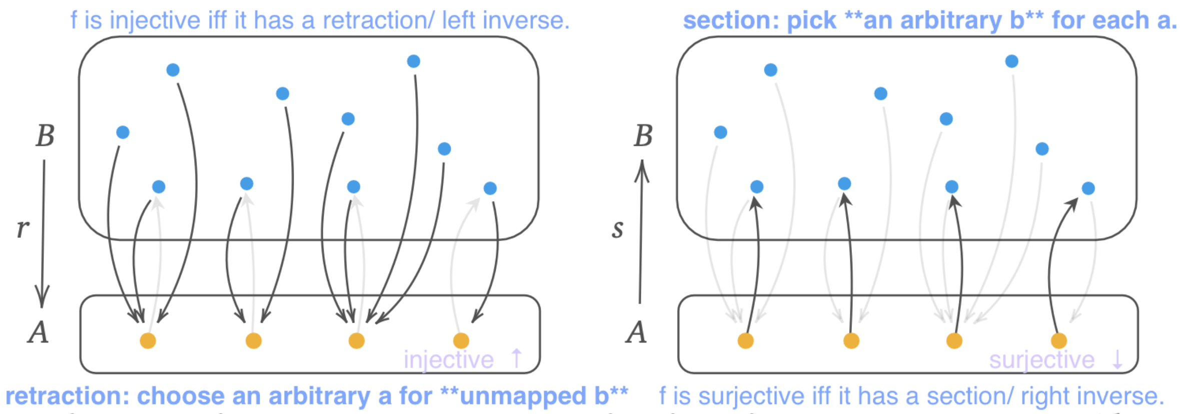
6. Section s

the action of choosing something from subspace to the topological space

$A \to B$ maps label $a \in A$ to a particular representative element $s(a) \in B$

Examples are

- Species to particular animals



7. (r, s) pair

$retraction_{BtoA} \circ section_{AtoB} = id_A$

= *Surjection* ○ *Injection pair*

It can be seen as mapping from selects

- a representative element $b \rightarrow$ label a in subspace; and then this label a in subspace $\rightarrow$ to its representative element b

See more at 2017p2q9.

Note on inverse of f

Note 1:

$f : A \rightarrow B$ is injective iff it has a left inverse / retraction.

However, its inverse $f^{-1} : B \rightarrow A$ may **not** be well-defined function for injective only function f

- need to be pick something for undefined b in B, fulfilling totality of f^{-1}

Let a_0 be arbitrary element from A.

```
f^{-1}(b) =
{
   = a , if f(a) = b
   = a_0, if b is not mapped to any a in f.
}
```

Note 2: $f : B \rightarrow A$ is surjective iff it has a right inverse / section.

However, its inverse $f^{-1} : A \rightarrow B$ may **not** be well-defined function for surjective only function f

- need to disambiguate f^{-1} by choosing a arbitrary element a for each b, fulfilling functionality of f^{-1}

```
f^{-1}(b) =
{
  min { a | f(a) = b }, if A is finite set
  choose a_0, if A is infinite set (supported by axiom of choice)
}
```

Aside:

min for infinite set may *not* a good choice!

- like $R \rightarrow 1$? As only for all finite sets (integer), $\leq$ is well founded, so min is not appropriate for infinite set R (real number).
- So see Axiom of Choice.

8. Axiom of Choice

A helper axiom for choosing surjective function!

Informally put, the axiom of choice says that given any collection of sets, each containing at least one element, it is possible to construct a new set by arbitrarily choosing one element from each set, even if the collection is infinite. (Wiki)

9. Idempotence of function

Idempotent function $e : B \rightarrow B$

- such that $e \circ e = e$
- Meaning: once something is brought into a standardised, normal form, it should not change if normalised again.

Example:

- Normalisation procedures
  - Absolute value function $|x| : Z \rightarrow Z$,
  - Sorting algorithms
- Map a set X to its **closure** under some property ClP (X)
  - The **closure** of a set under an operation is the smallest superset of A which is **closed** under the operation. [Set A is **closed** under some operation if applying the operation to elements of A yields an element of A.]
  - e.g. for an arbitrary relation R, taking the transitive closure of Cltrans(R) should be a no-op.

- In distributed system, multiple retries in not reliable links requires idempotence for naive deduplication.

10. (r,s) and e

[From example sheet]

$s : A \rightarrow B$ and $r : B \rightarrow A$

- Any section-retraction pair gives rise to an idempotent function

Proof:

RTP: when e = s∘r, e is an idempotent function, which is e ∘ e = e.

$(s \circ r) \circ (s \circ r) = s \circ (r \circ s) \circ r = s \circ id_A \circ r = s \circ r. \square$

- Any idempotent function can be split into a section-retraction pair.

Proof:

Require to show that for every idempotent $e : B \to B$, there exists a set A (called a retract of B) and a pair $s : A \to B$ and $r : B \to A$ such that $s \circ r = e$.

By definition of section, A is a subset of B (subspace to space, see above picture in section),

there is a canonical section $s : A \to B$ that embeds A to its superset: $s(x \in A) = x \in B$.

Conversely, the retraction that maps B to A is the idempotent function e itself, with its codomain / image restricted to its range: $r(y \in B) = e(y)$. $A = \{e(y) | y \in B\}$.$\square$