

# Algorithms and Complexity

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- IA Algorithm I,II
- IB Complexity, Computation

## Language and Automata

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$L \subseteq \Sigma^*, \forall x \in L, \text{ the length } n = |x|$

Reference: Formal Language and Automata

## Reduction

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Reduction of  $L_1 \rightarrow L_2$  is a computable function  $f : \Sigma_1^* \rightarrow \Sigma_2^*$  s.t.

- $\forall x \in \Sigma_1^*. f(x) \in L_2 \Leftrightarrow x \in L_1$ .
- every string in  $L_1$  is only mapped by  $f$  to a string in  $L_2$ .
  - $f(x) \notin L_2 \Leftrightarrow x \notin L_1$ .

Polynomial time reducible  $L_1 \leq_P L_2$ .

- the string  $f(x)$  produced by the reduction  $f$  on input  $x$ 
  - must be bounded in length by  $p(n)$ .

$g$  is polynomial function  $L_2 \rightarrow L_3$

- transitive as  $g \circ f$  is polynomial reduction  $L_1 \rightarrow L_3$

Usage,

- $L_2$  is decidable  $\rightarrow L_1$  is decidable
  - by polynomial  $f(x) \in / \notin L_2$
- $L_1$  is not decidable  $\rightarrow L_2$  is not decidable
  - $L_1$ : Halting problem

## Complexity Class

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### Time Complexity

measures computation steps

Polynomial  $p$  is of form  $n^k$ ,  $k$  is a constant.

$$L \in \mathcal{P} = \bigcup_{k=1}^{\infty} \text{TIME}(n^k)$$

$$L \in \mathcal{NP} = \bigcup_{k=1}^{\infty} \text{NTIME}(n^k)$$

Prover  $M$  (nondeterministic TM):  $x \in L \rightarrow \text{certificate } c$ , where  $|c| < p(|x|)$

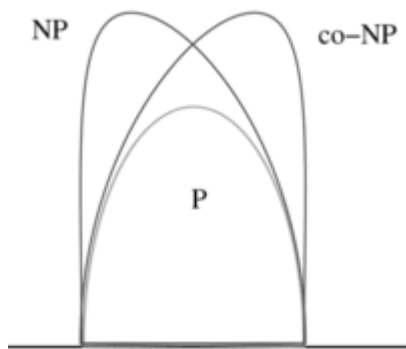
- solvable by a nondeterministic TM  $M$  within  $p(|x|)$

Verifier  $V$  (deterministic TM)

- $\exists c.(x, c)$  accepted by  $V$  running in polynomial time
- polynomially satisfiable / certificate of membership

Complement:  $L \in \text{co-}\mathcal{NP}$

- $\forall c.(x, c)$  are all not accepted by  $V$  in polynomial time
- polynomially falsifiable / certificates of disqualification
- Relationship with  $\mathcal{NP}$ ,  $\mathcal{P}$ : Unknown.
  - Intersection of  $NP$  and  $co-NP$ , at least  $P$



**Cook-Levin theorem**

$L \in \mathcal{NP}$ -hard

- if  $\forall A \in \mathcal{NP}, A \leq_P L$ .

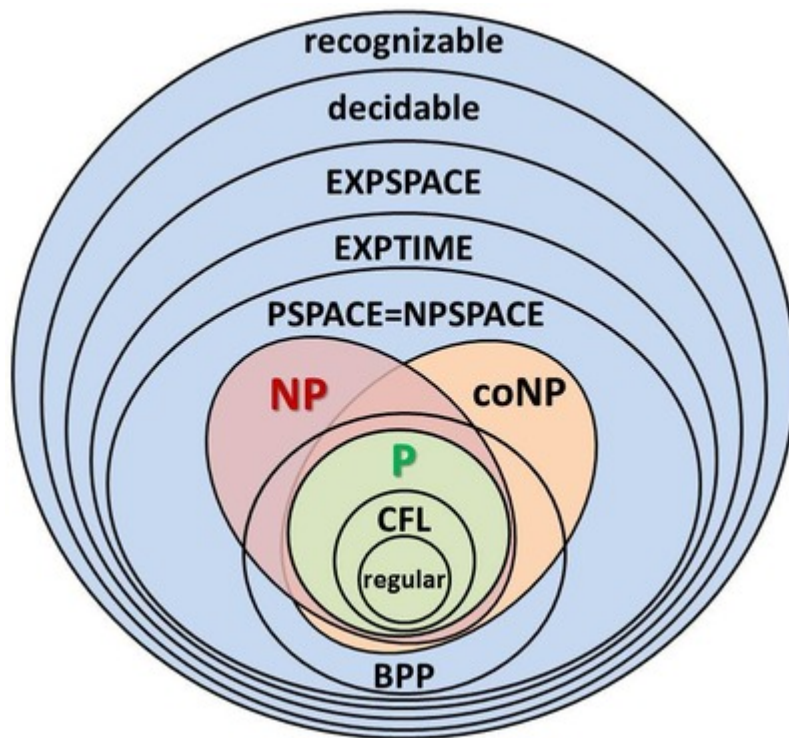
$L \in \mathcal{NP}$ -complete

- if  $L \in \mathcal{NP}$  and  $\mathcal{NP}$ -hard.

## Space Complexity

measures size of work tape

$$L \subseteq NL \subseteq P \subseteq NP \subseteq PSPACE \subseteq NPSPACE \subseteq EXP$$



- $L, P, PSPACE$  are all closed under complementation

$$NSPACE(f(n)) \subseteq TIME(k^{\log n + f(n)})$$

Savitch's Theorem

$$NSPACE(f(n)) \subseteq SPACE(f(n)^2), PSPACE = NPSPACE.$$

# Lists of Algorithms

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- IA Algorithm I,II
- IB Complexity
- II Randomized Algorithm, BioInfo

Decision problem: output starts with ?.

- Negation of decision problem swaps the accept/reject state in TM
  - $SAT = rej, \bar{SAT} = acc$

Optimization problem: output starts with *max / min*.

## Number Theory

$input \in \mathbb{N}, n := \#(bits\mathbb{B})$

Algorithm	Input	Output	Complexity	Note
Euclid's algo	$(x, y)$	$?x = 1$	$O(\log x + \log y)$	in #bits
Prime/COMPOSITE	$1\{0, 1\}^*$	Prime or Factor	$O(\sqrt{x})$	in #bits
Knapsack	$I = (v_i, w_i), W_M, V_m$	$? \exists I' \subseteq I. W \leq W_M$ $\wedge V \geq V_m$	$\mathcal{NP}$ -complete	$X3C <_p Knapsack$
Schedule	/	/	$\mathcal{NP}$ -complete	$Knapsack <_p Schedule$

## Boolean / nCNF

Variables  $X = \{x_1, x_2, \dots\}$

Expression  $\phi : X,$

- CNF  $\phi \equiv C_1 \wedge \dots \wedge C_m$
- 3CNF  $\phi'$ 
  - each clause  $C_i$  is  $\leq 3$  literals disjunction
  - $\phi$  conversion to  $\phi'$  in P.

Assignment  $T : X \rightarrow \mathbb{B}$

- CVP,  $l : X \rightarrow \mathbb{B} \cup \{\wedge, \vee, \neg\}$

Algorithm	Input	Output	Complexity	Note
Evaluation	$\phi, T$	$? \mathbb{T}.$	$O(n^2)$	<i>each rule</i> $O(n)$ remove one variable
<b>SAT</b>	$\phi$	$? \exists T. T(X) = \mathbb{T}$	$O(2^n n^2)$	$(\# T); \mathcal{NP}$ -complete
VAL	$\phi$	$? \forall T. T(X) = \mathbb{T}$	$O(2^n n^2)$	$\neg \phi_{SAT}$ [negate both IO] <i>co-NP</i> -complete
CVP	<i>DiG</i>	<i>Circuit Value</i> $\mathbb{B}$	$\mathcal{P}$	linear T by topological sort
CNF-SAT	$\phi_{CNF}$	Same as SAT	$\mathcal{NP}$ -complete	$SAT <_p CNF - SAT$
3SAT	$\phi_{3CNF}$	Same as SAT	$\mathcal{NP}$ -complete	$CNF-SAT <_p 3SAT$

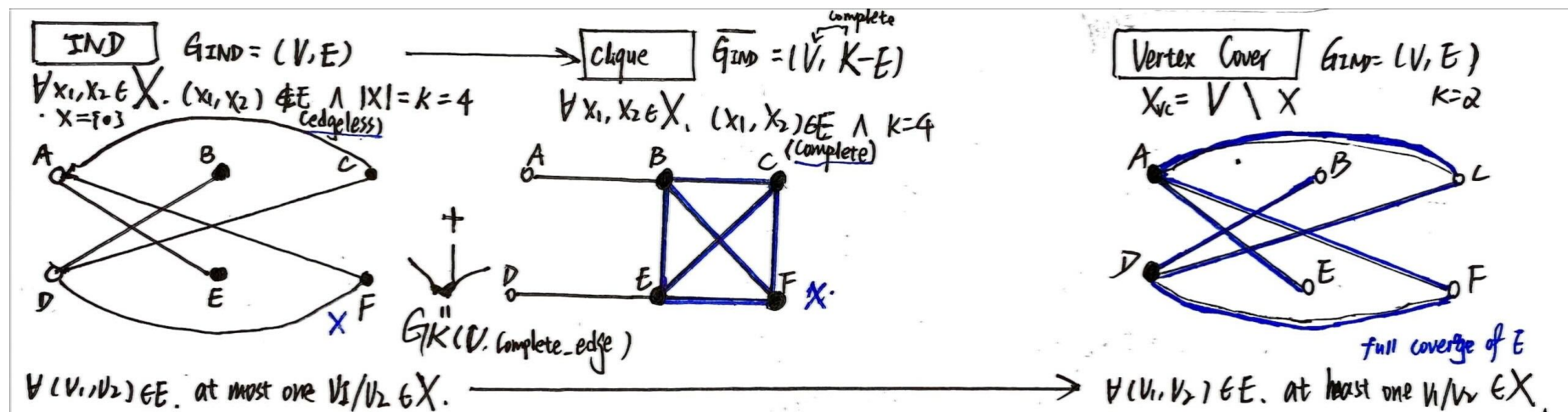
## Graph Theory

$G : (V, E)$ , Directed Acyclic Graph *DiAG*, Undirected Graph *UnG*,

Node / Vertex  $v \in V$ , the number of nodes  $n = |V|$

Algorithm	Input	Output	Complexity	Note
Reachability	$DiG, v_1, v_2$	$? \exists p. path(v_1 \rightarrow v_2).$	$O(n^2); S(n)$	marked V, neighbours
HAMiltonian	$G$	$? \exists cycle. path(v_1 \rightarrow all! v_i \rightarrow v_1)$	$O(n!)$ $\mathcal{NP}$ -complete	$3SAT <_p HAM$

Algorithm	Input	Output	Complexity	Note
TSP	$G, C : V \times V \rightarrow \mathbb{N}$	order/enum for $V$ HAM with min Cost	$O(n!)/O(2^n n^2)$ $\mathcal{NP}$ -complete	$\Omega(n \log n)$ $HAM <_p TSP$
Isomorphism	$G_1, G_2$	$\exists f. (v_1, v_2) \in E_1 \Leftrightarrow f(v_1), f(v_2) \in E_2$	$O(n!)$	all possible bijections
k-colourability	$G$	assignment of colours	$k = 2, \mathcal{P}$ $\mathcal{NP}$ -complete	$3SAT <_p 3color$
INDependent Set	$UnG, k =  X $	$\exists X \subseteq V. \forall x_i. (x_1, x_2) \notin E$ or $\forall (v_1, v_2) \in E.$ $v_1, v_2$ at most one $\in X$ .	$\mathcal{NP}$ -complete	$3SAT <_p IND$
Clique	$UnG, k =  X $	$\exists X \subseteq V. \forall x_i. (x_1, x_2) \in E$	$\mathcal{NP}$ -complete	$\bar{G}_{IND}, X_{IND}$
Vertex Cover	$UnG, k =  X $	$\exists X \subseteq V. \forall (v_1, v_2) \in E.$ $v_1 \in X \vee v_2 \in X$ (at least 1)	$\mathcal{NP}$ -complete	$G_{IND}, V - X_{IND}$



Set Theory

Algorithm	Input	Output	Complexity	Note
Bipartite	$B, G, M \subseteq B \times G$	$? \exists M'. \forall b \in B, g \in G. (b, g) \in M'$ and pairwise disjoint	$\mathcal{P}$	
3D Matching	$X, Y, Z, M$	similar as above	$\mathcal{NP}$ -complete	$3SAT <_p 3DM$
eXact Cover by 3-Sets	$U(3n), S(3) \subset \mathbb{P}(U)$	$? \exists S^*$ pairwise disjoint and full coverage	$\mathcal{NP}$ -complete	$U =_{3DM} X \cup Y \cup Z$
Set Cover	$U, S \subset \mathbb{P}(U), n$	$? \exists S^*$ full coverage	$\mathcal{NP}$ -complete	$(U_{X3C}, S_{X3C}, \frac{ U_{X3C} }{3})$ $E(G_{VC}), E(v_i)   deg(v) > 0$