Differential Drive Robot's Equations Of Motion

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1 Roadmap

Let's derive equations relating a desired twist of the robot's body frame to each wheel's angular velocities, $\dot{\phi_L}$ and $\dot{\phi_R}$. We would like to know how to take a desired twist of the body frame and compute velocities for each wheel. We also want to be able to take an arbitrary pair of wheel velocities and compute the resulting twist of the body frame. A good guide to twists, adjoints, and other screw theory used in these calculations can be found in chapter 3 of Modern Robotics. There are two main steps to doing this derivation.

1) Use the adjoint to map the desired twist of the robot's body frame to the corresponding twist of each wheel frame. So we have a twist of each wheel frame expressed in terms of the variables that make up the body frame twist.

2) We now need to constrain the motion by two assumptions. We will assume the robot's wheels will not slip and not slide. This allows us to relate $\dot{\phi_L}$ and $\dot{\phi_R}$ to the robot's body frame twist.

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Let r = wheel's radius
d = 0.5 the distance between each wheel
ϕ be the body's angle in the world frame
x and y be the body frame's center
x_l and y_l be the origin of the left wheel's frame
x_r and y_r be the origin of the right wheel's frame
ϕ_L and ϕ_R be each wheel's angle of rotation

The robot's body frame twist is then $\text{Twist}_{body} = \begin{bmatrix} \dot{\theta} \\ \dot{x} \\ \dot{y} \end{bmatrix}$

2 Derive Wheel Frame Twists

Write out the needed adjoint matrices

$$\text{Adjoint}_{left-wheel-body} = \begin{bmatrix} 1 & 0 & 0 \\ -d & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad Adjoint_{body-left-wheel} = \begin{bmatrix} 1 & 0 & 0 \\ d & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$Adjoint_{body-right-wheel} = \begin{bmatrix} 1 & 0 & 0 \\ -d & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad Adjoint_{right-wheel-body} = \begin{bmatrix} 1 & 0 & 0 \\ d & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Use the adjoint mappings to express how each wheel frame moves in terms of the body twist variables.

$$\text{Twist}_{left-wheel} = \begin{bmatrix} \dot{\theta_L} \\ \dot{x_L} \\ \dot{y_L} \end{bmatrix} = Adjoint_{left-wheel-body} * Twist_{body}$$

$$\begin{bmatrix} \dot{\theta_L} \\ \dot{x_L} \\ \dot{y_L} \end{bmatrix} = \begin{bmatrix} \dot{\theta} \\ -d\dot{\theta} + \dot{x} \\ \dot{y} \end{bmatrix}$$

So, we have an expression showing how the left wheel moves if the body frame follows the twist defined in terms of $\begin{bmatrix} \dot{\theta} \\ \dot{x} \\ \dot{y} \end{bmatrix}$

Now do the same for the right wheel to derive $\begin{bmatrix} \dot{\theta_R} \\ \dot{x_R} \\ \dot{y_R} \end{bmatrix} = \begin{bmatrix} \dot{\theta} \\ d\dot{\theta} + \dot{x} \\ \dot{y} \end{bmatrix}$

3 Constrain the Wheel's Motion

We first assume that the wheel does not slip. This means it will not rotate without also causing a proportional translation. Mathematically, $r\dot{\theta_L} = \dot{x_L}$ and $r\dot{\theta_R} = \dot{x_R}$. The wheel will slip a little in real life but this is a pretty reasonable assumption about how the wheels move.

We will also assume that the wheel cannot slide. This means that the wheel can only move in the direction in which it is oriented. The wheel will slide a little in real life depending on the surface it is on but this too is a pretty reasonable assumption. Mathematically, given how our frames are defined $\dot{y_L} = 0$ and $\dot{y_R}$ $= 0$.

Focusing in on the left wheel and using the assumptions from above, we know that $\dot{x_L}$ = r$\dot{\theta_L}$. We also know that $\dot{y_L} = 0$. This allows us to obtain the following expression for the left wheel's twist.

$$\begin{bmatrix} \dot{\theta_L} \\ \dot{x_L} \\ \dot{y_L} \end{bmatrix} = \begin{bmatrix} \dot{\theta} \\ r\dot{x_L} \\ \dot{y_L} \end{bmatrix} = \begin{bmatrix} \dot{\theta} \\ -d\dot{\theta} + \dot{x} \\ 0 \end{bmatrix}$$

Looking at the equations of the first two rows and some algebra leads us to

$\dot{\theta_L} = -d\dot{\theta}/r + \dot{x}/r$ or in matrix form,

$$\dot{\theta_L} = \begin{bmatrix} -d/r & 1/r & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta} \\ \dot{x} \\ \dot{y} \end{bmatrix}$$

Doing the same for the right wheel leads us to

$$\dot{\theta_R} = \begin{bmatrix} d/r & 1/r & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta} \\ \dot{x} \\ \dot{y} \end{bmatrix}$$

Expressing these two equations in matrix form creates

$$\begin{bmatrix} \dot{\theta_L} \\ \dot{\theta_R} \end{bmatrix} = \begin{bmatrix} -d/r & 1/r & 0 \\ d/r & 1/r & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta} \\ \dot{x} \\ \dot{y} \end{bmatrix}$$

Great! We have a mapping between the desired twist of the body frame and the wheel velocities. Given a twist of the body frame, we can easily compute what each wheel velocity should be. Now lets invert these equations so that given an arbitrary pair of wheel velocities, we can compute what the resulting body frame twist is. For convenience, let's relabel the system as A = BC. Since B is not square, we can't invert it. We will have to use the Moore-Penrose pseudo inverse. A little matrix algebra leads us to

$$B^{\dagger}\mathrm{A} = \mathrm{r} \begin{bmatrix} -0.5d & 0.5d \\ 0.5 & 0.5 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta_L} \\ \dot{\theta_R} \end{bmatrix} = \begin{bmatrix} \dot{\theta} \\ \dot{x} \\ \dot{y} \end{bmatrix}$$

There we go. We have derived expression for converting between a desired twist of the body frame and the wheel velocities.