UNIVERSITY OF COPENHAGEN
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NIELS BOHR INSTITUTE

Extracurricular project

De Fysiske Fag

Reflection and transmission of light

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Abstract

This report presents a theoretical investigation of the reflection of light, beginning with material that is taught at undergraduate level, and continuing on to an extended theory section, which presents a general solution to the differential equation that describes the movement of electrons being accelerated by electromagnetic waves, written as an infinite Fourier series. Furthermore the radiation due to these moving charges is calculated, from the inhomogeneous wave equation, and finally we present a macroscopic model of the imaginary part of the refractive index. Furthermore this report presents the frequency dependence of the $n\omega$ components of the reflected wave, measured at the source of the incident waves.

Contents

1 Introduction

Reflection and refraction are two phenomena that are very close to our everyday life, as we observe them constantly. However, even though we are so well acquainted with these phenomena, we bear little understanding of how they work, on a fundamental level. This project aims to shed some light on the underlying mechanisms behind the reflection and refraction of light. Over and above this, we will explore some finer details that arise from the theory that we commonly use to describe reflection and refraction qualitatively.

An interesting and yet counter intuitive implication from the theory that will be discussed in this report, is that over and above the common reflected and transmitted waves there is another transmitted wave. This wave travels parallel to the interface between to media, at double the frequency of the incident wave. In fact, there are additional waves at an odd number times the incident frequency, that travel with the common reflected wave, and waves at an even number times the incident frequency, that travel parallel to the interface. This can be seen schematically in Figure 1.1.

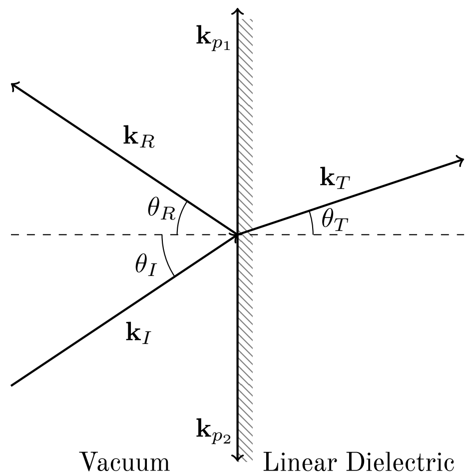

Figure 1.1: *Depiction of the plane of incidence of the transmission and reflection of electromagnetic waves at the barrier between two different media, as well as the double-frequency waves parallel to the interface.*

You may be wondering why we do not notice these effects in our everyday lives. This comes down to a fundamental quality of light; that the magnitude of the magnetic excitation in electromagnetic waves is a factor $c = 3 \times 10^8 \frac{\text{m}}{\text{s}}$ smaller than the electric excitation. Therefore the effects due to the magnetic excitation are insignificant compared to the effects caused by the electric excitation* – it so happens that these higher frequency transmission-effects are due to the magnetic force.

Another topic that will be discussed in this report is the refractive index, which is a constant of proportionality between the velocity at which electromagnetic waves propagate through media and through vacuum. For example the refractive index of air is $n \approx 1.0003$ (Ren et al. 2018), which means that electromagnetic waves propagate slightly slower through air than through vacuum. The refractive index however, also holds information about the angles of reflection and refraction. Additionally, if we let the index of refraction be a complex number, it holds information about how far electromagnetic waves penetrate metallic substances, before being reflected.

This report begins with a "Basic Theory" chapter, Chapter 2, where we will discuss the commonly used theories, that describe electromagnetic phenomena, such as, for example, Maxwell's equations, and the Drude-Lorentz model. This will be followed by Chapter 3, where we will investigate a model that is commonly used to describe the movement of electrons due to electromagnetic waves, and provide a solution to this, using a Fourier Series. Thereafter we will discuss a macroscopic model for the imaginary part of the refractive index. And finally, this report presents an investigation and discussion of the electromagnetic waves that are caused by the movement of electrons that are accelerated by the incident waves.

*assuming $v \ll c$, where v is the velocity of the particle being accelerated by the electric and magnetic fields.

2 Basic Theory

In this chapter we will discuss theories that are well known, and are commonly used for the description of electromagnetic phenomena. Beginning with Maxwell's Equations in vacuum and in linear media, and continuing to the Drude-Lorentz model, and its prediction of the index of refraction. Finally we will discuss the radiation due to arbitrarily accelerating charges.

2.1 Maxwell's Equations in Vacuum

The study of electromagnetic waves begins with Maxwell's equations (Griffiths 2017):

$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$	Gauss' Law
$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$	Faraday's Law
$\nabla \cdot \mathbf{B} = 0$	Divergence of $\mathbf{B}$
$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}$	Ampére's law

Table 2.1: *Maxwell's equations in Vacuum*

However, in the case of electromagnetic waves, there are neither charges, nor currents, thus reducing the equations to

$$\nabla \cdot \mathbf{E} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{2.1}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \tag{2.2}$$

And, applying the curl to Equations 2.1 and 2.2, and inserting, we yield

$$\nabla^2 \mathbf{E} = \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} \tag{2.3}$$

$$\nabla^2 \mathbf{B} = \frac{1}{c^2}\frac{\partial^2 \mathbf{B}}{\partial t^2} \tag{2.4}$$

Where I've used the following identity:

$$\nabla \times (\nabla \times \ ) = \nabla(\nabla \cdot \ ) - \nabla^2$$

Thus the speed of light is given by

$$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$$

Equations 2.3 and 2.4 are three dimensional wave equations. Given the three dimensional wave equation

$$\nabla^2 \varphi = \frac{1}{v^2}\frac{\partial^2 \varphi}{\partial t^2}$$

The general solution is given by

$$\varphi = f(\mathbf{k} \cdot \mathbf{r} \pm \omega t)$$

where

$$\omega = kv$$

and $\hat{\mathbf{k}}$ is the direction in which the electromagnetic waves propagates.

For the sake of simplicity we often study planar waves, which can be written as

$$\mathbf{E}(\mathbf{r}, t) = \tilde{\mathbf{E}} e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)} \tag{2.5}$$

$$\mathbf{B}(\mathbf{r}, t) = \tilde{\mathbf{B}} e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)} \tag{2.6}$$

Where $\tilde{\mathbf{E}}$ and $\tilde{\mathbf{B}}$ denote the complex amplitudes, and $\mathbf{k}$ denotes the direction of propagation. The physical electric and magnetic fields can be obtained by taking the real part of Equations 2.5 and 2.6

2.1.1 Coupled E- and B-fields

When we look at where we started, with Maxwell's Equations, it's clear that the $\mathbf{E}$- and $\mathbf{B}$ field are coupled, however it is not entirely clear how they are coupled.

We'll begin by looking at the solutions to the wave equation as in Equation 2.5 and 2.6. Firstly, both the $\mathbf{E}$- and $\mathbf{B}$-field must have zero divergence:

$$\nabla \cdot \mathbf{E} = \nabla \cdot \left(\tilde{\mathbf{E}} e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}\right)$$

$$= i(\mathbf{k} \cdot \tilde{\mathbf{E}}) e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}$$

$$\nabla \cdot \mathbf{B} = i(\mathbf{k} \cdot \tilde{\mathbf{B}}) e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}$$

As these are equal to zero, we can conclude that both $\tilde{\mathbf{E}}$ and $\tilde{\mathbf{B}}$ are orthogonal to $\mathbf{k}$. The solutions to the wave equation must also satisfy the remaining Maxwell's Equations, thus

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{B} = \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}$$

Here we'll need another identity:

$$\nabla \times (\varphi \mathbf{V}) = \varphi(\nabla \times \mathbf{V}) - \mathbf{V} \times (\nabla\varphi)$$

In this case $\mathbf{V}$ is uniform, which means that

$$\tilde{\mathbf{B}} \times \nabla e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} = \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}$$

this is satisfied when

$$i\mathbf{k} \times \tilde{\mathbf{B}} = \frac{i\omega}{c^2}\tilde{\mathbf{E}}$$

To continue we need to remember a very important property of light – the speed of light is constant, thus

$$c = \frac{\omega}{k}$$

which means that

$$\hat{\mathbf{k}} \times \tilde{\mathbf{B}} = \frac{1}{c}\tilde{\mathbf{E}}$$

This implies that the excitation in the $\mathbf{E}$- and $\mathbf{B}$-field always are orthogonal to each other (when dealing with electromagnetic waves), and that their amplitudes have a simple relation

$$B = \frac{E}{c}$$

2.2 Maxwell's Equations in linear media

In order to understand refraction and reflection of electromagnetic waves, we need to understand Maxwell's equations in linear media, and how they behave, when there are no free charges, and the only free currents are those induced by the oscillating electric field:

$$\nabla \cdot \mathbf{D} = 0$$
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{2.7}$$
$$\nabla \cdot \mathbf{B} = 0$$
$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J}_f \tag{2.8}$$

In linear media we know that:

$$\mathbf{D} = \varepsilon \mathbf{E}$$
$$\mathbf{H} = \frac{1}{\mu}\mathbf{B}$$

Classically we'd employ Ohm's Law here:

$$\mathbf{J} = \sigma \mathbf{E} \tag{2.9}$$

Inserting these and taking the curl of Equations (2.7) and (2.8):

$$\nabla^2 \mathbf{E} = \mu\varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} + \mu\sigma\frac{\partial \mathbf{E}}{\partial t}$$
$$\nabla^2 \mathbf{B} = \mu\varepsilon \frac{\partial^2 \mathbf{B}}{\partial t^2} + \mu\sigma\frac{\partial \mathbf{B}}{\partial t}$$

Note that these equations are very similar to the wave equations in a vacuum, with the difference being that the propagation velocity now is

$$v = \frac{1}{\sqrt{\mu\varepsilon}}$$

As $\mu > \mu_0$ and $\varepsilon > \varepsilon_0$ for most cases, the speed of light in media tends to be smaller than the speed of light in a vacuum. However at some frequencies, the speed of light could be greater than the speed of light in a vacuum, however this is only a narrow frequency interval, which means it still isn't possible for *information* to propagate at faster-than-light velocities. Additionally there is the first order temporal term in the new wave equation. Luckily planar waves still satisfy this partial differential equation, however $\tilde{\mathbf{k}}$ must be complex*:

$$\mathbf{E}(\mathbf{r}, t) = \tilde{\mathbf{E}} e^{i(\tilde{\mathbf{k}}\cdot\mathbf{r}-\omega t)}$$
$$\mathbf{B}(\mathbf{r}, t) = \tilde{\mathbf{B}} e^{i(\tilde{\mathbf{k}}\cdot\mathbf{r}-\omega t)}$$

Where

$$\tilde{k}^2 = \mu\varepsilon\omega^2 + i\mu\sigma\omega \tag{2.10}$$

Which implies that $\tilde{k}$ kan be written in terms of a real and an imaginary part:

$$\tilde{k} = k + i\chi$$

Which adds a decaying factor to the solutions†:

$$\mathbf{E}(\mathbf{r}, t) = \tilde{\mathbf{E}} e^{-\boldsymbol{\chi}\cdot\mathbf{r}} e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$$
$$\mathbf{B}(\mathbf{r}, t) = \tilde{\mathbf{B}} e^{-\boldsymbol{\chi}\cdot\mathbf{r}} e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$$

This is the absorption of light in media, as in Figure 2.1.

*This is analogous to the complex eigenvalues one gets for the damped harmonic oscillator.

†provided χ is positive, which is the case as $\tilde{k}^2$ lies in the first quadrant.

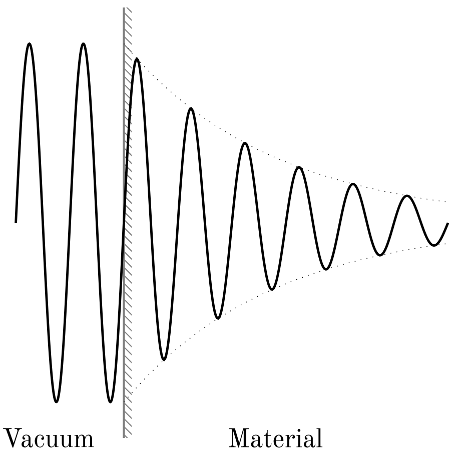

Figure 2.1: *Absorbtion of electromagnetic waves in conductive linear media.*

In materials like glass, where the electric conductivity is very low[‡], $\tilde{k}^2$ has a negligible imaginary part (for visible light), which means that light is barely absorbed by glass, which gives it its see-through appearance[§].

But let us look closer at Equation (2.10) (Sólyom 2009):

$$\tilde{k}^2 = \frac{\omega^2}{c^2}\left(\frac{\epsilon}{\epsilon_0} + i\frac{\sigma}{\epsilon_0 \omega}\right) \tag{2.11}$$

If these waves would have been propagating in vacuum, we would not have had the imaginary part in Equation (2.11). In that case we would have defined the index of refraction, n, to be

$$n^2 = \epsilon_r$$

But seeing as we have this extra imaginary term, we should define the complex index of refraction as:

$$n = \sqrt{\tilde{\epsilon}_r^b + i\frac{\sigma}{\epsilon_0 \omega}} \tag{2.12}$$

$\tilde{\epsilon}_r^b$ is the contribution of the bound charges, to the relative permitivity. The real part of the index of refraction is a measure for the velocity of propagation of electromagnetic waves in media: $v = \frac{c}{n}$, where v is the velocity inside the medium with index of refraction n. The imaginary part of the index of refraction is related to the skin depth, as we will see in Section 3.5. The free electrons in metals react to the electric and magnetic excitations in the field, and as per Lenz' law, will counteract the incoming electromagnetic wave, causing the magnitude of said wave to decrease exponentially – the skin depth is the depth at which the amplitudes of the electric and magnetic waves has decreased by a factor e.

However, Equation (2.12) isn't the whole story: σ is not just a constant, it is frequency dependent. To find the frequency dependence of the conductivity, we need to study the optical properties of free electrons. The forces on free electrons being accelerated by light are the electric force and the frictional force, due to collisions (Ashcroft and Mermin 1976):

$$\ddot{\mathbf{x}} = \frac{q\mathbf{E}(t)}{m} - \frac{\dot{\mathbf{x}}}{\tau} \tag{2.13}$$

where τ is the characteristic time between collisions of electrons with the surrounding grid, and is found to be between 10^{-15}s and 10^{-14}s at room temperature (Sólyom 2009). The characteristic time is a measure for how much friction the electron experiences over a period of time. As, if τ becomes smaller, the term proportional to $\dot{\mathbf{x}}$ in Equation 2.13 will increase, which means that the frictional term grows. This also makes physical sense, as if τ decreases, the electron will collide more often, and therewith be decelerated more often.

The characteristic time is usually used when one discusses direct current, and therefore some may argue against the use of this term in alternating current setups (such a cases with electromagnetic waves). If the drift velocity of electrons at room temperature is multiplied with τ we will find that in this time the electrons move a distance that is much smaller than the grid size. However, the electrons also have thermal energy in the form of thermal velocities. Often one calculates the root mean squared veloctу of electrons(Sólyom 2009), by treating them as an ideal gas:

$$v_{th} = \left(\frac{3k_B T}{m_E}\right)^{1/2}$$

which at room temperature is around $10^5\,\frac{\mathrm{m}}{\mathrm{s}}$, which makes the mean free path of electrons comparable to an Ångstrøm, and therewith to the gridsize.

The electric field oscillates with angular frequency ω. After all transient behaviour has passed, the solution to Equation (2.13) will be a harmonic oscillation:

$$\mathbf{v}(t) = \mathbf{v}_A(\omega)e^{-i\omega t}$$

[‡] For glass, the electric conductivity is between $\sigma = 10^{-9}\,\frac{\mathrm{S}}{\mathrm{m}}$ and $\sigma = 10^{-14}\,\frac{\mathrm{S}}{\mathrm{m}}$

[§] Naturally there are other factors at play, such as geometry – coal is not see-through though it effectively does not conduct electricity.

Which implies that

$$-i\omega \mathbf{v}_A e^{i\omega t} = \frac{q\mathbf{E}_0 e^{i\omega t}}{m} - \frac{\mathbf{v}_A e^{i\omega t}}{\tau}$$

which means that

$$\mathbf{v}_A(\omega) = \frac{\tau q \mathbf{E}}{m(1-i\omega\tau)}$$

This means the current density is

$$\mathbf{J}(\omega) = Nq\mathbf{v}_A(\omega) = \frac{Nq^2\tau\mathbf{E}}{m(1-i\omega\tau)}$$

This implies we can write the frequency dependent conductivity (AC-conductivity) as

$$\sigma(\omega) = \frac{Nq^2\tau}{m(1-i\omega\tau)} = \frac{\sigma_0}{1-i\omega\tau}$$

Where σ_0 is the Drude DC-conductivity. Thus the complex index of refraction as

$$n = \sqrt{\tilde{\epsilon}_r^b + i\frac{\sigma_0}{\epsilon_0\omega(1-i\omega\tau)}}$$

For now we can write the contribution of the free electrons to the relative permittivity as

$$\tilde{\epsilon}_r^f(\omega) = i\frac{\sigma_0}{\epsilon_0\omega(1-i\omega\tau)}$$

We will get back to the bound electrons contribution in Section 3.4.1.

2.3 Continuity

Due to the fact that the solution to partial differential equations is dependent of boundary conditions, we need to establish what said conditions are in our case. I have put them here, purely so that I can refer to them in later sections. The derivation of these boundary conditions goes beyond the scope of this project, but they can be obtained by writing Maxwell's equations in their integral form as in (Griffiths 2017):

$$\begin{aligned}
\left(\tilde{\mathbf{E}}_I + \tilde{\mathbf{E}}_R\right)_{x,y} &= \left(\tilde{\mathbf{E}}_T\right)_{x,y} \\
\frac{1}{\mu_1}\left(\tilde{\mathbf{B}}_I + \tilde{\mathbf{B}}_R\right)_{x,y} &= \frac{1}{\mu_2}\left(\tilde{\mathbf{B}}_T\right)_{x,y} \\
\varepsilon_1\left(\tilde{\mathbf{E}}_I + \tilde{\mathbf{E}}_R\right)_{z} &= \varepsilon_2\left(\tilde{\mathbf{E}}_T\right)_{z} \\
\left(\tilde{\mathbf{B}}_I + \tilde{\mathbf{B}}_R\right)_{z} &= \left(\tilde{\mathbf{B}}_T\right)_{z}
\end{aligned} \tag{2.14}$$

For an interface on the xy-plane, and planar electromagnetic waves travelling parallel to $\hat{\mathbf{z}}$.

2.4 Reflection and Transmission

In this section we will discuss the standard method of describing the reflected and transmitted waves o

Let us consider a light source far away from a sheet of glass (Griffiths 2017). The electromagnetic waves from the source approach the glass, and once they have entered the glass the charged particles within the glass – the electrons – are accelerated by the electric and magnetic fields. As will be discussed in Section 2.5, accelerating charged particles radiate. This radiation from the electron combines with the sources radiation, which is measured as a wave propagating at a lower velocity whilst inside the medium– the *transmitted* wave.

The radiation due to the acceleration of electrons is also sent back, towards the source and is what makes up the *reflected* wave.

Let us consider the following figure

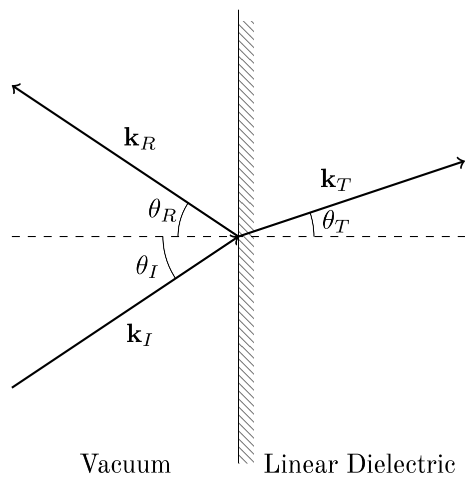

Figure 2.2: *Depiction of the plane of incidence of the transmission and reflection of electromagnetic waves at the barrier between two different media.*

We have now argued for the existence of both the reflected and the transmitted wave, now we will go into more details. The incoming wave, at an angle, θ_I, from the normal vector to the plane, will create a reflected wave as well as a transmitted wave:

$$\mathbf{E}_R = \tilde{\mathbf{E}}_R e^{i(\mathbf{k}_R\cdot\mathbf{r}-\omega t)}$$
$$\mathbf{E}_T = \tilde{\mathbf{E}}_T e^{i(\mathbf{k}_T\cdot\mathbf{r}-\omega t)}$$

For linear media it holds that the frequency of a wave remains unchanged, therefore

$$k_I v_I = k_R v_R = k_T v_T$$

Which implies

$$k_I = k_R = \frac{v_T}{v_I} k_T$$

Because the incident and reflected waves are in the same medium. Due to the previously mentioned continuity of the electromagnetic field, the following must be valid:

$$\begin{aligned}\mathcal{A}_1 \left(\tilde{\mathbf{E}}_I e^{i(\mathbf{k}_I\cdot\mathbf{r}-\omega t)} + \tilde{\mathbf{E}}_R e^{i(\mathbf{k}_R\cdot\mathbf{r}-\omega t)}\right)& \\ = \mathcal{A}_2 \left(\tilde{\mathbf{E}}_T e^{i(\mathbf{k}_T\cdot\mathbf{r}-\omega t)}\right)&\end{aligned}$$

$$\begin{aligned}\mathcal{B}_1 \left(\tilde{\mathbf{B}}_I e^{i(\mathbf{k}_I\cdot\mathbf{r}-\omega t)} + \tilde{\mathbf{B}}_R e^{i(\mathbf{k}_R\cdot\mathbf{r}-\omega t)}\right)& \\ = \mathcal{B}_2 \left(\tilde{\mathbf{B}}_T e^{i(\mathbf{k}_T\cdot\mathbf{r}-\omega t)}\right)&\end{aligned}$$

Where $\mathcal{A}_i$ and $\mathcal{B}_i$ are matrices determined by Equation (2.14). This must hold for all t, and seeing as the only temporal and spacial dependence comes from the exponential function, these must be equal too[¶], therefore

$$\mathbf{k}_I \cdot \mathbf{r} = \mathbf{k}_R \cdot \mathbf{r} = \mathbf{k}_T \cdot \mathbf{r}$$

This in turn implies that

$$k_I \sin\theta_I = k_R \sin\theta_R = k_T \sin\theta_T$$

And because $k_I = k_R$:

$$\theta_I = \theta_R$$

The angle of incidence is equal the angle of reflection, and

$$\frac{\sin\theta_T}{\sin\theta_I} = \frac{v_I}{v_T}$$

Snell's Law. Now it would be interesting to find an expression for how much of the electromagnetic waves is reflected, and how much of it is transmitted, that is, we would like to find the ratios between E_I, E_R and E_T. This too can be done using Equation (2.14). Let us once again suppose the interface lies on the xy-plane, and that the waves' polarisation lies within said plane. This then leads to following equalities

$$\begin{aligned}\tilde{E}_R &= \left(\frac{\alpha-\beta}{\alpha+\beta}\right)\tilde{E}_I \\ \tilde{E}_T &= \left(\frac{2}{\alpha+\beta}\right)\tilde{E}_I\end{aligned}$$

Where

$$\begin{aligned}\alpha &= \frac{\cos\theta_T}{\cos\theta_I} = \frac{\sqrt{1-\left(\frac{n_1\sin\theta_I}{n_2}\right)^2}}{\cos\theta_I} \\ \beta &= \frac{\mu_1 n_2}{\mu_2 n_1}\end{aligned}$$

From this result we can define the reflection, R, and transitission, T, coefficients[‖]:

$$\begin{aligned}R &= \left(\frac{\alpha-\beta}{\alpha+\beta}\right)^2 \\ T &= \alpha\beta\left(\frac{2}{\alpha+\beta}\right)^2\end{aligned} \tag{2.15}$$

A plot of these quantities as a function of the angle of incidence for the transition from air to glass can be seen in Figure 2.3

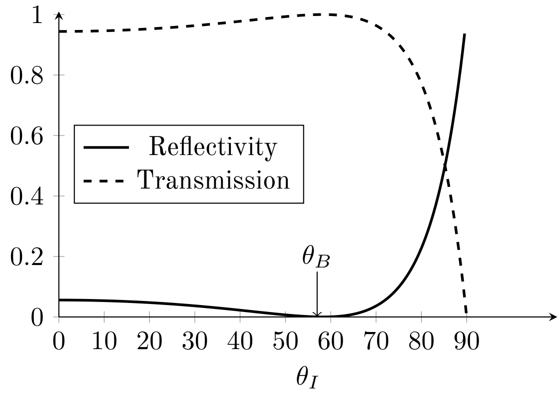

Figure 2.3: *Reflectivity and Transmittance from an air to glass. θ_B denotes Brewster's angle.*

The fact that

$$T + R = 1$$

makes intuitive sense, as it says that the intensity of light that hits the interface is equal to the intensity

[¶]assuming ω remains unchanged, which is valid for *linear media*.

[‖]These are the ratio between the reflected and incident intensities, as well as between the transmitted and incident intensities, respectively.

of light that is reflected plus the intensity of light transmitted – conservation of energy.

Equation (2.15) also implies that there is an angle where $R = 0$, this is Brewster's angle, θ_B.

2.5 Radiation of accelerating charges

In the derivation in the previous section, the fact that there will be a reflected and refracted wave seemed to appear from nowhere, in this section, I will begin to discuss the source of these waves.

Imagine an electron in a vacuum, that is standing still (Feynman et al. 1963). Its (electric) field lines will be straight, radial lines. Now we quickly move the electron slightly to the side. We know that the electric and magnetic fields propagate at the speed of light, that means that there will be a sphere where the electric field "update", this sphere's radius grows at the speed of light, which means everything outside of the sphere has the old field lines, and inside the sphere we measure the new field lines. In some cases this change of electric field at the surface of the sphere is coupled with a change in the magnetic field. This is the case when charged particles accelerate: accelerating charges radiate electromagnetic waves, due to the fact that information (in this case about the whereabouts of the particle saved in the electric and magnetic fields) propagates at the speed of light.

Generally the electric field of an arbitrarily moving electron(Feynman et al. 1963) can be written as follows:

$$\mathbf{E}(\mathbf{r}) = \frac{q}{4\pi\epsilon_0}\left(\frac{\hat{\boldsymbol{\mathscr{r}}}'}{\mathscr{r}'^2} + \frac{\mathscr{r}'}{c}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\hat{\boldsymbol{\mathscr{r}}}'}{\mathscr{r}'^2}\right) + \frac{1}{c^2}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\hat{\boldsymbol{\mathscr{r}}}'\right) \tag{2.16}$$

The $'$ demarks that the electric field at the point $\mathbf{r}$ is not decided by $\mathscr{r}$, which is the distance to the electron *now*, but rather by the retarded position of the electron $\mathscr{r}'$, which is not taken at t but at $t' = t - \frac{\mathscr{r}}{c}$.

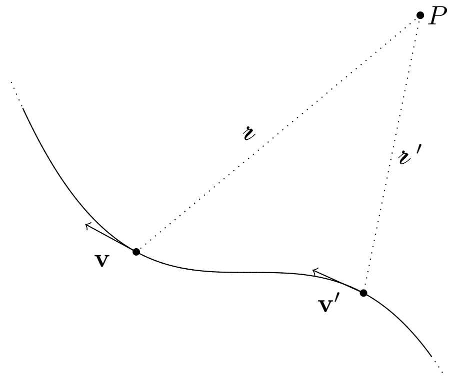

Figure 2.4: *We measure the electric field at point P, generated by an electron moving along the line. $\mathscr{r}$ and $\mathbf{v}$ denote the current position vector and velocity vector respectively, $\mathscr{r}'$ and $\mathbf{v}'$ denote the retarded position and velocity vector, respectively. That is the position and velocity at time $t - \mathscr{r}/c$*

Due to the fact that we tend to measure the electromagnetic fields far away from the source, the first two terms in Equation 2.16 will have died off, in comparison to the third term, which dies off more slowly. Here we have assumed that the electron is moving much slower than the speed of light. The only source of radiation from moving point charges is the *acceleration*:

$$\mathbf{E}(\mathbf{r}) = \frac{q}{4\pi\epsilon_0 c^2}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\hat{\boldsymbol{\mathscr{r}}}'$$

Here it becomes clear that it is only the acceleration that changes the direction of $\hat{\mathscr{r}}'$ that contributes to the electric field – that is linear acceleration which is parallel to $\boldsymbol{\mathscr{r}}'$ is not a source for radiation. Thus this can be written as

$$\mathbf{E}(\mathbf{r}) = \frac{q a'_{\perp}}{4\pi\epsilon_0 c^2 \mathscr{r}'}\hat{\mathbf{a}}'$$

Where $a'_{\perp}$ is the acceleration in the plane whose normal vector is $\hat{\boldsymbol{\mathscr{r}}}'$.

The magnetic field is orthogonal to both $\hat{\boldsymbol{\mathscr{r}}}$ and $\mathbf{E}$, and is scaled by a factor c^{-1}, thus

$$\mathbf{B} = \frac{1}{c}\mathbf{E}\times\hat{\boldsymbol{\mathscr{r}}}$$

The orthogonality and scaling follow directly from Maxwell's equations.

3 Continued Theory

Now that we have discussed the basic theory, we can continue to the heart of this report. In this chapter we will discuss the exact motion of electrons being accelerated by planar electromagnetic waves, how this motion can be used to predict the index of refraction, by using a very exact microscopic model, as well as a macroscopic model. Finally we will investigate the electromagnetic waves that are produced by the electrons that are accelerated by electromagnetic waves – the reflected, refracted and the parallel-transmitted waves.

3.1 The motion of electrons

In Section 2.2 we used Ohm's Law. However, Ohm's Law is a simplification that hides the fact that there are also currents that are caused by the movement of charges due to the magnetic field. In reality Equation (2.9) should have been:

$$\mathbf{J} = \sigma \left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right)$$

where $\mathbf{v}$ is the velocity of the charges. At $t < 0$ (so before the waves hit the medium) $\mathbf{v} = \mathbf{0}$, at $t = 0$ the free charges are accelerated by the Coulomb force, and at $t > 0$ the charges are accelerated by both the Coulomb and Lorentz forces.

Unfortunately the rest of the derivation in Section 2.2 is impossible, before we know more about the velocity of the electrons, therefore in the following two sections, we will solve the equations of motion for the electron being accelerated by electromagnetic radiation.

3.1.1 The bound electron

Materials that do not conduct electricity (insulators), such as glass, do not have any free electrons, which means that the reflection and transmission of light is purely due to the acceleration of *bound* electrons. This section explores the movement of a bound electron due to an electromagnetic wave, using the mass-on-a-spring model (Feynman et al. 1963):

$$m\left(\frac{\mathrm{d}^2 x(t)}{\mathrm{d}t^2} + \omega_0^2 x(t)\right) = F$$

where m is the mass of the particle being accelerated, and F is the force being exerted on said particle. The ω_0 term comes from the fact that we model the electron as a mass on a spring. This may seem like an invalid assumption, as in this case the equilibrium position of the electron is at $x = 0$, which is where the nucleus lies. This is due to the fact that we are modelling the electron as a sphere with uniform charge distribution. This sphere is what can be thought of as the average position of the electron. When a force is exerted on this sphere, the position will shift, as can be seen in Figure 3.1

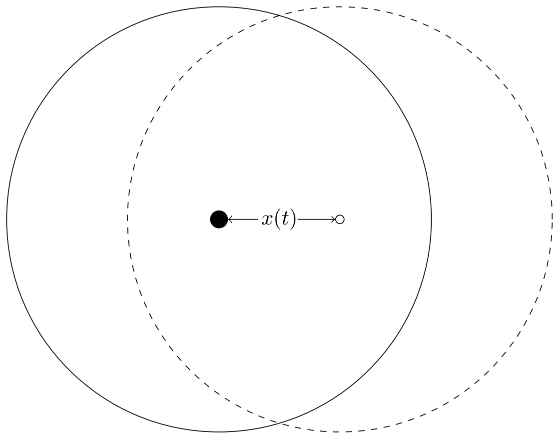

Figure 3.1: *The circle with a solid outline represents the perimeter of the electron-sphere at equilibrium, the small filled black circle in the centre of that is the nucleus. A force displaces the electron sphere to the dashed line– the distance between the nucleus and the new centre is $x(t)$.*

The electric forces that the electron sphere experiences due to this shift from the equilibrium position are very similar to the forces due to a spring, as the force should increase linearly with increasing diplacement. The eigenfrequency, ω_0, is related to the spring constant and the mass of the electron, $\omega_0^2 = \frac{k}{m}$, where k is the spring coefficient and m is the mass of the object which experiences the spring force. ω_0 is the frequency the system would oscillate at, if we were to remove the external force when the system is as it is shown in Figure 3.1.

In the case of electromagnetic fields the forces that disturb the electron are the Coulomb and Lorentz forces:

$$m\left(\frac{\mathrm{d}^2 \mathbf{x}(t)}{\mathrm{d}t^2} + \omega_0^2 \mathbf{x}(t)\right) = q\left(\mathbf{E} + \frac{\mathrm{d}\mathbf{x}(t)}{\mathrm{d}t} \times \mathbf{B}\right)$$

Suppose now that

$$\left(\mathbf{E} = Ee^{i\omega t}\hat{\mathbf{x}}\right) \quad \text{and} \quad \left(\mathbf{B} = Be^{i\omega t}\hat{\mathbf{y}}\right)$$

That would mean that

$$\frac{\mathrm{d}\mathbf{x}(t)}{\mathrm{d}t} \times \mathbf{B} = v_1 B\hat{\mathbf{z}} - v_3 B\hat{\mathbf{x}}$$

Which makes the differential equations:

$$\begin{aligned}\ddot{x}_1 &= -\Lambda e^{i\omega t}\dot{x}_3 - \omega_0^2 x_1 + \Gamma e^{i\omega t} \\ \ddot{x}_2 &= \quad 0 \\ \ddot{x}_3 &= \quad \Lambda e^{i\omega t}\dot{x}_1 - \omega_0^2 x_3\end{aligned}$$

where

$$\Gamma = \frac{qE}{m}, \quad \Lambda = \frac{qB}{m}$$

The motion of the charge is described by the real part of the solution.

It immediately becomes apparent that there will be acceleration in the x_1-x_3 plane – instead of there being linear motion parallel to the x_1-axis, as would be the case if we hadn't included the $\mathbf{v} \times \mathbf{B}$ term: we now have planar motion.

This is a coupled second order inhomogeneous differential equation. We will begin by approximating a solution, and thereafter we will continue to the general solution.

It is clear that $E \gg |\mathbf{v} \times \mathbf{B}|$, as $B = \frac{E}{c}$, therefore the term with Λ is negligible compared to Γ. ω_0 for electrons is often in the 10^{14}Hz regime (Vial et al. 2005), thus the term with ω_0 will also be dominant. This implies that the motion of the x_1 coordinate is primarily influenced by the electric field and the harmonic-oscillator-term. Therefore:

$$\ddot{x}_1(t) \approx \Gamma e^{i\omega t} - \omega_0^2 x_1$$

The solution to this is (Feynman et al. 1963):

$$x_1(t) = \frac{qE}{m(\omega_0^2 - \omega^2)} e^{i\omega t} \tag{3.1}$$

which makes

$$\dot{x}_1(t) = \frac{i\omega qE}{m(\omega_0^2 - \omega^2)} e^{i\omega t}$$

and finally, we plug this into the differential equation for $x_3(t)$:

$$\ddot{x}_3(t) = \frac{i\omega\Lambda\Gamma}{(\omega_0^2 - \omega^2)} e^{2i\omega t} - \omega_0^2 x_3 \tag{3.2}$$

The solution to this is once again

$$\begin{aligned}x_3(t) &= \qquad x_A e^{2i\omega t} \\ \ddot{x}_3(t) &= -4\omega x_A e^{2i\omega t}\end{aligned}$$

Putting these into Equation 3.2 yields

$$x_3(t) = \frac{i\omega q^2 EB}{m^2(\omega_0^2 - \omega^2)(\omega_0^2 - 4\omega^2)} e^{2i\omega t} \tag{3.3}$$

A plot to show the qualitative behaviour of this solution is shown in Figure 3.2.

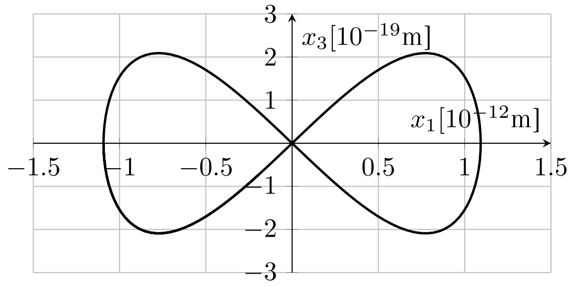

Figure 3.2: *Trajectory of the bound electron.*

We will now continue to the general solution. The terms in Equation 3.1 and Equation 3.3 are the dominant terms in an infinite Fourier series, but before we give the full solution, we should determine whether or not it is valid to express the solution as a Fourier series. To do this we need to look at the Dirichlet conditions (Riley and Hobson 2011); A function $f(t)$ can be written as a Fourier series if:

1. the function is periodic
2. $f(t)$ is single valued and is continuous, except at a finite number of finite discontinuities per period
3. the function may only have a finite number of maxima and minima within a period
4. the integral of one period of the absolute value of $f(t)$ must converge

We can argue that points *2.-4.* are fulfilled due to the fact that this is a physical problem. Point *1.* presents a slight problem, due to the fact that the general solution to the periodically-driven, damped harmonic oscillator has a non-periodic term, which is the transient term. By periodic I mean that $f(t) = f(t+T)$, where T is the period; this does not hold for the transient term, due to its amplitude's temporal decay. However, this term dies off rather quickly, and therefore if we assume that the transients have passed, we can readily use the Fourier series to express our solution.

The Fourier series of $x_1(t)$ consists of infinitely many $\alpha_n e^{ni\omega t}$ terms, where n is odd, and $x_3(t)$ consists of infinitely many $\beta_n e^{ni\omega t}$ terms, where n is even. These terms are given by*:

$$\begin{aligned}\alpha_n &= \frac{(n-1)!\omega^{n-1}\Lambda^{n-1}\Gamma}{\prod_{\ell=1}^{n}(\omega_0^2-\ell^2\omega^2)} \\ \beta_n &= \frac{i(n-1)!\omega^{n-1}\Lambda^{n-1}\Gamma}{\prod_{\ell=1}^{n}(\omega_0^2-\ell^2\omega^2)}\end{aligned} \tag{3.4}$$

Before we continue it is worth mentioning, that the complex conjugates should be included in the infinite Fourier series, luckily the derivations of the coefficients for the complex conjugate terms are analogous to those for α_n and β_n (we just need to let $i \to -i$), which implies that the full Fourier series consists of $\alpha_n e^{ni\omega t} + \alpha_n^* e^{-ni\omega t}$ and $\beta_n e^{ni\omega t} + \beta_n^* e^{-ni\omega t}$, where the asterisk denotes the complex conjugation, therefore we will omit these from the remainder of the report, as the coefficients can readily be calculated by conjugating the coefficients given in this report.

Interestingly enough the approximation we started off with, agrees perfectly with the first two terms in the specific solution we just obtained. What this means is that the system does not have a closed feedback loop, in the sense that the terms missing from our previous approximation, do not affect the first harmonic in the x_1 direction, nor the second harmonic in the x_2 direction, which is the first reason, why the approximation is so useful. The second reason is because the higher order terms fall by approximately a factor of c, which makes them difficult to measure, impossible to see with the naked eye. This rapid decrease in magnitude of the Fourier coefficients is due to the Λ^{n-1} in α_n and β_n in Equation 3.4.

If the damping due to radiation (Griffiths 2017) is to be included, then this is the differential equation†:

$$\begin{aligned}\ddot{x}_1 &= -\Lambda e^{i\omega t}\dot{x}_3 - \omega_0^2 x_1 - \gamma \dddot{x}_1 + \Gamma e^{i\omega t} \\ \ddot{x}_2 &= \quad 0 \\ \ddot{x}_3 &= \quad \Lambda e^{i\omega t}\dot{x}_1 - \omega_0^2 x_3 - \gamma \dddot{x}_3\end{aligned}$$

And the following are the coefficients of the solution:

$$\begin{aligned}\alpha_n &= \frac{(n-1)!\omega^{n-1}\Lambda^{n-1}\Gamma}{\prod_{\ell=1}^{n}(\omega_0^2-\ell^2\omega^2-i\gamma\omega^3\ell^3)} \\ \beta_n &= \frac{i(n-1)!\omega^{n-1}\Lambda^{n-1}\Gamma}{\prod_{\ell=1}^{n}(\omega_0^2-\ell^2\omega^2-i\gamma\omega^3\ell^3)}\end{aligned} \tag{3.5}$$

If we again ignore the higher order terms, and only consider the dominant terms, that is α_1 and β_2, we will see that the ratio of their amplitudes will be

$$\frac{\beta_2}{\alpha_1} = \frac{q\omega B}{m(\omega_0^2 - 4\omega^2 - 8i\gamma\omega^3)}$$

As we will see in Section 3.6 these coefficients are related to the amplitude of the electric and magnetic excitations in the reflected/transmitted wave. Therefore the emitted light at frequency ω will be greater than the emitted light with frequency 2ω, by a factor, that is proportional to B and $\frac{\omega}{\omega_0^2-4\omega^2-9i\gamma\omega^3}$.

During this section we used a model that describes the bound electron as a mass-on-spring – we modelled the motion of the electron as a driven and damped harmonic oscillation. The question that arises here is the validity of the γ term, as it is one that we added by hand, because we know that the radiation which is sent out by the electron due to its acceleration will cause the system to loose energy, and thus cause a damping effect. However, it is not clear whether this term should depend on the $\dddot{x}_i$ or $\dot{x}_i$. In (Griffiths 2017) it is mentioned that one can use *any* odd derivative of x_i. But the frequency dependency of the Fourier coefficients depends on which derivate is chosen. Had we chosen the first derivative, then γ term in would have appeared as a first order term, instead of a third order term.

3.1.2 The free electron

This section goes hand-in-hand with the previous section, except that we now are describing the motion of free electrons, instead of bound electrons.

Free electrons are electrons that are in the conduction band of atoms, which physically means they are not orbiting any particular atom, but rather behave as a kind of gas with in the material. Typically when one speaks of free electrons, they are in metals, as these are conductors. When electromagnetic waves

*The derivation of these formulae and the reason for the odd- and evenness of n respectively for x_1 and x_3 can be found in Appendix A.1.

†One may argue against the $\dddot{x}_i$ in $\ddot{x}_i$, arguing that it is not necessarily $\dddot{x}_i$ that should be used, but one of the other coordinates third temporal derivates, but if we imagine the oscillation to be in one dimension (say if we forget about the $\mathbf{B}$), then the dissipation of $\hat{\mathbf{x}}_1$ will be dependent of $\dddot{x}_1$, thus we can argue that the oscillators dissipations are independent of each other.

interacts with these, there is also an acceleration of free electrons. This is similar to our previous model, however the ω_0 terms are negligible now, as we consider the electrons to be *free*:

$$\begin{aligned}\ddot{x}_1 &= -\Lambda e^{i\omega t}\dot{x}_3 + \Gamma e^{i\omega t} - \tau^{-1}\dot{x}_1\\ \ddot{x}_2 &= \quad 0\\ \ddot{x}_3 &= \quad \Lambda e^{i\omega t}\dot{x}_1 - \tau^{-1}\dot{x}_3\end{aligned}$$

Where τ is the characteristic time between collisions, as in Section 2.2. Here the dominant terms are

$$\begin{aligned}x_1(t) &= -\frac{qE}{m(\omega^2 - i\omega\tau^{-1})}e^{i\omega t}\\ x_3(t) &= -\frac{iq^2E^2}{cm^2(4\omega^2 - i\omega\tau^{-1})(\omega^2 - 2i\omega\tau^{-1})}e^{2i\omega t}\end{aligned}$$

Due to the fact the differential equations for free electrons is the same as for bounded electrons with the condition that $\omega_0 = 0$ we can merely reuse the solution from the previous section. Additionally we have added the $\tau^{-1}\dot{x}_i$ term, which is similar to the damping term from before:

$$\alpha_n = \frac{(n-1)!\omega^{n-1}\Lambda^{n-1}\Gamma}{\prod_{\ell=1}^{n}\left(-\ell^2\omega^2 + i\ell\omega\tau^{-1}\right)}$$

$$\beta_n = \frac{i(n-1)!\omega^{n-1}\Lambda^{n-1}\Gamma}{\prod_{\ell=1}^{n}\left(-\ell^2\omega^2 + i\ell\omega\tau^{-1}\right)}$$

once again bearing in mind that n is odd for α_n and even for β_n. Again we can include the damping due to radiation, which yields following coefficients:

$$\alpha_n = \frac{(n-1)!\omega^{n-1}\Lambda^{n-1}\Gamma}{\prod_{\ell=1}^{n}\left(-\ell^2\omega^2 - i\gamma\omega^3\ell^3 + i\ell\omega\tau^{-1}\right)}$$

$$\beta_n = \frac{i(n-1)!\omega^{n-1}\Lambda^{n-1}\Gamma}{\prod_{\ell=1}^{n}\left(-\ell^2\omega^2 - i\gamma\omega^3\ell^3 + i\ell\omega\tau^{-1}\right)}$$

What this in reality means is that both the free and the bound electrons will, as we predicted previously, oscillate in a plane, whose normal is parallel to the magnetic field. The component of the oscillation that is parallel to the **E**-field will have odd powered harmonics, whereas the oscillation parallel to the normal of the interface will have even powered harmonics. This implies that when electromagnetic waves hit the interface between two media, there will be waves with double frequency that are sent parallel to the interface, as we will see in Section 3.6

An interesting question to ask at this point is, is it possible for the third harmonic to be greater than the second harmonic, of the light that is orthogonal to the interface? We now have the means to calculate the answer:

$$\frac{\alpha_3}{\alpha_1} = \frac{2\omega^2\Lambda^2}{(\omega_0^2 - 9\omega^2)(\omega_0^2 - 4\omega^2)} \geq 1$$

This can be achieved for these frequencies:

$$\omega = \frac{1}{12}\sqrt{4\Lambda^2 + 26\omega_0^2 \pm 2\sqrt{4\Lambda^4 + 52\Lambda^2\omega_0^2 + 25\omega_0^4}}$$

Naturally, when $\Lambda \ll \omega_0$ this will reduce to $\omega = \frac{1}{2}\omega_0$ or $\omega = \frac{1}{3}\omega_0$. However, the greater Λ becomes, the greater becomes the interval, where $\alpha_3 > \alpha_1$.

This presents us with a problem, either the electromagnetic field is too weak, and the only possibility of measuring that $\alpha_3 > \alpha_1$ is at 1/2 or 1/3 of the resonance frequency, in which case the mass-on-a-spring model no longer is valid. The other possibility is that $\alpha_3 > \alpha_1$ for a range of frequencies around 1/2 and 1/3 of the resonance frequency, but in order for this to be possible, we need a powerful electromagnetic field. In fact the electromagnetic field would need to be so powerful that it would ionise the atoms.

This is a natural consequence of the solution that we found to the differential equation, and this is clearly a weakness of the model, as the more terms we add to the Fourier series the fewer frequencies are available to us, that we can give to the solution without it becoming invalid. This shows the importance of γ in this model. If the factor γ is included the ratio α_3/α_1 is bound‡, which allows us to move is towards 1/2 and 1/3 of the resonance frequency, and possibly will allow us to measure that $\alpha_3/\alpha_1 > 1$.

The model that was used for the free electrons in metals was quite similar to the model used for bound electrons, except that we set $\omega_0 = 0$ and that we added at term that is dependent of τ^{-1}. This term is commonly included in the Lorentz-Drude model, and accounts for the frictional forces the electron experiences, due to collisions with the atomic-grid.

However, this characteristic time, τ, is independent of the motion of the electron. For example, when electrons are accelerated by electromagnetic waves, their harmonic motion's amplitude is far smaller

‡provided γ is large enough.

than the distance between vertices in the grid, which implies that there are far fewer collisions than what is accounted for. For direct current circuits the mechanism that is responsible for this frictional force is often referred to as *scattering*, and therefore one often uses the word *scattering* to describe the frictional mechanism in the Lorentz-Drude model for alternating current. However, one should perhaps think of the mechanism as a classical frictional force, that acts on a medium of evenly distributed electron-wave-functions.

Otherwise one may think of the collisions of the electrons as an effect due to their thermal motion (we have previously calculated that the distance they travel during τ due to their thermal motion is comparable to the gridsize), which may imply that the model needs to be reformulated, such that it is not the "drift" velocity, but the thermal velocity that causes the damping. This could for example be done by not having a velocity dependent frictional term, but a temperature dependent term. This term would be constant in time, and therefore change the appearance of the differential equation.

3.2 Divergence of the Fourier coefficients

A very important question to ask is; does the Fourier series converge? That is, does $\alpha_n \to 0$ as $n \to \infty$. If it were not the case that the Fourier series converges, then it would not be a valid to say that we merely can look at the first few terms in the series, as these no longer would be dominant. Therefore the convergence of the series is essential, when discussing whether or not it is valid to write the solution to equations of motion of the electron as a (truncated) Fourier series.

When discussing the divergence of series, one often compares the n. term to the $n+1$. term (in this case the $n+1$. term is zero, so we'll compare the n. term to the $n+2$. term):

$$\frac{\alpha_n}{\alpha_{n-2}} = \frac{(n-1)(n-2)\omega^2\Lambda^2}{(\omega_0^2 - n^2\omega^2)(\omega_0^2 - (n-1)^2\omega^2)}$$

We can neglect the damping term here, as this is won't cause the series to diverge. We would quite like for this to be smaller than one as $n \to \infty$, so that the series converges, which requires that

$$\frac{(n-1)(n-2)\omega^2\Lambda^2}{(\omega_0^2 - n^2\omega^2)(\omega_0^2 - (n-1)^2\omega^2)} \leq 1$$

Letting $n \gg 1$:

$$\frac{n^2\omega^2\Lambda^2}{n^4\omega^4} = \frac{\Lambda^2}{n^2\omega^2} \leq 1$$

As ω and Λ are constants the series will always converge. However, it is possible that this term is temporarily larger than one, which means that the coefficients will grow as a function of n, until higher orders of n "catch" up. This can happen for large Λ. The ratio between α_{n-2} and α_n also explode if $(\omega_0 \approx n\omega)$ or $(\omega_0 \approx (n-1)\omega)$. As mentioned previously, the explosion of α_n and α_{n-2} is bound by including the damping effect due to radiation.

3.3 Validity of the mass-on-a-spring model

The mass-on-a-spring model assumes that there is a force that holds the electron bound to the nucleus, irrespective of the distance. However, in reality if the distance between the electron and atom becomes too large, say more than an Ångstrøm, the electron will no longer be bound. Thus this model is only valid for electric fields, that aren't strong enough to ionise the atoms, nor is it valid at or around the resonant frequency (frequencies) of the electron. Due to the difference of about seven orders of magnitude between the movement parallel to the electric field, in comparison to the movement parallel to the normal to the interface, we can assume that it is only the the electric fields force that could ionise the atom. Thus

$$\frac{qE}{m(\omega_0^2 - \omega^2)}$$

may never exceed 10^{-10}m. Let's us assume $\omega \ll \omega_0$ and $\omega_0 \approx 10^{14}$Hz (Vial et al. 2005), then E may not exceed $3 \times 10^8 \text{Vm}^{-1}$. However, if $\omega \gg \omega_0$ then E may not exceed $6 \times 10^7 \left(\frac{\omega}{\omega_0}\right)^2 \text{Vm}^{-1}$. However, at these limits one is constrained by other issues, such as the fact these high voltages cause a discharge in the air. The upper limits here are solely for the bound electrons, whereas these discharges presumably happen because of the free electrons (it requires far less energy to move free charges than to unbind bound electrons and move those).

3.4 Index of refraction

3.4.1 Bound electrons

Let us begin with the induced dipole moment of a single atom (Feynman et al. 1963):

$$\mathbf{p} = q\mathbf{x}_e$$

Where $\mathbf{x}_e$ denotes the distance between the electron and its equilibrium position, thus the first order dipole moment of the atom is

$$\begin{aligned}\mathbf{p} =& \frac{q^2 E e^{i\omega t}}{m(\omega_0^2 - \omega^2 - i\gamma\omega^3)}\hat{\mathbf{x}} + \\ & \frac{i\omega q^3 E B e^{2i\omega t}}{m^2(\omega_0^2 - \omega^2 - i\gamma\omega^3)(\omega_0^2 - 4\omega^2 - 8i\gamma\omega^3)}\hat{\mathbf{z}}\end{aligned}$$

However, the movement parallel to $\hat{\mathbf{z}}$ will not contribute to the index of refraction, as the electromagnetic waves that are caused by this movement are primarily *parallel* to the interface, and have double the frequency of the incident electromagnetic waves. Thus we will only investigate the $\hat{\mathbf{x}}$ component of the dipole moment:

$$\mathbf{p} = \frac{q^2}{m(\omega_0^2 - \omega^2 - i\gamma\omega^3)}\mathbf{E}$$

$\mathbf{p}$ can also be written as

$$\mathbf{p} = \varepsilon_0 \alpha(\omega)\mathbf{E}$$

where $\alpha(\omega)$ is the atomic polarisability. In our case this would be

$$\alpha(\omega) = \frac{q^2}{m\varepsilon_0(\omega_0^2 - \omega^2 - i\gamma\omega^3)}$$

$\mathbf{p}$ is the polarisation per atom though, so the total polarisability would be

$$\mathbf{P} = N\varepsilon_0\alpha(\omega)\mathbf{E}$$

N denotes the number of atoms or molecules. $N\alpha(\omega)$ is now in place of χ_e, which is the electric susceptibility, which implies that the refractive index, n is (Feynman et al. 1963):

$$n = \sqrt{1 + N\alpha(\omega)} \approx 1 + \frac{1}{2}\frac{Nq^2}{m\varepsilon_0(\omega_0^2 - \omega^2 - i\gamma\omega^3)}$$

which makes κ (the imaginary part of n):

$$\kappa \approx \frac{Nq^2\gamma\omega^3}{2m\epsilon_0((\omega_0^2 - \omega^2)^2 + \gamma^2\omega^6)}$$

At this stage it is commonly stated that electrons have numerous resonant frequencies. All of these different resonances will contribute to the index of refraction, making the imaginary part of n:

$$\kappa \approx \frac{Nq^2\omega^3}{2m\varepsilon_0}\sum_{k=1}^{M}\frac{\gamma_k f_k}{(\omega_{0,k}^2 - \omega^2)^2 + \gamma_k^2\omega^6}$$

f_k denotes that there may be numerous electrons with ω_k and γ_k per atom/molecule. Additionally, we can write the bound electrons contribution to the relative permittivity as

$$\tilde{\varepsilon}_r^b = 1 + \frac{Nq^2}{m\varepsilon_0}\sum_{k=1}^{M}\frac{f_k}{(\omega_{0,k}^2 - \omega^2 - i\gamma_k\omega^3)}$$

3.4.2 Free electrons

The method we used to calculate the contribution to the index of refraction from the bound electrons can unfortunately not be used for the free electrons, as these alone cannot make a dipole. Of course the atoms in the grid are positively charged, but to calculate the dipole moment would require us to choose a reference point (the positive charge at the base of the dipole moment vector), however this would have to be an arbitrary atom.

However, as in Section 2.2 we can use the Drude-Lorentz model. This model does not include the damping force, due to the radiation, thus we need to include that, using the solution we found in Section 3.1.2. This solution tells us that the (first order approximation) velocity becomes

$$\mathbf{v}(t) = \frac{q\mathbf{E}\tau e^{i\omega t}}{m(1 - i\omega\tau - \omega^2\gamma\tau)}$$

And again putting this into $\mathbf{J} = Nq\mathbf{v}$ yields that the frequency dependent conductivity is

$$\sigma(\omega) = \frac{Nq^2\tau}{m(1 - i\omega\tau - \omega^2\gamma\tau)} = \frac{\sigma_0}{1 - i\omega\tau - \omega^2\gamma\tau}$$

Thus changing the free electrons' contribution to the relative permittivity to

$$\tilde{\varepsilon}_r^f = \frac{\sigma_0}{\epsilon_0\omega(1 - i\omega\tau - \omega^2\gamma\tau)}$$

3.4.3 Combined effects

We can now conclude that the relative permittivity from both the free and bound electrons is

$$\begin{aligned}\tilde{\varepsilon}_r =& \tilde{\varepsilon}_r^f + \tilde{\varepsilon}_r^b \\ =& 1 + \frac{\sigma_0}{\varepsilon_0}\left(\frac{1}{\omega(1 - i\omega\tau - \omega^2\gamma\tau)}\right. \\ & \left. + \frac{1}{\tau}\sum_{k=1}^{M}\frac{f_k}{\omega_{0,k} - \omega^2 - i\gamma_k\omega^3}\right)\end{aligned}$$

The index of refraction will be the square root of $\tilde{\varepsilon}_r$.

Now that is quite powerful – we can predict the behaviour of the complex index of refraction for materials, and with that predict the penetration depth.

However, this requires that we either know all $\omega_{0,k}$, γ_k etc. values, or that we fit the values to experimental data. Though this is possible, it would be far more interesting to have a prediction, that relies on no free parameters.

3.5 Macroscopic approach

The method I will be discussing in this section was inspired by notes written by my supervisor, Steen Hansen.

An issue with the approach discussed in the previous sections is twofold: firstly, there are free parameters in this model, which means that the functions for the real and imaginary parts of the refractive index must be fitted to experimental data. The second issue is the dependence on τ, the characteristic time between collisions. This constant was first introduced to the direct current Drude-Lorentz model, where there are electrons moving with respect to the grid, where it makes sense to speak of collisions. This is a material constant, that is dependent on temperature and the specific geometry. Additionally the model depends on $\omega_{0,k}$, γ_k and f_k, so three constants per electron-energy-level. All of these need to be measured, before the model can be effective.

Therefore it would be extremely beneficial to derive an equation for the imaginary part of the refractive index, using a model that takes said issues into account. This can, for example, be done using are more macroscopic approach. We begin with the conservation of energy: the energy that is required to accelerate the electrons, must be reaped from the electric and magnetic fields. If we ignore the effects due to magnetic fields, this work will be given by

$$W = \int_{t_1}^{t_2} q\mathbf{E} \cdot \mathbf{v} \mathrm{d}t$$

This should be equal zero, after a whole period, as the electron returns to its initial state. However, there is a phase shift, between when the energy is delivered to the electron, and when the energy returns to the electromagnetic fields. Therefore, though it is true that the net energy of the system is unchanged, the energy is converted from energy in the electromagnetic fields, to the kinetic energy of the electron. The electron is accelerated eight times throughout the oscillation: from zero the v_E, back to zero, to $-v_E$ and back to zero, and this is repeated once more (with the opposite sign). Therefore the total energy that is delivered to the electron must be

$$U_E = 4m_E v_E^2, \quad v_E = \frac{qE_0}{m_E\omega}$$

for the free electrons. Let us suppose each electron occupies some area $a = d^2$, where d is the typical distance between atoms in the grid. That is to say that it can absorb energy from the electromagnetic field within that area. At this stage it may be useful to think of the electrons as a type of continuum, and not as a point particle. The energy delivered by the electromagnetic field per area per unit time is given by the Poynting vector, whose average magnitude over a period is:

$$\langle S \rangle = \frac{E_0^2}{2c\mu_0}$$

Thus the average energy delivered by the electromagnetic field over a whole period is the S times the area, $a = d^2$, times the length of a period $T = \frac{2\pi}{\omega}$:

$$U_{EM} = \langle S \rangle \frac{2\pi d^2}{\omega}$$

This means that the fraction of energy delivered to the electron per oscillation is

$$\frac{U_E}{U_{EM}} = \frac{4c\mu_0 q^2}{m_E\omega\pi d^2} = L$$

But U_E can also be thought of as minus the spacial derivative of U_{EM} (in the direction of $\mathbf{S}$). Thus

$$\frac{\mathrm{d}U_{EM}}{\mathrm{d}z} = -\frac{L}{d} U_{EM}$$

Which means

$$U_{EM} = U_{EM,0} \exp\left(-\frac{L}{d} z\right)$$

Therefore we can define the skin depth as

$$\delta = dL^{-1} = \frac{m_E\omega\pi d^3}{4c\mu_0 q^2}$$

The skin depth is also related to the extinction coefficient ($\Im(n) = \kappa$):

$$\delta = \frac{\lambda}{4\pi\kappa} = \frac{c}{2\kappa\omega}$$

Which would make

$$\kappa = \frac{2c^2\mu_0 q^2}{m_E\omega^2\pi d^3}$$

This is only valid for $\lambda < d$. However, if the wave manages to interact with numerous layers of electrons, per period, the effect will be slightly different, as the maximal velocity will be a function of

$E(t)$. The energy left over in the electromagnetic field after one electron is

$$U_{EM} = U_{EM,0} - \Delta U_{EM}$$

But we've already established that

$$\Delta U_{EM} = L U_{EM}$$

which implies that

$$U_{EM} = \frac{U_{EM,0}}{1+L}$$

Which makes the improved extinction coefficient:

$$\kappa = \frac{2c^2\mu_0 q^2}{m_E \pi d^3}\frac{1}{\omega^2}\frac{1}{1+L}, \quad L = \frac{4c\mu_0 q^2}{m_E \pi d^2 \omega}$$

The benefit of this is that it shows the general behaviour of the extinction coefficient, without requiring a plethora of constants. However, due to its simplicity, this model does not predict the finer behaviour of the extinction coefficient, as can be seen in Figure 3.3

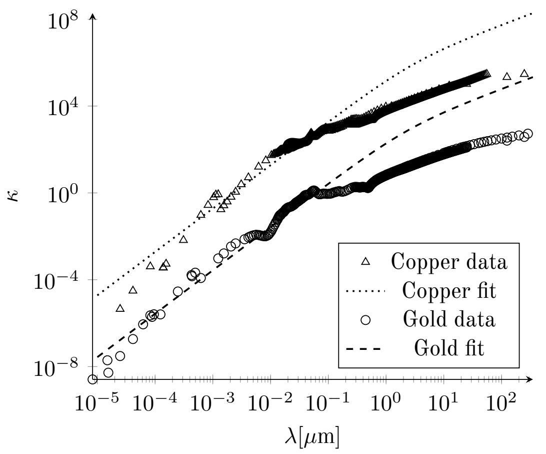

Figure 3.3: $\triangle$ *is copper (Hagemann 1974), and* $\circ$ *is gold (Olmon et al. 2012; Ordal et al. 1987). Note that the copper data has been multiplied by a factor of 1000, simply so that both data sets can be seen.*

Unfortunately the fits do not follow the data exactly, especially at higher wavelengths. However, the model does show potential: both the plots and the data have a high linear slope for low wavelength and a transition to a lower linear slope at higher wavelengths.

3.6 Maxwell's Equations

Now that we have an expression for the velocity of the electron, we can combine Maxwell's Equations with the fact that the current densities in the system will be

$$\mathbf{J} = Nq\mathbf{v}$$

where N is the number of electrons per unit volume. Now applying the curl to Maxwell's Equations, as before, and inserting the expression for the current density:

$$\begin{aligned} \Box^2\mathbf{E} &= Nq\mu\dot{\mathbf{v}} \\ \Box^2\mathbf{B} &= Nq\mu\left(\nabla\times\mathbf{v}\right) \end{aligned}$$

Where $\Box^2$ is the d'Alambert operator. And we're reminded that the first approximation of $\mathbf{v}$ is:

$$\begin{aligned} \mathbf{v} = {} & i\omega\alpha_1(\omega)e^{i\omega(t-r/c)}\hat{\mathbf{x}} \\ & +2i\omega\beta_2(\omega)e^{2i\omega(t-r/c)}\hat{\mathbf{z}} \end{aligned} \tag{3.6}$$

where

$$\begin{aligned} \alpha_1(\omega) &= \frac{qE}{m(\omega_0^2-\omega^2-i\gamma\omega^3)} \\ \beta_2(\omega) &= \frac{i\omega q^2 E^2}{cm^2(\omega_0^2-\omega^2-i\gamma\omega^3)(\omega_0^2-4\omega^2-8i\gamma\omega^3)} \end{aligned}$$

are the first coefficients from the solution we found in Section 3.1.1

Unfortunately things get rather odd if we assume that the waves are planar here, thus we will henceforth assume the incident waves are produced at position $\boldsymbol{\mathscr{r}} = (0,0,-\ell)$, which means that $r = |\mathbf{r}' - \boldsymbol{\mathscr{r}}| = \sqrt{x'^2+y'^2+(z'+\ell)^2}$[§] – we now have spherical waves. We can always let $\ell \to \infty$, to go back to planar waves. Now, we will be needing the temporal derivate of and curl of $\mathbf{v}$:

$$\begin{aligned} \dot{\mathbf{v}} &= -\omega^2\alpha_1(\omega)e^{i\omega t'}\hat{\mathbf{x}} + -4\omega^2\beta_2(\omega)e^{2i\omega t'}\hat{\mathbf{z}} \\ \nabla\times\mathbf{v} &= -\frac{\omega^2}{r'c}\Big(-4y'\beta_2 e^{2i\omega t'}\hat{\mathbf{x}} + (2x'\beta_2(\omega)e^{2i\omega t'} \\ &\qquad -(z'+\ell)e^{i\omega t'})\hat{\mathbf{y}} + y'\alpha_1(\omega)e^{i\omega t'}\hat{\mathbf{z}}\Big) \end{aligned}$$

where $t' = t - |\boldsymbol{\mathscr{r}} - \mathbf{r}'|/c$. Now the inhomogenous wave equations for the $\mathbf{E}$ and $\mathbf{B}$ will evidently comprise of both first and second harmonic terms. As these are independent of each other, I will immediately split the wave equation up, into equations

[§] $\mathbf{r}$ describes the position of the electron we are currently looking at – that is we are interested in the velocity at position $\mathbf{r}'$

where the sources only oscillate at the first or second harmonics. Beginning with the first harmonic:

$$\begin{aligned}
\Box^2 E_{x,1} &= -\omega^2 N q \mu \alpha_1(\omega) e^{i\omega t'} \\
\Box^2 E_{y,1} &= 0 \\
\Box^2 E_{z,1} &= 0 \\
\Box^2 B_{x,1} &= 0 \\
\Box^2 B_{y,1} &= \omega^2 \frac{Nq\mu}{rc}(z' + \ell)\alpha_1(\omega) e^{i\omega t'} \\
\Box^2 B_{z,1} &= -\omega^2 \frac{Nq\mu}{rc} y' \alpha_1(\omega) e^{i\omega t'}
\end{aligned} \tag{3.7}$$

Where the second subscript denotes that these are the wave equations for the first harmonics. The second harmonic equations are

$$\begin{aligned}
\Box^2 E_{x,2} &= 0 \\
\Box^2 E_{y,2} &= 0 \\
\Box^2 E_{z,2} &= -4\omega^2 N q \mu \beta_2(\omega) e^{2i\omega t'} \\
\Box^2 B_{x,2} &= 4\omega^2 \frac{Nq\mu y'}{rc} \beta_2(\omega) e^{2i\omega t'} \\
\Box^2 B_{y,2} &= -4\omega^2 \frac{Nq\mu x'}{rc} \beta_2(\omega) e^{2i\omega t'} \\
\Box^2 B_{z,2} &= 0
\end{aligned}$$

Note that there is cylindrical symmetry, with the z axis as the cylinders axis. This is due to the fact that $y\hat{\mathbf{x}} - x\hat{\mathbf{y}} = -\hat{\boldsymbol{\varphi}}$, where φ is the azimuth in cylindrical coordinates. Thus we conclude that there are waves propagating radially (on the xy-plane), with double the frequency of the incident wave.

To be clear, these inhomogeneous wave equations describe the electromagnetic waves that are radiated by the electrons oscillating in the medium. This means that the total electric and magnetic fields will be a superposition of the solution to these equations and the incident wave.

We do have to impose some conditions on this. The solution to the differential equation that describes the motion of the electrons assumes that the waves are planar, but we have just changed these to spherical. Thus the solution for the motion of the electrons is only valid if, for example we impose the condition that the oscillating charges lie in a cylinder with diameter d, additionally we cannot allow z' to become too large, because the electron model has not taken into account that the electromagnetic fields change due to the acceleration of the electron, that the amplitude of the electric and magnetic waves decay exponentially, due to the fact that each electron lends energy from the fields, as discussed in Section 3.5.

Unfortunately the solutions to the inhomogeneous wave equations are non-trivial – we will resort to Green's function, to solve this (Jackson 1962). We begin by discussing the general case:

$$\Box^2 \Psi(\boldsymbol{\rho}, t) = -4\pi f(\boldsymbol{\rho}, t)$$

The solution to this is given as

$$\Psi(\boldsymbol{\rho}, t) = \int \int G^{(\pm)}(\boldsymbol{\rho}, t, \mathbf{r}', t') f(\mathbf{r}', t') \mathrm{d}\mathbf{r}' \mathrm{d}t'$$

That is to say the solution is a convolution of the Green's function, specific to the problem at hand, and the source function. In the case of the d'Alambert operator, the Green's functions are given as

$$G^{(\pm)}(\boldsymbol{\rho}, t, \mathbf{r}', t') = \frac{\delta(t' - (t \mp (\boldsymbol{\rho} - \mathbf{r}')/c))}{|\boldsymbol{\rho} - \mathbf{r}'|}$$

Thus $G^{+}(\boldsymbol{\rho}, t, \mathbf{r}', t')$ is the *retarded* Green's function, and $G^{-}(\boldsymbol{\rho}, t, \mathbf{r}', t')$ is the *advanced* Green's function. Due to causality, we can deem the advanced Green's function unphysical, which means we only need to take the retarded Green's function into consideration, thus the solution becomes

$$\Psi(\mathbf{r}, t) = \int_{\mathcal{V}} \frac{f(\mathbf{r}', t_R)}{|\boldsymbol{\rho} - \mathbf{r}'|} \mathrm{d}\mathbf{r}' \tag{3.8}$$

where $t_R = t - \frac{|\boldsymbol{\rho} - \mathbf{r}'|}{c}$ and $\mathcal{V}$ is the volume where there are sources. But hang on, haven't we already taken the retardation into account in the inhomogeneous wave equations? Well, yes, we have taken a retardation into account, but there is another one: The retardation in the inhomogeneous wave equations exists due to the fact that it takes time for the light to reach each electron in the material, from the *source*. The retardation in the solution to the inhomogeneous wave equations takes into account, that it takes time for the light that is produced in the material to the reach the position at which we are measuring the waves.

At this stage things have become rather complicated, with a source point, points in the material, and points where we measure the field; the following figure is intended to clear this up:

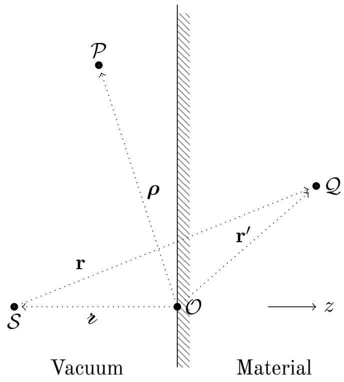

Figure 3.4: *A graphical representation as to explain the meaning of the variables* $\boldsymbol{\rho}$, $\boldsymbol{\mathscr{r}}$ *and* $\mathbf{r}'$. $\mathcal{O}$ *denotes the origin,* P *denotes the point at which we are measuring the electric and magnetic fields that are induced by the oscillating charge.* Q *is the current charge we are looking at (that is the charge at position* $\mathbf{r}'$ *in the integration in Equation* (3.8)*) and* S *is the position of the source of the incident waves.*

Thus

$$
\begin{aligned}
E_{x,1} &= -\omega^2 Nq\mu\alpha_1(\omega)\int_{\mathcal{V}} \frac{e^{i\omega t''}}{|\boldsymbol{\rho}-\mathbf{r}'|}\mathrm{d}\mathbf{r}' \\
E_{y,1} &= C_{y,1} \\
E_{z,1} &= C_{z,1} \\
B_{x,1} &= C_{x,1} \\
B_{y,1} &= \omega^2\frac{Nq\mu}{c}\alpha_1(\omega)\int_{\mathcal{V}} \frac{z'+\ell}{|\boldsymbol{\mathscr{r}}-\mathbf{r}'||\boldsymbol{\rho}-\mathbf{r}'|}e^{i\omega t''}\mathrm{d}\mathbf{r}' \\
B_{z,1} &= -\omega^2\frac{Nq\mu}{c}\alpha_1(\omega)\int_{\mathcal{V}} \frac{y'}{|\boldsymbol{\mathscr{r}}-\mathbf{r}'||\boldsymbol{\rho}-\mathbf{r}'|}e^{i\omega t''}\mathrm{d}\mathbf{r}'
\end{aligned}
$$

where $t'' = t-|\boldsymbol{\mathscr{r}}-\mathbf{r}'|/c-|\boldsymbol{\rho}-\mathbf{r}'|/c$. This doubly retarded time makes physical sense – the light travels from the souce, S, to the charge, Q, and thereafter to the measurement point P. And for the second harmonic

$$
\begin{aligned}
E_{x,2} &= C_{x,2} \\
E_{y,2} &= C_{y,2} \\
E_{z,2} &= -4\omega^2 Nq\mu\beta_2(\omega)\int_{\mathcal{V}} \frac{e^{2i\omega t''}}{|\boldsymbol{\rho}-\mathbf{r}'|}\mathrm{d}\mathbf{r}' \\
B_{x,2} &= 4\omega^2\frac{Nq\mu}{c}\beta_2(\omega)\int_{\mathcal{V}} \frac{y'}{|\boldsymbol{\mathscr{r}}-\mathbf{r}'||\boldsymbol{\rho}-\mathbf{r}'|}e^{2i\omega t''}\mathrm{d}\mathbf{r}' \\
B_{y,2} &= -4\omega^2\frac{Nq\mu}{c}\beta_2(\omega)\int_{\mathcal{V}} \frac{x'}{|\boldsymbol{\mathscr{r}}-\mathbf{r}'||\boldsymbol{\rho}-\mathbf{r}'|}e^{2i\omega t''}\mathrm{d}\mathbf{r}' \\
B_{z,2} &= C_{z,2}
\end{aligned}
\tag{3.9}
$$

Where $\ell \gg d > \delta$

The constants $C_{i,j}$ should be zero – we assume these are the only sources of radiation, which implies that the components of the electric and magnetic fields that do not have sources will remain zero.

The solution to the differential equation that describes the movement of electrons due to the acceleration from electromagnetic waves assumes that the incident waves are planar. However, in the beginning of this section we assumed that the waves were spherical. Therefore we can no longer have an infinite plane of oscillating charges – we need to set boundaries on $\mathcal{V}$. Another boundary we need to set is that the medium is very thin, due to the fact that we have not taken the loss of energy in the electromagnetic wave into account in the electron model. These conditions are portrayed in Figure 3.5:

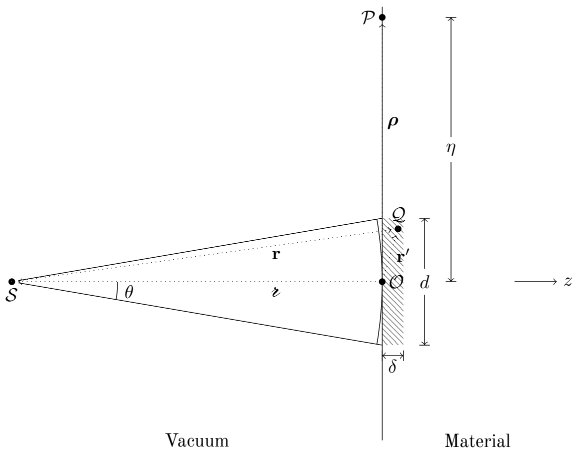

Figure 3.5: *A depiction meant to shed light on the different vectors and points used to calculate the magnitude of the electric and magnetic fields. The greyed area is the area where there are oscillating charges. Note that* $\mathscr{r} \gg d > \delta$

This allows us to define the volume where there are oscillating charges:

$$
\mathcal{V} = \left\{\mathbf{x}\in\mathbb{R}^3 : \left(x^2+y^2 = \frac{d^2}{4}\right) \text{ and } (0 \le z \le \delta)\right\}
$$

Approximating the integrals at S is possible, due to the symmetry. The calculation, which can be seen in Appendix A.3, yields the following approximation for the electric and magnetic fields:

$$\begin{aligned} E_{x,1} &\approx -\omega^2 Nq\mu\alpha_1(\omega) I_2^S e^{i\omega t} \\ B_{y,1} &\approx \frac{\omega^2}{c} Nq\mu\alpha_1(\omega) I_2^S e^{i\omega t} \\ I_2^S &= (1-\cos\theta)\frac{\pi c\ell}{i\omega}\left(e^{2i\omega(\ell+\delta)-e^{2i\omega\ell}}\right) \end{aligned} \tag{3.10}$$

where θ is the angle between $\hat{\boldsymbol{z}}$ and $\hat{\boldsymbol{z}} + \frac{d}{2}\hat{\mathbf{s}}$, and $\hat{\mathbf{s}}$ lies in the same plane as the interface between the media, as in Figure 3.5. Equation 3.10 is the reflected wave! The function I_2^S is deceiving, as one may assume that when ℓ grows, the magnitude of the electromagnetic waves grows. However $\cos\theta = \frac{\ell}{\sqrt{\ell^2+d^2/4}}$ and therefore

$$\ell(1-\cos\theta) = \ell\left(1 - \frac{\ell}{\sqrt{\ell^2+d^2/4}}\right)$$

which approaches zero when $\ell \to \infty$.

To find the total electric and magnetic fields, we just need to superpose this solution and the incident wave, which now is spherical. Note that the wave equations in Equations 3.9 for the magnetic field will be equal to zero, due to the y' and x', at the point S, therefore we do not need to worry about those waves, as they are not electromagnetic waves. Due to this symmetry we know that the only waves that reach the point S, that are transmitted by the oscillating electrons in the medium, will have frequency $n\omega$, where n is an odd number. We have just shown that this holds for ω, but this holds generally for $n\omega$, as we will see soon.

When $\theta \to 0$ as well as when $\delta \to 0$ the electric and magnetic fields will be equal to zero. This makes physical sense, because these limits mean that $\mathcal{V} \to 0$, in which case there is nothing to radiate electromagnetic waves.

These excitations are in phase and the magnetic field is smaller than the electric field, by a factor c. Additionally, we can see that the wave propagates in $-\hat{\mathbf{z}}$ direction, as expected (due to the different signs; $-\hat{\mathbf{x}} \times \hat{\mathbf{y}} = -\hat{\mathbf{z}}$). Interestingly enough, these waves will be proportional to the following frequency dependent value:

$$\zeta_1(\omega) = \frac{i\omega qE}{m(\omega_0^2 - \omega^2 - i\gamma\omega^3)}$$

Which is the coefficient of the velocity of charges that follow the first order approximation of the Fourier series we found in Section 3.1.1. Though this was done using a very different method from (Feynman et al. 1963), we agree on the frequency dependence of the electric field.

For a material with only one ω_0, the magnitude of the electric (and magnetic) field will be as shown in Figure 3.6, however, as materials usually have multiple ω_0s due to the fact that there are electrons at different energy levels, the frequency dependent coefficient will be a superposition of $\zeta_1^j(\omega)$s, where the j denotes that it is the jth $\omega_{0,j}$ we use when calculating $\zeta(\omega)$.

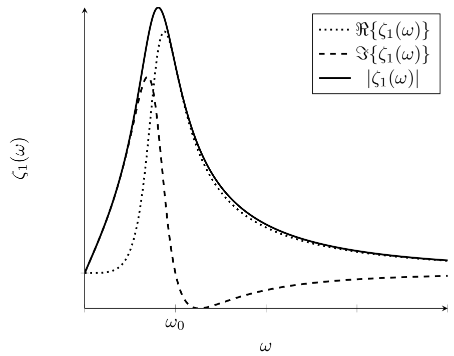

Figure 3.6: *Frequency dependence of the first harmonic of the reflected electric and magnetic waves, for (theoretical) materials which only have one ω_0. The dotted, dashed and solid lines are the real part, imaginary part and absolute value of $\zeta_1(omega)$*

We measure the real part of the electric wave, but the amplitude of this wave will be proportional to the absolute value of $\zeta_1(\omega)$. The maximum in Figure 3.6 makes physical sense – if we push electrons at their resonance frequency, they will accelerate more freely and thus radiate more, than at other frequencies.

We began this section with the first approximation of $\mathbf{v}$, as in Equation 3.6, and continued to calculate the first harmonic of the electric and magnetic fields and the point S. However, the $\hat{\mathbf{x}}$ component of the velocity also has higher order terms. Due to the fact that the only spacial and temporal dependence in the velocity is in the retarded time $t' = t - r/c$, we can readily generalise the result in Equation 3.10 to higher order frequencies.

We can generalise the inhomogeneous wave equation in Equation 3.7:

$$\begin{aligned}
\Box^2 E_{x,n} &= -n^2\omega^2 Nq\mu\alpha_n(\omega)e^{ni\omega t'} \\
\Box^2 E_{y,n} &= 0 \\
\Box^2 E_{z,n} &= 0 \\
\Box^2 B_{x,n} &= 0 \\
\Box^2 B_{y,n} &= n^2\omega^2 \frac{Nq\mu}{rc}(z' + \ell)\alpha_n(\omega)e^{ni\omega t'} \\
\Box^2 B_{z,n} &= -n^2\omega^2 \frac{Nq\mu}{rc} y'\alpha_n(\omega)e^{ni\omega t'}
\end{aligned}$$

The same method we used to solve the first order equations can be used to solve the nth order frequency equations, because the spacial and temporal dependency has not changed. Therefore we can conclude that the electric and magnetic fields measured at the points S are:

$$\begin{aligned}
E_{x,n} &\approx -n^2\omega^2 Nq\mu\alpha_n(\omega)I^{\mathcal{S}}_{2,n}e^{ni\omega t} \\
B_{y,n} &\approx n^2\frac{\omega^2}{c} Nq\mu\alpha_n(\omega)I^{\mathcal{S}}_{2,n}e^{ni\omega t} \\
I^{\mathcal{S}}_{2,n} &= (1-\cos\theta)\frac{\pi c\ell}{ni\omega}\left(e^{2ni\omega(\ell+\delta)} - e^{2ni\omega\ell}\right)
\end{aligned}$$

Once again, we need to remember that n is an *odd* number. This solution implies that the n-th order electric field at point S is proportional to $\zeta_n(\omega)$:

$$\zeta_n(\omega) = ni\omega\alpha_n(\omega)$$

For example the frequency dependence of the third harmonic at the point S is given by $\zeta_3(\omega)$, which is plotted in Figure 3.7:

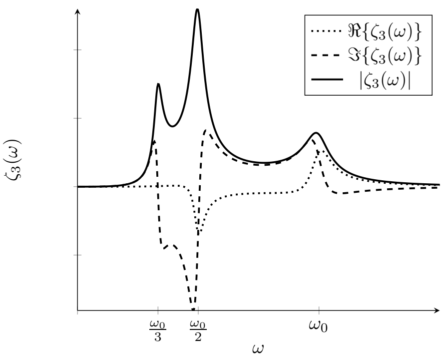

Figure 3.7: *Frequency dependence of the third harmonic of the reflected electric and magnetic waves, for (theoretical) materials which only have one ω_0. The dotted, dashed and solid lines are the real part, imaginary part and absolute value of $\zeta_3(\omega)$.*

Here will have three resonant frequencies, at ω_0, $\frac{1}{2}\omega_0$ and $\frac{1}{3}\omega_0$.

Unfortunately cannot do the same with the electric and magnetic fields at the point P, due to the lack of symmetry, which implies the integrals are far more complicated to solve. I have made a first order approximation of the integrals that give the second harmonics of the electric and magnetic waves at the point P. This first order approximation however results in an integral that evaluates to zero, this calculation can be seen in Appendix A.3.2. Therefore, if the first order approximation is dominant, that is to say that $\delta \ll \ell$, which is our assumption anyway, we can conclude that the 2ω component of the electric and magnetic fields at the point P are equal to zero. Once again, we could generalise this and claim that the $n\omega$, where n is even, components of the electric and magnetic waves at point P are equal to zero.

The solution to the ordinary differential equation that describes the movement of the electron, as in Section 3.1.1, assumes that the incoming waves are planar. This assumption was made early on, due to the fact that the system of differential equations are complicated enough to solve as they are, and having to solve these for spherical waves would have complicated matters further. However, in retrospect, it may have been a necessity to solve for the general spherical case, as the inhomogeneous wave equations in this section become nonsensical when we assume planar waves.

Therefore the inhomogeneous wave equations in this section are not valid for all values of ℓ (the distance between the interface and the incident wave source), but rather are only valid for very large values of ℓ. Over and above this, the conservation of energy is not exactly taken into account in the wave equations, due to the fact that we assume that the electromagnetic waves do not change in amplitude when passing through the matter. However, as we learnt in Section 2.2 some of the energy is indeed lost from the electromagnetic field, when said fields interact with matter. Therefore these inhomogeneous wave equations are additionally only valid for very thin layers of matter, thus implying δ must be a value comparable to or less than the wavelength of the incident wave.

The steps taken during this section were specifically done for the bound electrons, and therefore the $\alpha_n(\omega)$s and $\beta_n(\omega)$s used when discussing the frequency dependence of the electromagnetic waves at the point S were the coefficients for the bound electrons. However, there is no reason to specifically speak of the bound electrons, and therefore the same results are also valid for the free electrons, but we just need to remember to replace the $\alpha(\omega)$s and $\beta(\omega)$s of the bound electrons with those of the free electrons. However, the velocity field of the electrons was treated as a continuous vector function, but in reality the bound electrons cannot be described as continuous medium, therefore the curl of $\mathbf{v}$ is an odd concept. However, due to the fact that we said that $\ell \gg d$, we can assume that we are far enough away from the bound charges, that it is a good approximation to treat them as a type of continuum. For the free electrons, however, it is more valid to treat them as a continuum, thus implying the curl of $\mathbf{v}$ makes physics sense.

4 Conclusion

In Chapter 2, began with theories that are taught in undergraduate lectures, such as Maxwell's Equations and the Drude-Lorentz model. These were used to show the standard method for the calculation the index of refraction.

In Chapter 3 we went into further detail, first describing the motion of electrons being accelerated by electromagnetic waves, and thereafter using the solution to their equations of motion to describe the index of refraction more thoroughly. Additionally an expression for the extinction coefficient of light in conducting material was found, using a macroscopic model. However, this model lacked the finer details, that are measured experimentally.

Finally the function of the electron's motion was put back into Maxwell's Equations, which gave us inhomogeneous wave equations, that describe refraction and reflection, as well as higher frequency-effects, such as the double-frequency electromagnetic waves that are sent parallel to the interface. The solution to inhomogeneous wave equation for the reflected wave could be approximated, due the symmetry, and we found the same frequency dependence of the electric amplitude as (Feynman et al. 1963). Additionally we were able to establish the magnitude of the electric and magnetic waves around the source point of the incident radiation, at all odd numbers times the incident frequency.

Unfortunately the inhomogeneous wave equations describing the 2ω component of the electric and magnetic fields at the point P could not be solved, due to the lack of symmetry. However, a first order approximation inside the integral gave an integral that evaluates to zero. Due to the fact that we assume that $\delta \ll \ell$ we can argue that the first order term will dominate, and therefore, for very thin sheets, we can argue that the 2ω component of the electric and magnetic fields are zero at the point P, and similarly we may argue that this also holds for the $n\omega$, where n is even. However, further work needs to be done on this, before we can confirm this hypothesis.

The solutions that describe the electromagnetic waves emitted by the bound electrons in the medium can readily be generalised for the free electrons. In order to do this, we merely need to replace the Fourier coefficients, $\alpha_n(\omega)$ and $\beta_n(\omega)$, for the bound electrons with those found for the free electrons – the remaining steps are valid for both the bound and free electrons.

Early on in the report, the assumption was made, that the incident waves are planar, as this is an assumption one often makes. However, this proved to cause some difficulty, with the inhomogeneous wave equations, therefore it would be beneficial to calculate the velocity field of electrons, that are accelerated by spherical electromagnetic waves.

The macroscopic description of the extinction coefficient showed to work qualitatively, one may investigate how this model behaves if we start to add free parameters, perhaps beginning with the parameter τ, as this is an effect that manifests itself in the free electrons. The free electrons are what makes the optical properties of metals different to those of nonmetals, and therefore introducing τ would probably be the best way to improve the model.

The methods used to describe the reflected electromagnetic waves, are very powerful and general, in the sense that they are valid for all frequencies. However, the theory discussed in this report can naturally be improved on, for example, by considering the following points.

1. Generalising the solution to the ordinary differential equation that describes the motion of electrons to spherical waves, instead of planar waves.
2. Calculating the $n\omega$ component of the electric and magnetic waves, not only at the points P and S, but anywhere in the plane.
3. Establishing the theoretical prediction of the amplitude of the 3ω component. Unfortunately this prediction requires the values for ω_0, γ and τ for each electron energy level, specific to the material at hand.
4. Finding a general solution for the integrals in Appendix A.3.
5. Refining the macroscopic model for the extinction coefficient, such that it behaves well for higher wavelengths, and includes some of the finer details, that can be seen in experimental data.

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A Calculations

A.1 Fourier Series to solve the ODE

$$\begin{aligned}
\ddot{x}_1 &= -\Lambda e^{i\omega t}\dot{x}_3 - \omega_0^2 x_1 + \Gamma e^{i\omega t} \\
\ddot{x}_2 &= 0 \\
\ddot{x}_3 &= \Lambda e^{i\omega t}\dot{x}_1 - \omega_0^2 x_3
\end{aligned}$$

The solutions to these differential equations will almost certainly be periodic (assuming they are not chaotic) and C^∞, thus it fulfils the Dirichlet conditions– we can write the solution as a Fourier series.

$$\begin{aligned}
\tilde{x}_1(t) &= \alpha_1 e^{i\omega t} + \alpha_2 e^{2i\omega t} + \cdots + \alpha_n e^{ni\omega t} \\
x_2(t) &= x_{2,0} + v_2 t = 0 \\
\tilde{x}_3(t) &= \beta_1 e^{i\omega t} + \beta_2 e^{2i\omega t} + \cdots + \beta_n e^{ni\omega t}
\end{aligned}$$

Where $n \to \infty$. The actual coordinates will naturally be the real part of these equations. Let us plug these into the differential equations

$$\begin{aligned}
-\left(\alpha_1\omega^2 e^{i\omega t} + 4\alpha_2\omega^2 e^{2i\omega t} + \cdots + n^2\alpha_n\omega^2 e^{ni\omega t}\right) = -\Lambda e^{i\omega t}\left(i\omega\beta_1 e^{i\omega t} + 2i\omega\beta_2 e^{2i\omega t} + \cdots + ni\omega\beta_n e^{ni\omega t}\right) \\
+\Gamma e^{i\omega t} - \omega_0^2\left(\alpha_1 e^{i\omega t} + \alpha_2 e^{2i\omega t} + \cdots + \alpha_n e^{ni\omega t}\right)
\end{aligned} \tag{A.1}$$

$$\begin{aligned}
-\left(\omega^2\beta_1 e^{i\omega t} + 4\omega^2\beta_2 e^{2i\omega t} + \cdots + n^2\omega^2\beta_n e^{ni\omega t}\right) = \Lambda e^{i\omega t}\left(i\omega\alpha_1 e^{i\omega t} + 2i\omega\alpha_2 e^{2i\omega t} + \cdots + ni\omega\alpha_n e^{ni\omega t}\right) \\
-\omega_0^2\left(\beta_1 e^{i\omega t} + \beta_2 e^{2i\omega t} + \cdots + \beta_n e^{ni\omega t}\right)
\end{aligned} \tag{A.2}$$

Due to the mutual orthogonality of $\left\{e^{ni\omega t}, e^{mi\omega t}\right\}$ given that $n \neq m$, we can rewrite these as a set of equations, one for each harmonic. We begin with the first (leaving out the $e^{i\omega t}$ term as they all have that in common):

$$\begin{aligned}
-\alpha_1\omega^2 &= \Gamma - \omega_0^2\alpha_1 \to \alpha_1 = \frac{\Gamma}{\omega_0^2 - \omega^2} \\
-\beta_1\omega^2 &= \quad - \omega_0^2\beta_1 \to \beta_1 = 0
\end{aligned}$$

The second implication is because we are assuming that $\omega \neq \omega_0$. There will be constructive interference when $\omega = \omega_0$ which will create some movement which goes beyond the scope of this project. Now for the second order terms:

$$\begin{aligned}
-4\omega^2\alpha_2 &= -\Lambda i\omega\beta_1 - \omega_0^2\alpha_2 \to \alpha_2 = \frac{-\Lambda i\omega\beta_1}{\omega_0^2 - 4\omega^2} = 0 \\
-4\omega^2\beta_2 &= \quad \Lambda i\omega\alpha_1 - \omega_0^2\beta_2 \to \beta_2 = \frac{\Lambda i\omega\alpha_1}{\omega_0^2 - 4\omega^2}
\end{aligned}$$

Now I could use α_1 to explicitly state what β_2 is, but I will keep it in this form, to get a general form for α_n and β_n. Using the same steps we'll get

$$\begin{aligned}
\alpha_3 &= \frac{-2i\omega\Lambda\beta_2}{\omega_0^2 - 9\omega^2} \\
\beta_3 &= 0
\end{aligned}$$

We can already see some tendencies and because we have a beautifully behaving infinite sum, we can just put it all together:

$$\begin{aligned}
\alpha_n &= \frac{-(n-1)i\omega\Lambda\beta_{n-1}}{\omega_0^2 - n^2\omega^2} \\
\beta_n &= \frac{(n-1)i\omega\Lambda\alpha_{n-1}}{\omega_0^2 - n^2\omega^2}
\end{aligned}$$

Putting this together we get that

$$\alpha_n = \frac{(n-1)(n-2)\omega^2\Lambda^2\alpha_{n-2}}{(\omega_0^2 - n^2\omega^2)(\omega_0^2 - (n-1)^2\omega^2)}$$
$$\beta_n = \frac{(n-1)(n-2)\omega^2\Lambda^2\beta_{n-2}}{(\omega_0^2 - n^2\omega^2)(\omega_0^2 - (n-1)^2\omega^2)}$$

These recursive definitions of course rely on base cases, we have already calculated the base cases for α:

$$\alpha_1 = \frac{\Gamma}{\omega_0^2 - \omega^2}$$
$$\alpha_2 = 0$$

And the base cases for β are:

$$\beta_1 = 0$$
$$\beta_2 = \frac{i\omega\Lambda\Gamma}{(\omega_0^2 - 4\omega^2)(\omega_0^2 - \omega^2)}$$

Finally we can compress this information into one equation for each

$$\alpha_n = \frac{(n-1)!\omega^{n-1}\Lambda^{n-1}\Gamma}{\prod_{\ell=1}^{n}(\omega_0^2 - \ell^2\omega^2)} \tag{A.3}$$
$$\beta_n = \frac{i(n-1)!\omega^{n-1}\Lambda^{n-1}\Gamma}{\prod_{\ell=1}^{n}(\omega_0^2 - \ell^2\omega^2)}$$

Where n is odd for α and even for β. Thus we conclude that the solutions are

$$x_1(t) = \sum_{n=1}^{\infty} \frac{(2n-2)!\omega^{2n-2}\Lambda^{2n-2}\Gamma}{\prod_{\ell=1}^{n}(\omega_0^2 - \ell^2\omega^2)} \cos((2n-1)\omega t)$$
$$x_3(t) = -\sum_{n=1}^{\infty} \frac{(2n-1)!\omega^{2n-1}\Lambda^{2n-1}\Gamma}{\prod_{\ell=1}^{n}(\omega_0^2 - \ell^2\omega^2)} \sin(2n\omega t) \tag{A.4}$$

However, charged particles radiate energy, which means as the particle accelerates, the oscillation is damped. This dampening is proportional to $\dddot{x}$ (Griffiths 2017). Thus giving us these differential equations

$$\begin{aligned}
\ddot{x}_1 &= -\Lambda e^{i\omega t}\dot{x}_3 - \omega_0^2 x_1 - \gamma\dddot{x}_1 + \Gamma e^{i\omega t} \\
\ddot{x}_2 &= \quad 0 \\
\ddot{x}_3 &= \quad \Lambda e^{i\omega t}\dot{x}_1 - \omega_0^2 x_3 - \gamma\dddot{x}_3
\end{aligned}$$

This would turn Equations A.1 and A.2 into:

$$\begin{aligned}
-\left(\alpha_1\omega^2 e^{i\omega t} + 4\alpha_2\omega^2 e^{2i\omega t} + \cdots + n^2\alpha_n\omega^2 e^{ni\omega t}\right) &= -\Lambda e^{i\omega t}\left(i\omega\beta_1 e^{i\omega t} + 2i\omega\beta_2 e^{2i\omega t} + \cdots + ni\omega\beta_n e^{ni\omega t}\right) \\
+\Gamma e^{i\omega t} - \omega_0^2\left(\alpha_1 e^{i\omega t} + \alpha_2 e^{2i\omega t} + \cdots + \alpha_n e^{ni\omega t}\right) &+ \gamma\left(\alpha_1 i\omega^3 e^{i\omega t} + 8i\alpha_2\omega^3 e^{2i\omega t} + \cdots + n^3\omega^3 i\alpha_n e^{ni\omega t}\right)
\end{aligned}$$

$$\begin{aligned}
-\left(\omega^2\beta_1 e^{i\omega t} + 4\omega^2\beta_2 e^{2i\omega t} + \cdots + n^2\omega^2\beta_n e^{ni\omega t}\right) &= \Lambda e^{i\omega t}\left(i\omega\alpha_1 e^{i\omega t} + 2i\omega\alpha_2 e^{2i\omega t} + \cdots + ni\omega\alpha_n e^{ni\omega t}\right) \\
-\omega_0^2\left(\beta_1 e^{i\omega t} + \beta_2 e^{2i\omega t} + \cdots + \beta_n e^{ni\omega t}\right) &+ \gamma\left(\beta_1 i\omega^3 e^{i\omega t} + 8i\beta_2\omega^3 e^{2i\omega t} + \cdots + n^3\omega^3 i\beta_n e^{ni\omega t}\right)
\end{aligned}$$

This looks like it might complicate things awfully, but luckily we can merely replace ω_0^2 with $\omega_0^2 - i\gamma\omega^3 m^3$ in Equations A.3 to A.4, thus:

$$\alpha_n = \frac{(n-1)!\omega^{n-1}\Lambda^{n-1}\Gamma}{\prod_{\ell=1}^{n}(\omega_0^2 - \ell^2\omega^2 - i\gamma\omega^3\ell^3)}$$
$$\beta_n = \frac{i(n-1)!\omega^{n-1}\Lambda^{n-1}\Gamma}{\prod_{\ell=1}^{n}(\omega_0^2 - \ell^2\omega^2 - i\gamma\omega^3\ell^3)}$$
$$x_1(t) = \sum_{n=1}^{\infty}\frac{(2n-2)!\omega^{2n-2}\Lambda^{2n-2}\Gamma}{\prod_{\ell=1}^{n}(\omega_0^2 - \ell^2\omega^2 - i\gamma\omega^3\ell^3)}\cos((2n-1)\omega t)$$
$$x_3(t) = -\sum_{n=1}^{\infty}\frac{(2n-1)!\omega^{2n-1}\Lambda^{2n-1}\Gamma}{\prod_{\ell=1}^{n}(\omega_0^2 - \ell^2\omega^2 - i\gamma\omega^3\ell^3)}\sin(2n\omega t)$$

A.2 Electromagnetic waves radiated by electrons following the solution to the differential equations in Section 3.1.1

Now that we have found a solution to the differential equation discussed in Section 3.1 we can begin to discuss the electromagnetic waves that are created by these oscillations. The expression we have for the lower terms of the retarded acceleration of the electron is

$$\ddot{x}_1(t) = \frac{-\omega^2 qEe^{i\omega(t-r/c)}}{m(\omega_0^2 - \omega^2 - i\gamma\omega^3)}$$
$$\ddot{x}_3(t) = \frac{-4i\omega^3 q^2 EBe^{2i\omega(t-r/c)}}{m^2(\omega_0^2 - \omega^2 - i\gamma\omega^3)(\omega_0^2 - 4\omega^2 - 8i\gamma\omega^3)}$$

We will begin by discussing the electromagnetic waves due to the oscillation in the x_1 direction. The electric field is given as:

$$\mathbf{E} = \frac{-\eta\omega^2 q^2 Ee^{i\omega t}}{2\varepsilon_0 mc^2(\omega_0^2 - \omega^2 - i\gamma\omega^3)}\int_0^{\infty}\frac{\rho e^{-i\omega \mathscr{r}/c}}{\mathscr{r}}\mathrm{d}\rho$$

So we need to evaluate

$$\int_0^{\infty}\frac{\rho e^{-i\omega \mathscr{r}/c}}{\mathscr{r}}\mathrm{d}\rho$$

But

$$\mathscr{r}^2 = \rho^2 + z^2$$

thus,

$$\mathscr{r}\,\mathrm{d}\mathscr{r} = \rho\mathrm{d}\rho$$

as z is a constant, therefore

$$\int_0^{\infty}\frac{\rho e^{-i\omega \mathscr{r}/c}}{\mathscr{r}}\mathrm{d}\rho = \int_0^{\infty} e^{-i\omega \mathscr{r}/c}\mathrm{d}\mathscr{r} = \left[\frac{ic}{\omega}e^{-i\omega \mathscr{r}/c}\right]_0^{\infty}$$

Now usually we'd struggle with the $e^{i\infty}$, but in this case we will just set it to be zero (Feynman et al. 1963). This means that the total field is

$$\mathbf{E} = \frac{-i\eta\omega q^2 Ee^{i\omega t}}{2\varepsilon_0 mc(\omega_0^2 - \omega^2 - i\gamma\omega^3)}\hat{\mathbf{x}}_1$$

A.3 Integrals

A.3.1 Far from the plane, along the normal

Suppose we are at the point S. Here $\boldsymbol{\rho} = \boldsymbol{\mathscr{z}}$, which makes the first integral that we need to evaluate:

$$I_1^{\mathcal{S}} = \int_{\mathcal{V}} \frac{e^{2i\omega|\boldsymbol{\mathscr{z}} - \mathbf{r}'|/c}}{|\boldsymbol{\mathscr{z}} - \mathbf{r}'|} \mathrm{d}\mathbf{r}'$$

But due to the fact that S lies very far away from $\mathcal{V}$ we can assume that this merely is a truncated cone (and thus do the integral in spherical coordinates with centre S). $\delta \ll \ell$, and $d \ll \ell$, which implies that this is a valid assumption. Therefore:

$$I_1^{\mathcal{S}} = \int_0^{\theta} \int_0^{2\pi} \int_{\ell}^{\ell+\delta} \frac{e^{2i\omega r/c}}{r} r^2 \sin\theta' \mathrm{d}r \mathrm{d}\varphi \mathrm{d}\theta'$$

We get

$$I_1^{\mathcal{S}} = 2\pi(1-\cos\theta) \frac{c(2i\omega r - c)e^{2i\omega r/c}}{4\omega^2} \Bigg|_{\ell}^{\ell+\delta} \approx (1-\cos\theta)\frac{\pi c}{i\omega}\left((\ell+\delta)e^{2i\omega(\ell+\delta)/c} - \ell e^{2i\omega\ell/c}\right)$$

Next is:

$$\int_{\mathcal{V}} \frac{(z'+\ell)e^{2i\omega|\boldsymbol{\mathscr{z}} - \mathbf{r}'|/c}}{|\boldsymbol{\mathscr{z}} - \mathbf{r}'|^2} \approx (1-\cos\theta)\frac{\pi c\ell}{i\omega}\left(e^{2i\omega(\ell+\delta)} - e^{2i\omega\ell}\right) = I_2^{\mathcal{S}} \approx I_1^{\mathcal{S}}$$

Assuming that $\ell \gg \delta$ and that $\omega\ell \gg c$. The integrals that involve a x' and y' will evaluate to zero, due to the fact that we are integrating φ from 0 to 2π. Therefore no light will be measured with double the frequency at the point S. Note that in the limits where $\delta \to 0$ or $\theta \to 0$ there will be no electric field, which also makes physical sense. Additionally if δ equal to the wavelength of the light, then there will be no light either, due to interference.

A.3.2 Far from the plane, parallel to the interface. 2ω component

Now the point P, this one we will do in cylindrical coordinates. the z-axis as in Figure 3.5, and therefore s will be the radial component from the disc of oscillating charges to the point P, where we are measuring, we conclude that :

$$|\boldsymbol{\mathscr{z}} - \mathbf{r}'| = \sqrt{(z+\ell)^2 + s^2}$$
$$|\boldsymbol{\rho} - \mathbf{r}'| = \sqrt{z^2 + (\cos\varphi s + \eta)^2 + \sin^2\varphi s^2}$$

Which makes the integrals

$$I_1^{\mathcal{P}} = \int_0^{2\pi}\int_0^{\delta}\int_0^{\frac{d}{2}} \frac{e^{2i\omega\sqrt{(z+\ell)^2+s^2}/c} e^{2i\omega\sqrt{z^2+(\cos\varphi s+\eta)^2+\sin^2\varphi s^2}/c}}{\sqrt{z^2+(\cos\varphi s+\eta)^2+\sin^2\varphi s^2}} s \mathrm{d}s\mathrm{d}z\mathrm{d}\varphi$$

$$I_2^{\mathcal{P}} = \int_0^{2\pi}\int_0^{\delta}\int_0^{\frac{d}{2}} \frac{e^{2i\omega\sqrt{(z+\ell)^2+s^2}/c} e^{2i\omega\sqrt{z^2+(\cos\varphi s+\eta)^2+\sin^2\varphi s^2}}/c}{\sqrt{z^2+(\cos\varphi s+\eta)^2+\sin^2\varphi s^2}\sqrt{(z+\ell)^2+s^2}} s^2\cos\varphi \mathrm{d}s\mathrm{d}z\mathrm{d}\varphi$$

$$I_3^{\mathcal{P}} = \int_0^{2\pi}\int_0^{\delta}\int_0^{\frac{d}{2}} \frac{e^{2i\omega\sqrt{(z+\ell)^2+s^2}/c} e^{2i\omega\sqrt{z^2+(\cos\varphi s+\eta)^2+\sin^2\varphi s^2}/c}}{\sqrt{z^2+(\cos\varphi s+\eta)^2+\sin^2\varphi s^2}\sqrt{(z+\ell)^2+s^2}} s^2\sin\varphi \mathrm{d}s\mathrm{d}z\mathrm{d}\varphi$$

As before $\ell, \eta \gg \delta$ and $\ell, \eta \gg d$, which implies that

$$\begin{aligned}\sqrt{z^2 + (\cos\varphi s + \eta)^2 + \sin^2\varphi s^2} &\approx \eta\left(1 + \frac{\cos\varphi s}{\eta}\right)\\ \sqrt{(z+\ell)^2 + s^2} &\approx \ell\left(1 + \frac{z}{\ell}\right)\end{aligned}$$

and thus

$$\begin{aligned}e^{2i\omega\sqrt{z^2+(\cos\varphi s+\eta)^2+\sin^2\varphi s^2}/c} &\approx e^{2i\omega\eta/c}e^{2i\omega\cos\varphi s/c}\\ e^{2i\omega\sqrt{(z+\ell)^2+s^2}/c} &\approx e^{2i\omega\ell/c}e^{2i\omega z/c}\end{aligned}$$

Which makes the integrals

$$\begin{aligned}I_1^{\mathcal{P}} &\approx \frac{e^{2i\omega(\eta+\ell)/c}}{\eta}\int_0^{2\pi}\int_0^{\delta}\int_0^{\frac{d}{2}} e^{2i\omega(z+\cos\varphi s)/c}\left(1-\frac{\cos\varphi s}{\eta}\right)s\mathrm{d}s\mathrm{d}z\mathrm{d}\varphi = 0\\ I_2^{\mathcal{P}} &\approx \frac{e^{2i\omega(\eta+\ell)/c}}{\ell\eta}\int_0^{2\pi}\int_0^{\delta}\int_0^{\frac{d}{2}} e^{2i\omega(z+\cos\varphi s)/c}\left(1-\frac{z}{\ell}\right)\left(1-\frac{\cos\varphi s}{\eta}\right)s^2\cos\varphi\mathrm{d}s\mathrm{d}z\mathrm{d}\varphi = 0\\ I_3^{\mathcal{P}} &\approx \frac{e^{2i\omega(\eta+\ell)/c}}{\ell\eta}\int_0^{2\pi}\int_0^{\delta}\int_0^{\frac{d}{2}} e^{2i\omega(z+\cos\varphi s)/c}\left(1-\frac{z}{\ell}\right)\left(1-\frac{\cos\varphi s}{\eta}\right)s^2\sin\varphi\mathrm{d}s\mathrm{d}z\mathrm{d}\varphi = 0\end{aligned}$$

because $(1+x)^{-1} \approx 1 - x$ for small x. Due to the fact that all integrals give zero, we should probably take higher order terms into account

$$\sqrt{z^2 + (\cos\varphi s + \eta)^2 + \sin^2\varphi s^2} = \eta\left(1+\frac{\cos\varphi s}{\eta}\right)\sqrt{1+\frac{z^2+\sin^2\varphi s^2}{(\cos\varphi s+\eta)^2}} \approx \eta\left(1+\frac{\cos\varphi s}{\eta}\right)\left(1+\frac{z^2+\sin^2\varphi s^2}{2(\cos\varphi s+\eta)^2}\right)$$

$$\sqrt{(z+\ell)^2+s^2} = \ell\left(1+\frac{z}{\ell}\right)\sqrt{1+\frac{s^2}{(z+\ell)^2}} \approx \ell\left(1+\frac{z}{\ell}\right)\left(1+\frac{s^2}{2(z+\ell)^2}\right)$$

as $\sqrt{1+x} \approx 1 + \frac{x}{2}$ for small x values. These can be be rewritten as

$$\begin{aligned}\eta + \cos\varphi s + \frac{z^2+\sin^2\varphi s^2}{2(\cos\varphi s+\eta)} &= \eta\left(1+\frac{\cos\varphi s}{\eta}+\frac{z^2+\sin^2\varphi s^2}{2\eta(\cos\varphi s+\eta)}\right)\\ \ell + z + \frac{s^2}{2(z+\ell)} &= \ell\left(1+\frac{z}{\ell}+\frac{s^2}{2\ell(z+\ell)}\right)\end{aligned}$$

This implies that the integral becomes

$$I_1^{\mathcal{P}} \approx \frac{e^{2i\omega(\eta+\ell)/c}}{\eta}\int_0^{2\pi}\int_0^{\delta}\int_0^{\frac{d}{2}} e^{2i\omega(z+\cos\varphi)/c}e^{2i\omega/c\left(\frac{z^2+\sin^2\varphi s^2}{2(\cos\varphi s+\eta)}+\frac{s^2}{2(z+\ell)}\right)}\left(1-\frac{\cos\varphi s}{\eta}-\frac{z^2+\sin^2\varphi s^2}{2\eta(\cos\varphi s+\eta)}\right)s\mathrm{d}s\mathrm{d}z\mathrm{d}\varphi$$

And similar integrals for $I_2^{\mathcal{P}}$ and $I_3^{\mathcal{P}}$