

Algorithms and Complexity

- IA Algorithm I,II
- IB Complexity, Computation

Language and Automata

$L \subseteq \Sigma^*, \forall x \in L, \text{ the length } n = |x|$

Reference: Formal Language and Automata

Reduction

Reduction of $L_1 \rightarrow L_2$ is a computable function $f : \Sigma_1^* \rightarrow \Sigma_2^*$ s.t.

- $\forall x \in \Sigma_1^*. f(x) \in L_2 \Leftrightarrow x \in L_1.$
- every string in L_1 is only mapped by f to a string in L_2 .

Polynomial time reducible $L_1 \leq_P L_2$.

- the string $f(x)$ produced by the reduction f on input x
- must be bounded in length by $p(n)$.
- transitive as $g \circ f$ is polynomial reduction $L_1 \rightarrow L_3$
- g is polynomial function $L_2 \rightarrow L_3$

Usage,

- L_2 is decidable $\rightarrow L_1$ is decidable
- by $f(x) \in / \notin L_2$
- L_1 is not decidable $\rightarrow L_2$ is not decidable
- L_1 : Halting problem

Complexity Class

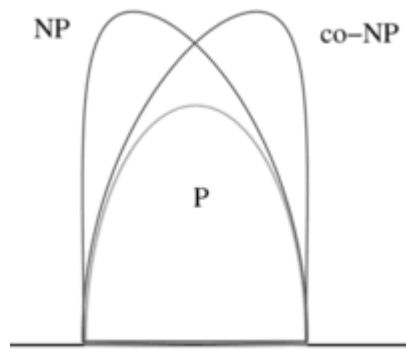
Polynomial p is of form n^k , k is a constant.

$$L \in \mathcal{P} = \bigcup_{k=1}^{\infty} \text{TIME}(n^k)$$

$$L \in \mathcal{NP} = \bigcup_{k=1}^{\infty} \text{NTIME}(n^k)$$

- Prover $M : x \in L \rightarrow \text{certificate } c$ and $|c| < p(|x|)$
- solvable by a nondeterministic TM M within $p(|x|)$
- Verifier V
- $\exists c. (x, c)$ is accepted by V running in polynomial time
- verifiable by a deterministic TM V in polynomial time
- polynomially satisfiable / certificate of membership (L)
- Complement: $L \in \text{co-}\mathcal{NP}$

- $\forall c. (x, c)$ are all not accepted by V in polynomial time
- polynomially falsifiable / certificates of disqualification
- Relationship with \mathcal{NP} , \mathcal{P} : Unknown.



$L \in \mathcal{NP}$ -hard

- if $\forall A \in \mathcal{NP}, A \leq_P L$.

$L \in \mathcal{NP}$ -complete

- if $L \in \mathcal{NP}$ and \mathcal{NP} -hard.

Lists of Algorithms

- IA Algorithm I,II
- IB Complexity
- II Randomized Algorithm, BioInfo

Decision problem: output starts with ?.

Optimization problem: output starts with *max* / *min*.

Number Theory

input $\in \mathbb{N}$, $n := \#(\text{bits}\mathbb{B})$

Algorithm	Input	Output	Complexity	Note
Euclid's algo	(x, y)	$?x = 1$	$O(\log x + \log y)$	in #bits
Prime/COMPOSITE	$1\{0, 1\}^*$	Prime or Factor	$O(\sqrt{x})$	in #bits
Knapsack	$I = (v_i, w_i), W_M, V_m$	$? \exists I' \subseteq I. W \leq W_M$ $\wedge V \geq V_m$	\mathcal{NP} -complete	$X3C <_p it$
Schedule	/	/	\mathcal{NP} -complete	$Knapsack <_p it$

Boolean / nCNF

Variables $X = \{x_1, x_2, \dots\}$

Expression ϕ ,

- CNF $\phi \equiv C_1 \wedge \dots \wedge C_m$
- 3CNF ϕ'
 - each clause C_i is ≤ 3 literals disjunction

- ϕ conversion to ϕ' in P.

Assignment $T : X \rightarrow \mathbb{B}$

- CVP, $l : X \rightarrow \mathbb{B} \cup \{\wedge, \vee, \neg\}$

Algorithm	Input	Output	Complexity	Note
Evaluation	ϕ, T	$?T$.	$O(n^2)$	<i>each rule</i> $O(n)$ remove one variable
SAT	ϕ	$? \exists T. T(X) = \mathbb{T}$.	$O(2^n n^2)$	$(\# T); \mathcal{NP}$ -complete
VAL	ϕ	$? \forall T. T(X) = \mathbb{T}$.	$O(2^n n^2)$	$\neg \phi$: not SAT
CVP	<i>DiG</i>	<i>Circuit Value</i> \mathbb{B}	\mathcal{P}	linear T by topological sort
CNF-SAT	ϕ_{CNF}	Same as SAT	\mathcal{NP} -complete	$SAT <_p$ it
3SAT	ϕ_{3CNF}	Same as SAT	\mathcal{NP} -complete	$CNF - SAT <_p$ it

Graph

$G : (V, E)$, Directed Acyclic Graph *DiAG*, Undirected Graph *UnG*,

Node / Vertex $v \in V$, the number of nodes $n = |V|$

Algorithm	Input	Output	Complexity	Note
Reachability	<i>DiG</i> , v_1, v_2	$? \exists p.path(v_1 \rightarrow v_2)$.	$O(n^2); S(n)$	marked V, neighbours
HAMiltonian	G	$? \exists cycle.path(v_1 \rightarrow all!v_i \rightarrow v_1)$	$O(n!)$ \mathcal{NP} -complete	$3SAT <_p$ it
TSP	$G, C : V \times V \rightarrow N$	order/enum for V HAM with min Cost	$O(n!)/O(n^2 2^n)$ \mathcal{NP} -complete	$\Omega(n \log n)$ $HAM <_p$ it

Algorithm	Input	Output	Complexity	Note
Isomorphism	G_1, G_2	$? \exists f. (v_1, v_2) \in E_1 \Leftrightarrow f(v_1), f(v_2) \in E_2$	$O(n!)$	all possible bijections
k-colourability	G	assignment of colours	$k = 2, \mathcal{P}$ \mathcal{NP} -complete	$3SAT <_p k = 3$
INDependent Set	UnG	$X \subseteq V, \forall x_i. (x_1, x_2) \notin E$	\mathcal{NP} -complete	$3SAT <_p k = 3$
Clique	G	$X \subseteq V, \forall x_i. (x_1, x_2) \in E$	\mathcal{NP} -complete	\bar{G}_{IND}, k_{IND}
Vertex Cover	UnG	$? X \subseteq V. \forall (v_1, v_2) \in E. v_1 \in X \vee v_2 \in X.$	\mathcal{NP} -complete	$G_{IND}, n - k_{IND}$

Sets

Algorithm	Input	Output	Complexity	Note
Bipartite	$B, G, M \subseteq B \times G$	$? \exists M'. \forall b \in B, g \in G. (b, g) \in M'$ and pairwise disjoint	\mathcal{P}	
3D Matching	X, Y, Z, M		\mathcal{NP} -complete	$3SAT <_p it$
eXact Cover by 3-Sets	$U(3n), S(3) \subset \mathbb{P}(U)$	$? \exists S^*$ pairwise disjoint and full coverage	\mathcal{NP} -complete	$U = X \cup Y \cup Z$
Set Cover	$U, S \subset \mathbb{P}(U), n$	$? \exists S^*$ full coverage	\mathcal{NP} -complete	$(U_{X3C}, S_{X3C}, \frac{ U_{X3C} }{3})$ $E(G_{VC}), E(v_i) deg(v) > 0$