University of Copenhagen

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Condensed Matter Theory

Coulomb Drag in Bilayer-Graphene

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Abstract

This report studies Coulomb Drag in Bilayer Graphene. The drag conductivity is derived using the Boltzmann Equation first for isotropic systems, and then generalised for non-isotropic systems. Thereafter the expression for the drag conductivity is applied to two layers of Graphene, which results in integrals that cannot be solved analytically and must instead be calculated numerically. The temperature dependence as well as the chemical potential dependence of the resulting drag resistivity is in agreement with the literature.

Contents

1 Introduction

In this report we study Coulomb Drag, which is an effect that can be measured in a system which consists of two sub-systems that are coupled weakly, for instance, the two layers of Graphene. Let us now send a current through one of the sub-systems and measure the induced electric field in the other. The ratio between these two quantities is the Coulomb Drag Resistivity.

For isotropic systems, such as Graphene in the low temperature limit (*i.e* in the limit where we can approximate the dispersion as linear), the Drag Resistivity is a scalar — the induced electric field is parallel to the applied current. However, for non-isotropic systems this does not hold in general.

We begin by using standard kinetic theory to obtain an expression for the drag resistivity in isotropic and non-isotropic systems. From there we focus specifically on low-temperature Graphene and perform numeric calculations to discover the behaviour of the resistivity as a function of temperature, frequency and momentum.

A lengthy Appendix is provided after the Bibliography where the details of the important calculations are given, so that they can be omitted from the main text, to provide for a more compact and uninterrupted read. Additionally the code for the numeric calculation is provided.

Coulomb Drag

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In order to obtain a general expression for the Coulomb Drag Resistivity we will use the Boltzmann Equation, first we will follow the standard procedure for isotropic systems, and thereafter generalise this for non-isotropic systems, which results in an additional contribution to the resistivity that is not seen in isotropic systems

2.1 Boltzmann Equation

In equilibrium the distribution function for fermions is the Fermi-Dirac distribution, f. However, when the equilibrium is disturbed by for instance interactions we need to modify the distribution function — we will denote the non-equilibrium distribution function by $\tilde{f}$. The way we determine the true distribution function is by using a continuity equation; the Boltzmann Equation (Bruus and Flensberg, 2016, pp. 271):

$$\frac{\mathrm{d}\tilde{f}}{\mathrm{d}t} = \boldsymbol{v}_{\boldsymbol{k}} \cdot \nabla_{\boldsymbol{r}} \tilde{f} + \dot{\boldsymbol{k}} \cdot \nabla_{\boldsymbol{k}} \tilde{f} + \frac{\partial \tilde{f}}{\partial t} \tag{2.1}$$

where $\boldsymbol{v}_{\boldsymbol{k}} = \nabla_{\boldsymbol{k}} \varepsilon_{\boldsymbol{k}} = \nabla_{\boldsymbol{k}} \xi_{\boldsymbol{k}}$. This describes the so-called *flow-force* part of the Boltzmann Equation; if we were to include collisions we would modify it to

$$\frac{\mathrm{d}\tilde{f}}{\mathrm{d}t} = \left(\frac{\partial \tilde{f}}{\partial t}\right)_{\mathrm{flow-force}} - \left(\frac{\partial \tilde{f}}{\partial t}\right)_{\mathrm{collisions}} \tag{2.2}$$

In steady state the flow-force and collision terms cancel each other, such that the Boltzmann Equation becomes

$$\left(\frac{\partial \tilde{f}}{\partial t}\right)_{\mathrm{collisions}} = \boldsymbol{v}_{\boldsymbol{k}} \cdot \nabla_{\boldsymbol{r}} \tilde{f} + \dot{\boldsymbol{k}} \cdot \nabla_{\boldsymbol{k}} \tilde{f} + \frac{\partial \tilde{f}}{\partial t} \tag{2.3}$$

By linearising[1] we obtain the *linearised Boltzmann Equation* (Jauho and Smith, 1993):

$$\left(\frac{\partial \tilde{f}}{\partial t}\right)_{\text{coll.}} = \dot{\boldsymbol{k}} \cdot \frac{\partial f}{\partial \boldsymbol{k}} \tag{2.4}$$

Semi-classically we can think of $\dot{\boldsymbol{k}}$ as a force acting on the electronic system:

$$\dot{\boldsymbol{k}} = \frac{e\nabla_{\boldsymbol{r}}\phi}{\hbar} = -\frac{e\boldsymbol{E}}{\hbar} \tag{2.5}$$

where the final equality naturally only holds in the case that the only force acting on the system is from an electric field.

The collision term is an integral, which tells us about the number of transitions per unit time: we multiply the transition rate, $w_{\text{f}\leftarrow\text{i}}$ by the availability of the electron/holes, $f_{\text{i}}(1 - f_{\text{f}})$:

$$\left(\frac{\partial \tilde{f}_{\boldsymbol{k}}}{\partial t}\right)_{\text{coll.}} = -\sum_{\boldsymbol{k}'}\left[\tilde{f}_{\boldsymbol{k}}(1-\tilde{f}_{\boldsymbol{k}'})w_{\boldsymbol{k}'\leftarrow\boldsymbol{k}} - \tilde{f}_{\boldsymbol{k}'}(1-\tilde{f}_{\boldsymbol{k}})w_{\boldsymbol{k}\leftarrow\boldsymbol{k}'}\right] \tag{2.6}$$

We can introduce a quantity that tells us about how much the true distribution function differs from the Fermi-Dirac distribution; the *deviation* (Jauho and Smith, 1993):

$$\tilde{f} - f = f(1-f)\psi \tag{2.7}$$

In the case that we drive a current through the system, due to an electric field $\boldsymbol{E} \parallel \hat{\boldsymbol{x}}$ this deviation function is (Hwang et al., 2011):

$$\psi = \frac{\tau e v_x E_x}{T} \tag{2.8}$$

where τ is the momentum relaxation time in the system.

2.2 Coupled Boltzmann Equations

If we have two independent electronic systems we can label their respective distribution functions as $\tilde{f}_a$ and $\tilde{f}_p$ (we will soon learn why a and p are suitable for differentiating between the systems). The two systems can be coupled by the collision term. This is the case in two Graphene layers, whose electronic

[1] *i.e* assuming that $\tilde{f}$ is close to f and that it is homogenous

systems are coupled by their Coulomb interaction. The (coupled & linearised) Boltzmann Equation hence reads

$$\dot{\boldsymbol{k}}_a \cdot \frac{\partial f^{(a)}}{\partial \boldsymbol{k}_a} = \left(\frac{\partial \tilde{f}_a}{\partial t}\right)_{\text{coll.}} = -\frac{\tilde{f}_a - f_a}{\tau} \tag{2.9}$$

$$\dot{\boldsymbol{k}}_p \cdot \frac{\partial f^{(p)}}{\partial \boldsymbol{k}_p} = \left(\frac{\partial \tilde{f}_p}{\partial t}\right)_{\text{coll.}} = -\frac{\tilde{f}_p - f_p}{\tau} + S[\psi^a, \psi^p = 0] \tag{2.10}$$

where the first term in the collision integral is from the *relaxation time approximation*[2] and we can adapt the form of our collision integral to account for two systems (Jauho and Smith, 1993; Hwang et al., 2011):

$$S = -\sum_{\sigma_a \sigma_{a'} \sigma_{a'}} \int \frac{\mathrm{d}^2 \boldsymbol{k}_a}{(2\pi)^2} \frac{\mathrm{d}^2 \boldsymbol{k}_{a'}}{(2\pi)^2} \, w_{a'p' \leftarrow ap} \, f^a f^p (1 - f^{a'})(1 - f^{p'}) \left[\psi_a + \psi_p - \psi_{a'} - \psi_{p'}\right] \times \delta(\varepsilon_a + \varepsilon_p - \varepsilon_{a'} - \varepsilon_{p'}) \tag{2.11}$$

due to momentum conservation it must be the case that $\boldsymbol{k}'_p = \boldsymbol{k}_a + \boldsymbol{k}_p - \boldsymbol{k}_{a'}$ which accounts for the fact that there is no $\boldsymbol{k}_{p'}$ integral above. We have assumed that $w_{a'p' \leftarrow ap} = w_{ap \leftarrow a'p'}$ which certainly holds according to Fermi's Golden Rule. We will write the transition rate as

$$w_{\boldsymbol{k}_a, \boldsymbol{k}_p, \boldsymbol{q}, \omega} = 2\pi \left|V_{\text{RPA}}(q, \omega)\right|^2 F_{\boldsymbol{k}_a, \boldsymbol{q}} F_{\boldsymbol{k}_p, -\boldsymbol{q}}, \qquad F_{\boldsymbol{k}, \boldsymbol{q}} \equiv \left|\psi^\dagger_{\boldsymbol{k}+\boldsymbol{q}} \psi_{\boldsymbol{k}}\right|^2 \tag{2.12}$$

where V_{RPA} is the RPA-screened interaction, and F is the wave-function overlap.

2.3 Drag Resistivity — Isotropic

Let us now apply an electric field to system-a parallel to the x-axis[3] — we assume that system-p is not affected *directly* by this field. This will drive a current through the active system, and the collision integral will couple the two systems causing a smaller electric field in the passive system. We assume that $|E_p| \ll |E_a|$, which implies that $\psi_p, \psi_{p'} \ll \psi_a, \psi_{a'}$.

The current through system ℓ is related to the (local) velocity of the system

$$\boldsymbol{j}_\ell = -e \sum_{s\boldsymbol{k}} \boldsymbol{v}^\ell_{s\boldsymbol{k}} \tilde{f}^\ell_{s\boldsymbol{k}} \tag{2.13}$$

[2] The relaxation time approximation assumes that any perturbation from the equilibrium distribution will die out exponentially with characteristic time τ.

[3] System-a is the *active* system, whereas p is the passive system

We define the drag-resistivity as the ratio between the driven current and the induced electric field:

$$\rho_D^{\alpha\beta} \equiv \frac{E_p^\alpha}{j_a^\beta} \tag{2.14}$$

Generally ρ_D is a tensor because we cannot be certain that the electric field in the passive system is parallel to the driven current, however, often the off-diagonal terms are neglected (Jauho and Smith, 1993; Hwang et al., 2011); especially in systems where we can assume isotropy and hence can assume the off-diagonal terms are zero.

By means of the relaxation-time-approximation Boltzmann Equation we can relate the passive system's current to the equilibrium distribution function and the inter-layer collision integral:

$$\boldsymbol{j}_p = -e\sum_{\boldsymbol{k}_p} \boldsymbol{v}_{\boldsymbol{k}_p} \tilde{f}_{\boldsymbol{k}_p} = -e\tau \sum_{\boldsymbol{k}} \boldsymbol{v}_{\boldsymbol{k}_p} \left(e\boldsymbol{E}_p \cdot \nabla_{\boldsymbol{k}_p} f_{\boldsymbol{k}_p} + S[\psi^a, \psi^p = 0] \right) \tag{2.15}$$

We used that the equilibrium current is equal to zero, *i.e* the term with $f_{\boldsymbol{k}_p}$ does not enter above — only its derivative. The active layer has a non-zero current, as this is the current that we are driving, but we do not short circuit the passive layer, which implies that there is no current flowing through the passive layer.

If the dispersion only depends on the norm of $\boldsymbol{k}$, see Appendix A.1, then the first terms above simplifies such that we only need to consider vectors parallel to $\boldsymbol{E}_p$;

$$j_p^\alpha = -e\tau_p \sum_{\boldsymbol{k}_p} \left[e(\boldsymbol{v}_{\boldsymbol{k}_p})^\alpha E_p^\alpha (\nabla_{k_\alpha} f_{\boldsymbol{k}_p})^\alpha + \boldsymbol{v}_{\boldsymbol{k}_p} S[\psi^a, \psi^p = 0] \right] \tag{2.16}$$

The first term relates the current in the passive layer to the induced electric field in the passive layer, therefore this term is due to the local conductivity of the layer, σ^p. However, the second term couples the two layers and hence this term is responsible for the Coulomb drag. S is proportional to E_a so the second term gives us an inter-layer Ohm's law.

We can express the Coulomb Drag Conductivity in terms of susceptibility functions, see Appendix A.2:

Coulomb Drag Conductivity

$$\sigma_D^{\alpha\beta} = \frac{1}{\pi T} \int \frac{\mathrm{d}^2\boldsymbol{q}}{(2\pi)^2} \int_0^\infty \mathrm{d}\omega \, \frac{|V(q,\omega)|^2}{\sinh^2(\omega/2T)} \mathrm{Im}\left[\boldsymbol{\Gamma}_p(-\boldsymbol{q}, -\omega)\right]^\alpha \mathrm{Im}\left[\boldsymbol{\Gamma}_a(\boldsymbol{q}, \omega)\right]^\beta \tag{2.17}$$

where the susceptibility vector of layer λ is defined as

$$\boldsymbol{\Gamma}_\lambda(\boldsymbol{q},\omega) = \frac{e\tau_\lambda}{2}\int \frac{\mathrm{d}^2\boldsymbol{k}}{(2\pi)^2}\,\frac{F_{\boldsymbol{k},\boldsymbol{q}}(\boldsymbol{v}_{\boldsymbol{k}}-\boldsymbol{v}_{\boldsymbol{k}+\boldsymbol{q}})(f_{\boldsymbol{k}}-f_{\boldsymbol{k}+\boldsymbol{q}})}{\omega+\varepsilon_{\boldsymbol{k}}-\varepsilon_{\boldsymbol{k}+\boldsymbol{q}}+i\delta} \tag{2.18}$$

It should be noted that a central assumption in the derivation above is that $|V(q,\omega)| = |V(-q,-\omega)|$, which is not generally true; specifically if V described a screened potential in a system with a non-isotropic dispersion.

Due to the fact that we are not short-circuiting the passive-layer there will be no current through it, thus the two contributions to the passive-layer current must cancel:

$$\sigma_p E_p^x = -\sigma_D^{xx} E_a^x, \qquad \sigma_p \equiv e^2\tau_p \sum_{s\boldsymbol{k}} (v_{s\boldsymbol{k}}^p)^x(-\nabla_{\boldsymbol{k}} f_{s\boldsymbol{k}}^p) \tag{2.19}$$

We are usually interested in the parallel drag resistivity:

$$\rho_D \equiv \rho_D^{xx} = \frac{E_p^x}{j_a^x} = -\frac{\sigma_D E_a^x}{\sigma_p}\frac{1}{j_a^x} = -\frac{\sigma_D}{\sigma_p}\frac{E_a^x}{\sigma_a E_a^x + \sigma_D E_p^x} = -\frac{\sigma_D}{\sigma_a\sigma_p - \sigma_D^2} \approx -\frac{\sigma_D}{\sigma_p\sigma_a} \tag{2.20}$$

where $\sigma_D \equiv \sigma_D^{xx}$. The σ_D^2 term in the denominator can be neglected because it is due to an additional induced current in the active layer due to a non-zero electric field in the passive layer; it is a second-order effect. The expression above is in agreement with (Hwang et al., 2011).

2.4 Drag Resistivity — Non-Isotropic

We would now like to extend our theory of the drag resistivity so that we can also describe systems that are non-isotropic. That is, if it is no longer the case that $\epsilon_{\boldsymbol{k}} = \epsilon_{|\boldsymbol{k}|}$ then the first term in Equation 2.16 becomes slightly more complicated, because E_p and ∇f obtain an index different from β which is to be summed over — because the dot product in Equation 2.15 no longer simplifies so that we obtain Equation 2.16. Additionally, non-isotropy will affect the screened potential, which will imply that we cannot assume that $|V(-q)| = |V(q)|$.

It is easiest to do this using tensorial notation. Let us denote the scattering rate from (a,p) to (a',p') as

$$R_{a,p;a',p'} = 2\pi g\left|w_{a,p;a',p'}\right|^2 \tilde{f}_a\tilde{f}_p(1-\tilde{f}'_a)(1-\tilde{f}'_p)\delta_{\boldsymbol{k}_a+\boldsymbol{k}_p,\boldsymbol{k}'_a+\boldsymbol{k}'_p}\times \\ \delta(\epsilon_a+\epsilon_p-\epsilon'_a-\epsilon'_p)\delta_{s_a s'_a}\delta_{s_p s'_p} \tag{2.21}$$

Splitting W up into a symmetric and an anti-symmetric part:

$$\left|w_{a,p;a',p'}\right|^2 = \left(\underbrace{\frac{\left|w_{a,p;a',p'}\right|^2 + \left|w_{a',p';a,p}\right|^2}{2}}_{\left|w_{\text{sym}}\right|^2} + \underbrace{\frac{\left|w_{a,p;a',p'}\right|^2 - \left|w_{a',p';a,p}\right|^2}{2}}_{\left|w_{\text{asym}}\right|^2} \right) \tag{2.22}$$

this splits up the drag resistivity into two parts, one from the symmetric part of w and one from the anti-symmetric part. We can then write the collision integral in terms of this scattering rate

$$S[\psi^a, \phi^p = 0](d) = \sum_{a,p,a',p'} R_{a,p;a',p'}(\delta_{a,d} - \delta_{a',d}) \tag{2.23}$$

and hence also the drag current:

$$\boldsymbol{j}_D = -e\tau \sum_{d,a,p,a',p'} \boldsymbol{v}_d \; R_{a,p;a',p'}(\delta_{p,d} - \delta_{p',d}) \tag{2.24}$$

Then we look at small displacements from the equilibrium distribution function as we previously did, neglect equilibrium current terms (because these will be zero) and we obtain an expression for the symmetric and anti-symmetric parts of the drag conductivity:

$$\sigma_D^{\alpha\beta} = \sigma_{D,\text{sym}}^{\alpha\beta} + \sigma_{D,\text{asym}}^{\alpha\beta} \tag{2.25}$$

where

$$\begin{aligned}
\sigma_{D,\text{sym}}^{\alpha\beta} &= \frac{(e\tau)^2 \pi g}{T} \sum_{apa'p'} \left|w_{\text{sym}}\right|^2 (v_p - v'_p)^\alpha (v_a - v'_a)^\beta f_a f_p (1 - f'_a)(1 - f'_p) \delta_{\text{en.,mom.,qu.}} \\
\sigma_{D,\text{asym}}^{\alpha\beta} &= \frac{(e\tau)^2 \pi g}{T} \sum_{apa'p'} \left|w_{\text{asym}}\right|^2 (v_p - v'_p)^\alpha \left(\tanh\left(\frac{\epsilon_a}{2T}\right) v_a + \tanh\left(\frac{\epsilon'_a}{2T}\right) v'_a \right)^\beta \times \\
&\qquad f_a f_p (1 - f'_a)(1 - f'_p) \delta_{\text{en.,mom.,qu.}}
\end{aligned} \tag{2.26}$$

and we have that

$$j_D^\alpha = \sum_\beta \sigma_D^{\alpha\beta} E_a^\beta \tag{2.27}$$

The symmetric part of the drag conductivity is just the isotropic conductivity, but the assumption of non-isotropy introduced an additional term, which we cannot write in terms of the non-linear drag susceptibilities we had previously.

We still have the Fermi-Dirac distributions that are multiplied with the energy-conservation Dirac-delta, so we still get one non-linear drag susceptibility — namely in the passive layer. However, in the active layer we obtain a modified susceptibility

$$\mathbf{\Lambda}_a(\boldsymbol{q},\omega) = \frac{e\tau_a}{2}\int \frac{\mathrm{d}^2\boldsymbol{k}}{(2\pi)^2}\,\frac{F_{\boldsymbol{k},\boldsymbol{q}}\left(\tanh\left(\frac{\beta\epsilon_{\boldsymbol{k}}}{2}\right)\boldsymbol{v}_{\boldsymbol{k}} + \tanh\left(\frac{\beta\epsilon_{\boldsymbol{k}+\boldsymbol{q}}}{2}\right)\boldsymbol{v}_{\boldsymbol{k}+\boldsymbol{q}}\right)(f_{\boldsymbol{k}} - f_{\boldsymbol{k}+\boldsymbol{q}})}{\omega + \varepsilon_{\boldsymbol{k}} - \varepsilon_{\boldsymbol{k}+\boldsymbol{q}} + i\delta} \tag{2.28}$$

and so

$$\sigma^{\alpha\beta}_{D,\mathrm{asym}} = \frac{1}{T}\int \frac{\mathrm{d}\boldsymbol{k}_a}{(2\pi)^3}\frac{\mathrm{d}\boldsymbol{k}_p}{(2\pi)^3}\frac{\mathrm{d}\boldsymbol{q}}{(2\pi)^3}\mathrm{d}\omega\;\frac{\left|V_{\mathrm{asym}}(q,\omega)\right|^2}{\sinh^2(\beta\omega/2)}\mathrm{Im}\left[\mathbf{\Gamma}_p(-\boldsymbol{q},-\omega)\right]^\alpha \mathrm{Im}\left[\mathbf{\Lambda}_a(,\boldsymbol{q},\omega)\right]^\beta \tag{2.29}$$

3 Graphene

3.1 General Remarks

The carbon-atoms in graphene reside on a honey-comb lattice:

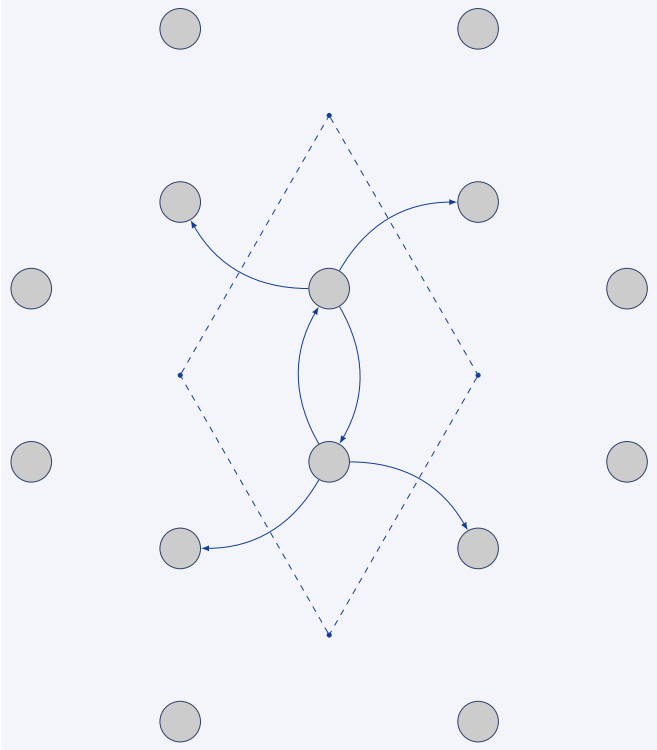
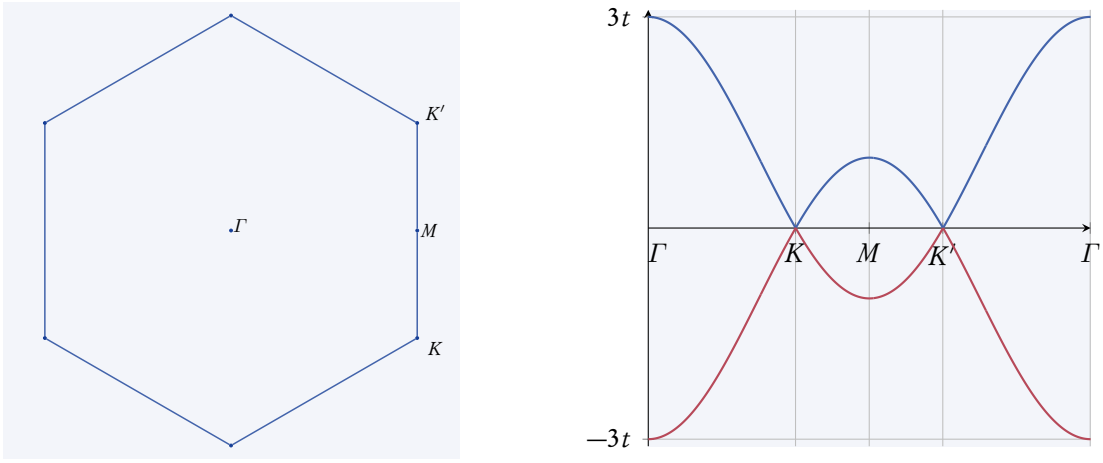


Fig. 3.1.: *An up-close figure of the Graphene honey-comb lattice, with nearest-neighbour hopping. The dashed lines denote the border of a primitive unit-cell.*

In the simplest model electrons can hop to nearest-neighbour sites, and the two hopping energies are equal due to the mirror-symmetry of the lattice. By finding the eigenvalues of the tight-binding Hamiltonian we obtain the energies

$$\varepsilon_{\pm} = \varepsilon_0 \pm |t| \sqrt{1 + 4\cos^2\left(\frac{ak_x}{2}\right) + 4\cos\left(\frac{ak_x}{2}\right)\cos\left(\frac{\sqrt{3}ak_y}{2}\right)} \tag{3.1}$$

where a is the shortest distance between alike[1] atoms, t is the hopping energy and ε_0 is the on-site energy. Below is the reciprocal unit cell as well as the dispersion plotted along a curve in momentum space



(a) Reciprocal Unit Cell (b) Dispersion

Fig. 3.2.: *Reciprocal Lattice (left) of Graphene with four special point in momentum space* Γ, K, M *and* K'. *Dispersion along a curve in momentum space showing the two Dirac Points at* K *and* K' *(right).*

The literature shows that $t \approx 3$eV, see Castro Neto et al., 2009, whereas room-temperature is ~ 30meV, so even at high temperatures we are very close to the Dirac Points, and hence we can linearise the dispersion relation, giving us

$$\epsilon_{K,K',\pm} = \pm v\,|\boldsymbol{k}| \tag{3.2}$$

where $\boldsymbol{k}$ is the momentum measured with respect to K or K'. This tells us that even at room temperature we can approximate the Graphene energy dispersion to be linear and *isotropic*, which is an important assumption in the forthcoming analysis.

3.2 Drag Resistivity

Now that we have obtained a general expression for the Coulomb Drag Resistivity, we can attempt to calculate this for a specific system — Graphene. However, the calculation is simplified by using

[1] there are two atoms in the unit cell

the expression for the Coulomb drag conductivity one obtains from a diagrammatic approach, see Narozhny et al., 2012; Kamenev and Oreg, 1995:

$$\sigma_D^{\alpha\beta} = \frac{1}{16\pi T}\sum_{\boldsymbol{q}}\int \frac{\mathrm{d}\omega}{\sinh^2\frac{\omega}{2T}}\Gamma_a^\beta(\omega,\boldsymbol{q})\Gamma_p^\alpha(\omega,\boldsymbol{q})\left|\mathcal{D}_{12}^R\right|^2 \tag{3.3}$$

where $\mathcal{D}_{12}^R$ is the retarded propagator of the inter-layer interaction and Γ is the imaginary part of the non-linear susceptibility. Where now

$$\begin{aligned}\boldsymbol{\Gamma} = \int \frac{\mathrm{d}\varepsilon}{2\pi}\Bigg[&\left(\tanh\left(\frac{\varepsilon-\mu}{2T}\right)-\tanh\left(\frac{\omega+\varepsilon-\mu}{2T}\right)\right)\boldsymbol{\gamma}_{12}(\varepsilon,\omega)\\ &+\left(\tanh\left(\frac{\varepsilon-\mu}{2T}\right)-\tanh\left(\frac{-\omega+\varepsilon-\mu}{2T}\right)\right)\boldsymbol{\gamma}_{21}(\varepsilon,-\omega)\Bigg]\end{aligned} \tag{3.4}$$

The subscripts 1 and 2 refer to the spatial coordinates $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ respectively and

$$\boldsymbol{\gamma}_{12}(\varepsilon,\omega) \equiv \int \mathrm{d}\boldsymbol{r}_3\ \left[G_{12}^R(\varepsilon+\omega)-G_{12}^A(\varepsilon+\omega)\right]G_{23}^R(\varepsilon)\hat{\boldsymbol{J}}_3 G_{31}^A(\varepsilon) \tag{3.5}$$

Where $\boldsymbol{J}_3$ is the current operator (which depends on $\boldsymbol{r}_3$).

> We assume to be in the *ballistic regime*, which implies that our impurity-scattering rate is small and hence
>
> $$G_s^R(\varepsilon,\boldsymbol{k})G_s^A(\varepsilon,\boldsymbol{k}) \approx 2\pi\tau_\varepsilon\delta(\varepsilon-\varepsilon_{s\boldsymbol{k}}) \tag{3.6}$$
> $$\mathrm{Im}G_s^R(\varepsilon,\boldsymbol{k}) \approx -\pi\delta(\varepsilon-\varepsilon_{s\boldsymbol{k}}) \tag{3.7}$$

In this limit we can show that Equations 3.3, 3.4 and 3.5 are equivalent to the Boltzmann Equation result in Equation 2.17, see Appendix A.3.

3.2.1 Non-Linear Drag Susceptibility

The full derivation can be found in Appendix A.4. In the ballistic regime we get that

$$\boldsymbol{\gamma}(\varepsilon,\omega,\boldsymbol{q}) \approx -Nev\tau_\varepsilon \sum_{s,s'=\pm1} s\int \mathrm{d}^2k\ \boldsymbol{n}_{\boldsymbol{k}}\left|\lambda_{\boldsymbol{k},\boldsymbol{k}+\boldsymbol{q}}^{ss'}\right|^2\delta(\varepsilon-svk)\delta(\varepsilon+\omega-s'v\left|\boldsymbol{k}+\boldsymbol{q}\right|) \tag{3.8}$$

where $\boldsymbol{n}_{\boldsymbol{k}} \equiv \frac{\boldsymbol{k}}{k}$, $N=4$ and

$$\left|\lambda_{\boldsymbol{k},\boldsymbol{k}'}^{ss'}\right|^2 \equiv \frac{1}{2}\left(1+ss'\frac{\boldsymbol{k}\cdot\boldsymbol{k}'}{kk'}\right) \tag{3.9}$$

is the inner product between two Graphene wave-functions at different momenta and bands (which are denoted by s and s'). v is the Fermi-velocity and τ_ε is the energy-dependent scattering time.

The Dirac-δ guarantee energy conservation and allow us to simplify the integrals significantly. Due to the isotropy of the system we can assume that $\boldsymbol{\gamma} \parallel \boldsymbol{q}$, and that

$$\begin{aligned} \boldsymbol{\gamma}(\varepsilon, \omega, \boldsymbol{q}) &= -ev\tau_\varepsilon \left[1 + \frac{\omega^2 - v^2q^2}{4\varepsilon(\varepsilon+\omega)}\right] g_0 \boldsymbol{q}, \\ g_0 &\equiv \sum_{ss'} s \int \mathrm{d}^2k \, \frac{\boldsymbol{k}\cdot\boldsymbol{q}}{k} \delta(\varepsilon - svk)\delta(\varepsilon + \omega - s'v\,|\boldsymbol{k}+\boldsymbol{q}|) \end{aligned} \tag{3.10}$$

We can calculate g_0 exactly by changing variables appropriately and we obtain

$$g_0 = \frac{4\,|\varepsilon+\omega|\,\theta_0(\varepsilon,\omega,q)}{\sqrt{(v^2q^2-\omega^2)\left([2\varepsilon+\omega]^2 - v^2q^2\right)}} \tag{3.11}$$

where θ_0 ensures that $|\varepsilon - vq| < s\,|\varepsilon+\omega| < |\varepsilon+vq|$, and can be written as

$$\begin{aligned} \theta_0(\varepsilon,\omega,q) = \theta(vq - |2\varepsilon+\omega|)[\theta(-\omega-vq) - \theta(\omega-vq)] \\ + \theta(vq-|\omega|)[\theta(2\varepsilon+\omega-vq) - \theta(-vq-2\varepsilon-\omega)] \end{aligned} \tag{3.12}$$

Now we can combine our results and use θ_0 to simplify the integration limits. We obtain:

Non-Linear Drag Susceptibility

$$\boldsymbol{\Gamma}(\omega,\boldsymbol{q}) = \frac{2e\tau\boldsymbol{q}}{\pi} \begin{cases} \sqrt{r^2-1}\displaystyle\int_0^1 \mathrm{d}z\, \frac{z\sqrt{1-z^2}}{z^2-r^2} I(z,W,Q,x) & \text{for } W > Q \\ -\sqrt{1-r^2}\displaystyle\int_1^\infty \mathrm{d}z\, \frac{z\sqrt{z^2-1}}{z^2-r^2} I(z,W,Q,x) & \text{for } W < Q \end{cases} \tag{3.13}$$

where

$$\begin{aligned} I(z,W,Q,x) \equiv \tanh\left[\frac{Qz-W-x}{2}\right] - \tanh\left[\frac{Qz+W-x}{2}\right] - \\ \tanh\left[\frac{Qz-W+x}{2}\right] + \tanh\left[\frac{Qz+W+x}{2}\right] \end{aligned} \tag{3.14}$$

and we have defined the dimensionless quantities

Dimensionless Quantities used in Equation 3.13

$$W = \frac{\omega}{2T}, \qquad Q = \frac{vq}{2T}, \qquad E = \frac{\varepsilon}{2T}, \qquad x = \frac{\mu}{T}, \qquad r = \frac{W}{Q} \tag{3.15}$$

Unfortunately the integral in Equation 3.13 cannot be solved analytically, so one must resort to numerical integration. Below we see the value of the integral from Equation 3.13:

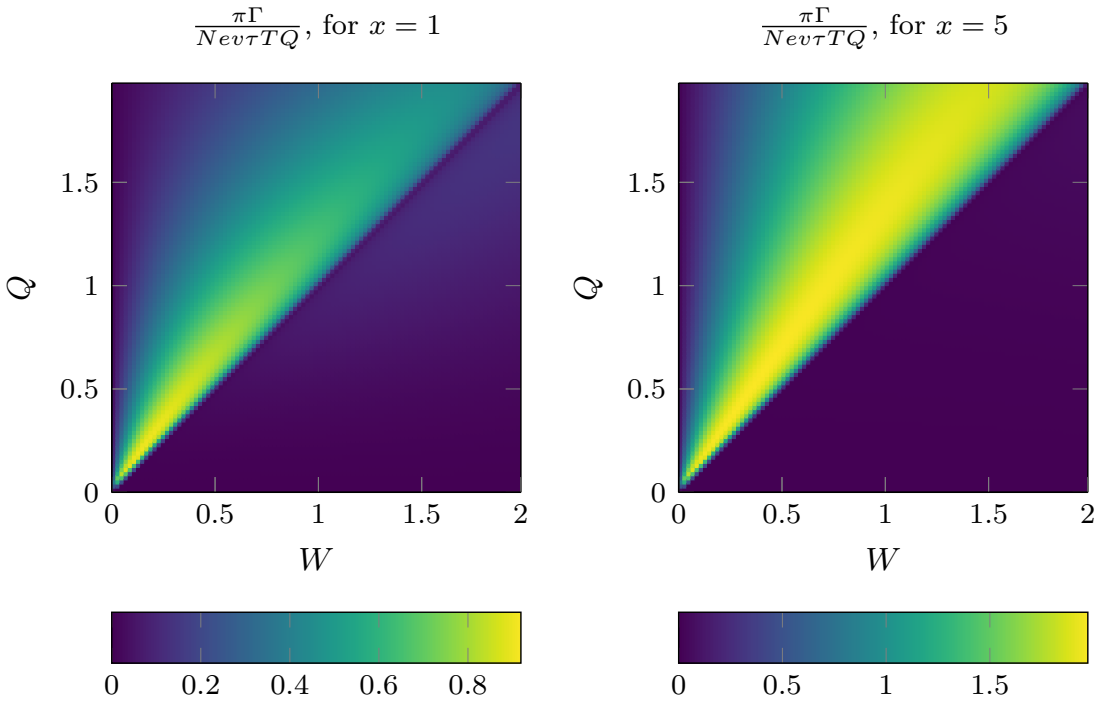


Fig. 3.3.: *The dimensionless integral from Equation 3.13 for two different values of μ.*

The line $Q = W$, which corresponds to $\omega = vq$ can be thought of a the linear dispersion of Graphene – it is only for $vq > \omega$ that the non-linear susceptibility is non-trivial (this is oversimplifying because the non-linear susceptibility is not exactly equal to zero below the line).

Below we plot the function $\frac{\pi\Gamma}{Nev\tau TQ}$ as a function of Q for different x:

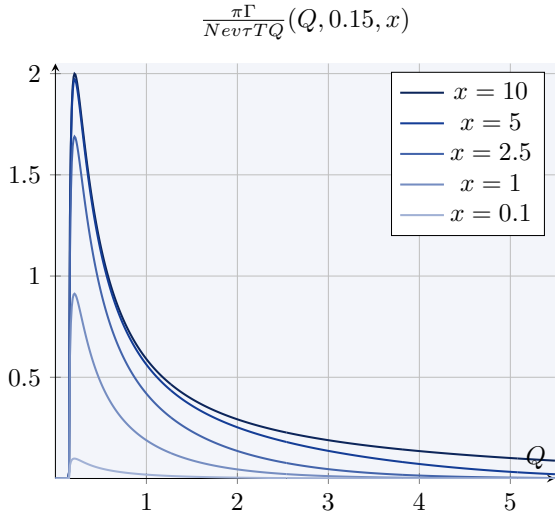


Fig. 3.4.: $\frac{\pi\Gamma}{Nev\tau TQ}(Q, 0.15, x)$ *as a function of* Q *with varying values of* x*. Compare to Fig. 6 in Narozhny et al., 2012.*

3.2.2 Drag Resistivity Numerics

At this stage we are ready to calculate the drag conductivity numerically. In the limit where the interlayer distance is small and $\alpha = e^2/v \ll 1$, which corresponds to the limit of weak interaction (Narozhny et al., 2012) we can approximate the interaction propagator as $\mathcal{D}^R = -\frac{2\pi e^2}{q}$ and hence obtain

$$\sigma_D(x_1, x_2) = \frac{4\alpha^2 e^2 T^2 \tau^2}{\pi^2} \iint_0^\infty \mathrm{d}Q\mathrm{d}W \, \frac{Qg(W, Q, x_1)g(W, Q, x_2)}{\sinh^2 W} \tag{3.16}$$

Let us look at the part of σ_D that is unitless:

$$f(x_1, x_2) = \frac{4}{\pi^2} \iint_0^\infty \mathrm{d}Q\mathrm{d}W \, \frac{Qg(W, Q, x_1)g(W, Q, x_2)}{\sinh^2 W} \tag{3.17}$$

and now consider the regular conductivity of Graphene, Narozhny et al., 2012:

$$\sigma_p = \sigma_a = e^2 T \tau h_0 \left(\frac{\mu}{T}\right), \qquad h_0(x) = \frac{2}{\pi} \int_{-\infty}^{\infty} \mathrm{d}z \, \frac{|z|}{\cosh^2\left[z + \frac{x}{2}\right]} \tag{3.18}$$

For identical layers we let $x = x_1 = x_2$ and define

$$r_0(x) = \frac{f_0(x, x)}{h_0(x)^2} \tag{3.19}$$

which is the unitless part of ρ_D for identical layers:

$$\rho_D(x) = \frac{\alpha^2}{e^2} r_0(x) \tag{3.20}$$

Below we show the dimensionless drag coefficient, r_0, as a function of x, compare to Figure 2 in Narozhny et al., 2012.

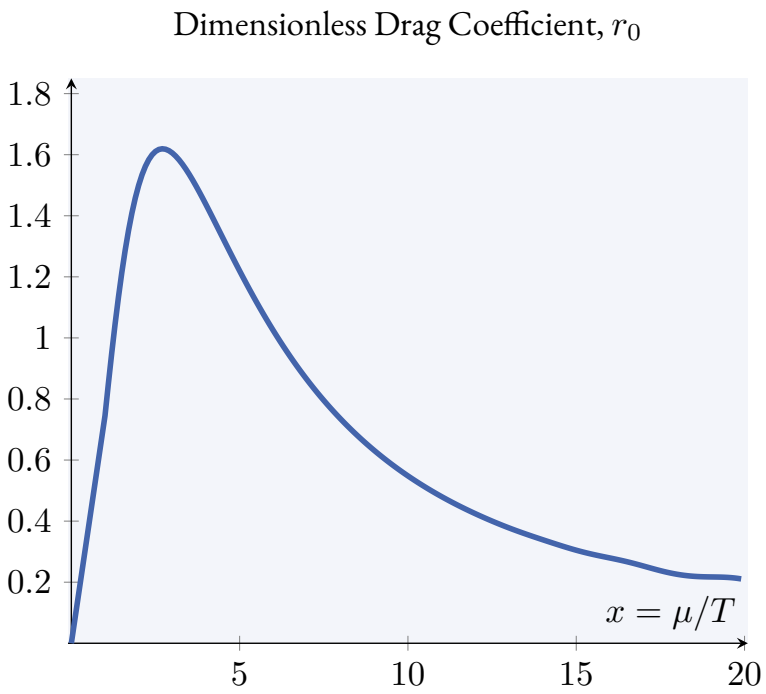


Fig. 3.5.: *Dimensionless drag coefficient r_0 as a function of chemical potential.*

The C++ code that makes the plot above can be found in Appendix A.5, it is a continuation of the code that plotted Figure 3.3— Γ is simply the function `G` in the C++ code.

Conclusion

4

This report was intended as an introduction to Coulomb Drag in isotropic systems. We derived an expression for the Drag Conductivity in isotropic systems using the Boltzmann Equation, which was shown to be equivalent to the diagrammatic result obtained for Graphene, which holds in the *ballistic regime*; the regime where the impurity-scattering rate is small. The non-linear drag susceptibility, which is a component in the integral that described the drag conductivity was evaluated for Graphene; the delta functions that ensure energy conservation could be performed by hand which saved computational power. However, as is the case in many many-body problems, the final result needed to be integrated numerically, which was done by means a of simple Riemannian sum in C++ (which ran faster than python). The resulting non-linear susceptibility and drag resistivities had the correct dependence on temperature and the chemical potential, in agreement with Narozhny et al., 2012.

Due to the additional contribution in the expression for the drag resistivity that appears for non-isotropic systems it would be interesting to study non-isotropic systems both from a theoretical and an experimental point of view.

Additionally the Riemannian sum which is performed numerically quite heavy, which implies an alternative method, such as Monte-Carlo integration may be appropriate while performing the calculations.

Bibliography

Bruus, Henrik and Karsten Flensberg (2016). *Many-Body Quantum Theory in Condensed Matter Physics. An Introduction*. Oxford University Press (cit. on p. 2).

Castro Neto, A. H., F. Guinea, N. M. R. Peres, K. S. Novoselov, and A. K. Geim (Jan. 2009). "The electronic properties of graphene". In: *Rev. Mod. Phys.* 81 (1), pp. 109–162 (cit. on p. 10).

Hwang, E., Rajdeep Sensarma, and Sankar Das Sarma (Nov. 2011). "Coulomb drag in monolayer and bilayer graphene". In: *Physical Review B* 84 (cit. on pp. 3–6).

Jauho, Antti-Pekka and Henrik Smith (Feb. 1993). "Coulomb Drag Between Parallel Two-Dimensional Electron Systems". In: *Phys. Rev. B* 47 (8), pp. 4420–4428 (cit. on pp. 3–5).

Kamenev, Alex and Yuval Oreg (Sept. 1995). "Coulomb drag in normal metals and superconductors: Diagrammatic approach". In: *Phys. Rev. B* 52 (10), pp. 7516–7527 (cit. on p. 11).

Narozhny, B. N., M. Titov, I. V. Gornyi, and P. M. Ostrovsky (May 2012). "Coulomb drag in graphene: Perturbation theory". In: *Phys. Rev. B* 85 (19), p. 195421 (cit. on pp. 11, 14–16).

Appendix

A

A.1 Interlayer Integral

We are interested in an integral of the form

$$\int \frac{\mathrm{d}^d \boldsymbol{k}}{(2\pi)^d} \boldsymbol{v}_{\boldsymbol{k}} (\boldsymbol{E} \cdot \nabla_{\boldsymbol{k}} f_{\boldsymbol{k}}) = \int \frac{\mathrm{d}^d \boldsymbol{k}}{(2\pi)^d} \boldsymbol{v}_{\boldsymbol{k}} (\boldsymbol{E} \cdot \boldsymbol{v}_{\boldsymbol{k}}) \frac{\partial f}{\partial \varepsilon_{\boldsymbol{k}}} \tag{A.1}$$

Now let us assume that $\varepsilon_{\boldsymbol{k}} = \varepsilon_{|\boldsymbol{k}|}$ in which case $\boldsymbol{v}_{\boldsymbol{k}} = \hat{\boldsymbol{k}} \varepsilon'_{|\boldsymbol{k}|}$. Let us fix the axes such that $\boldsymbol{E} \parallel \hat{\boldsymbol{x}}$, which means

$$\left(E_x \int \frac{\mathrm{d}^d \boldsymbol{k}}{(2\pi)^d} (\varepsilon'_{|\boldsymbol{k}|})^2 \hat{k}_\alpha \hat{k}_x \frac{\partial f}{\partial \varepsilon_{\boldsymbol{k}}} \right) \parallel \hat{\boldsymbol{x}} \tag{A.2}$$

The $\parallel \hat{\boldsymbol{x}}$ is due to the fact that everything is even in $\boldsymbol{k}$ except $\hat{\boldsymbol{k}}_\alpha$ and $\hat{\boldsymbol{k}}_x$ so the only nonzero component is when $\alpha = x$; the integral is parallel to the electric field.

A.2 Drag Conductivity

The interlayer collision integral can be written as

$$S[\psi_a, \psi_p = 0] = -2\pi \iint \frac{\mathrm{d}^2 \boldsymbol{k}_a}{(2\pi)^2} \frac{\mathrm{d}^2 \boldsymbol{q}}{(2\pi)^2} \int \mathrm{d}\omega \ |V(q,\omega)|^2 F_{\boldsymbol{k}_a, \boldsymbol{q}} F_{\boldsymbol{k}_p, -\boldsymbol{q}} f_{\boldsymbol{k}_a} (1 - f_{\boldsymbol{k}_a + \boldsymbol{q}}) f_{\boldsymbol{k}_p} (1 - f_{\boldsymbol{k}_p - \boldsymbol{q}}) \times \left(\psi_{\boldsymbol{k}_a} - \psi_{\boldsymbol{k}'_a} \right) \delta(\varepsilon_{\boldsymbol{k}_a} + \varepsilon_{\boldsymbol{k}_p} - \varepsilon_{\boldsymbol{k}_a + \boldsymbol{q}} - \varepsilon_{\boldsymbol{k}_p - \boldsymbol{q}}) \tag{A.3}$$

due to the ω integral we can rewrite the δ-function as

$$\int \mathrm{d}\omega \ g(\omega) \delta(\varepsilon_{\boldsymbol{k}_a} + \varepsilon_{\boldsymbol{k}_p} - \varepsilon_{\boldsymbol{k}'_a} - \varepsilon_{\boldsymbol{k}'_p}) = \int \mathrm{d}\omega \ g(\omega) \delta(\omega + \varepsilon_{\boldsymbol{k}_a} - \varepsilon_{\boldsymbol{k}'_a}) \delta(\omega - \varepsilon_{\boldsymbol{k}_p} + \varepsilon_{\boldsymbol{k}'_p}) \tag{A.4}$$

Additionally the δ-functions ensure that $\Delta\varepsilon_a = -\omega = -\Delta\varepsilon_p$ which implies that we can rewrite

$$f_{\boldsymbol{k}_a} (1 - f_{\boldsymbol{k}_a + \boldsymbol{q}}) = (f_{\boldsymbol{k}_a} - f_{\boldsymbol{k}_a + \boldsymbol{q}}) n_{\mathrm{B}}(\omega), \qquad f_{\boldsymbol{k}_p} (1 - f_{\boldsymbol{k}_p - \boldsymbol{q}}) = (f_{\boldsymbol{k}_p} - f_{\boldsymbol{k}_p - \boldsymbol{q}}) n_{\mathrm{B}}(-\omega) \tag{A.5}$$

But the Bose-Einstein distributions can be written as a hyperbolic sine squared:

$$n_{\mathrm{B}}(\omega)n_{\mathrm{B}}(-\omega) = \frac{1}{-4\sinh^2(\omega/2T)} \tag{A.6}$$

giving us

$$S[\psi_a, \psi_p = 0] = \frac{2\pi e\tau_a E_a^x}{T}\int \frac{\mathrm{d}^2\boldsymbol{k}_a}{(2\pi)^2}\frac{\mathrm{d}^2\boldsymbol{q}}{(2\pi)^2}\int \mathrm{d}\omega \frac{|V(q,\omega)|^2}{4\sinh^2(\omega/2T)} F_{\boldsymbol{k}_a,\boldsymbol{q}}F_{\boldsymbol{k}_p,-\boldsymbol{q}}(f_{\boldsymbol{k}_a} - f_{\boldsymbol{k}_a+\boldsymbol{q}})(f_{\boldsymbol{k}_p} - f_{\boldsymbol{k}_p-\boldsymbol{q}})\times \\ (\boldsymbol{v}_{\boldsymbol{k}_a} - \boldsymbol{v}_{\boldsymbol{k}_a+\boldsymbol{q}})^x\,\delta(\omega + \varepsilon_{\boldsymbol{k}_a} - \varepsilon_{\boldsymbol{k}_a+\boldsymbol{q}})\delta(\omega - \varepsilon_{\boldsymbol{k}_p} + \varepsilon_{\boldsymbol{k}_p-\boldsymbol{q}}) \tag{A.7}$$

where I have used that $\psi = \frac{e\tau \boldsymbol{v}\cdot\boldsymbol{E}}{T}$

We are interested in

$$\boldsymbol{j}_D = -e\tau_p \int \frac{\mathrm{d}^2\boldsymbol{k}_p}{(2\pi)^2}\boldsymbol{v}_{\boldsymbol{k}_p} S[\psi_a, \psi_p = 0] \tag{A.8}$$

which is the part of the current in the passive layer due to the electric field in the active layer. By writing this all out and using transforming the integrals such that we integrate over all momenta and include a delta function that ensures momentum conservation we can show that the α^{th} component of Equation A.8 can be written as

$$\boldsymbol{j}_D^\alpha = -\frac{2\pi e^2\tau_a\tau_p E_a^x}{T}\int \frac{\mathrm{d}^2\boldsymbol{k}_a}{(2\pi)^2}\frac{\mathrm{d}^2\boldsymbol{k}_p}{(2\pi)^2}\frac{\mathrm{d}^2\boldsymbol{q}}{(2\pi)^2}\int \mathrm{d}\omega \frac{|V(q,\omega)|^2}{4\sinh^2(\omega/2T)} F_{\boldsymbol{k}_a,\boldsymbol{q}}F_{\boldsymbol{k}_p,-\boldsymbol{q}}(f_{\boldsymbol{k}_a} - f_{\boldsymbol{k}_a+\boldsymbol{q}})(f_{\boldsymbol{k}_p} - f_{\boldsymbol{k}_p-\boldsymbol{q}})\times \\ \left(\boldsymbol{v}_{\boldsymbol{k}_p} - \boldsymbol{v}_{\boldsymbol{k}_p-\boldsymbol{q}}\right)^\alpha (\boldsymbol{v}_{\boldsymbol{k}_a} - \boldsymbol{v}_{\boldsymbol{k}_a+\boldsymbol{q}})^x\,\delta(\omega + \varepsilon_{\boldsymbol{k}_a} - \varepsilon_{\boldsymbol{k}_a+\boldsymbol{q}})\delta(\omega - \varepsilon_{\boldsymbol{k}_p} + \varepsilon_{\boldsymbol{k}_p-\boldsymbol{q}}) \tag{A.9}$$

generally, however, we are only interested in the x-component thereof. By defining

$$\gamma_\lambda(\boldsymbol{q},\omega) = \frac{e\tau_\lambda}{2}\int \frac{\mathrm{d}^2\boldsymbol{k}}{(2\pi)^2}\,\frac{F_{\boldsymbol{k},\boldsymbol{q}}(\boldsymbol{v}_{\boldsymbol{k}} - \boldsymbol{v}_{\boldsymbol{k}+\boldsymbol{q}})(f_{\boldsymbol{k}} - f_{\boldsymbol{k}+\boldsymbol{q}})}{\omega + \varepsilon_{\boldsymbol{k}} - \varepsilon_{\boldsymbol{k}+\boldsymbol{q}} + i\delta} \tag{A.10}$$

we can simplify things to

$$\boldsymbol{j}_D^x = \frac{E_a^x}{\pi T}\int \frac{\mathrm{d}^2\boldsymbol{q}}{(2\pi)^2}\int \mathrm{d}\omega\,\frac{|V(q,\omega)|^2}{\sinh^2(\omega/2T)}\mathrm{Im}\left[\vec{\chi}_p(-\boldsymbol{q},-\omega)\right]^x \mathrm{Im}\left[\vec{\chi}_a(\boldsymbol{q},\omega)\right]^x \tag{A.11}$$

A.3 From Boltzmann to the Diagrammatic Result

We wish to show that the two expressions for the non-linear drag susceptibility are the same. Let us begin with the expression obtained from the kinetic theory and work towards the result from the Diagrammatic Approach:

$$\mathbf{\Gamma}_\lambda(\boldsymbol{q},\omega) = \frac{e\tau_\lambda}{2}\sum_{\boldsymbol{k}} \frac{F_{\boldsymbol{k},\boldsymbol{q}}(\boldsymbol{v}_{\boldsymbol{k}}-\boldsymbol{v}_{\boldsymbol{k}+\boldsymbol{q}})(f_{\boldsymbol{k}}-f_{\boldsymbol{k}+\boldsymbol{q}})}{\omega+\varepsilon_{\boldsymbol{k}}-\varepsilon_{\boldsymbol{k}+\boldsymbol{q}}+i\delta} \tag{A.12}$$

We first notice that the Diagrammic result only has one $\boldsymbol{v}$ (which is in the $\boldsymbol{j}$), so let us take the imaginary part of $\mathbf{\Gamma}$ and recast the second $\boldsymbol{v}$-term to a second summation over $\boldsymbol{k}$:

$$\begin{aligned}\operatorname{Im}\mathbf{\Gamma}_\lambda = \frac{e\pi\tau_\lambda}{2}\Bigg[&\sum_{\boldsymbol{k}} F_{\boldsymbol{k},\boldsymbol{q}}(f_{\boldsymbol{k}}-f_{\boldsymbol{k}+\boldsymbol{q}})\boldsymbol{v}_{\boldsymbol{k}}\delta(\omega+\varepsilon_{s\boldsymbol{k}}-\varepsilon_{s'\boldsymbol{k}+\boldsymbol{q}}) \\ &-\sum_{\boldsymbol{k}} F_{\boldsymbol{k},\boldsymbol{q}}(f_{\boldsymbol{k}}-f_{\boldsymbol{k}+\boldsymbol{q}})\boldsymbol{v}_{\boldsymbol{k}+\boldsymbol{q}}\delta(\omega+\varepsilon_{s\boldsymbol{k}}-\varepsilon_{s'\boldsymbol{k}+\boldsymbol{q}})\Bigg]\end{aligned} \tag{A.13}$$

by letting $\boldsymbol{k} \rightsquigarrow -\boldsymbol{k}-\boldsymbol{q}$ (and $s \leftrightarrow s'$) we can make the two terms look more alike

$$\begin{aligned}\operatorname{Im}\mathbf{\Gamma}_\lambda = \frac{e\pi\tau_\lambda}{2}\Bigg[&\sum_{\boldsymbol{k}} F_{\boldsymbol{k},\boldsymbol{q}}(f_{\boldsymbol{k}}-f_{\boldsymbol{k}+\boldsymbol{q}})\boldsymbol{v}_{\boldsymbol{k}}\delta(\omega+\varepsilon_{s\boldsymbol{k}}-\varepsilon_{s'\boldsymbol{k}+\boldsymbol{q}}) \\ &+\sum_{\boldsymbol{k}} F_{-\boldsymbol{k}-\boldsymbol{q},\boldsymbol{q}}(f_{\boldsymbol{k}}-f_{\boldsymbol{k}+\boldsymbol{q}})\boldsymbol{v}_{\boldsymbol{k}}\delta(-\omega+\varepsilon_{s\boldsymbol{k}}-\varepsilon_{s'\boldsymbol{k}+\boldsymbol{q}})\Bigg]\end{aligned} \tag{A.14}$$

Which we can write as

$$\operatorname{Im}\mathbf{\Gamma}_\lambda = \frac{\pi\tau_\lambda}{2}\sum_{\boldsymbol{k}} \boldsymbol{j}_{\boldsymbol{k}}(f_{\boldsymbol{k}}-f_{\boldsymbol{k}+\boldsymbol{q}})\left[F_{\boldsymbol{k},\boldsymbol{q}}\delta(\omega+\epsilon_{s\boldsymbol{k}}-\epsilon_{s'\boldsymbol{k}+\boldsymbol{q}}) + F_{-\boldsymbol{k}-\boldsymbol{q},\boldsymbol{q}}\delta(-\omega+\epsilon_{s\boldsymbol{k}}-\epsilon_{s'\boldsymbol{k}+\boldsymbol{q}})\right] \tag{A.15}$$

Now, let us introduce a new variable, ϵ which we integrate over, such that we can split each of the δ-functions into two:

$$\int \mathrm{d}\epsilon\ \cdots\delta(\omega+\epsilon_{s\boldsymbol{k}}-\epsilon_{s'\boldsymbol{k}+\boldsymbol{q}}) = \int \mathrm{d}\epsilon\ \cdots\delta(\epsilon-\epsilon_{s\boldsymbol{k}})\delta(\epsilon+\omega-\epsilon_{s'\boldsymbol{k}+\boldsymbol{q}}) \tag{A.16}$$

which means we can write

$$\begin{aligned}\operatorname{Im}\mathbf{\Gamma}_\lambda = \frac{\pi\tau_\lambda}{2}\sum_{\boldsymbol{k}} \boldsymbol{j}_{\boldsymbol{k}}(f[\epsilon]-f[\epsilon+\omega])\Big[&F_{\boldsymbol{k},\boldsymbol{q}}\delta(\epsilon-\epsilon_{s\boldsymbol{k}})\delta(\epsilon+\omega-\epsilon_{s'\boldsymbol{k}+\boldsymbol{q}}) \\ &+ F_{-\boldsymbol{k}-\boldsymbol{q},\boldsymbol{q}}\delta(\epsilon-\epsilon_{s\boldsymbol{k}})\delta(\epsilon-\omega-\epsilon_{s'\boldsymbol{k}+\boldsymbol{q}})\Big]\end{aligned} \tag{A.17}$$

Now it would be quite handy to simplify the F-part:

$$F_{\boldsymbol{k},\boldsymbol{q}} \equiv \left|\psi^\dagger_{\boldsymbol{k}+\boldsymbol{q}}\psi_{\boldsymbol{k}}\right|^2 = \left|\psi^\dagger_{\boldsymbol{k}}\psi_{\boldsymbol{k}+\boldsymbol{q}}\right|^2 \overset{\text{Isotropy}}{=} \left|\psi^\dagger_{-\boldsymbol{k}}\psi_{-\boldsymbol{k}-\boldsymbol{q}}\right|^2 = F_{-\boldsymbol{k}-\boldsymbol{q},\boldsymbol{q}} \tag{A.18}$$

leaving

$$\text{Im }\boldsymbol{\Gamma}_\lambda = \frac{\pi\tau_\lambda}{2}\sum_{\boldsymbol{k}} \boldsymbol{j}_{\boldsymbol{k}} F_{\boldsymbol{k},\boldsymbol{q}}\delta(\epsilon-\epsilon_{s\boldsymbol{k}})\left[(f[\epsilon]-f[\epsilon+\omega])\delta(\epsilon+\omega-\epsilon_{s'\boldsymbol{k}+\boldsymbol{q}}) + (f[\epsilon]-f[\epsilon-\omega])\delta(\epsilon-\omega-\epsilon_{s'\boldsymbol{k}+\boldsymbol{q}})\right] \tag{A.19}$$

Now, notice how

$$f[\epsilon-\omega/2]-f[\epsilon+\omega/2] = \frac{\sinh(\omega/2T)}{\cosh(\epsilon/T)+\cosh(\omega/2T)} \tag{A.20}$$

and

$$\tanh\left(\frac{\epsilon-\omega/2}{2T}\right) - \tanh\left(\frac{\epsilon+\omega/2}{2T}\right) = -\frac{2\sinh(\omega/2T)}{\cosh(\epsilon/T)+\cosh(\omega/2T)} \tag{A.21}$$

and hence

$$f[\epsilon]-f[\epsilon+\omega] = -\frac{1}{2}\left[\tanh\left(\frac{\epsilon}{2T}\right) - \tanh\left(\frac{\epsilon+\omega}{2T}\right)\right] \tag{A.22}$$

and so

$$\begin{aligned}\text{Im }\boldsymbol{\Gamma}_\lambda = -\frac{\pi\tau_\lambda}{4}\int \mathrm{d}\epsilon \sum_{\boldsymbol{k}} \boldsymbol{j}_{\boldsymbol{k}} F_{\boldsymbol{k},\boldsymbol{q}}\delta(\epsilon-\epsilon_{s\boldsymbol{k}})\times \\ \left[\left(\tanh\frac{\epsilon}{2T}-\tanh\frac{\epsilon+\omega}{2T}\right)\delta(\epsilon+\omega-\epsilon_{s'\boldsymbol{k}+\boldsymbol{q}}) + \left(\tanh\frac{\epsilon}{2T}-\tanh\frac{\epsilon-\omega}{2T}\right)\delta(\epsilon-\omega-\epsilon_{s'\boldsymbol{k}+\boldsymbol{q}})\right]\end{aligned} \tag{A.23}$$

Let us now define $\boldsymbol{\gamma}(\epsilon;\omega,\boldsymbol{q})$:

$$\boldsymbol{\gamma}(\epsilon;\omega,\boldsymbol{q}) = -\tau\sum_{\boldsymbol{k}}\boldsymbol{j}_{\boldsymbol{k}}\sum_{ss'} sF^{ss'}_{\boldsymbol{k},\boldsymbol{q}}\delta(\epsilon-\epsilon_{s\boldsymbol{k}})\delta(\epsilon+\omega-\epsilon_{s'\boldsymbol{k}+\boldsymbol{q}}) \tag{A.24}$$

which simplifies things

$$\text{Im }\boldsymbol{\Gamma}_\lambda = \frac{\pi}{4}\int \mathrm{d}\epsilon\ \left[\left(\tanh\frac{\epsilon}{2T}-\tanh\frac{\epsilon+\omega}{2T}\right)\boldsymbol{\gamma}(\epsilon;\omega,\boldsymbol{q}) + \left(\tanh\frac{\epsilon}{2T}-\tanh\frac{\epsilon-\omega}{2T}\right)\boldsymbol{\gamma}(\epsilon;-\omega,-\boldsymbol{q})\right] \tag{A.25}$$

by moving the two factors of 4 from the definition of $\mathbf{\Gamma}$ to the definition of σ we obtain the factor of 16 in the denominator as expected, and the factor $\frac{1}{2\pi^2}$ difference between the above and Narozhny's result is due to the factors in the Green's functions. Additionally by including the chemical potential we obtain desired expression.

A.4 Graphene

Let us start with

$$\boldsymbol{\gamma}(\varepsilon,\omega,\boldsymbol{q}) \approx -Nev\tau_\varepsilon \sum_{ss'} s \int \mathrm{d}^2k\, \boldsymbol{n_k} \left|\lambda^{ss'}_{\boldsymbol{k},\boldsymbol{k}+\boldsymbol{q}}\right|^2 \delta(\varepsilon - svk)\delta(\varepsilon+\omega-s'v\left|\boldsymbol{k}+\boldsymbol{q}\right|) \tag{A.26}$$

because $s^2 = (s')^2 = 1$ the Dirac-deltas tells us that

$$k^2 = \frac{\varepsilon^2}{v^2}, \qquad \left|\boldsymbol{k}+\boldsymbol{q}\right|^2 = \frac{(\varepsilon+\omega)^2}{v^2} \tag{A.27}$$

and therefore

$$\boldsymbol{k}\cdot\boldsymbol{q} = \frac{1}{2}\left[\left|\boldsymbol{k}+\boldsymbol{q}\right|^2 - k^2 - q^2\right] = \frac{1}{2v^2}\left[(\varepsilon+\omega)^2 - \varepsilon^2 - v^2q^2\right] = \frac{\omega^2+2\omega\varepsilon - v^2q^2}{2v^2} \tag{A.28}$$

and hence we can rewrite λ:

$$\left|\lambda^{ss'}_{\boldsymbol{k},\boldsymbol{k}+\boldsymbol{q}}\right|^2 = \frac{1}{2}\left[1 + ss'\frac{\boldsymbol{k}\cdot(\boldsymbol{k}+\boldsymbol{q})}{k\left|\boldsymbol{k}+\boldsymbol{q}\right|}\right] = 1 + \frac{\omega^2 - v^2q^2}{4\varepsilon(\varepsilon+\omega)} \tag{A.29}$$

which implies that

$$\boldsymbol{\gamma}(\varepsilon,\omega,\boldsymbol{q}) = -Nev\tau_\varepsilon\left[1+\frac{\omega^2-v^2q^2}{4\varepsilon(\varepsilon+\omega)}\right]\boldsymbol{g}, \tag{A.30}$$

$$\boldsymbol{g} \equiv \sum_{ss'} s\int \mathrm{d}^2k\, \boldsymbol{n_k}\delta(\varepsilon-svk)\delta(\varepsilon+\omega-s'v\left|\boldsymbol{k}+\boldsymbol{q}\right|)$$

The angular part of the integral is trivial if we measure the angle of $\boldsymbol{k}$ w.r.t $\boldsymbol{q}$ in which case the angular integral cancels everything, leaving only a contribution parallel to $\boldsymbol{q}$. Thus it must be the case that

$$\boldsymbol{g} = A\boldsymbol{q}, \quad \Rightarrow \quad \boldsymbol{g} = \boldsymbol{q}\frac{\boldsymbol{q}\cdot\boldsymbol{g}}{q^2} \tag{A.31}$$

we have that

$$\boldsymbol{g}\cdot\boldsymbol{q} = \sum_{ss'} s\int \mathrm{d}^2k\, \frac{\boldsymbol{k}\cdot\boldsymbol{q}}{k}\delta(\varepsilon-svk)\delta(\varepsilon+\omega-s'v\left|\boldsymbol{k}+\boldsymbol{q}\right|) \tag{A.32}$$

once again using our expression for the dot product

$$\boldsymbol{g} = \boldsymbol{q}\left(\frac{\omega^2 + 2\varepsilon\omega - v^2q^2}{2v^2q^2}\right) g_0, \qquad g_0 \equiv \sum_{ss'} s \int \frac{\mathrm{d}^2k}{k} \delta(\varepsilon - svk)\delta(\varepsilon + \omega - s'v\,|\boldsymbol{k} + \boldsymbol{q}|) \tag{A.33}$$

let $\theta \to Q = |\boldsymbol{k} + \boldsymbol{q}| = \sqrt{k^2 + q^2 + 2kq\cos\theta}$ so

$$\theta = \arccos\frac{Q^2 - k^2 - q^2}{2kq}, \qquad \mathrm{d}\theta = -\frac{2Q\mathrm{d}Q}{\sqrt{4k^2q^2 - (k^2 + q^2 - Q^2)^2}} \tag{A.34}$$

so

$$\begin{aligned} g_0 &= \sum_{ss'} s \int_0^\infty \mathrm{d}k \int_{|\boldsymbol{k}-\boldsymbol{q}|}^{|\boldsymbol{k}+\boldsymbol{q}|} \mathrm{d}Q\, \frac{2Q\delta(\varepsilon - svk)\delta(\varepsilon + \omega - s'vQ)}{\sqrt{4k^2q^2 - (k^2 + q^2 - Q^2)^2}} \\ &= \sum_{s'} \int_{|\varepsilon/v - q|}^{|\varepsilon/v + q|} \mathrm{d}Q\, \frac{4Q\delta(\varepsilon + \omega - s'vQ)}{\sqrt{4\varepsilon^2q^2v^{-2} - (\varepsilon^2v^{-2} + q^2 - Q^2)^2}} \end{aligned} \tag{A.35}$$

So we need $|\varepsilon - vq| < |\varepsilon + \omega| < |\varepsilon + vq|$. Notice that when we set $Q = v^{-1}(\varepsilon + \omega)$ we get

$$\frac{4\,|\varepsilon + \omega|}{\sqrt{(v^2q^2 - \omega^2)\left([2\varepsilon + \omega]^2 - v^2q^2\right)}} \tag{A.36}$$

where I've included the $|\cdot|$ because Q is a positive quantity. We must now determine the conditions on ε, ω and q that are required for the Dirac-delta to be satisfied. It turns out that $|\varepsilon - vq| < \frac{|\varepsilon+\omega|}{s'} < |\varepsilon + vq|$ is equivalent to

$$\theta_0(\varepsilon, \omega, q) = \theta(vq - |2\varepsilon + \omega|)[\theta(-\omega - vq) - \theta(\omega - vq)] + \theta(vq - |\omega|)[\theta(2\varepsilon + \omega - vq) - \theta(-vq - 2\varepsilon - \omega)] \tag{A.37}$$

thus we get that

$$g_0 = \frac{4\,|\varepsilon + \omega|\,\theta_0(\varepsilon, \omega, q)}{\sqrt{(v^2q^2 - \omega^2)\left([2\varepsilon + \omega]^2 - v^2q^2\right)}} \tag{A.38}$$

Now we just need to put it all together:

$$\boldsymbol{\gamma} = -\boldsymbol{q}Nev\tau_\varepsilon\left[1 + \frac{\omega^2 - v^2q^2}{4\varepsilon(\varepsilon + \omega)}\right]\left(\frac{\omega^2 + 2\varepsilon\omega - v^2q^2}{2v^2q^2}\right)\frac{4\,|\varepsilon + \omega|\,\theta_0(\varepsilon, \omega, q)}{\sqrt{(v^2q^2 - \omega^2)\left([2\varepsilon + \omega]^2 - v^2q^2\right)}} \tag{A.39}$$

Which can be simplified

$$\boldsymbol{\gamma} = -\boldsymbol{q} N e v \tau_\varepsilon \sqrt{\frac{[2\varepsilon+\omega]^2 - v^2q^2}{v^2q^2-\omega^2}} \left(\frac{\omega^2 + 2\varepsilon\omega - v^2q^2}{2\varepsilon v^2 q^2}\right) \operatorname{sgn}(\varepsilon+\omega)\theta_0(\varepsilon,\omega,q) \tag{A.40}$$

Using that

$$\theta_0(\varepsilon,\omega,q) = -\theta_0(-\varepsilon,-\omega,q) \tag{A.41}$$

we get that

$$\boldsymbol{\gamma}(\varepsilon,\omega,\boldsymbol{q}) = \boldsymbol{\gamma}(-\varepsilon,-\omega,-\boldsymbol{q}) \tag{A.42}$$

which allows us to rewrite (the Fourier transform of) Equation 3.13 as

$$\boldsymbol{\Gamma}(\omega,q) = -\frac{Ne v \boldsymbol{q}}{2\pi} \int \mathrm{d}\varepsilon\ \tau_\varepsilon I(\varepsilon,\omega) F(\varepsilon,\omega,q)\theta_0(\varepsilon,\omega,q) \tag{A.43}$$

where

$$F(\varepsilon,\omega,q) \equiv \sqrt{\frac{[2\varepsilon+\omega]^2 - v^2q^2}{v^2q^2-\omega^2}} \left(\frac{\omega^2 + 2\varepsilon\omega - v^2q^2}{2\varepsilon v^2 q^2}\right) \operatorname{sgn}(\varepsilon+\omega), \tag{A.44}$$

$$I(\varepsilon,\omega) \equiv \tanh\frac{\varepsilon-\mu}{2T} - \tanh\frac{\varepsilon+\omega-\mu}{2T} - \tanh\frac{\varepsilon+\mu}{2T} + \tanh\frac{\varepsilon+\omega+\mu}{2T} \tag{A.45}$$

At $\mu = 0$ we get $I = 0$ which implies that the drag conductivity is zero at the Dirac Point — this is due to particle-hole symmetry

Now the goal is to convert this into a dimensionless form, so that we understand scales in our numeric calculation. A natural energy scale is $2T$, so by defining $W = \frac{\omega}{2T}$, $Q = \frac{vq}{2T}$, $E = \frac{\varepsilon}{2T}$ and $M = \frac{\mu}{2T}$ we obtain ($\mathrm{d}\varepsilon = 2T\mathrm{d}E$)

$$F = \sqrt{\frac{(2E+W)^2 - Q^2}{Q^2-W^2}} \left(\frac{W^2 + 2EW - Q^2}{4TEQ^2}\right) \operatorname{sgn}(E+W) \tag{A.46}$$

$$I = \tanh(E-M) - \tanh(E+W-M) - \tanh(E+M) + \tanh(E+W+M) \tag{A.47}$$

Now let us change variables once more: Let

$$z = \frac{1}{Q}(2E+W), \qquad \mathrm{d}z = \frac{2\mathrm{d}E}{Q} = \frac{2}{Q}\frac{\mathrm{d}\varepsilon}{2T} \to \mathrm{d}\varepsilon = TQ\mathrm{d}z \tag{A.48}$$

this gives us

$$TQF = \frac{(Q - Wz)\sqrt{\frac{z^2-1}{1-(W/Q)^2}}}{2(W - Qz)} = \frac{1}{2}\frac{(Q - Wz)(W + Qz)}{W^2 - Q^2z^2}\sqrt{\frac{z^2-1}{1-(W/Q)^2}} \\ = \frac{1}{2}\frac{WQ + (Q^2 - W^2)z + WQz^2}{W^2 - Q^2z^2}\sqrt{\frac{z^2-1}{1-(W/Q)^2}} \tag{A.49}$$

The integration in Equation A.43 goes over $\mathbb{R}$ and we have that I is even under $(\varepsilon, \omega) \rightsquigarrow (-\varepsilon, -\omega)$, and θ_0 is odd. Thus the only parts of F that survive are the parts that are odd in z:

$$TQF = \frac{1}{2}\frac{z\sqrt{(1-r^2)\,(z^2-1)}}{(r^2 - z^2)} \tag{A.50}$$

where we now only integrate z from zero to infinity. Additionally we get that

$$I = \tanh\left[\frac{Qz - W - x}{2}\right] - \tanh\left[\frac{Qz + W - x}{2}\right] \\ - \tanh\left[\frac{Qz - W + x}{2}\right] + \tanh\left[\frac{Qz + W + x}{2}\right] \tag{A.51}$$

where $x = 2M$. Now it just remains to see what θ_0 does to the integration limits. I find that if $z > 1$ we have that $\theta_0 = 1$ for $Q > |W|$ and for $z < 1$ I find that $\theta_0 = -\mathrm{sgn}(W)\theta(|W| - Q)$. But we take $W > 0$, so that means that we obtain

$$\mathbf{\Gamma}(\omega, \boldsymbol{q}) = \frac{2e\tau\boldsymbol{q}}{\pi}\begin{cases} \sqrt{r^2-1}\int_0^1 \mathrm{d}z\ \frac{z\sqrt{1-z^2}}{z^2-r^2} I(z, W, Q, x) & \text{for } W > Q \\ -\sqrt{1-r^2}\int_1^\infty \mathrm{d}z\ \frac{z\sqrt{z^2-1}}{z^2-r^2} I(z, W, Q, x) & \text{for } W < Q \end{cases} \tag{A.52}$$

A.5 Integrator

```
#include <iostream>
#include <fstream>
#include <math.h>
using namespace std;

double I(double z,double W,double Q,double x){ //I as defined in the report
  return tanh((z*Q + W + x)/2)-tanh((z*Q + W - x)/2)+
  tanh((z*Q - W - x)/2)-tanh((z*Q - W + x)/2);
}

double J(double z,double W,double Q,double x){ //the part of the integrand
  double ret = 0;          // that is not I
```

```
  if(abs(W) > Q){
    ret = sqrt(pow(W,2)/pow(Q,2)-1) * z*sqrt(1-pow(z,2))
    /(pow(z,2)-pow(W,2)/pow(Q,2));
  }
  if(abs(W) < Q){
    ret = -sqrt(1-pow(W,2)/pow(Q,2)) * z*sqrt(pow(z,2)-1)
    /(pow(z,2)-pow(W,2)/pow(Q,2));
  }
  return ret;
}

// if W>Q we just integrate z from 0 to 1
// if W<Q we change variables s.t. z-> 1/(1-t) and we need to multiply
// the measure by 1/(1-t)^2

double G(double W,double Q,double x,double N){
  double zmin = 0.00000000001; //NaN if we have [0,1] limits (1/(1-1))
  double zmax = 0.99999999999;
  double dz = (zmax-zmin)/N; //We are just doing a Riemann Sum
  double sum = 0;
  if(abs(W) > Q) //maybe replace with a switch ?
  {
    for(double z = zmin; z < zmax; z += dz)
    {
      sum+=I(z,W,Q,x)*J(z,W,Q,x)*dz;
    }
  }
  if(abs(W) < Q)
  {
    for(double z = zmin; z < zmax; z += dz)
    {
      sum+=I(1/(1-z),W,Q,x)*J(1/(1-z),W,Q,x)*dz/pow((1-z),2);
    }
  }
  if(abs(W) == Q)
  {
    return 0;
  }
  return sum;
}


double integrand(double x1,double Q,double W,double N){
  double s = sinh(W);
  double g = G(W,Q,x1,N);
  return (Q/(pow(s,2)))*pow(g,2); //integrand, notice the 1/sinh^2
```

```
}               // suppression for large W

double hintegrand(double z,double x){ //for the regular conductivity
  return abs(z)/(pow(cosh(z+x/2),2));
}

double h0(double x){ //regular conductivity
  double N = 100000;
  double dt = 1/N;
  double sum = 0;
  for(double t=0.000000001; t<0.9999999999;t+=dt){ //integrate from
                  //minus infty to plus infty
    sum += (hintegrand(t/(1-pow(t,2)),x)+hintegrand(-t/(1-pow(t,2)),x))
    *(1+pow(t,2))/(pow((1-pow(t,2)),2))*dt;
  }
  return sum;
}

double sigmaD(double x1,double N,double M){ //drag conductivity
  double dQ = 1; double dW = 1;

  double sum = 0;
  for(double q=0.000001;q<1000;q+=dQ){ //momentum cutoff...
    for(double w=0.000001;w<100;w+=dW){ //only up to W=100
      sum += integrand(x1,q,w,N)*dQ*dW; //due to sinh^2
    }                 //suppresion, see above
  }
  return sum;
}

int main(int argc, char *argv[]){
  ofstream myfile;
  myfile.open ("graphenecpp2.dat");

  myfile<< 0 << " " << 0 << endl;

  for(double x=1;x<20;x+=0.1){
    cout << x << endl;//just so you know how far and fast the code is.
    double s = sigmaD(x,100,100);
    double h = h0(x);
    myfile<< x << " " << s/pow(h,2) << endl;//save to file for plotting
  }
  myfile.close();
}
```

Colophon

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