ELEMENTARY SET THEORY

An SAD3 Formalisation of the Appendix of "General Topology" by John L. Kelley

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0.1 The Classification Axiom Scheme

Let $a, b, c, d, e, r, s, t, x, y, z$ stand for *classes*.
Let $a \in x$ stand for a is an *element* of x.

Axiom (I). *For each x for each y $x = y$ iff for each z $z \in x$ iff $z \in y$.*

[set/-s]

Definition (1). *A* set *is a class x such that for some y $x \in y$.*

0.2 Elementary Algebra of Classes

Definition (2). $x \cup y = \{set\ u \mid u \in x\ or\ u \in y\}$.

Definition (3). $x \cap y = \{set\ u \mid u \in x\ and\ u \in y\}$.

Let the *union* of x and y stand for $x \cup y$. Let the *intersection* of x and y stand for $x \cap y$.

Theorem (4a). *$z \in x \cup y$ iff $z \in x$ or $z \in y$.*

Theorem (4b). *$z \in x \cap y$ iff $z \in x$ and $z \in y$.*

Theorem (5a). $x \cup x = x$.

Theorem (5b). $x \cap x = x$.

Theorem (6a). $x \cup y = y \cup x$.

Theorem (6b). $x \cap y = y \cap x$.

Theorem (7a). $(x \cup y) \cup z = x \cup (y \cup z)$.

Theorem (7b)**.** $(x \cap y) \cap z = x \cap (y \cap z)$.

Theorem (8a)**.** $x \cap (y \cup z) = (x \cap y) \cup (x \cap z)$.

Theorem (8b)**.** $x \cup (y \cap z) = (x \cup y) \cap (x \cup z)$.

Let $a \notin b$ stand for a is not an element of b.

Definition (10)**.** $\sim x = \{set\ u \mid u \notin x\}$. *Let the* complement *of* x *stand for* $\sim x$.

Theorem (11)**.** $\sim (\sim x) = x$.

Theorem (12a)**.** $\sim (x \cup y) = (\sim x) \cap (\sim y)$.

Theorem (12b)**.** $\sim (x \cap y) = (\sim x) \cup (\sim y)$.

Definition (13)**.** $x \sim y = x \cap (\sim y)$.

Theorem (14)**.** $x \cap (y \sim z) = (x \cap y) \sim z$.

Definition (15)**.** $0 = \{set\ u \mid u \neq u\}$. *Let the* void class *stand for* 0. *Let* zero *stand for* 0.

Theorem (16)**.** $x \notin 0$.

Theorem (17a)**.** $0 \cup x = x$.

Theorem (17b)**.** $0 \cap x = 0$.

Definition (18)**.** $\mathcal{U} = \{set\ u \mid u = u\}$. *Let the* universe *stand for* $\mathcal{U}$.

Theorem (19)**.** $x \in \mathcal{U}$ *iff* x *is a set.*

Theorem (20a)**.** $x \cup \mathcal{U} = \mathcal{U}$.

Theorem (20b)**.** $x \cap \mathcal{U} = x$.

Theorem (21a)**.** $\sim 0 = \mathcal{U}$.

Theorem (21b)**.** $\sim \mathcal{U} = 0$.

Definition (22)**.** $\bigcap x = \{setu \mid \text{ for each } y \text{ if } y \in x \text{ then } u \in y\}$. *Let the* intersection *of* x *stand for* $\bigcap x$.

Definition (23)**.** $\bigcup x = \{setu \mid \text{ for some } y (y \in x \text{ and } u \in y)\}$. *Let the* union *of* x *stand for* $\bigcup x$.

Theorem (24a)**.** $\bigcap 0 = \mathcal{U}$.

Theorem (24b)**.** $\bigcup 0 = 0$.

Definition (25)**.** *A* subclass *of* y *is a class* x *such that each element of* x *is an element of* y. *Let* $x \subset y$ *stand for* x *is a subclass of* y. *Let* x *is* contained *in* y *stand for* $x \subset y$.

Proposition. $0 \subset 0$ *and* $0 \notin 0$.

Theorem (26a). $0 \subset x$.

Theorem (26b). $x \subset \mathcal{U}$.

Theorem (27). $x = y$ *iff* $x \subset y$ *and* $y \subset x$.

Theorem (28). *If* $x \subset y$ *and* $y \subset z$ *then* $x \subset z$.

Theorem (29). $x \subset y$ *iff* $x \cup y = y$.

Theorem (30). $x \subset y$ *iff* $x \cap y = x$.

Theorem (31a). *If* $x \subset y$ *then* $\bigcup x \subset \bigcup y$.

Theorem (31a). *If* $x \subset y$ *then* $\bigcap y \subset \bigcap x$.

Theorem (32a). *If* $x \in y$ *then* $x \subset \bigcup y$.

Theorem (32b). *If* $x \in y$ *then* $\bigcap y \subset x$.

0.3 Existence of Sets

Axiom (III). *If* x *is a set then there is a set* y *such that for each* z *if* $z \subset x$ *then* $z \in y$.

Theorem (33). *If* x *is a set and* $z \subset x$ *then* z *is a set.*

Theorem (34a). $0 = \bigcap \mathcal{U}$.

Theorem (34b). $\mathcal{U} = \bigcup \mathcal{U}$.

Theorem (35). *If* $x \neq 0$ *then* $\bigcap x$ *is a set.*

Definition (36). $2^x = \{set\ y \mid y \subset x\}$.

Theorem (37). $\mathcal{U} = 2^{\mathcal{U}}$.

Theorem (38a). *If* x *is a set then* 2^x *is a set.*

Proof. Let x be a set. Take a set y such that for each z if $z \subset x$ then $z \in y$ (by III). Then $2^x \subset y$. □

Theorem (38b). *If* x *is a set then* $y \subset x$ *iff* $y \in 2^x$.

Definition. $\mathrm{R} = \{set\ x \mid x \notin x\}$.

Lemma. R *is not a set.*

Theorem (39). $\mathcal{U}$ *is not a set.*

Definition (40). $\{x\} = \{set\ z \mid\ if\ x \in \mathcal{U}\ then\ z = x\}$. *Let the* singleton *of x stand for $\{x\}$.*

Theorem (41). *If x is a set then for each y $y \in \{x\}$ iff $y = x$.*

Theorem (42). *If x is a set then $\{x\}$ is a set.*

Proof. Let x be a set. Then $\{x\} \subset 2^x$. 2^x is a class. $\square$

Theorem (43). *$\{x\} = \mathcal{U}$ iff x is not a set.*

Theorem (44a). *If x is a set then $\bigcap\{x\} = x$.*

Theorem (44b). *If x is a set then $\bigcup\{x\} = x$.*

Theorem (44c). *If x is not a set then $\bigcap\{x\} = 0$.*

Theorem (44d). *If x is not a set then $\bigcup\{x\} = \mathcal{U}$.*

Axiom (IV). *If x is a set and y is a set then $x \cup y$ is a set.*

Definition (45). $\{x, y\} = \{x\} \cup \{y\}$. *Let the* unordered pair *of x and y stand for $\{x, y\}$.*

Theorem (46a). *If x is a set and y is a set then $\{x, y\}$ is a set.*

Theorem (46b). *If x is a set and y is a set then $z \in \{x, y\}$ iff $z = x$ or $z = y$.*

Theorem (46c). *$\{x, y\} = \mathcal{U}$ iff x is not a set or y is not a set.*

Theorem (47a). *If x, y are sets then $\bigcap\{x, y\} = x \cap y$.*

Theorem (47b). *If x, y are sets then $\bigcup\{x, y\} = x \cup y$.*

Proof. Let x, y be sets. $\bigcup\{x, y\} \subset x \cup y$. $x \cup y \subset \bigcup\{x, y\}$. $\square$

Theorem (47c). *If x is not a set or y is not a set then $\bigcap\{x, y\} = 0$.*

Theorem (47d). *If x is not a set or y is not a set then $\bigcup\{x, y\} = \mathcal{U}$.*

0.4 Ordered Pairs: Relations

Definition (48). $(x, y) = \{\{x\}, \{x, y\}\}$. *Let the* ordered pair *of x and y stand for (x, y).*

Theorem (49a). *(x, y) is a set iff x is a set and y is a set.*

Theorem (49b). *If (x, y) is not a set then $(x, y) = \mathcal{U}$.*

Theorem (50). *If x and y are sets then $\bigcup(x, y) = \{x, y\}$ and $\bigcap(x, y) = \{x\}$ and $\bigcup\bigcap(x, y) = x$ and $\bigcap\bigcap(x, y) = x$ and $\bigcup\bigcup(x, y) = x \cup y$ and $\bigcap\bigcup(x, y) = x \cap y$.*

Theorem. *If x is not a set or y is not a set then $\bigcup\bigcap(x,y) = 0$ and $\bigcap\bigcap(x,y) = \mathcal{U}$ and $\bigcup\bigcup(x,y) = \mathcal{U}$ and $\bigcap\bigcup(x,y) = 0$.*

Definition (51)**.** $1^{st}z = \bigcap\bigcap z$. *Let the* first coordinate *of z stand for $1^{st}z$.*

Definition (52)**.** $2^{nd}z = (\bigcap\bigcup z) \cup ((\bigcup\bigcup z) \sim \bigcup\bigcap z)$. *Let the* second coordinate *of z stand for $2^{nd}z$.*

Theorem (53)**.** $2^{nd}\mathcal{U} = \mathcal{U}$.

Theorem (54a)**.** *If x and y are sets then $1^{st}(x,y) = x$.*

Theorem (54b)**.** *If x and y are sets then $2^{nd}(x,y) = y$.*

Proof. Let x and y be sets. $2^{nd}(x,y) = (\bigcap\bigcup(x,y)) \cup ((\bigcup\bigcup(x,y)) \sim \bigcup\bigcap(x,y)) = (x \cap y) \cup ((x \cup y) \sim x) = y$. $\square$

Theorem (54c)**.** *If x is not a set or y is not a set then $1^{st}(x,y) = \mathcal{U}$ and $2^{nd}(x,y) = \mathcal{U}$.*

Theorem (55)**.** *If x and y are sets and $(x,y) = (r,s)$ then $x = r$ and $y = s$.*