

## Convolution of Two Causal Exponential Sequences

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**Case #1.**  $y[n] = x_1[n] * x_2[n]$  where  $x_1[n] = a^n u[n]$  and  $x_2[n] = b^n u[n]$  and  $a \neq b$ .

$$y[n] = (a^n u[n]) * (b^n u[n])$$

$$y[n] = \sum_{m=-\infty}^{\infty} (a^m u[m]) (b^{n-m} u[n-m])$$

For  $u[m]$  to be 1,  $m \geq 0$ . For  $u[n-m]$  to be 1,  $n-m \geq 0$  or equivalently  $m \leq n$ .

For  $n \geq 0$ , the limits of summation are  $0 \leq m \leq n$ .

$$y[n] = \sum_{m=0}^n a^m b^{n-m} \quad \text{for } n \geq 0$$

The term of  $n$  inside the summation does not depend on  $m$  and can be pulled out.

$$y[n] = b^n \sum_{m=0}^n \left(\frac{a}{b}\right)^m \quad \text{for } n \geq 0$$

We can simplify the above summation using an identity on page 887 in Roberts' *Signals and Systems* book:

$$y[n] = b^n \frac{1 - \left(\frac{a}{b}\right)^{n+1}}{1 - \frac{a}{b}} \quad \text{for } n \geq 0$$

We can multiply numerator and denominator by  $b$ , and multiply the  $b^n$  term through numerator to obtain the following result:

$$y[n] = \frac{b^{n+1} - a^{n+1}}{b - a} u[n]$$

**Case #2.**  $y[n] = x_1[n] * x_2[n]$  where  $x_1[n] = b^n u[n]$  and  $x_2[n] = b^n u[n]$ .

The first four steps are the same as in the above. We can then substitute  $a = b$ :

$$y[n] = b^n \sum_{m=0}^n 1^m \quad \text{for } n \geq 0$$

With  $1^m = 1$ , we are summing 1 for  $(n+1)$  times when  $n \geq 0$ , which gives us

$$y[n] = (n+1)b^n u[n]$$