ELEMENTARY SET THEORY

An SAD3 Formalisation of the Appendix of "General Topology" by John L. Kelley Relations and Preliminaries

October 26, 2018

0.1 Preliminaries

[prove off]
Let x, y, z stand for *classes.*
[object/-s]

Signature (Ontology)**.** *An object is a notion. Let a, b, c, d, e stand for objects.*

Let $a \in x$ stand for a is an *element* of x.

Axiom. *Every element of x is an object.*

Axiom (I)**.** *For each x for each y $x = y$ iff for each z $z \in x$ iff $z \in y$.*

[set/-s]

Definition (1)**.** *A* set *is a class that is an object.*

Definition (2)**.** $x \cup y = \{object\ u \mid u \in x\ or\ u \in y\}$.

Definition (3)**.** $x \cap y = \{object\ u \mid u \in x\ and\ u \in y\}$.

Definition (25)**.** *A* subclass *of y is a class x such that each element of x is an element of y. Let $x \subset y$ stand for x is a subclass of y. Let x is* contained *in y stand for $x \subset y$.*

Theorem (27)**.** *$x = y$ iff $x \subset y$ and $y \subset x$.*

Theorem (28)**.** *If $x \subset y$ and $y \subset z$ then $x \subset z$.*

Signature (48)**.** *(a, b) is an object.*

Definition (48a)**.** *An* ordered pair *is an object c such that there exist objects a and b such that $c = (a, b)$.*

Axiom (55)**.** *If $(a, b) = (c, d)$ then $a = c$ and $b = d$.*

0.2 Relations

[relation/-s]

Definition (56)**.** *A* relation *is a class r such that every element of r is an ordered pair.*

Let r, s, t stand for relations.

Definition (57)**.** $r \circ s = \{(x, z) \mid x, z$ *are objects and there exists b such that* $(x, b) \in s$ *and* $(b, z) \in r\}$.

Theorem (58)**.** $(r \circ s) \circ t = r \circ (s \circ t)$.

Proof. $(r \circ s) \circ t \subset r \circ (s \circ t)$ and $r \circ (s \circ t) \subset (r \circ s) \circ t$. □

[/prove]

Theorem (59a)**.** $r \circ (s \cup t) = (r \circ s) \cup (r \circ t)$.

Proof. $r \circ (s \cup t) \subset (r \circ s) \cup (r \circ t)$. $(r \circ s) \cup (r \circ t) \subset r \circ (s \cup t)$. □

Theorem (59b)**.** $r \circ (s \cap t) \subset (r \circ s) \cap (r \circ t)$.

Definition (60)**.** $r^{-1} = \{(b, a) \mid a, b$ *are objects and* $(a, b) \in r\}$. *Let the* relation inverse *to r stand for r^{-1}.*

Lemma. r^{-1} *is a relation.*

Theorem (61)**.** $(r^{-1})^{-1} = r$.

Proof. $r \subset (r^{-1})^{-1}$. $(r^{-1})^{-1} \subset r$. □

Lemma (62a)**.** *Assume $r \subset s$. Then $r^{-1} \subset s^{-1}$.*

Lemma (62b)**.** $(r \circ s)^{-1} \subset (s^{-1}) \circ (r^{-1})$.

Lemma. $(s^{-1}) \circ (r^{-1}) \subset (r \circ s)^{-1}$.

Proof. $((s^{-1}) \circ (r^{-1}))^{-1} \subset ((r^{-1})^{-1}) \circ ((s^{-1})^{-1})$ (by 62b) . $((s^{-1}) \circ (r^{-1}))^{-1} \subset r \circ s$ (by 61) . $(((s^{-1}) \circ (r^{-1}))^{-1})^{-1} \subset (r \circ s)^{-1}$ (by 62a). $(s^{-1}) \circ (r^{-1}) \subset (r \circ s)^{-1}$ (by 61). □

Theorem (62)**.** $(r \circ s)^{-1} = (s^{-1}) \circ (r^{-1})$.

Proof. $(r \circ s)^{-1} \subset (s^{-1}) \circ (r^{-1})$. $(s^{-1}) \circ (r^{-1}) \subset (r \circ s)^{-1}$. □