Knuth's Exercise in Technical Writing in ForTheL (LaTeXVersion)

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Preliminaries on Sets (Types)

Signature 1. *A* type *is a set. Let A, B, C stand for sets. Let T stand for a type. Let $a : t$ stand for a is an element of t. Let $a \in t$ stand for a is an element of t.*

Let $x \neq y$ stand for $x! = y$.

Signature 2. *The* empty set *is a set that has no elements. Let $\emptyset$ stand for the empty set.*

Definition 1. *A* subset of *B is a set A such that every element of A is an element of B. Let $A \subseteq B$ stand for A is a subset of B. Let A* is contained in *B stand for A is a subset of B. Let B* contains *A stand for A is a subset of B.*

Preliminaries on Natural Numbers

[synonym number/numbers]

Signature 3. *A* natural number *is a notion.*

Definition 2. $\mathbb{N}$ *is the set of natural numbers. Let i,j,k,l,m denote natural numbers.*

Signature 4. 0 *is a natural number.*

Signature 5. 1 *is a natural number.*

Signature 6. *$k + l$ is a natural number.*

Signature 7. *$k * l$ is a natural number.*

Axiom 1. *$k- < -k + 1$.*

Axiom 2. *If $k \neq 0$ then there is l such that $k = l + 1$.*

Axiom 3. *$k + 0 = k$.*

Axiom 4. *$0 * l = 0$.*

Axiom 5. *$(k + 1) * l = (k * l) + l$.*

Signature 8. *$k \leq l$ is an atom.*

Axiom 6. *$k \leq k$.*

Axiom 7. *If $i \leq j \leq k$ then $i \leq k$.*

Axiom 8. *If not $l \leq m$ then $m + 1 \leq l$.*

Axiom 9. *If $i \leq j$ and $k \leq l$ then $i + k \leq j + l$.*

Axiom 10. *If $i \leq j$ and $k \leq l$ then $i * k \leq j * l$.*

Axiom 11. *(Archimedean Property) Assume that for every $k \in \mathbb{N}$ $i+(k*l) \leq j + (k * m)$. Then $l \leq m$.*

0.0.1 Preliminaries on Functions and Sequences

Axiom 12 (Functional Extensionality)**.** *Let f, g be functions such that $Dom(f) = Dom(g)$. If $f[i] = g[i]$ for all $i \in Dom(f)$ then $f = g$.*

Definition 3. *A* sequence of length *l is a function a such that $Dom(a) = \{i \in \mathbb{N} \mid 1 \leq i \leq l\}$.*

Definition 4. *$\mathbb{N}^l = \{a \mid a$ is a sequence of length l such that $a[i] \in \mathbb{N}$ for all $i \in Dom(a)\}$.*

We fix a dimension or length:

Signature 9. *n is a natural number. Let a, b, c, p denote elements of $\mathbb{N}^n$.*

Signature 10. *$a++b$ is a sequence of length n such that*

$$(a++b)[i] = a[i] + b[i]$$

for all $i \in Dom(a)$.

Signature 11. *0^n is a sequence of length n such that $0^n[i] = 0$ for all $i \in Dom(0^n)$.*

Lemma 1. *$c++0^n$ is a sequence of length n and $c = c++(0^n)$.*

Proof. For all $i \in Dom(c)$ $c[i] = (c++0^n)[i]$. $Dom(c) = Dom(c++0^n)$. $\square$

Signature 12. *$k**a$ is a sequence of length n such that $(k**a)[i] = k*a[i]$ for all $i \in Dom(a)$.*

Lemma 2. *$(l+1)**p = (l**p)++p$.*

Proof. $(l**p)++p \in \mathbb{N}^n$. Let us show that for all elements $i of Dom(p)$ we have

$$((l+1)**p)[i] = ((l**p)++p)[i].$$

end. $\square$

Preliminaries on Special Subsets of $\mathbb{N}^n$

Signature 13. *A* submonoid *is a subset Q of $\mathbb{N}^n$ such that $0^n \in Q$ and $a++b \in Q$ for all $a, b \in Q$.*

Lemma 3. *Let Q be a submonoid and $p \in Q$. Then for all natural numbers k $k**p \in Q$.*

Proof. (Proof by induction) Let k be a natural number.

(1) Case $k = 0$. Then $k**p = 0^n \in Q$. end.

(2) Case $k \neq 0$. Take l such that $k = l+1$. $l-<-k$ and $l**p \in Q$. Then $k**p = (l+1)**p = (l**p)++p \in Q$. end. $\square$

Definition 5. *$A(n) = \{a \in \mathbb{N}^n \mid$ for all i, j (if $1 \leq i \leq j \leq n$ then $a[i] \leq a[j])\}$.*

Signature 14. *Let $P \subseteq \mathbb{N}^n$. The* submonoid generated by P *is a submonoid Q such that P is contained in Q and Q is contained in every submonoid that contains P. Let P^* stand for the submonoid generated by P.*

Definition 6. *Let $A, B \subseteq \mathbb{N}^n$. $A+++B = \{a++b \mid a \in A$ and $b \in B\}$.*

The Lemma

Lemma 4. *Let* $P, C \subseteq \mathbb{N}^n$*. Let* $C \neq \emptyset$ *and* $C +++ (P^*) \subseteq A(n)$*. Then we have* $C, P \subseteq A(n)$*.*

Proof. We have $C \subseteq A(n)$. Indeed for all $c \in C$

$$c = c ++ 0^n \in (C +++ (P^*)) \subseteq A(n).$$

Let us show that $P \subseteq A(n)$. Let $p \in P$. Let i, j be natural numbers such that $1 \leq i \leq j \leq n$. Take $c \in C$. For all $k \in \mathbb{N}$ we have

$$c ++ (k ** p) \in C +++ (P^*) \subseteq A(n)$$

and

$$c[i] + (k * p[i]) \leq c[j] + (k * p[j]).$$

$p[i] \leq p[j]$ [using the Archimedean Property]. end. $\square$