Numbers for Rudin

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0.1 Set-Theoretic Terminology

Let A, B stand for sets. Let $x \in A$ denote x is an element of A. Let x is *in* A denote x is an element of A. Let $x \notin A$ denote x is not an element of A.

Signature 1 (1.3) *The* empty set *is the set that has no elements. Let* $\emptyset$ *denote the empty set.*

Definition 1 *A is* nonempty *iff A has an element.*

Definition 2 *A subset of B is a set A such that every element of A is an element of B. Let $A \subseteq B$ stand for A is a subset of B. Let $B \supseteq A$ stand for A is a subset of B.*

Definition 3 *A proper subset of B is a subset A of B such that there is an element of B that is not in A.*

Proposition 1 *$A \subseteq A$.*

Proposition 2 *If $A \subseteq B$ and $B \subseteq A$ then $A = B$.*

Definition 4 $A \cup B = \{x \mid x \in A \lor x \in B\}$.

1 The real field

[number/-s]

Signature 2 *A* real number *is a notion.*

Signature 3 $\mathbb{R}$ *is the set of* real numbers. *Let x, y, z denote real numbers.*

Signature 4 (1.12 A1) *$x + y$ is a real number. Let the* sum *of x and y denote $x + y$.*

Signature 5 (1.12 M1) *$x \cdot y$ is a real number. Let the product of x and y denote $x \cdot y$.*

Signature 6 (1.5) *$x < y$ is an atom. Let $x > y$ stand for $y < x$. Let $x \leq y$ stand for $x < y \vee x = y$. Let $x \geq y$ stand for $y \leq x$.*

Axiom 1 (1.5(i)) *$x < y \wedge x \neq y \wedge \neg y < x$ or $\neg x < y \wedge x = y \wedge \neg y < x$ or $\neg x < y \wedge x \neq y \wedge y < x$.*

Axiom 2 (1.5(ii)) *If $x < y$ and $y < z$ then $x < z$.*

Proposition 3 *$x \leq y$ iff not $x > y$.*

Axiom 3 (1.12 A2) *$x + y = y + x$.*

Axiom 4 (1.12 A3) *$(x + y) + z = x + (y + z)$.*

Signature 7 (1.12 A4) *0 is a real number such that for every real number x $x + 0 = x$.*

Signature 8 (1.12 A5) *$-x$ is a real number such that $x + (-x) = 0$.*

Axiom 5 (1.12 M2) *$x \cdot y = y \cdot x$.*

Axiom 6 (1.12 M3) *$(x \cdot y) \cdot z = x \cdot (y \cdot z)$.*

Signature 9 (1.12 M4) *1 is a real number such that $1 \neq 0$ and for every real number x $1 \cdot x = x$.*

Signature 10 (1.12 M5) *Assume $x \neq 0$. $1/x$ is a real number such that $x * (1/x) = 1$.*

Axiom 7 (1.12 D) *$x \cdot (y + z) = (x \cdot y) + (x \cdot z)$.*

Proposition 4 (Dist1) *$(y \cdot x) + (z \cdot x) = (y + z) \cdot x$.*

Proposition 5 (1-14 a) *If $x + y = x + z$ then $y = z$.*

Proof Assume $x + y = x + z$. Then

$$y = (-x + x) + y = -x + (x + y) = -x + (x + z) = (-x + x) + z = z.$$

$\square$

Proposition 6 *If $x + y = x$ then $y = 0$.*

Proposition 7 *If $x + y = 0$ then $y = -x$.*

Proposition 8 (1.14 d) *$-(-x) = x$.*

Proposition 9 (1.15 a) *If $x \neq 0$ and $x \cdot y = x \cdot z$ then $y = z$.*

Proof Let $x \neq 0$ and $x \cdot y = x \cdot z$.

$y = 1 \cdot y = ((1/x) \cdot x) \cdot y = (1/x) \cdot (x \cdot y) = (1/x) \cdot (x \cdot z) = ((1/x) \cdot x) \cdot z = 1 \cdot z = z.$

□

Proposition 10 *If* $x \neq 0$ *and* $x \cdot y = x$ *then* $y = 1$.

Proposition 11 *If* $x \neq 0$ *and* $x \cdot y = 1$ *then* $y = 1/x$.

Proposition 12 *If* $x \neq 0$ *then* $1/(1/x) = x$.

Proposition 13 (1.16 a) $0 \cdot x = 0$.

Proposition 14 *If* $x \neq 0$ *and* $y \neq 0$ *then* $x \cdot y \neq 0$.

Proposition 15 $(-x) \cdot y = -(x \cdot y)$.

Proof $(x \cdot y) + (-x \cdot y) = (x + (-x)) \cdot y = 0 \cdot y = 0.$ □

Proposition 16 $-x = -1 \cdot x$.

Proposition 17 (1.16 d) $(-x) \cdot (-y) = x \cdot y$.

Proof $(-x) \cdot (-y) = -(x \cdot (-y)) = -((-y) \cdot x) = -(-(y \cdot x)) = y \cdot x = x \cdot y.$ □

Let $x - y$ stand for $x + (-y)$. Let $\frac{x}{y}$ stand for $x \cdot (1/y)$.

2 The real ordered field

Axiom 8 (1.17 i) *If* $y < z$ *then* $x + y < x + z$ *and* $y + x < z + x$.

Axiom 9 (1.17 ii) *If* $x > 0$ *and* $y > 0$ *then* $x \cdot y > 0$.

Definition 5 x *is* positive *iff* $x > 0$.

Definition 6 x *is* negative *iff* $x < 0$.

Proposition 18 (1.18 a) $x > 0$ *iff* $-x < 0$.

Proposition 19 (1.18 b) *If* $x > 0$ *and* $y < z$ *then* $x * y < x * z$.

Proof Let $x > 0$ and $y < z$. $z - y > y - y = 0$. $x \cdot (z - y) > 0$. $x \cdot z = (x \cdot (z - y)) + (x \cdot y)$. $(x \cdot (z - y)) + (x \cdot y) > 0 + (x \cdot y)$ (by 1.17 i). $0 + (x \cdot y) = x \cdot y$. □

Proposition 20 (1.18 bb) *If $x > 0$ and $y < z$ then $y \cdot x < z \cdot x$.*

Proposition 21 (1.18 d) *If $x \neq 0$ then $x \cdot x > 0$.*

Proposition 22 (1.18 dd) *$1 > 0$.*

Proposition 23 *$x < y$ iff $-x > -y$.*

Proof $x < y \Leftrightarrow x - y < 0$. $x - y < 0 \Leftrightarrow (-y) + x < 0$. $(-y) + x < 0 \Leftrightarrow (-y) + (-(-x)) < 0$. $(-y) + (-(-x)) < 0 \Leftrightarrow (-y) - (-x) < 0$. $(-y) - (-x) < 0 \Leftrightarrow -y < -x$. $\square$

Proposition 24 (1.18 c) *If $x < 0$ and $y < z$ then $x \cdot y > x \cdot z$.*

Proof Let $x < 0$ and $y < z$. $-x > 0$. $(-x) \cdot y < (-x) \cdot z$ (by 1.18 b). $-(x \cdot y) < -(x \cdot z)$. $\square$

Proposition 25 (1.18 cc) *If $x < 0$ and $y < z$ then $y \cdot x > z \cdot x$.*

Proposition 26 (Next) *$x + 1 > x$.*

Proposition 27 *$x - 1 < x$.*

Proposition 28 *If $0 < x$ then $0 < 1/x$.*

[prove off]

Proposition 29 *Assume $0 < x < y$. Then $1/y < 1/x$.*

Proof Case $1/x < 1/y$. Then

$$1 = x \cdot (1/x) = (1/x) \cdot x < (1/x) \cdot y = y \cdot (1/x) < y \cdot (1/y) = 1.$$

Contradiction. end.

Case $1/x = 1/y$. Then

$$1 = x * (1/x) < y * (1/y) = 1.$$

Contradiction. end.

Case $1/y < 1/x$. end. $\square$

3 Upper and lower bounds

[/prove]

Definition 7 *Let E be a subset of $\mathbb{R}$. An* upper bound *of E is a real number b such that for all elements x of E $x \leq b$.*

Definition 8 *Let E be a subset of $\mathbb{R}$. E is* bounded above *iff E has an upper bound.*

Definition 9 *Let E be a subset of $\mathbb{R}$. A* lower bound *of E is a real number b such that for all elements x of E $x \geq b$.*

Definition 10 *Let E be a subset of $\mathbb{R}$. E is* bounded below *iff E has a lower bound.*

Definition 11 *Let E be a subset of $\mathbb{R}$ such that E is bounded above. A* least upper bound *of E is a real number a such that a is an upper bound of E and for all x if $x < a$ then x is not an upper bound of E.*

Definition 12 *Let E be a subset of $\mathbb{R}$ such that E is bounded below. A* greatest lower bound *of E is a real number a such that a is a lower bound of E and for all x if $x > a$ then x is not a lower bound of E.*

Axiom 10 *Assume that E is a nonempty subset of $\mathbb{R}$ such that E is bounded above. Then E has a least upper bound.*

Definition 13 *. Let E be a subset of $\mathbb{R}$. $E^- = \{-x \mid x \in E\}$.*

Lemma 1 *Let E be a subset of $\mathbb{R}$. x is an upper bound of E iff $-x$ is a lower bound of E^-.*

Theorem 1 *Assume that E is a nonempty subset of $\mathbb{R}$ such that E is bounded below. Then E has a greatest lower bound.*

Proof Take a lower bound a of E. $-a$ is an upper bound of E^-. Take a least upper bound b of E^-. Let us show that $-b$ is a greatest lower bound of E. $-b$ is a lower bound of E. Let c be a lower bound of E. Then $-c$ is an upper bound of E^-. end. $\square$

4 The rational numbers

Signature 11 *A* rational number *is a real number. Let p, q, r stand for rational numbers.*

Definition 14 $\mathbb{Q}$ *is the set of rational numbers.*

$\mathbb{Q}$ is a subfield of $\mathbb{R}$:

Lemma 2 $\mathbb{Q} \subseteq \mathbb{R}$.

Axiom 11 $p+q$, $p \cdot q$, 0, $-p$, 1 *are rational numbers.*

Axiom 12 *Assume* $p \neq 0$. $1/p$ *is a rational number.*

Axiom 13 *There exists a subset* A *of* $\mathbb{Q}$ *such that (*A *is bounded above and* x *is a least upper bound of* A*).*

Theorem 2 $\mathbb{R} = \{x \in \mathbb{R} \mid$ *there exists* $A \subseteq \mathbb{Q}$ *such that* A *is bounded aboveand* x *is a least upper bound*

5 Integers

[integer/-s]

Signature 12 *An* integer *is a rational number. Let* a, b *stand for integers.*

Definition 15 $\mathbb{Z}$ *is the set of integers.*

$\mathbb{Z}$ is a discrete subring of $\mathbb{Q}$:

Axiom 14 $a+b$, $a * b$, 0, $-a$, 1 *are integers.*

Axiom 15 *There is no integer* a *such that* $0 < a < 1$.

Axiom 16 *There exist* a, b *such that* $a \neq 0 \wedge p = \frac{b}{a}$.

Theorem 3 (Archimedes1) $\mathbb{Z}$ *is not bounded above.*

Proof Assume the contrary. $\mathbb{Z}$ is nonempty. Take a least upper bound b of $\mathbb{Z}$. Let us show that $b-1$ is an upper bound of $\mathbb{Z}$. Let $x \in \mathbb{Z}$. $x+1 \in \mathbb{Z}$. $x+1 \leq b$. $x = (x+1) - 1 \leq b - 1$. end. □

Theorem 4 (Archimedes2) *There is an integer* a *such that* $x \leq a$.

Proof x is not an upper bound of $\mathbb{Z}$ (by Archimedes1). Take $a \in \mathbb{Z}$ such that not $a \leq x$. Then $x \leq a$. □

6 The natural numbers

Definition 16 $\mathbb{N}$ *is the set of positive integers. Let* m, n *stand for positive integers.*

Definition 17 $\{x\} = \{y \in \mathbb{R} \mid y = x\}$.

Lemma 3 $\mathbb{Z} = (\mathbb{N}^- \cup 0) \cup \mathbb{N}$.

Theorem 5 (Induction Theorem) *Assume* $A \subseteq \mathbb{N}$ *and* $1 \in A$ *and for all* $n \in A$ $n+1 \in A$. *Then* $A = \mathbb{N}$.

Proof Let us show that every element of $\mathbb{N}$ is an element of A. Let $n \in \mathbb{N}$. Assume the contrary. Define $F = \{j \in \mathbb{N} \mid j \notin A\}$. F is nonempty. F is bounded below. Take a greatest lower bound a of F. Let us show that $a + 1$ is a lower bound of F. Let $x \in F$. $x - 1 \in \mathbb{Z}$.

Case $x - 1 < 0$. Then $0 < x < 1$. Contradiction. end.

Case $x - 1 = 0$. Then $x = 1$ and $1 \notin A$. Contradiction. end.

Case $x-1 > 0$. Then $x-1 \in \mathbb{N}$. $x-1 \in F$. $a \leq x-1$. $a+1 \leq (x-1)+1 = x$. end. end.

Then $a + 1 > a$ (by Next). Contradiction. end. □

7 Archimedian properties

Theorem 6 (1.20 a) *Let* $x > 0$. *Then there is a positive integer* n *such that* $n \cdot x > y$.

Proof Take an integer a such that $a > \frac{y}{x}$. Take a positive integer n such that $n > a$. $n > \frac{y}{x}$ and $n \cdot x > (\frac{y}{x}) * x = y$. □

Theorem 7 (1.20 b) *Let* $x < y$. *Then there exists* $p \in \mathbb{Q}$ *such that* $x < p < y$.

Proof Assume $x < y$. Then $y - x > 0$. Take a positive integer n such that $n \cdot (y - x) > 1$ (by 1.20 a). [prove off] Take an integer m such that $m - 1 \leq n \cdot x < m$. Then

$$n{\cdot}x < m = (m-1)+1 <= (n{\cdot}x)+1 < (n{\cdot}x)+(n{\cdot}(y-x)) = n{\cdot}(x+(y-x)) = n{\cdot}y.$$

[/prove] $m \leq (n \cdot x) + 1 < n \cdot y$. Let us show that $m < n \cdot y$.

Case $m < (n \cdot x) + 1$. end.

Case $m = (n \cdot x) + 1$. end. end. $\frac{m}{n} < \frac{n \cdot y}{n}$. Indeed $m < n \cdot y$ and $1/n > 0$. Then

$$x = \frac{n \cdot x}{n} < \frac{m}{n} < \frac{n \cdot y}{n} = y.$$

Let $p = \frac{m}{n}$. Then $p \in \mathbb{Q}$ and $x < p < y$. □