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Contents

1 Introduction

When modelling prices of financial assets, modelling the volatility process itself plays a crucial role. In the derivates world, log-prices are often modelled as continuous semi-martingales. For a given asset with log-price Y_t, such a process takes the form

$$dY_t = \mu_t dt + \sigma_t dW_t \tag{1}$$

where μ_t is a drift term and W_t is a one-dimensional Brownian motion. The term σ_t represents the volatility process, and it is a very important ingredient in the model (vol is rough). However, modelling the volatility process itself is also a complex task. Different approaches to estimate σ_t is used in the literature. In the simplest models, the volatility is either constant or a deterministic function of time. In more popular stochastic volatility models, the volatility process is modelled as a stochastic process itself. In notable models such as the Hull and White model, the Heston model, and the SABR model, the volatility σ_t is modelled as a continuous Brownian semi-martingale (vol is rough). The stochastic volatility models seem way more realistic but generated prices from these models are in many cases not consistent with observed prices. (33, 46)

Various authors have proposed other kinds of models to model the volatility. A popular one is stochastic volatility models driven by fractional Brownian motion. A well-known example of such a fractional stochastic volatility model is the one proposed by Comte and Renault (1998) who models the dynamics of the volatility σ_t of an asset as:

$$Y_t = \ln \sigma_t, \quad dY_t = -\gamma Y_t dt + \theta dB_t^H \tag{2}$$

where B^H is a fractional Brownian motion with Hurst exponent H. The model was first introduced to model long range dependence effect observed in financial time series. The long range dependence is modelled by choosing $1 > H > \frac{1}{2}$ (comte and renault 1998). In recent literature starting with (vol is rough), it has been suggested to use the fractional stochastic volatility models with $H < \frac{1}{2}$ for modelling volatility. Processes driven by a fractional Brownian motion with $H < \frac{1}{2}$ are referred to as 'rough processes' since these fractional Brownian motion have trajectories rougher than a standard Brownian motion (rough vol), and (vol is rough) concludes that volatility is rough. It is important to note that (vol is rough) unlike previous literature rely on the behaviour of volatility estimators over short intraday time scales in order to asses the 'roughness'.

(rough vol) challenges the conclusions of (vol is rough). (rough vol) simulates from models where the true spot volatility is known and shows that measures of roughness for realized volatility based on the data from these simulations are in many cases much rougher than those of the underlying true spot volatility. This difference solely lies in the estimation error, and challenges the use of high-frequency volatility estimators when measuring the roughness.

In this thesis, we will take a similar approach to (rough vol) and investigate further if volatility appears to be rough even when the true model does not exhibit rough behaviour.

2 Fractional Brownian motion

A fractional Brownian motion (fBm) $(B_t^H)_{t\in\mathbb{R}}$ with Hurst parameter $H \in (0,1)$ is a centered continuous Gaussian process with covariance function

$$\mathbb{E}\left[B_t^H B_s^H\right] = \frac{1}{2}\left(t^{2H} + S^{2H} - |t-s|^{2H}\right) \tag{3}$$

for $s, t \geq 0$. Note that B_t^H reduces to an ordinary Brownian motion for $H = \frac{1}{2}$ (specsim). The incremental process of a fractional Brownian motion is called fractional Gaussian noise, and it is a stationary discrete-time process. We define the fractional Gaussian noise $X = \{X_k : k = 0, 1, ...\}$ by

$$X_k := B_{t_{k+1}}^H - B_{t_k}^H. \tag{4}$$

With this definition, we immediately see that every X_k is normally distribution with mean 0 and variance

$$\begin{aligned}\mathbb{E}\left[(B_{t_{k+1}}^H - B_{t_k}^H)^2\right] &= Var(B_{t_{k+1}}^H) + Var(B_{t_k}^H) - 2Cov(B_{t_{k+1}}^H, B_{t_k}^H) \\ &= t_{k+1}^{2H} + t_k^{2H} - \left(t_{k+1}^{2H} + t_k^{2H} - (t_{k+1} - t_k)^{2H}\right) = (t_{k+1} - t_k)^{2H}\end{aligned}$$

using the covariance function (3). Thus, the fractional Gaussian noise is standard normal distributed when the time step is 1.

2.1 Simulation of fractional Brownian motion

Throughout this thesis we will be simulating from the fractional Brownian motion by using a spectral method which can be used for stationary processes. The idea is to analyse the stochastic process in the spectral domain rather than the time domain. The spectral density is computed as follows for frequencies $-\pi \leq \lambda \leq \pi$:

$$f(\lambda) := \sum_{j=-\infty}^{\infty} \gamma(j) \exp(ij\lambda) \tag{5}$$

where $\gamma(\cdot)$ represent the autocovariance function. It can be shown that the spectral density of fractional Gaussian noise is given by

$$f(\lambda) = 2\sin(\pi H)\Gamma(2H+1)(1-\cos\lambda)[|\lambda|^{-2H-1} + B(\lambda, H)] \tag{6}$$

where $\Gamma(\cdot)$ denotes the Gamma function and

$$B(\lambda, H) := \sum_{j=1}^{\infty} \left((2\pi j + \lambda)^{-2H-1} + (2\pi j - \lambda)^{-2H-1}\right) \tag{7}$$

for $-\pi \leq \lambda \leq \pi$. The infinite sum makes direct numerical evaluation almost impossible. However, a useful approximation of the sum is made by (Paxson 12). They show that by using

$$\tilde{B}_3(\lambda, H) :=$$

$$\sum_{j=1}^{3}\left(\left(a_j^+\right)^{-2H-1} + \left(a_j^-\right)^{-2H-1}\right) + \frac{\left(a_3^+\right)^{-2H} + \left(a_3^-\right)^{-2H} + \left(a_4^+\right)^{-2H} + \left(a_4^-\right)^{-2H}}{8H\pi}$$

where $a_j^{\pm} = 2\pi j \pm \lambda$, $f(\lambda)$ is approximated quite well.

Now, consider a stationary Gaussian discrete-time process $X = \{X_n : n = 0, ..., N-1\}$ where N is the required sample size. The spectral theorem states that it can be represented in terms of the spectral density as

$$X_n \stackrel{d}{=} \int_0^{\pi} \sqrt{\frac{f(\lambda)}{\pi}} \cos(n\lambda) dB_1(\lambda) - \int_0^{\pi} \sqrt{\frac{f(\lambda)}{\pi}} \sin(n\lambda) dB_2(\lambda) \tag{8}$$

where B_1 and B_2 are two mutually independent Brownian motions (specsim). We wish to approximate (8). The integrand is replaced by a simpler function. Fix some integer l and set $t_k = \pi k/l$ for $k = 0, ..., l$. Now, define a simple function $\xi_n^{(l)}$ on $[0, \pi]$ for $0 \leq n \leq N-1$ by

$$\xi_n^{(\ell)}(\lambda) = \sqrt{\frac{f(t_1)}{\pi}} \cos(nt_1) \mathbf{1}_{\{0\}}(\lambda) + \sum_{k=0}^{l-1} \sqrt{\frac{f(t_{k+1})}{\pi}} \cos(nt_{k+1}) \mathbf{1}_{(t_k, t_{k+1}]}(\lambda). \tag{9}$$

The first integral in (8) can be approximated by $\int_0^{\pi} \xi_n^{(\ell)}(\lambda) dB_1(\lambda)$. Since $\xi_n^{(\ell)}$ is a simple function the stochastic integral can be computed as

$$\int_0^{\pi} \xi_n^{(l)}(\lambda) dB_1(\lambda) = \sum_{j=0}^{\ell-1} \xi_n^{(l)}(B(t_{j+1}) - B(t_j)). \tag{10}$$

Thus, we obtain

$$\int_0^{\pi} \xi_n^{(l)}(\lambda) dB_1(\lambda) = \sum_{j=0}^{\ell-1} \sum_{k=0}^{\ell-1} \sqrt{\frac{f(t_{k+1})}{\pi}} \cos(nt_{k+1}) \mathbf{1}_{(t_k, t_{k+1}]}(B_1(t_{j+1}) - B_1(t_j))$$

$$= \sum_{k=0}^{l-1} \sqrt{\frac{f(t_{k+1})}{\pi}} \cos(nt_{k+1})(B_1(t_{k+1}) - B_1(t_k))$$

$$= \sum_{k=0}^{\ell-1} \sqrt{\frac{f(t_{k+1})}{\pi}} \cos(nt_{k+1}) U_k^{(0)} \sqrt{\frac{\pi}{l}} = \sum_{k=0}^{\ell-1} \sqrt{\frac{f(t_{k+1})}{\ell}} \cos(nt_{k+1}) U_k^{(0)}$$

where $U_k^{(0)}$ is an i.i.d. standard normal random variable for $k = 0, ..., \ell - 1$. The $U_k^{(0)} \sqrt{\pi/\ell}$ represent the Brownian motion increments which per definition are normally distributed with mean 0 and variance $t_{k+1} - t_k$.

The second integral in (8) can be approximated in a similar way by replacing the cosine terms with sine terms. Thus, we obtain the following approximation of X_n:

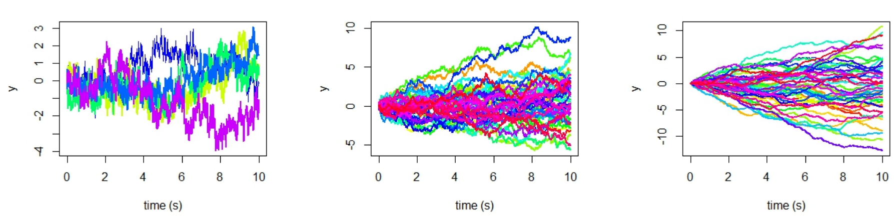
$$\hat{X}_n^{(\ell)} := \sum_{k=0}^{\ell-1} \sqrt{\frac{f(t_{k+1})}{\ell}} \left(\cos(n t_{k+1}) U_k^{(0)} - \sin(n t_{k+1}) U_k^{(1)} \right). \tag{11}$$

The two vectors $U^{(0)}$ and $U^{(1)}$ are mutually independent since B_1 and B_2 are independent as well. In order to calculate $\hat{X}_n^{(\ell)}$ efficiently we will be using the fast Fourier transform (FFT). To this end, we define the sequence $(a_k)_{k=0,\ldots,2\ell-1}$ by

$$a_k := \begin{cases} 0 & k = 0; \\ \frac{1}{2}\left(U_{k-1}^{(0)} + i U_{k-1}^{(1)} \right) \sqrt{\frac{f(t_k)}{\ell}} & k = 1, \ldots, \ell - 1; \\ U_{k-1}^{(0)} \sqrt{\frac{f(t_k)}{\ell}} & k = \ell; \\ \frac{1}{2}\left(U_{2\ell-k-1}^{(0)} - i U_{2\ell-k-1}^{(1)} \right) \sqrt{\frac{f(t_{2\ell-k})}{\ell}} & k = \ell + 1, \ldots, 2\ell - 1. \end{cases}$$

It is shown in appendix that the Fourier transform of a_k is indeed real and equals $\hat{X}_n^{(\ell)}$. From this approximated fractional Gaussian noise we can generate the fBm.

(Specsim) shows that the finite-dimensional distributions of $\hat{X}^{(\ell)}$ converge in probability to the corresponding finite-dimensional distributions of X as $\ell \to \infty$. The rate of convergence is, however, quite slow. Therefore, we will be using a $\ell \geq 30000$ when simulating from the fBm by the spectral method in this thesis in order to make sure that our simulations are reliable. Figure 1 shows three simulations of fractional Brownian motions with different Hurst parameters generated by the spectral method.



(a) H=0.2 (b) H=0.5 (c) H=0.8

Figure 1: Fractional Brownian motions generated by the spectral method.

3 Roughness of a path

Determining the roughness of realized volatility plays a crucial role in order to make proper model specification and design estimators. We will be working with high frequency data. We observe only a singe price path, and we need to measure the roughness of this path. We will closely follow the roughness indexes described in (roughvol) amd (vol is rough).