

THE OUTPUT RESISTANCE OF A POWER SUPPLY: AN INTRODUCTORY RESISTANCE EXERCISE

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1 INTRODUCTION

Sources of electricity produce a voltage potential across their terminals called an electromotive force (*emf*) or an open circuit voltage, V_{∞} . In practice, when a closed circuit is made, a current I will be drawn and the voltage at the terminals, V called *terminal voltage* will fall below V_{∞} . A plot of what the terminal voltage V vs. current I may look like is shown in 1.

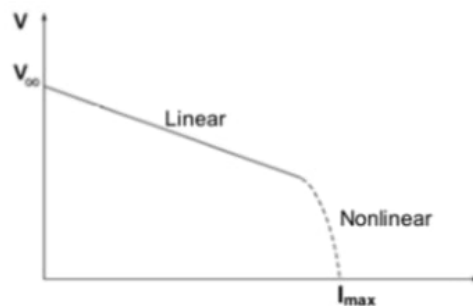


Figure 1: Terminal voltage vs. current

For most cases, many powers sources will exhibit a linear variation of R for small current values, with nonlinear behaviour at higher currents. The linear part of the curve can be described by:

$$V = V_{\infty} - RI$$

where R is the *output resistance of the powers source*. In this linear regime, according to Thevenin's theorem, the power source is completely represented by the equivalent circuit shown below.

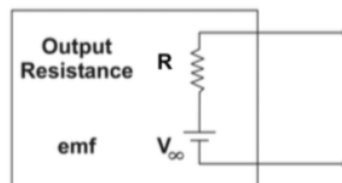


Figure 2: Equivalent circuit of an electric power source.

The output resistance (R) can be determined by attaching different external resistances of the load (R_l) to the power source, and measuring the current and voltage with a multimeter. 3 shows two possible ways of doing this. Both would be equivalent **if** the multimeter were ideal. However, in this exercise we will measure with real, not ideal, multimeters.

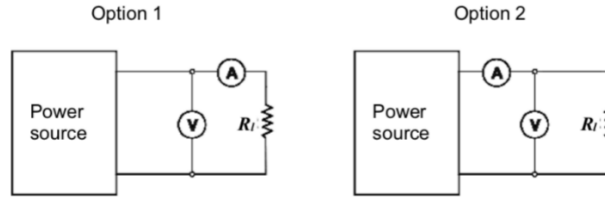


Figure 3: Possible circuits for determining the output resistance of a power source

2 PRE-LAB EXERCISES

Without connecting circuits and making measurements, we can expect the the voltmeter and ammeter readings to differ between the two options due to current and voltage leakage. More specifically, in option 1, we can expect the current to differ as the voltmeter allows some current to pass through. The amount of leakage will depend on the load resistance. Similarly, in option 2, we can expect that the voltage will differ as the ammeter will draw some voltage. Its leakage will too depend on the load resistance.

To calculate the internal resistances of the voltmeter and the ammeter we can use basic current and voltage division. For the voltmeter, we can calculate the current that we expect to pass through the ammeter and solve for the voltmeter resistance.

2.1 AMMETER INTERNAL RESISTANCE DERIVATION

Starting from Ohm's law: $V = I(R_A + R_L)$ Solve for R_A :

$$\frac{V}{I} = R_A + R_L \Rightarrow R_A = \frac{V}{I} - R_L$$

2.2 VOLTMETER INTERNAL RESISTANCE DERIVATION

Starting from Ohm's Law: $V = I \left(\frac{R_V R_L}{R_V + R_L} \right)$ Solve for R_V :

$$\begin{aligned} \frac{V}{I} &= \frac{R_V R_L}{R_V + R_L} \Rightarrow \frac{V}{I} (R_V + R_L) = R_V R_L \Rightarrow \frac{V}{I} R_V + \frac{V}{I} R_L = R_V R_L \\ \frac{V}{I} R_L &= R_V \left(R_L - \frac{V}{I} \right) \Rightarrow R_V = \frac{\frac{V}{I} R_L}{R_L - \frac{V}{I}} = \frac{V R_L}{I R_L - V} \end{aligned}$$

3 THE EXPERIMENT

We began by measuring the resistance values of the provided resistors. For the subsequent circuit experiments, we selected the two highest and two lowest resistance values to represent the load resistance conditions for the voltmeter and ammeter configurations, respectively. The measured values with their associated uncertainties are presented below.

Table 1: Resistor Values and Uncertainties

	R_{l1}	R_{l2}	R_{l3}	R_{l4}	R_{l5}	R_{l6}
Value (Ω)	100.32	219.91	461.3	2.6760 k	26.814 k	101.57 k
Uncertainty (Ω)	± 0.25	± 0.49	± 0.142	± 0.0005 k	± 0.006 k	± 0.25 k

3.1 CIRCUIT OPTION 1

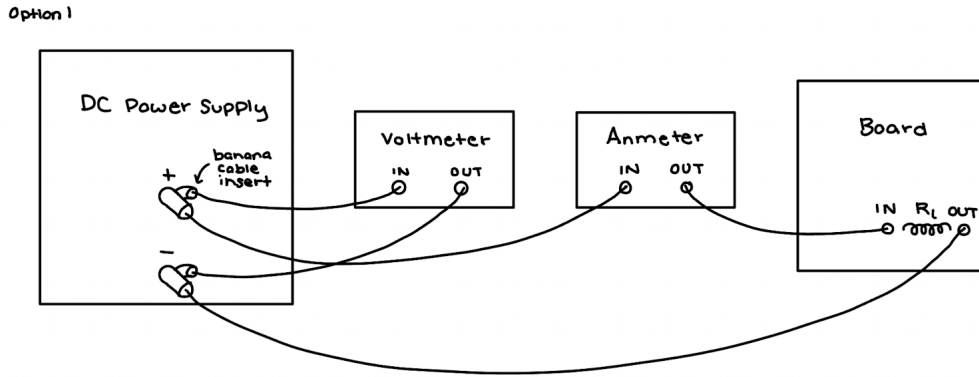


Figure 4: Circuit option 1 setup

Table 2: Circuit 1 Readings & Results

Resistance R_{li} (Ω)	Uncertainty ΔR_{li} (Ω)	Voltage V (V)	Uncertainty ΔV (V)	Current I (mA)	Uncertainty ΔI (mA)	Ammeter Res. R_A (Ω)	Uncertainty ΔR_A (Ω)
100.32	± 0.25	6.499	± 0.005	63.67	± 0.18	1.78	± 0.39
219.91	± 0.49	6.500	± 0.005	29.315	± 0.064	1.85	± 0.71
26.814 k	± 60	6.501	± 0.005	0.241	± 0.051	161.10	± 5708.77
101.57 k	± 250	6.501	± 0.005	0.063	± 0.005	1620.48	± 8193.92
Average:						446.3	± 3475.9

From the experimental data, a linear regression was applied to determine the slope m_2 and its associated uncertainty. The analysis yielded a slope value of $-0.046 \pm 0.004 \Omega$, where the uncertainty was derived through the standard error propagation for linear fitting parameters (see Appendix C).

The χ^2 analysis and examination of residuals suggest a potential underestimation of the experimental uncertainties. The resistance values R_V used in this fit were calculated from voltage, current, and lead resistance measurements, with their uncertainties propagated according to the method detailed in Appendix B.

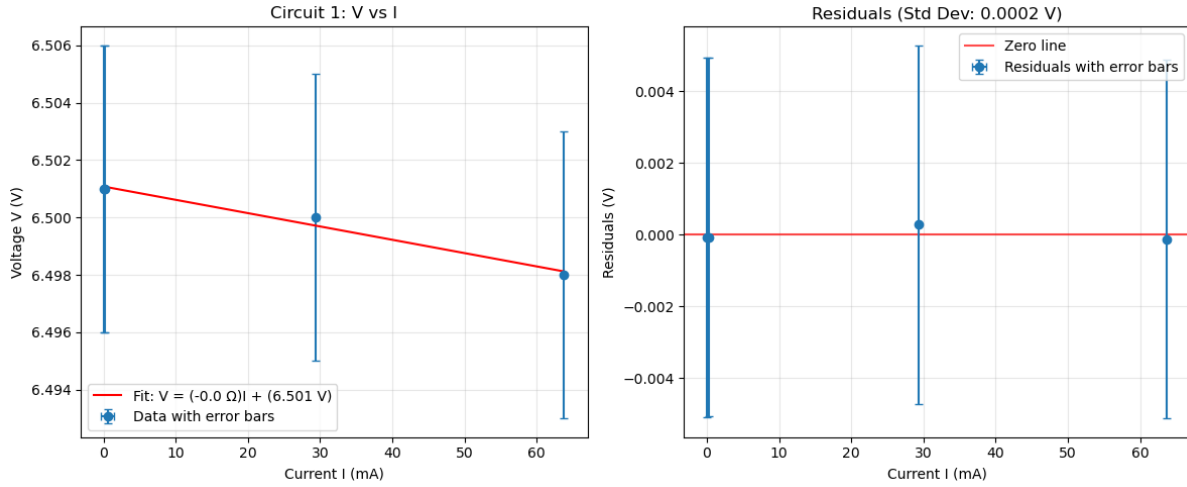


Figure 5: Linear regression of voltage versus current measurements for the voltmeter circuit configuration. The slope of $-0.046 \pm 0.004 \Omega$ represents the output resistance of the power source, while the intercept of $6.501 \pm 0.000 \text{ V}$ corresponds to the open-circuit voltage. The near-zero slope indicates minimal internal resistance in the power supply. The fit quality ($\chi^2 = 0.003$, $\chi^2/\nu = 0.001$, $p = 0.9987$) suggests potential underestimation of measurement uncertainties. Reading uncertainty was used for both V and I since they were greater than statistical uncertainties.

3.2 CIRCUIT OPTION 2

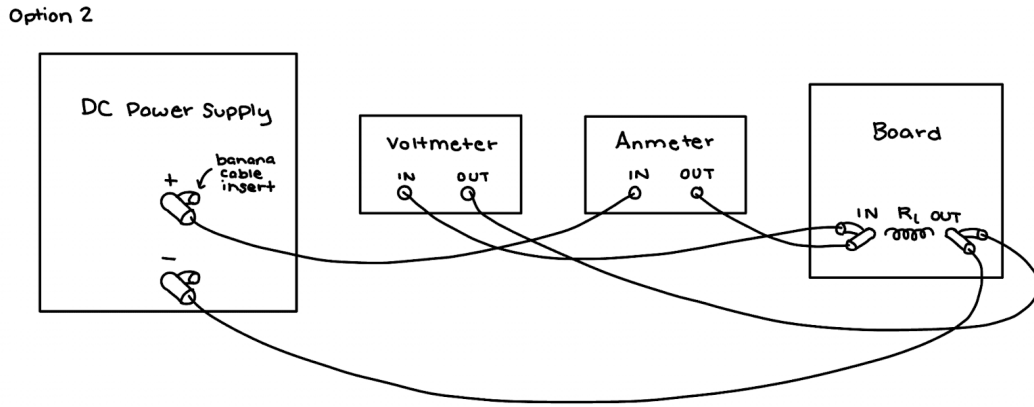


Figure 6: Circuit option 2 setup

From the experimental data, a linear regression was applied to determine the slope m_2 and its associated uncertainty. The analysis yielded a slope value of $-1.813 \pm 0.003 \Omega$, where the uncertainty was derived through the standard error propagation for linear fitting parameters (see Appendix C).

The χ^2 analysis and examination of residuals suggest a potential underestimation of the experimental uncertainties. The resistance values R_A used in this fit were calculated from voltage and current measurements, with their uncertainties propagated according to the method detailed in Appendix A.

Table 3: Circuit 2 Readings & Results

Resistance R_{li} (Ω)	Uncertainty ΔR_{li} (Ω)	Voltage V (V)	Uncertainty ΔV (V)	Current I (mA)	Uncertainty ΔI (mA)	Voltmeter Res. R_V (Ω)	Uncertainty ΔR_V (Ω)
100.32	± 0.25	6.386	± 0.005	63.60	± 0.18	-1.13×10^5	$\pm 4.94 \times 10^5$
219.91	± 0.49	6.448	± 0.005	29.322	± 0.064	7.05×10^6	$\pm 7.27 \times 10^8$
26.814 k	± 59	6.501	± 0.005	0.243	± 0.051	1.18×10^7	$\pm 1.09 \times 10^9$
101.57 k	± 250	6.501	± 0.005	0.065	± 0.005	6.53×10^6	$\pm 3.29 \times 10^7$
Average:						6.31×10^6	$\pm 4.62 \times 10^8$

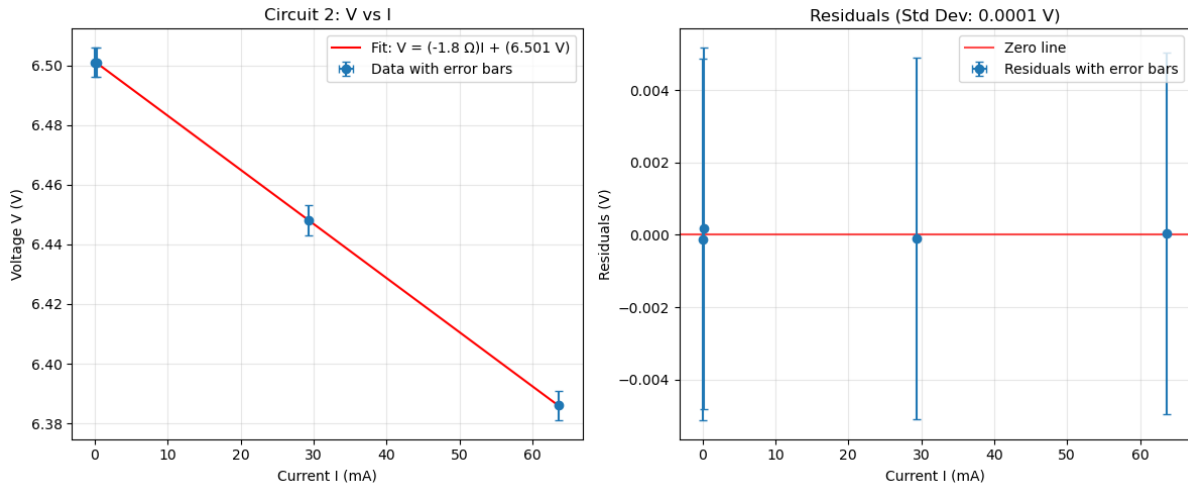


Figure 7: Linear regression of voltage versus current measurements showing an ideal voltage source behavior. The slope of $-1.813 \pm 0.003 \Omega$ represents the effective circuit resistance, while the intercept of $6.501 \pm 0.000 \text{ V}$ corresponds to the source voltage. The fit quality ($\chi^2 = 0.003$, $\chi^2/\nu = 0.001$, $p = 0.9987$) suggests potential underestimation of measurement uncertainties. Reading uncertainty was used for both V and I since they were greater than statistical uncertainties.

4 ANALYSIS

4.1 INTERNAL RESISTANCE USING R_V

The internal resistance of the power source can be derived from the relationship between the measured resistance m_1 , the voltmeter resistance R_V , and the internal resistance $R_{internal}$. The measured resistance m_1 represents the parallel combination of the voltmeter resistance and the internal resistance:

$$m_1 = \frac{R_V \cdot R_1}{R_V + R_1}$$

Solving this equation for R_1 :

$$\begin{aligned} m_1(R_V + R_1) &= R_V \cdot R_1 \\ m_1 R_V + m_1 R_1 &= R_V \cdot R_1 \\ m_1 R_V &= R_V \cdot R_1 - m_1 R_1 \\ m_1 R_V &= R_1(R_V - m_1) \end{aligned}$$

Thus, the explicit expression for the internal resistance is:

$$R_1 = \frac{m_1 \cdot R_V}{R_V - m_1}$$

We obtained a value of $-4.6000e - 02 \pm -4.7631e + 00\Omega$ and our uncertainty propagation can be found in D

4.2 INTERNAL RESISTANCE USING R_A

The internal resistance of the power source can also be derived from the relationship between the measured resistance m_2 , the calculated resistance R_A , and the internal resistance $R_{internal}$. The measured resistance m_2 represents the parallel combination of the resistance R_A and the internal resistance:

$$m_2 = \frac{R_A \cdot R_2}{R_A + R_2}$$

Solving this equation for R_2 :

$$\begin{aligned} m_2(R_A + R_2) &= R_A \cdot R_2 \\ m_2 R_A + m_2 R_2 &= R_A \cdot R_2 \\ m_2 R_A &= R_A \cdot R_2 - m_2 R_2 \\ m_2 R_A &= R_2(R_A - m_2) \end{aligned}$$

Thus, the explicit expression for the internal resistance is:

$$R_2 = \frac{m_2 \cdot R_A}{R_A - m_2}$$

We obtained a value of $1.8057e + 00 \pm -1.9848e + 01\Omega$ and our uncertainty propagation can be found in E

4.3 INTERNAL RESISTANCE RESULTS CONCLUSION

The internal resistance results differ when using R_V and R_A because of the ammeter and voltmeter's internal resistances. Theoretically, the option 1 circuit should be more accurate as the readings would be closer to the true source current and voltage. The second option circuit should be less accurate because the ammeter would cause a slight voltage drop and the voltmeter would draw small current. Our results do not reflect this regarding the internal resistance using R_V —for which a physically impossible negative value was determined—likely due to measurement or calculation error. The voltmeter's internal resistance may not be large enough relative to R_1 , meaning that as the resistances for R_{li} increase, their uncertainties also increase significantly. The error propagation calculations may have further skewed the results as they utilize average values for the calculations, meaning the high load resistors created greater average values. The slope, m , is also error-prone, which could be an additional source of error.

A UNCERTAINTY DERIVATION FOR RESISTANCE MEASUREMENT R_ASTEP 1: UNCERTAINTY IN DIVISION $\frac{V}{I}$

$$z = \frac{V}{I} \Rightarrow \left(\frac{\sigma_z}{z}\right)^2 = \left(\frac{\sigma_V}{V}\right)^2 + \left(\frac{\sigma_I}{I}\right)^2 \Rightarrow \sigma_{term1} = \frac{V}{I} \cdot \sqrt{\left(\frac{\sigma_V}{V}\right)^2 + \left(\frac{\sigma_I}{I}\right)^2}$$

STEP 2: UNCERTAINTY IN SUBTRACTION $R_A = term1 - R_{li}$

$$R_A = A - B \Rightarrow \sigma_{R_A} = \sqrt{\sigma_A^2 + \sigma_B^2} \Rightarrow \sigma_{R_A} = \sqrt{\sigma_{term1}^2 + \sigma_{R_{li}}^2}$$

COMPLETE UNCERTAINTY PROPAGATION

$$\sigma_{R_A} = \sqrt{\left[\frac{V}{I} \cdot \sqrt{\left(\frac{\sigma_V}{V}\right)^2 + \left(\frac{\sigma_I}{I}\right)^2}\right]^2 + \sigma_{R_{li}}^2} \Rightarrow \sigma_{R_A} = \sqrt{\left(\frac{V}{I}\right)^2 \cdot \left[\left(\frac{\sigma_V}{V}\right)^2 + \left(\frac{\sigma_I}{I}\right)^2\right] + \sigma_{R_{li}}^2}$$

Where: $\sigma_V = dV, \sigma_I = dI, \sigma_{R_{li}} = dR_{li}$

B UNCERTAINTY DERIVATION FOR RESISTANCE MEASUREMENT R_VSTEP 1: UNCERTAINTY IN NUMERATOR $N = V \cdot R_{li}$

$$N = V \cdot R_{li} \Rightarrow \left(\frac{\sigma_N}{N}\right)^2 = \left(\frac{\sigma_V}{V}\right)^2 + \left(\frac{\sigma_{R_{li}}}{R_{li}}\right)^2 \Rightarrow \sigma_N = N \cdot \sqrt{\left(\frac{\sigma_V}{V}\right)^2 + \left(\frac{\sigma_{R_{li}}}{R_{li}}\right)^2}$$

STEP 2: UNCERTAINTY IN $D_1 = I \cdot R_{li}$

$$D_1 = I \cdot R_{li} \Rightarrow \left(\frac{\sigma_{D_1}}{D_1}\right)^2 = \left(\frac{\sigma_I}{I}\right)^2 + \left(\frac{\sigma_{R_{li}}}{R_{li}}\right)^2 \Rightarrow \sigma_{D_1} = D_1 \cdot \sqrt{\left(\frac{\sigma_I}{I}\right)^2 + \left(\frac{\sigma_{R_{li}}}{R_{li}}\right)^2}$$

STEP 3: UNCERTAINTY IN DENOMINATOR $D = D_1 - V$

$$D = D_1 - V \Rightarrow \sigma_D = \sqrt{\sigma_{D_1}^2 + \sigma_V^2}$$

STEP 4: UNCERTAINTY IN QUOTIENT $R_V = \frac{N}{D}$

$$R_V = \frac{N}{D} \Rightarrow \left(\frac{\sigma_{R_V}}{R_V}\right)^2 = \left(\frac{\sigma_N}{N}\right)^2 + \left(\frac{\sigma_D}{D}\right)^2 \Rightarrow \sigma_{R_V} = R_V \cdot \sqrt{\left(\frac{\sigma_N}{N}\right)^2 + \left(\frac{\sigma_D}{D}\right)^2}$$

FINAL COMBINED EXPRESSION

$$\sigma_{R_V} = \frac{V \cdot R_{li}}{I \cdot R_{li} - V} \cdot \sqrt{\left(\frac{\sigma_V}{V}\right)^2 + \left(\frac{\sigma_{R_{li}}}{R_{li}}\right)^2 + \left(\frac{\sigma_{I \cdot R_{li}}}{I \cdot R_{li}}\right)^2 + \left(\frac{\sigma_V}{I \cdot R_{li} - V}\right)^2}$$

Where: $\sigma_V = dV$, $\sigma_I = dI$, $\sigma_{R_{li}} = dR_{li}$

C LINEAR REGRESSION WITH UNCERTAINTY PROPAGATION

LINEAR MODEL AND DEFINITIONS

$$y = mx + b \quad \text{where } N = \text{number of data points}$$

KEY QUANTITIES

$$\Delta = N \sum x_i^2 - \left(\sum x_i \right)^2$$
$$S_x = \sum x_i, \quad S_y = \sum y_i, \quad S_{xx} = \sum x_i^2, \quad S_{xy} = \sum x_i y_i$$

PARAMETER ESTIMATION

$$m = \frac{N \cdot S_{xy} - S_x \cdot S_y}{\Delta} \Rightarrow \text{slope}$$
$$b = \frac{S_y - m \cdot S_x}{N} \Rightarrow \text{intercept}$$

RESIDUALS AND VARIANCE

$$\hat{y}_i = b + mx_i \Rightarrow \text{predicted values}$$
$$r_i = y_i - \hat{y}_i \Rightarrow \text{residuals}$$
$$\sigma_y^2 = \sqrt{\frac{\sum r_i^2}{N - 2}} \Rightarrow \text{standard error of estimate}$$

PARAMETER UNCERTAINTIES

$$\sigma_m = \sqrt{\frac{\sigma_y^2 \cdot N}{\Delta}} \Rightarrow \text{slope uncertainty}$$
$$\sigma_b = \sqrt{\frac{\sigma_y^2 \cdot S_{xx}}{\Delta}} \Rightarrow \text{intercept uncertainty}$$

CHI-SQUARED ANALYSIS

$$\chi^2 = \sum \left(\frac{y_i - \hat{y}_i}{\sigma_{y_i}} \right)^2 \Rightarrow \text{goodness of fit}$$
$$\chi_\nu^2 = \frac{\chi^2}{N - 2} \Rightarrow \text{reduced chi-squared}$$
$$p = 1 - F(\chi^2, N - 2) \Rightarrow \text{p-value}$$

Where: σ_{y_i} are the individual y-uncertainties, F is the chi-squared CDF

D UNCERTAINTY DERIVATION FOR RESISTANCE MEASUREMENT R₁STEP 1: UNCERTAINTY IN MULTIPLICATION $m_1 \cdot R_V$

$$z = m_1 \cdot R_V \Rightarrow \left(\frac{\sigma_z}{z}\right)^2 = \left(\frac{\sigma_{m_1}}{m_1}\right)^2 + \left(\frac{\sigma_{R_V}}{R_V}\right)^2 \Rightarrow \sigma_{mult1} = m_1 \cdot R_V \cdot \sqrt{\left(\frac{\sigma_{m_1}}{m_1}\right)^2 + \left(\frac{\sigma_{R_V}}{R_V}\right)^2}$$

STEP 2: UNCERTAINTY IN SUBTRACTION $R_V - m_1$

$$z = R_V - m_1 \Rightarrow \sigma_z = \sqrt{\sigma_{R_V}^2 + \sigma_{m_1}^2} \Rightarrow \sigma_{sub2} = \sqrt{\sigma_{R_V}^2 + \sigma_{m_1}^2}$$

STEP 3: UNCERTAINTY IN DIVISION $\frac{m_1 \cdot R_V}{R_V - m_1}$

$$z = \frac{A}{B} \Rightarrow \left(\frac{\sigma_z}{z}\right)^2 = \left(\frac{\sigma_A}{A}\right)^2 + \left(\frac{\sigma_B}{B}\right)^2 \Rightarrow \sigma_{R_1} = \frac{m_1 \cdot R_V}{R_V - m_1} \cdot \sqrt{\left(\frac{\sigma_{mult1}}{m_1 \cdot R_V}\right)^2 + \left(\frac{\sigma_{sub2}}{R_V - m_1}\right)^2}$$

COMPLETE UNCERTAINTY PROPAGATION

$$\sigma_{R_1} = \frac{m_1 \cdot R_V}{R_V - m_1} \cdot \sqrt{\left[\sqrt{\left(\frac{\sigma_{m_1}}{m_1}\right)^2 + \left(\frac{\sigma_{R_V}}{R_V}\right)^2} \right]^2 + \left[\frac{\sqrt{\sigma_{R_V}^2 + \sigma_{m_1}^2}}{R_V - m_1} \right]^2}$$

$$\sigma_{R_1} = \frac{m_1 \cdot R_V}{R_V - m_1} \cdot \sqrt{\left(\frac{\sigma_{m_1}}{m_1}\right)^2 + \left(\frac{\sigma_{R_V}}{R_V}\right)^2 + \left(\frac{\sqrt{\sigma_{R_V}^2 + \sigma_{m_1}^2}}{R_V - m_1}\right)^2}$$

Where: $\sigma_{m_1} = \text{dm1}$, $\sigma_{R_V} = \text{dR}_V$

E UNCERTAINTY DERIVATION FOR RESISTANCE MEASUREMENT R₂STEP 1: UNCERTAINTY IN MULTIPLICATION $m_2 \cdot R_A$

$$z = m_2 \cdot R_A \Rightarrow \left(\frac{\sigma_z}{z}\right)^2 = \left(\frac{\sigma_{m_2}}{m_2}\right)^2 + \left(\frac{\sigma_{R_A}}{R_A}\right)^2 \Rightarrow \sigma_{mult1} = m_2 \cdot R_A \cdot \sqrt{\left(\frac{\sigma_{m_2}}{m_2}\right)^2 + \left(\frac{\sigma_{R_A}}{R_A}\right)^2}$$

STEP 2: UNCERTAINTY IN SUBTRACTION $R_A - m_2$

$$z = R_A - m_2 \Rightarrow \sigma_z = \sqrt{\sigma_{R_A}^2 + \sigma_{m_2}^2} \Rightarrow \sigma_{sub2} = \sqrt{\sigma_{R_A}^2 + \sigma_{m_2}^2}$$

STEP 3: UNCERTAINTY IN DIVISION $\frac{m_2 \cdot R_A}{R_A - m_2}$

$$z = \frac{A}{B} \Rightarrow \left(\frac{\sigma_z}{z}\right)^2 = \left(\frac{\sigma_A}{A}\right)^2 + \left(\frac{\sigma_B}{B}\right)^2 \Rightarrow \sigma_{R_2} = \frac{m_2 \cdot R_A}{R_A - m_2} \cdot \sqrt{\left(\frac{\sigma_{mult1}}{m_2 \cdot R_A}\right)^2 + \left(\frac{\sigma_{sub2}}{R_A - m_2}\right)^2}$$

COMPLETE UNCERTAINTY PROPAGATION

$$\sigma_{R_2} = \frac{m_2 \cdot R_A}{R_A - m_2} \cdot \sqrt{\left[\sqrt{\left(\frac{\sigma_{m_2}}{m_2}\right)^2 + \left(\frac{\sigma_{R_A}}{R_A}\right)^2} \right]^2 + \left[\frac{\sqrt{\sigma_{R_A}^2 + \sigma_{m_2}^2}}{R_A - m_2} \right]^2}$$

$$\sigma_{R_2} = \frac{m_2 \cdot R_A}{R_A - m_2} \cdot \sqrt{\left(\frac{\sigma_{m_2}}{m_2}\right)^2 + \left(\frac{\sigma_{R_A}}{R_A}\right)^2 + \left(\frac{\sqrt{\sigma_{R_A}^2 + \sigma_{m_2}^2}}{R_A - m_2}\right)^2}$$

Where: $\sigma_{m_2} = \text{dm}_2, \sigma_{R_A} = \text{dR}_A$