Question 1 Solution

1. A constant rate of inflation has been assumed. Estimate inflation rate from increases in actual cost of college:

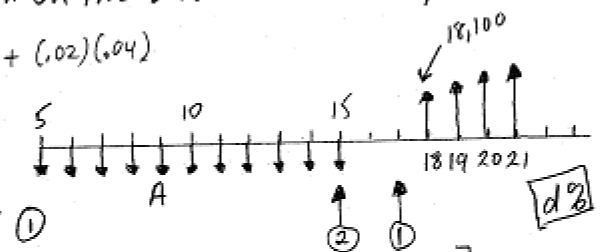
$\frac{18,825}{18,100} = 1.040 \quad \frac{19575}{18,825} = 1.040 \quad \frac{20,360}{19,575} = 1.040$

Therefore, a 4 % rate of inflation has been assumed by the parents.
Determine the actual rate of return on the bond mutual fund,

$d = i + j + ij = .02 + .04 + (.02)(.04)$

$= 0.0608$

$\Rightarrow 6.08\%$



How much is required at Year 17 ? 1

$PV_{17} = 18,100\,(P|A,\ 6.08,\ 4,\ 4) = 18,100\left[\frac{1-(1+j)^n(1+d)^{-n}}{d-j}\right]$

$= 18,100\,(3.66128) = 66,269$

How much is required at Year 15 ? 2

$PV_{15} = 66,269\,\overset{0.88865}{(P|F\ 6.08,\ 2)} = \frac{66,269}{(1+0.0608)^2} = 58,890$

What deposit (11 of them from 5 to 15) is required ?

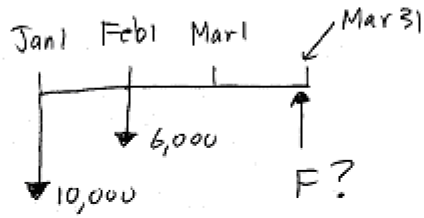
$A = 58,890\,(A|F\ 6.08,\ 11) = 58,890\left[\frac{d}{(1+d)^n - 1}\right]$

$= 58,890\,(0.066512)$

$= \boxed{\$3,917}$

Question 2 Solution

2. (a) "Savings Advantage"
2.4% compounded quarterly
on minimum quarterly balance



Minimum balance = 10,000

Quarterly interest rate $= \frac{2.4\%}{4} = 0.6\%$

$$FV_{MAR31} = 10{,}000(F|P\ 0.6\%, 1) + 6000 = 16{,}060.00$$

"Money Multiplier"
pays 2% compounded quarterly from time of deposit.

Since deposits are made monthly, find monthly interest rate.

Let i_m = monthly interest rate.

$$i_{eff} = \left(F|P \tfrac{r}{m}, m\right) - 1 = \left(1 + \tfrac{.02}{4}\right)^4 - 1 = \left(1 + i_m\right)^{12} - 1$$

$$\Rightarrow i_m = \sqrt[3]{1 + \tfrac{0.02}{4}} - 1 = 0.16639$$

$$FV_{MAR31} = 10{,}000(F|P .166\%, 3) + 6{,}000(F|P .166\%, 2)$$
$$= 10{,}000(1.0050) + 6{,}000(1.003331)$$
$$= 16{,}069.98$$

With one account, choose $\boxed{\text{Money Multiplier}}$ and be $\boxed{\$9.98}$ ahead

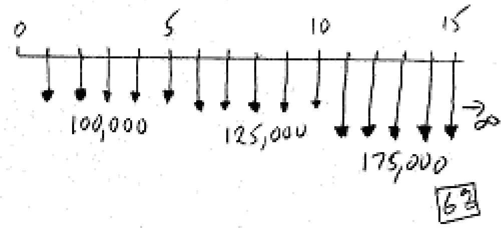
(b) Put the \$10,000 in "Savings Advantage" and the \$6,000 in Money Multiplier.

$$FV_{MAR31} = 10{,}000(F|P\ 0.6\%, 1) + 6000(F|P\ .166\%, 2)$$
$$= 10{,}000(1.06) + 6000(1.003331)$$
$$= 16{,}079.98$$

and be $\boxed{\$10.00}$ ahead of (a).

Question 3 Solution

(a)



$$PV = 100{,}000\,(P|A\ 6,5) + 125{,}000\,(P|F\ 6,5)(P|A\ 6,5) + \frac{175{,}000}{0.06}\,(P|F\ 6,10)$$

$$= 100{,}000\,(4.2124) + 125{,}000\,(0.7473)(4.2124) + \frac{175{,}000}{0.06}\,(0.5584)$$

$$= \boxed{\$2{,}443{,}397}$$

(b) With a service life of 35 years, the perpetuity $\frac{175{,}000}{0.06}$ is replaced with $175{,}000\,(P|A\ 6,25)$ (Years 11→35)

$$PV = 100{,}000\,(4.2124) + 125{,}000\,(0.7473)(4.2124) + 175{,}000\,(12.7834)(0.5584)$$

$$= \$2{,}063{,}925$$

Therefore, $\boxed{\$379{,}473}$ less is required.

Question 4 Solution

To realize a 6% annual rate of return over 5 years, the bond sale plus interest must have a future value of

$$FV = 103{,}000\,(F|P\;6,5) = 103{,}000\,(1.3382) = 137{,}835$$

Over 5 years, the bond will pay $100{,}000 \times \frac{0.05}{2} = \$2500$ interest every six months. There will be 10 interest payments. If she invests these at 2% compounded semi-annually (or 1% every six months), the future value will be:

$$FV_{INT} = 2500\,(F|A\;1,10) = 2{,}500\,(10.4622) = 26{,}156$$

$$FV = SP + FV_{INT}$$

where SP is selling price of the bond at EOY 5

$$\therefore\quad SP = FV - FV_{INT}$$

$$= 137{,}835 - 26{,}156 = \$111{,}679$$

The bond must be sold (or converted to stock) for at least $\boxed{\$111{,}679}$ to realize the 6% required rate of return.

Note: The bond is selling for a premium of $3,000 above face value of $100,000 because interest rates have recently fallen. This makes the 5% return more attractive and bond buyers bid up the price due to increased demand.

<u>Question 5 Solution</u>

$$PV_A = -10{,}000 + 2000\overset{4.3295}{(P|A\ 5,5)} + 2000\overset{8.2369}{(P|G\ 5,5)}$$

$$= 15{,}133$$

$$PV_B = -2{,}000 + 2X\overset{4.3295}{(P|A\ 5,5)} - 0.25X\overset{8.2369}{(P|G\ 5,5)} - \underbrace{\overset{0.8638}{(P|F\ 5,3)}1.5X}_{\text{TO MAKE UP FOR MISSING TERM AT } t=3.}$$

$$= -2{,}000 + 5.3041X$$

$$PV_C = 2Y\overset{4.3295}{(P|A\ 5,5)} - Y\overset{1.8594}{(P|A\ 5,2)}\overset{0.8638}{(P|F\ 5,3)}$$

$$= 7.0529Y$$

Equivalence implies $PV_A = PV_B = PV_C$

$$15{,}133 = -2000 + 5.3041X = 7.0529Y \Rightarrow \boxed{\begin{matrix} X = 3{,}230 \\ Y = 2{,}146 \end{matrix}}$$

Question 6 Solution

(a) Determine size of mortgage \$3,000 per month can buy.
Amortization period is 25 years or 300 months.
Interest rate is 6% compounded monthly or 0.5% per month.
Let M = value of mortgage \$3000/mo. can purchase

$$A = P(A|P\ 0.5\%, 300) \Rightarrow M = 3000\,(P|A\ 0.5\%, 300)$$
$$= 3000\left[\frac{(1+i)^n - 1}{i(1+i)^n}\right]\Bigg|_{\substack{i = 0.05 \\ n = 300}}$$
$$= 3000\,(155.2069)$$
$$= 465{,}621$$

Let V = value of house

$$M = .75V \Rightarrow V = \frac{M}{.75} = \frac{465{,}621}{.75} = \$620{,}827$$

and the down payment = $(V - M)$ = \$155,207

(b) They have made 36 X \$3,000 in payments over 3 years = \$108,000
The amount of principal to be repaid after 3 years (300-36 = 264 payments remaining)

$$= 3000\,(P|A\ 0.5\%, 264) = 3000(146.3969) = \$439{,}191$$

Therefore, of the \$108K in payments, \$26,430 (465,621 − 439,191) went to reduce the debt; therefore, the rest, \$81,570, went to interest payments.

(c) Value of their house = 620,827 (F|P 3%, 3) = 678,395
therefore, they own $\dfrac{678{,}395 - 439{,}191}{678{,}395} = 35\%$ of the house

Question 7 Solution

The construction costs are already in constant millions of dollars, so they can be discounted at the real interest rate of 12%. The beginning-of-year cash flows $50, 110, 120, 130, 140$ can be split into a lump sum, a uniform sequence, and a gradient sequence. Remember that the n-period gradient series has *end* of period cash flows $0, G, 2G, \ldots (n-1)G$, which would be $0, 0, G, 2G, \ldots$ at the beginning of the period. Therefore the constant component is $0, 110, 110, 110, 110$, or 110 per year for four years ($n = 4$) with the first payment in one year. The end-of-period gradient with $n = 4$ is $0, G, 2G, 3G$, which is $0, 0, G, 2G, 3G$ at the beginning of period.

$$PV[\text{construction}] = 50 + 110(P|A, i, 4) + 10(P|G, i, 4) \quad (54)$$

$$= 50 + 110 \cdot 3.037 + 10 \cdot 4.127 = 425.34 \quad (55)$$

(a) The capital recovery charge is the level annual payment stream (today's constant dollars) that would be equivalent in present value to the capital cost of building the road (at the required rate of return).

The capital recovery period runs from the start of year 6 to the end of year 40 (35 years). To compute the PV of real (constant-dollar) cash flows, we must use the real interest rate. First find the PV at the beginning of year 6 of a year-6 constant dollars received annually at the end of year for 35 years.

$$PV_6[CR] = a(P|A, i, 35) \quad (56)$$

This is the PV at the beginning of year 6, in year-6 dollars. The construction cost PV was calculated at the beginning of year 1, so to compare we must move the capital recovery charge back 5 years, and then set $PV_1[CR] = PV_1[construction]$. Since the year-6 cash flow is expressed in year-6 dollars, it must be moved using the nominal interest rate d.

$$PV_1[CR] = PV_6[CR](P|F, d, 5) = a(P|F, i, 5)(P|A, i, 35) \quad (57)$$

$$a = \frac{PV_1[CR]}{(P|F, d, 5)(P|A, i, 35)} \quad (58)$$

$$= \frac{1}{0.514 \cdot 8.176} PV_1[construction] \quad (59)$$

$$= 101.21 \quad (60)$$

The capital recovery cost is \$101.21m per year (today's constant dollars) throughout the operating period.

(b) For the project to be worthwhile, the revenue must equal the capital cost plus the future expenses on a present-value basis at the required rate of return. The PV of construction costs was found in (a) so we need only find the PV of operating costs.

The constant to convert between present value and constant (year-1) dollar end-of-year cash flows for years 6-40 was found in (a) when computing capital recovery. Since the \$50m maintenance cost is expressing in year-6 nominal dollars, it must first be converted to year-1 dollars by multiplying by $(1+j)^{-5}$. That stream of constant-dollar cash flows in years 6-40 has a PV of $50(1+j)^{-5}(P|F, i, 5)(P|A, i, 35) = 50 \cdot 0.906 \cdot 0.514 \cdot 8.176 = 190.37$.

Equivalently, we could have calculated the PV at year 6 in year-6 dollars as $(P|A, i, 35)$ and moved that back to year-1 dollars in year 1 using the nominal rate $(P|F, d, 5)$. The nominal rate would be used

Question 7 Solution cont'd

because we're changing both the timing and the inflation base year of the cash flows. The two approaches are equivalent because $(P|F,d,n) = (1+d)^{-n} = (1+i)^{-n}(1+j)^{-n} = (1+j)^{-n}(P|F,i,n)$.

The PV of revenue must exceed PV of capital plus expenses, or $425.34 + 190.37$ which is \$615.71m.

(c) The required PV of revenue was calculated in (b) above. When calculating the capital recovery in (a), the present value of a constant-dollar stream at the end of years 6-40 was calculated. The equivalent annual constant (year-1) dollar revenue stream r is

$$r = \frac{PV[revenue]}{(P|F,d,5)(P|A,i,35)} = \frac{615.71}{0.514 \cdot 8.176} = 146.51 \tag{61}$$

The revenue for the first year of operation must be \$146.51m in year-1 dollars. It should not be suprising that this is equal to the constant-dollar capital recovery 101.21 plus constant-dollar operating expenses $50(1+j)^{-5} = 45.28$.

Since that is constant year-1 dollars, it must be adjusted by inflation to get year-6 nominal dollars as requested. To convert, multiply by $(1+j)^5 = 1.104$ to get \$161.76m in nominal dollars.

(d) In the case where revenue increases at a rate g ($g > j$ is good for the company), end-of-year nominal operating revenue r follows the sequence $r, r(1+g), r(1+g)^2, \ldots$ starting in year 6. Since we are dealing with nominal dollars, discounting on the nominal interest rate is appropriate. The PV in nominal dollars at year 6 is

$$PV[revenue] = r(P|A,d,g,35) \tag{62}$$

Which can be moved to constant (year-1) dollars using the nominal interest rate.

$$r = \frac{PV[revenue]}{(P|F,d,5)(P|A,d,g,35)} = \frac{615.71}{0.514 \cdot 9.401} = 127.42 \tag{63}$$

The road must therefore take in \$127.42m (year-6 nominal dollars) in its first year if the revenue increases annually by 4%. This is less than the case where the revenue increases with inflation.

Question 8 Solution

(a) For the problem below, consider all rates (expected inflation, return) to be semi-annual since the cash flows occur semi-annually. They can be converted to annual when finished. Let j be the semi-annual inflation rate, and y_r, y_n the semi-annual yields of the real and nominal bonds respectively.

The real semi-annual yield for a bond is the discount rate y that sets the present value V of its inflation-adjusted cash flows equal to the current price P. To adjust a nominal cash flow at time t for inflation, we must divide by CPI_t/CPI_0. Since we're assuming a constant inflation rate, $CPI_t = (1+j)^t CPI_0$. For a par bond with N periods until maturity, the price is equal to face value F and the nominal coupon payments are $\frac{Fc}{2}$ so

$$V = P = F = \sum_{k=1}^{N} \frac{Fc_1}{2}(1+y_n)^{-k}(1+j)^{-k} + F(1+y_n)^{-N}(1+j)^{-N} \tag{11}$$

$$1 = \frac{c_n}{2}\sum_{k=1}^{N} u_n^{-k} + u_n^{-N} \quad u_n = (1+y_n)(1+j) \tag{12}$$

$$\tag{13}$$

For the real-return bond, it is the same as the nominal bond except the face value is no longer constant, rather $F_t = \frac{CPI_t}{CPI_0} = F_0(1+j)^t$.

$$V = P = F_0 = \sum_{k=1}^{N} \frac{F_t c_r}{2}(1+y_r)^{-k}(1+j)^{-k} + F_t(1+y_r)^{-N}(1+j)^{-N} \tag{14}$$

$$F_0 = \frac{c_r}{2}\sum_{k=1}^{N} F_0(1+j)^k(1+y_r)^{-k}(1+j)^{-k} + F_0(1+j)^N(1+y_r)^{-N}(1+j)^{-N} \tag{15}$$

$$F_0 = \frac{c_r}{2}\sum_{k=1}^{N} F_0(1+y_r)^{-k} + F_0(1+y_r)^{-N} \tag{16}$$

$$1 = \frac{c_r}{2}\sum_{k=1}^{N} u_r^{-k} + u_r^{-N} \quad u_r = 1 + y_r \tag{17}$$

$$\tag{18}$$

Both the RRBs and the nominal bonds have the form

$$1 = \frac{c}{2}\sum_{k=1}^{N} u^{-k} + u^{-N} \tag{19}$$

$$1 = \frac{cu}{2}\frac{1-u^{-N}}{1-u} + u^{-N} \tag{20}$$

$$1 - u^{-N} = \frac{cu}{2}\frac{1-u^{-N}}{1-u} \tag{21}$$

$$1 - u^{-1} = \frac{cu^{-1}}{2} \tag{22}$$

$$u = 1 + \frac{c}{2} \tag{23}$$

but with different values of u. To find the yield y for each, sub in the appropriate value of u:

Question 8 Solution cont'd

$$u_r = 1 + y_r = 1 + \frac{c_r}{2} \implies y_r = \frac{c_r}{2} \tag{24}$$

$$u_n = (1 + y_n)(1 + j) = 1 + \frac{c_n}{2} \implies y_n = \frac{\frac{c_n}{2} - j}{1 + j} \neq \frac{c_n}{2} \quad \text{unless } j = 0 \tag{25}$$

So the par *real-return bond* has a *real* semiannual yield exactly equal to half its coupon rate regardless of inflation. This is why it's called a real-return bond. The par nominal bond has a real semiannual yield that varies with inflation (less than half coupon rate if $j > 0$). Its *nominal* yield is always equal to its coupon rate, hence the term *nominal* bond.

Remember that these are all semiannual values, so must be converted to annual using $1 + i^{(1)} = (1 + i^{(n)})^n$. Also, if j were given as an annual rate you would have to find the semiannual equivalent.

(b) As shown above, the value of a risk-free real-return bond depends on the real interest rate i, but not inflation or the nominal rate. The yield of a nominal bond depends on the nominal interest rate d which includes inflation and the real rate. If the nominal interest rate is 8% and the real rate is 5% then the expected inflation rate is

$$1 + d = (1 + i)(1 + j) \tag{26}$$

$$j = \frac{1 + d}{1 + i} - 1 \tag{27}$$

$$= \frac{1.08}{1.05} - 1 \tag{28}$$

$$= 2.86\% \tag{29}$$

(c) The yield on the nominal bond should equal the nominal interest rate, so the nominal 15-year rate is 6.2%. The real interest rate is the yield on a real-return bond, so we have to solve for the yield of a bond priced at $P = 95\frac{1}{8}$ with a 5% coupon.

The value V of a real-return bond with face value F, annual coupon rate c paid semiannually, and real interest rate i is

$$V = \sum_{k=1}^{N} \frac{Fc}{2}(1 + i)^{-k} + F(1 + i)^{-N} \tag{30}$$

This can be solved numerically by guess-and-check followed by interpolation. Since the price is near (but below) par, the yield must be just a little above coupon rate. We know at par ($V = 1$) the yield is equal the coupon, so let's try 6%.

Yield guess	V	V-P
5%	1.0000	4.88%
6%	0.9020	-4.93%
5.5%	0.9513	0.01%

The real interest rate is (to the nearest decimal) 5.5%. From that, the inflation must be $j = \frac{1+i}{1+d} - 1 = 0.66\%$.

(d) If inflation expectations increase but real interest rates remain the same, then the nominal interest rate must also increase. Nominal bond yields should increase to match the nominal interest rate. If bond yields increase, prices decrease so nominal bonds get cheaper. On the other hand, the yield of a real-return bond depends only on real interest rates. An increase in inflation expectations does not change its yield, therefore its price should not change.

Question 9 Solution

Let the prime rate be P, and the total balance outstanding at the beginning of period t be B_t, composed of the federal part $B_{f,t}$ and provincial part $B_{p,t}$. The interest accrued during month $t > 6$ is $I_t = B_{f,t}\frac{P+0.025}{12} + B_{p,t}\frac{P+0.01}{12}$. The portion of the payment exceeding the interest $a_t - I_t$ goes towards principal, and is split proportionally between federal and provincial balances. As with any amortizing loan, if you make a payment of a_t at the end of the month t, the balance becomes

$$B_{t+1} = B_{f,t+1} + B_{p,t+1} = B_t + I_t - a_t \tag{31}$$

(a) If the payment covers the interest and a proportion of principal $f = \frac{B_{f,t}}{B_t}$ is allocated to the federal balance, then the federal part of the payment must be $I_{f,t} + f(a_t - I_t)$.

$$\begin{aligned} B_{f,t+1} =& B_{f,t} + I_{f,t} - [f(a_t - I_t) + I_{f,t}] & (32)\\ =& fB_t - f(a_t - I_t) & (33)\\ =& f(B_t - a_t + I_t) = fB_{t+1} & (34) \end{aligned}$$

You can see that the federal balance is still f times the total balance. The provincial part must be $1 - f$ times the balance. Both of those will go to zero if and only if the total balance goes to zero, ie. both will be paid off at the same time.

(b) The combined interest rate is $\frac{1}{12}(P + 0.025f + 0.010(1 - f)) = 0.358\%$ per month. Let the monthly payment for four years (48 months) be m_{48} and the monthly payment for nine-and-a-half years be m_{114}.

$$\begin{aligned} m_N(P|A, i, N) =& B_7 & (35)\\ m_N =& \frac{B_7}{(P|A, i, N)} & (36) \end{aligned}$$

Total payments for a series of N level payments m_N is just $m_N N$. Since $m_N N$ payments consist of interest plus principal and the pay off the principal B_7 entirely, the remaining amount $m_N N - B_7$ must have been interest.

Evaluate: $(P|A, i, 48) = 44.027$ $(P|A, i, 114) = 93.452$ $m_{48} = 227.13$ $m_{114} = 107.01$ $48m_{48} - B_7 = 902.24$ $114m_{114} - B_7 = 2199.10$. The total interest savings amount to \$1296.60.

(c) During the first six months, interest accrues only on the federal portion which totals $B_{f,t} = fB_t$ so the interest charged is $B_t\left(f\frac{P+0.025}{12} + (1 - f)0\right)$ per month. Dividing by B_t you get the effective rate for the entire balance, which converts to an effective annual rate of $(1 + f\frac{P+0.025}{12})^{12}$. If you can do better than that investing for six months, then it would be optimal to do so.

With a federal balance at 50% of total, and a prime rate of 3% this evaluates to 0.229% monthly, equivalent to 2.78% annually.

(d) If you receive a tax credit of 15% of the interest paid, then the interest rate is effectively only 85% as much after the government pays you back. That makes it cheaper to not pay the loan immediately, and so lowers the break-even point to invest. If your investment can exceed $0.85 \cdot 2.78\% = 2.36\%$ then it would be optimal to wait to repay since your investment returns will exceed your accrued interest.

Question 9 Solution cont'd

(e) After the first six months, you have to pay interest on the provincial and federal portions. That increases the combined monthly interest rate on the outstanding balance to

$$f\frac{P+0.025}{12} + (1-f)\frac{P+0.01}{12} = \frac{P+0.01+0.015f}{12} \tag{37}$$

Again assuming 50% federal-provincial split and a 3% prime rate, the combined monthly rate is 0.396%, equivalent to 4.85% annual effective. After the tax credit, this is 4.12%, so you should repay the loan unless you can earn more than that rate after tax.

(f) The amortization schedule is attached as a separate spreadsheet. For each component it shows the balance at the start of the month, the interest accrued (note provincial is zero for first six months), and the proportion of payment allocated to the component. If the payment is greater than interest, then the payment to each component is the interest on that component plus a proportional share of principal repaid (payment minus interest), ie the payment allocated to the federal part is $B_{f,t}i_f + f(a_t - I_t)$.

To create the Excel sheet, the payment was divided into interest and principal (payment minus interest) components. The principal component was further divided proportionally into provincial and federal parts based on the ratio of balance outstanding at the beginning of the month. The monthly payment was found using Excel's Goal Seek function to set the beginning of month (BOM) 31 total balance cell to zero by changing the payment cell. Try it - it's quite handy for problems where it's harder to find an equation to give you the solution you want.

Using equations, the parameters change after month 6 so you have to split into two time periods. For the first six months, just accumulate the interest for the federal portion

$$B_{f,7} = B_{f,1}(F|P, i_f, 6) = 2000 \cdot 1.0278 \tag{38}$$

$$B_{p,7} = B_{p,1}(F|P, 0, 6) = B_{p,1} = 4000 \tag{39}$$

For the remaining 24 months, it's already been shown that the interest is just the average of the two interest rates weighted by balance outstanding, and that if principal repayments are split evenly then the weights remain constant between provincial and federal balances.

$$f = \frac{B_{f,7}}{B_7} = 0.3395 \tag{40}$$

$$i = fi_f + (1-f)i_p = 0.375\% \tag{41}$$

$$\tag{42}$$

From here, it is just finding the level monthly payment a to pay off a loan with balance B_7:

$$B_7 = a(P|A, i, 30) \tag{43}$$

$$a = \frac{B_{p,7} + B_{f,7}}{(P|A, i, 24)} \tag{44}$$

$$= \frac{4000 + 2000 \cdot 1.0278}{22.911} = 264.30 \tag{45}$$

which is close enough (within rounding error) to the spreadsheet-derived value of $264.34.