ELEMENTARY SET THEORY

An SAD3 Formalisation of the Appendix of
"General Topology" by John L. Kelley
Functions and Preliminaries

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0.1 Preliminaries

[prove off]
Let x, y, z stand for *classes*.
[object/-s]

Signature (Ontology). *An object is a notion. Let a, b, c, d, e, u, v stand for objects.*

Let $a \in x$ stand for a is an *element* of x.

Axiom. *Every element of x is an object.*

Axiom (I). *For each x for each y $x = y$ iff for each z $z \in x$ iff $z \in y$.*

[set/-s]

Definition (1). *A* set *is a class that is an object.*

Definition (2). $x \cup y = \{object\ u \mid u \in x\ or\ u \in y\}$.

Definition (23). $\bigcup x = \{objectu \mid\ for\ some\ y(y \in x\ and\ u \in y)\}$. *Let the* union *of x stand for $\bigcup x$.*

Definition (25). *A* subclass *of y is a class x such that each element of x is an element of y. Let $x \subset y$ stand for x is a subclass of y. Let x is* contained *in y stand for $x \subset y$.*

Theorem (27). *$x = y$ iff $x \subset y$ and $y \subset x$.*

Theorem (28). *If $x \subset y$ and $y \subset z$ then $x \subset z$.*

Axiom (III). *If x is a set then there is a set y such that for each z if $z \subset x$ then $z \in y$.*

Theorem (33). *If x is a set and $z \subset x$ then z is a set.*

Definition (36)**.** $2^x = \{set\ y \mid y \subset x\}$.

Theorem (38a)**.** *If x is a set then 2^x is a set.*

Proof. Let x be a set. Take a set y such that for each z if $z \subset x$ then $z \in y$ (by III). Then $2^x \subset y$. □

Definition (40)**.** $\{a\} = \{a\}$.

Signature (48)**.** (a, b) *is an object.*

Definition (48a)**.** *An* ordered pair *is an object c such that there exist objects a and b such that $c = (a, b)$.*

Axiom (55)**.** *If $(a, b) = (c, d)$ then $a = c$ and $b = d$.*

[relation/-s]

Definition (56)**.** *A* relation *is a class r such that every element of r is an ordered pair.*

Let r, s, t stand for relations.

Definition (57)**.** $r \circ s = \{(x, z) \mid x, z \text{ are objects and there exists } b \text{ such that } (x, b) \in s \text{ and } (b, z) \in r\}$.

0.2 Functions (Maps)

Since "function" is predefined in SAD3, we use the word "map" instead.

[/prove] [map/-s]

Definition (63)**.** *A* map *is a relation f such that for each a, b, c if $(a, b) \in f$ and $(a, c) \in f$ then $b = c$. Let f, g stand for maps.*

Theorem (64)**.** *If f, g are maps then $f \circ g$ is a map.*

Definition (65)**.** $\mathrm{domain} f = \{object\ u \mid \text{ there exists an object } v \text{ such that } (u, v) \in f\}$.

Definition (66)**.** $\mathrm{range} f = \{object\ v \mid \text{ there exists an object } u \text{ such that } (u, v) \in f\}$.

Signature (68)**.** *Let $u \in \mathrm{domain} f$. The value of f at u is an object v such that $(u, v) \in f$. Let $f(u)$ stand for the value of f at u.*

Theorem (70)**.** *Let f be a map. Then $f = \{(u, f(u)) \mid u \in \mathrm{domain} f\}$.*

Theorem (71)**.** *Assume* $\mathrm{domain} f = \mathrm{domain} g$ *and for every element u of* $\mathrm{domain} f$ $f(u) = g(u)$. *Then $f = g$.*

Axiom (V)**.** *If f is a map and* $\mathrm{domain} f$ *is a set then* $\mathrm{range} f$ *is a set.*

Axiom (VI)**.** *If x is a set then $\bigcup x$ is a set.*

Definition (72)**.** $x \times y = \{(u, v) \mid u \in x \text{ and } v \in y\}$.

Theorem (73)**.** *Let u be an object. Let y be a set. Then $\{u\} \times y$ is a set.*

Proof. Define $f = \{(w, v) \mid w \in y \text{ and } v = (u, w)\}$. f is a map. $\mathrm{domain} f = y$. $\mathrm{range} f = \{u\} \times y$. □

Theorem (74)**.** *Let x, y be sets. Then $x \times y$ is a set.*

Proof. Define $f = \{(u, w) \mid u \in x \text{ and } w = \{u\} \times y\}$. f is a map. $\mathrm{domain} f = x$. $\mathrm{range} f$ is a set. $\mathrm{range} f = \{\text{set } z \mid \text{there exists } u \in x \text{ such that } z = \{u\} \times y\}$. $\bigcup(\mathrm{range} f)$ is a set. $\bigcup(\mathrm{range} f) \subset x \times y$. Let us show that $x \times y \subset \bigcup(\mathrm{range} f)$. Let $w \in x \times y$. Take an $u \in x$ and $v \in y$ such that $w = (u, v)$. $w \in \{u\} \times y \in \mathrm{range} f$. $w \in \bigcup \mathrm{range} f$. end. □

Theorem (75)**.** *Let f be a map. Let $\mathrm{domain} f$ be a set. Then f is a set.*

Proof. $f \subset \mathrm{domain} f \times \mathrm{range} f$. □

Definition (76)**.** $y^x = \{\mathit{map}\ f \mid \mathrm{domain} f = x \text{ and } \mathrm{range} f \subset y\}$.

Theorem (77)**.** *Let x, y be sets. Then y^x is a set.*

Proof. $y^x \subset 2^{x \times y}$. □

Definition (78)**.** f *is* on x *iff* $x = \mathrm{domain} f$.

Definition (79)**.** f *is* to y *iff* $\mathrm{range} f \subset y$.

Definition (80)**.** f *is* onto y *iff* $\mathrm{range} f = y$.