Time Value Of Money Operations

- The value of a given sum of money depends on when that money is received
- Interest – rental amount charged for the use of money

$$F_n = P + I_n$$

where

P = present value of a single sum of money

F_n = future value of P after *n* time periods

I_n = accumulated interest ($)

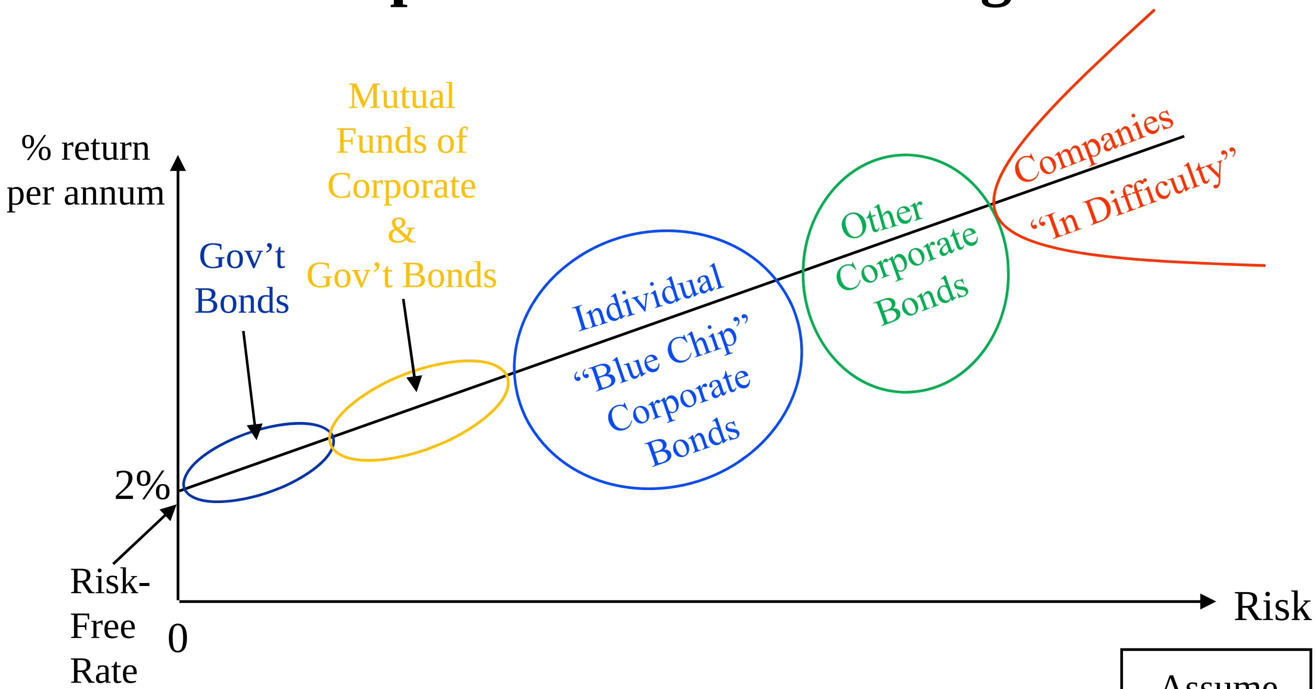
- i – annual interest rate (%)

  – change in value of a dollar over a one-year period

$$i = \frac{1}{(1-t)}(i_{PC} + i_R + i_I)$$

- Why do lenders charge interest?
  - Compensation for postponing consumption
  - Compensation for risk
  - Compensation for inflation
  - Only after-tax compensation is of value to investor
    - t is the tax rate on interest income

Compensation For Lending`



Assume marginal tax rate $t = 50\%$

$$i = \frac{1}{(1-t)}(i_{PC} + i_R + i_I)$$

<u>"Risk-Free" Gov't Bond Yield</u>

$$5\% = \frac{1}{1-0.5}(0.5\% + 0\% + 2.0\%)$$

<u>Low-Risk Interest Rate</u>

$$6\% = \frac{1}{1-0.5}(0.5\% + 0.5\% + 2.0\%)$$

<u>Risky Corporate Bond Yield</u>

$$15\% = \frac{1}{1-0.5}(0.5\% + 5.0\% + 2.0\%)$$

Interest Rate Calculation Approaches

$$F_n = P + I_n$$

Two approaches used to determine the value of I_n:

- Simple interest approach

  $I_n = P\ i\ n \longrightarrow F_n = P(1 + i\ n)$

- Compound interest approach

  $I_n = i\ F_{n-1} \longrightarrow F_n = F_{n-1}\ (1 + i)$

$1 000 loan over 2 years - interest rate 7% per annum

Simple Interest		Compound Interest
$F_2 = \$1\ 000\ [1+ 0.07(2)]$		$F_1 = 1\ 000\ (1+0.07)$
$= \$1\ 140.00$		$= 1\ 070$
		$F_2 = 1\ 070\ (1+0.07)$
		$=\$1\ 144.90$
	OR	$F_2 = 1\ 000\ (1.07)(1.07)$
		$= 1\ 000\ (1.07)^2$
		$= \$1\ 144.90$

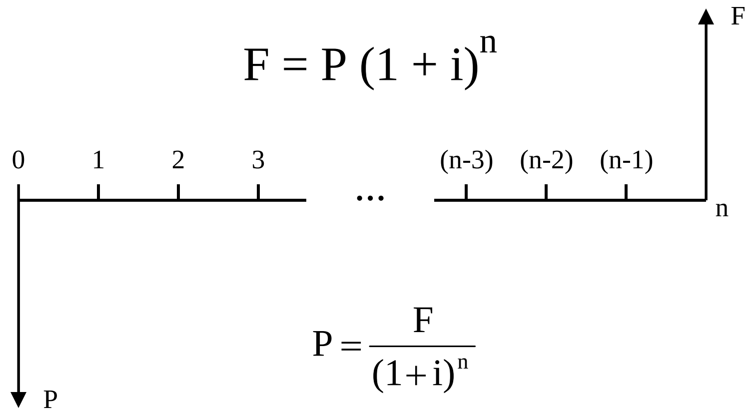
For Compound Interest at Year n

$$F_n = P(1 + i)^n$$

- assume compound interest unless otherwise stated

Single Sums of Money

$$F = P\,(1+i)^n$$



$$P = \frac{F}{(1+i)^n}$$

P – present value

– equivalent value of an amount of money at time 0

F – future value

– equivalent value of an amount of money at time n

Example 1: \$1 000 loan at 6% compounded annually. If the loan is to be repaid after 5 years, how much will be owed?

$$\begin{aligned} F &= P\,(1+i)^n \\ &= \$1\,000\,(1+0.06)^5 \\ &= \$1\,000\,(1.3382) \\ &= \$1\,338.20 \end{aligned}$$

Example 2: How much must be deposited today in order to accumulate \$2 000 in two years in a savings account?

$$\begin{aligned} P &= F\,(1+i)^{-n} \\ &= \frac{\$2\,000}{(1+0.06)^2} \\ &= \$1\,780 \end{aligned}$$

Future Value Factor

$$(F|P\ \ i,n) = (1 + i)^n$$

Find the future value given the present value.

$$\begin{aligned} F &= P\ (F|P\ i,n) \\ &= P\ (F|P\ 6,5) \\ &= \$1\ 000\ (1.3382) \\ &= \$1\ 338.20 \end{aligned}$$

Future Value Factor Table

(F|P i,n)

Period	1%	2%	3%	4%	5%	6%	7%	8%	9%	10%	12%	14%	15%	16%	18%	20%	24%	28%	32%	36%
1	1.0100	1.0200	1.0300	1.0400	1.0500	1.0600	1.0700	1.0800	1.0900	1.1000	1.1200	1.1400	1.1500	1.1600	1.1800	1.2000	1.2400	1.2800	1.3200	1.3600
2	1.0201	1.0404	1.0609	1.0816	1.1025	1.1236	1.1449	1.1664	1.1881	1.2100	1.2544	1.2996	1.3225	1.3456	1.3924	1.4400	1.5376	1.6384	1.7424	1.8496
3	1.0303	1.0612	1.0927	1.1249	1.1576	1.1910	1.2250	1.2597	1.2950	1.3310	1.4049	1.4815	1.5209	1.5609	1.6430	1.7280	1.9066	2.0972	2.3000	2.5155
4	1.0406	1.0824	1.1255	1.1699	1.2155	1.2625	1.3108	1.3605	1.4116	1.4641	1.5735	1.6890	1.7490	1.8106	1.9388	2.0736	2.3642	2.6844	3.0360	3.4210
5	1.0510	1.1041	1.1593	1.2167	1.2763	1.3382	1.4026	1.4693	1.5386	1.6105	1.7623	1.9254	2.0114	2.1003	2.2878	2.4883	2.9316	3.4360	4.0075	4.6526
6	1.0615	1.1262	1.1941	1.2653	1.3401	1.4185	1.5007	1.5869	1.6771	1.7716	1.9738	2.1950	2.3131	2.4364	2.6996	2.9860	3.6352	4.3980	5.2899	6.3275
7	1.0721	1.1487	1.2299	1.3159	1.4071	1.5036	1.6058	1.7138	1.8280	1.9487	2.2107	2.5023	2.6600	2.8262	3.1855	3.5832	4.5077	5.6295	6.9826	8.6054
8	1.0829	1.1717	1.2668	1.3686	1.4775	1.5938	1.7182	1.8509	1.9926	2.1436	2.4760	2.8526	3.0590	3.2784	3.7589	4.2998	5.5895	7.2058	9.2170	11.703
9	1.0937	1.1951	1.3048	1.4233	1.5513	1.6895	1.8385	1.9990	2.1719	2.3579	2.7731	3.2519	3.5179	3.8030	4.4355	5.1598	6.9310	9.2234	12.166	15.916
10	1.1046	1.2190	1.3439	1.4802	1.6289	1.7908	1.9672	2.1589	2.3674	2.5937	3.1058	3.7072	4.0456	4.4114	5.2338	6.1917	8.5944	11.805	16.059	21.646
11	1.1157	1.2434	1.3842	1.5395	1.7103	1.8983	2.1049	2.3316	2.5804	2.8531	3.4785	4.2262	4.6524	5.1173	6.1759	7.4301	10.657	15.111	21.198	29.439
12	1.1268	1.2682	1.4258	1.6010	1.7959	2.0122	2.2522	2.5182	2.8127	3.1384	3.8960	4.8179	5.3502	5.9360	7.2876	8.9161	13.214	19.342	27.982	40.037
13	1.1381	1.2936	1.4685	1.6651	1.8856	2.1329	2.4098	2.7196	3.0658	3.4523	4.3635	5.4924	6.1528	6.8858	8.5994	10.699	16.386	24.758	36.937	54.451
14	1.1495	1.3195	1.5126	1.7317	1.9799	2.2609	2.5785	2.9372	3.3417	3.7975	4.8871	6.2613	7.0757	7.9875	10.147	12.839	20.319	31.691	48.756	74.053
15	1.1610	1.3459	1.5580	1.8009	2.0789	2.3966	2.7590	3.1722	3.6425	4.1772	5.4736	7.1379	8.1371	9.2655	11.973	15.407	25.195	40.564	64.358	100.71
16	1.1726	1.3728	1.6047	1.8730	2.1829	2.5404	2.9522	3.4259	3.9703	4.5950	6.1304	8.1372	9.3576	10.748	14.129	18.488	31.242	51.923	84.953	136.96
17	1.1843	1.4002	1.6528	1.9479	2.2920	2.6928	3.1588	3.7000	4.3276	5.0545	6.8660	9.2765	10.761	12.467	16.672	22.186	38.740	66.461	112.13	186.27
18	1.1961	1.4282	1.7024	2.0258	2.4066	2.8543	3.3799	3.9960	4.7171	5.5599	7.6900	10.575	12.375	14.462	19.673	26.623	48.038	85.070	148.02	253.33
19	1.2081	1.4568	1.7535	2.1068	2.5270	3.0256	3.6165	4.3157	5.1417	6.1159	8.6128	12.055	14.231	16.776	23.214	31.948	59.567	108.89	195.39	344.53
20	1.2202	1.4859	1.8061	2.1911	2.6533	3.2071	3.8697	4.6610	5.6044	6.7275	9.6463	13.743	16.366	19.460	27.393	38.337	73.864	139.37	257.91	468.57
21	1.2324	1.5157	1.8603	2.2788	2.7860	3.3996	4.1406	5.0338	6.1088	7.4002	10.803	15.667	18.821	22.574	32.323	46.005	91.591	178.40	340.44	637.26
22	1.2447	1.5460	1.9161	2.3699	2.9253	3.6035	4.4304	5.4365	6.6586	8.1403	12.100	17.861	21.644	26.186	38.142	55.206	113.57	228.35	449.39	866.67
23	1.2572	1.5769	1.9736	2.4647	3.0715	3.8197	4.7405	5.8715	7.2579	8.9543	13.552	20.361	24.891	30.376	45.007	66.247	140.83	292.30	593.19	1178.6
24	1.2697	1.6084	2.0328	2.5633	3.2251	4.0489	5.0724	6.3412	7.9111	9.8497	15.178	23.212	28.625	35.236	53.108	79.496	174.63	374.14	783.02	1602.9
25	1.2824	1.6406	2.0938	2.6658	3.3864	4.2919	5.4274	6.8485	8.6231	10.834	17.000	26.461	32.918	40.874	62.668	95.396	216.54	478.90	1033.5	2180.0

Present Value Factor

$$(P|F\ i,n) = (1 + i)^{-n}$$

Find the present value given the future value.

$$\begin{aligned} P &= F\,(P|F\ i,n) \\ &= F\,(P|F\ 6,2) \\ &= \$2\,000\ (0.8900) \\ &= \$1\,780 \end{aligned}$$

Present Value Factor Table

(P|F i,n)

Period	1%	2%	3%	4%	5%	6%	7%	8%	9%	10%	12%	14%	15%	16%	18%	20%	24%	28%	32%	36%
1	.9901	.9804	.9709	.9615	.9524	.9434	.9346	.9259	.9174	.9091	.8929	.8772	.8696	.8621	.8475	.8333	.8065	.7813	.7576	.7353
2	.9803	.9612	.9426	.9246	.9070	.8900	.8734	.8573	.8417	.8264	.7972	.7695	.7561	.7432	.7182	.6944	.6504	.6104	.5739	.5407
3	.9706	.9423	.9151	.8890	.8638	.8396	.8163	.7938	.7722	.7513	.7118	.6750	.6575	.6407	.6086	.5787	.5245	.4768	.4348	.3975
4	.9610	.9238	.8885	.8548	.8227	.7921	.7629	.7350	.7084	.6830	.6355	.5921	.5718	.5523	.5158	.4823	.4230	.3725	.3294	.2923
5	.9515	.9057	.8626	.8219	.7835	.7473	.7130	.6806	.6499	.6209	.5674	.5194	.4972	.4761	.4371	.4019	.3411	.2910	.2495	.2149
6	.9420	.8880	.8375	.7903	.7462	.7050	.6663	.6302	.5963	.5645	.5066	.4556	.4323	.4104	.3704	.3349	.2751	.2274	.1890	.1580
7	.9327	.8706	.8131	.7599	.7107	.6651	.6227	.5835	.5470	.5132	.4523	.3996	.3759	.3538	.3139	.2791	.2218	.1776	.1432	.1162
8	.9235	.8535	.7894	.7307	.6768	.6274	.5820	.5403	.5019	.4665	.4039	.3506	.3269	.3050	.2660	.2326	.1789	.1388	.1085	.0854
9	.9143	.8368	.7664	.7026	.6446	.5919	.5439	.5002	.4604	.4241	.3606	.3075	.2843	.2630	.2255	.1938	.1443	.1084	.0822	.0628
10	.9053	.8203	.7441	.6756	.6139	.5584	.5083	.4632	.4224	.3855	.3220	.2697	.2472	.2267	.1911	.1615	.1164	.0847	.0623	.0462
11	.8963	.8043	.7224	.6496	.5847	.5268	.4751	.4289	.3875	.3505	.2875	.2366	.2149	.1954	.1619	.1346	.0938	.0662	.0472	.0340
12	.8874	.7885	.7014	.6246	.5568	.4970	.4440	.3971	.3555	.3186	.2567	.2076	.1869	.1685	.1372	.1122	.0757	.0517	.0357	.0250
13	.8787	.7730	.6810	.6006	.5303	.4688	.4150	.3677	.3262	.2897	.2292	.1821	.1625	.1452	.1163	.0935	.0610	.0404	.0271	.0184
14	.8700	.7579	.6611	.5775	.5051	.4423	.3878	.3405	.2992	.2633	.2046	.1597	.1413	.1252	.0985	.0779	.0492	.0316	.0205	.0135
15	.8613	.7430	.6419	.5553	.4810	.4173	.3624	.3152	.2745	.2394	.1827	.1401	.1229	.1079	.0835	.0649	.0397	.0247	.0155	.0099
16	.8528	.7284	.6232	.5339	.4581	.3936	.3387	.2919	.2519	.2176	.1631	.1229	.1069	.0930	.0708	.0541	.0320	.0193	.0118	.0073
17	.8444	.7142	.6050	.5134	.4363	.3714	.3166	.2703	.2311	.1978	.1456	.1078	.0929	.0802	.0600	.0451	.0258	.0150	.0089	.0054
18	.8360	.7002	.5874	.4936	.4155	.3503	.2959	.2502	.2120	.1799	.1300	.0946	.0808	.0691	.0508	.0376	.0208	.0118	.0068	.0039
19	.8277	.6864	.5703	.4746	.3957	.3305	.2765	.2317	.1945	.1635	.1161	.0829	.0703	.0596	.0431	.0313	.0168	.0092	.0051	.0029
20	.8195	.6730	.5537	.4564	.3769	.3118	.2584	.2145	.1784	.1486	.1037	.0728	.0611	.0514	.0365	.0261	.0135	.0072	.0039	.0021
25	.7798	.6095	.4776	.3751	.2953	.2330	.1842	.1460	.1160	.0923	.0588	.0378	.0304	.0245	.0160	.0105	.0046	.0021	.0010	.0005
30	.7419	.5521	.4120	.3083	.2314	.1741	.1314	.0994	.0754	.0573	.0334	.0196	.0151	.0116	.0070	.0042	.0016	.0006	.0002	.0001
40	.6717	.4529	.3066	.2083	.1420	.0972	.0668	.0460	.0318	.0221	.0107	.0053	.0037	.0026	.0013	.0007	.0002	.0001	*	*
50	.6080	.3715	.2281	.1407	.0872	.0543	.0339	.0213	.0134	.0085	.0035	.0014	.0009	.0006	.0003	.0001	*	*	*	*
60	.5504	.3048	.1697	.0951	.0535	.0303	.0173	.0099	.0057	.0033	.0011	.0004	.0002	.0001	*	*	*	*	*	*

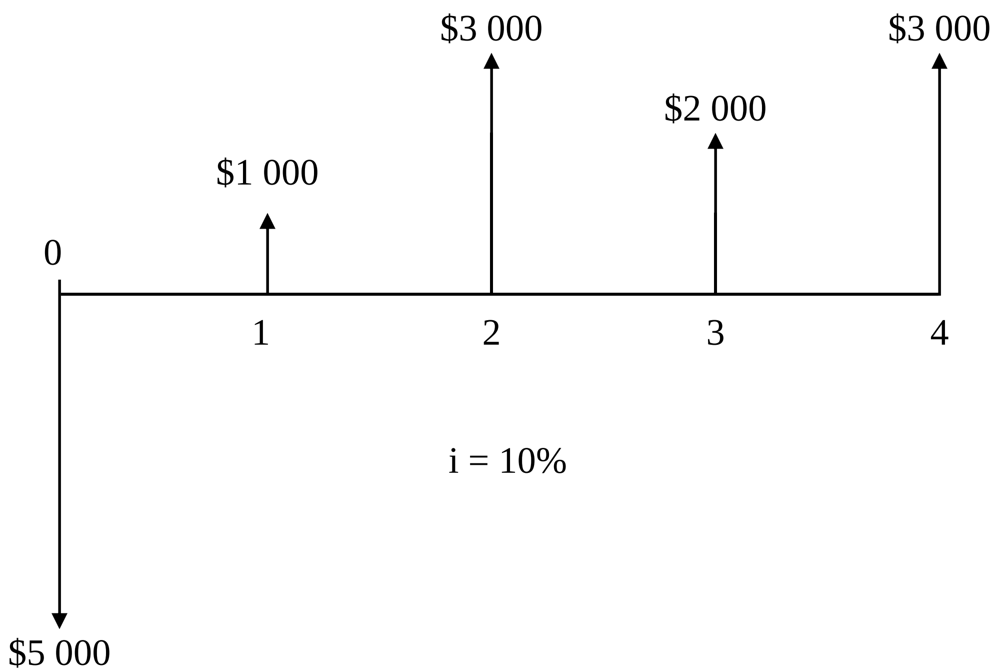
*The factor is zero to four decimal places.

N.B. Discrete versus continuous compounding factors.

Series of Cash Flows

Present Value of a series of cash flows:

$$P = A_0 + \frac{A_1}{(1+i)} + \frac{A_2}{(1+i)^2} + \ldots + \frac{A_{n-1}}{(1+i)^{n-1}} + \frac{A_n}{(1+i)^n}$$



$$\begin{aligned} P &= A_0 + \frac{A_1}{(1+i)} + \frac{A_2}{(1+i)^2} + \frac{A_3}{(1+i)^3} + \frac{A_4}{(1+i)^4} \\ &= -5\,000 + \frac{1\,000}{1.1} + \frac{3\,000}{(1.1)^2} + \frac{2\,000}{(1.1)^3} + \frac{3\,000}{(1.1)^4} \\ &= \$1\,939.90 \end{aligned}$$

Series of Cash Flows

Present Value of a series of n cash flows:

$$
\begin{aligned}
P \quad &= \quad A_0 + A_1(1+i)^{-1} + A_2(1+i)^{-2} + \ldots + A_k(1+i)^{-k} + \ldots \\
&\qquad + A_{n-1}(1+i)^{-(n-1)} + A_n(1+i)^{-n} \\
&= \quad A_0 + \sum_{k=1}^{n} A_k (1+i)^{-k} \\
&= \quad A_0 + \sum_{k=1}^{n} A_k (P|F\ i,k)
\end{aligned}
$$

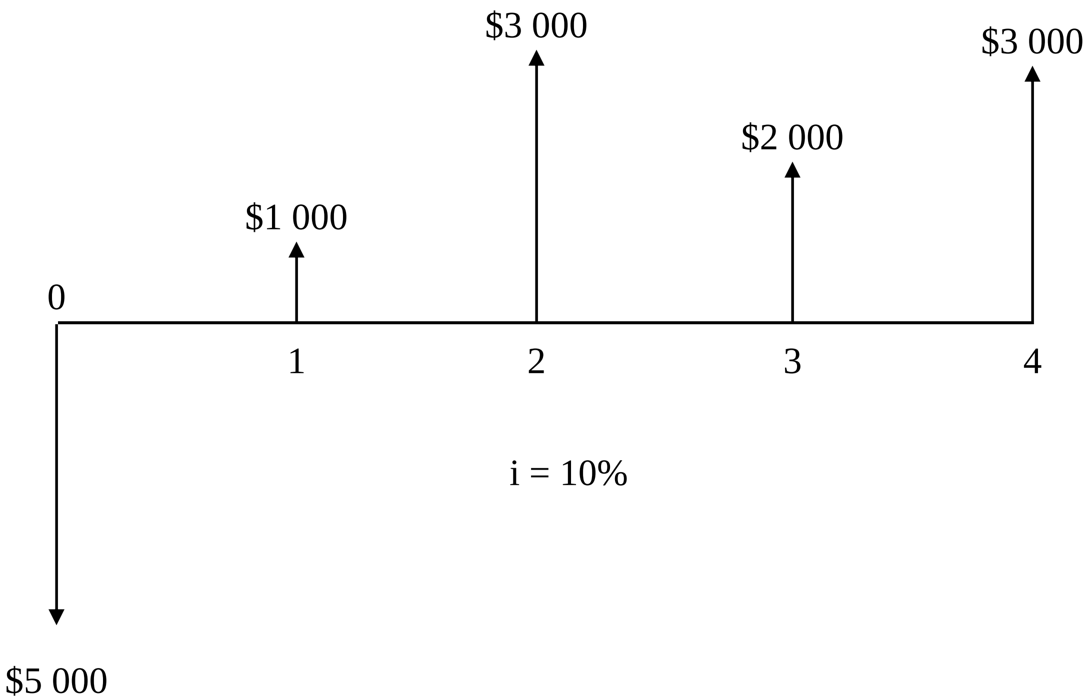
where

(P|F i,k) = Present Value Factor for Period k

Series of Cash Flows

Future value of a series of cash flows

$$F = A_0(1+i)^n + A_1(1+i)^{n-1} + ... + A_{n-1}(1+i) + A_n$$



$$F = A_0(1+i)^4 + A_1(1+i)^3 + A_2(1+i)^2 + A_3(1+i) + A_4$$
$$= -5\ 000(1.1)^4 + 1\ 000(1.1)^3 + 3\ 000(1.1)^2 + 2\ 000(1.1) + 3\ 000$$
$$= \$2\ 840.50$$

What is the Future Value of the Opportunity Cost investment?

What if the Future Value equals 0? Reject the investment?

Series of Cash Flows

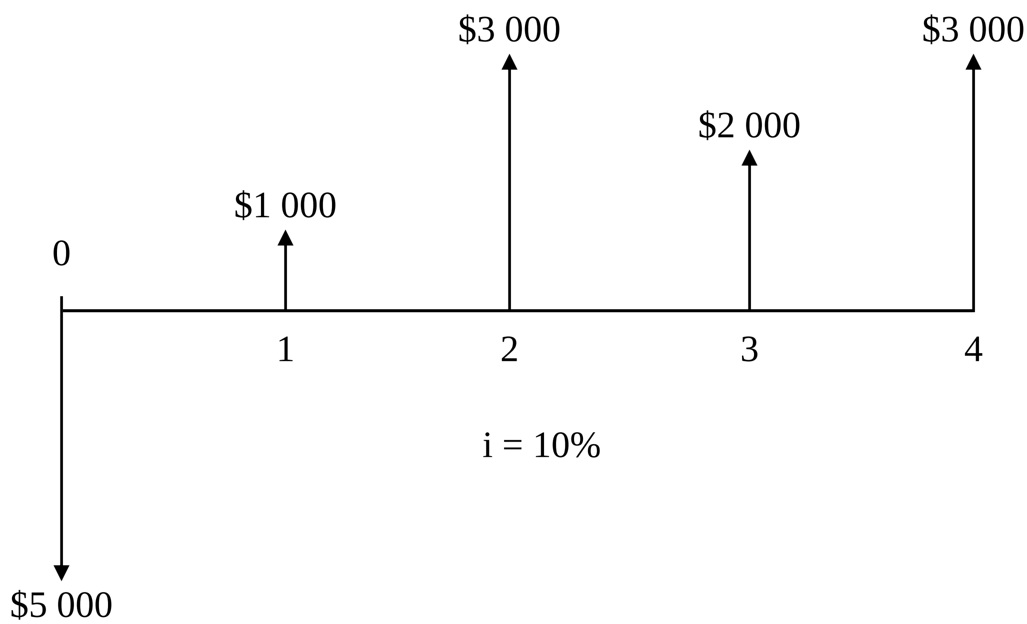
Future Value

$$
\begin{aligned}
F \quad = \quad & A_0(1+i)^n + A_1(1+i)^{n\text{-}1} + A_2(1+i)^{n\text{-}2} + \dots \\
& + A_k(1+i)^{n\text{-}k} + \dots + A_{n\text{-}1}(1+i) + A_n \\
= \quad & \sum_{k=0}^{n} A_k(1+i)^{n\text{-}k} \\
= \quad & \sum_{k=0}^{n} A_k \; (F \mid Pi, n-k)
\end{aligned}
$$

where

(F|P $i,n-k$) = Future Value Factor

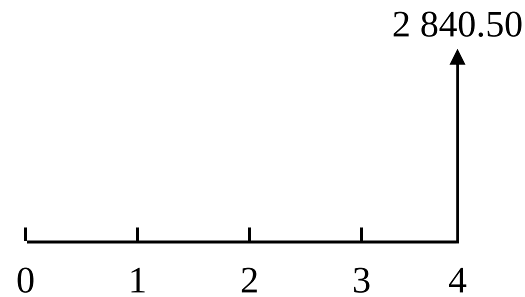
Present Value Versus Future Value



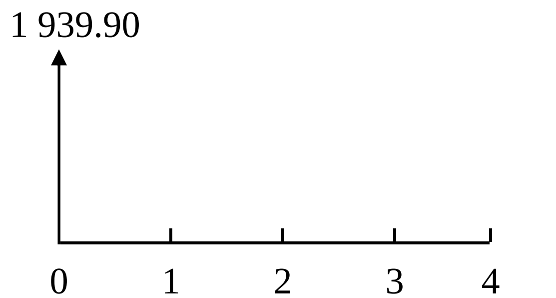
As calculated from the individual cash flows:

P (EOY 0) = $1 939.90
F (EOY 4) = $2 840.50

F (EOY 4) = P (EOY 0) (F|P 10,4)
= 1 939.90 (1.4641)
= $ 2 840.50



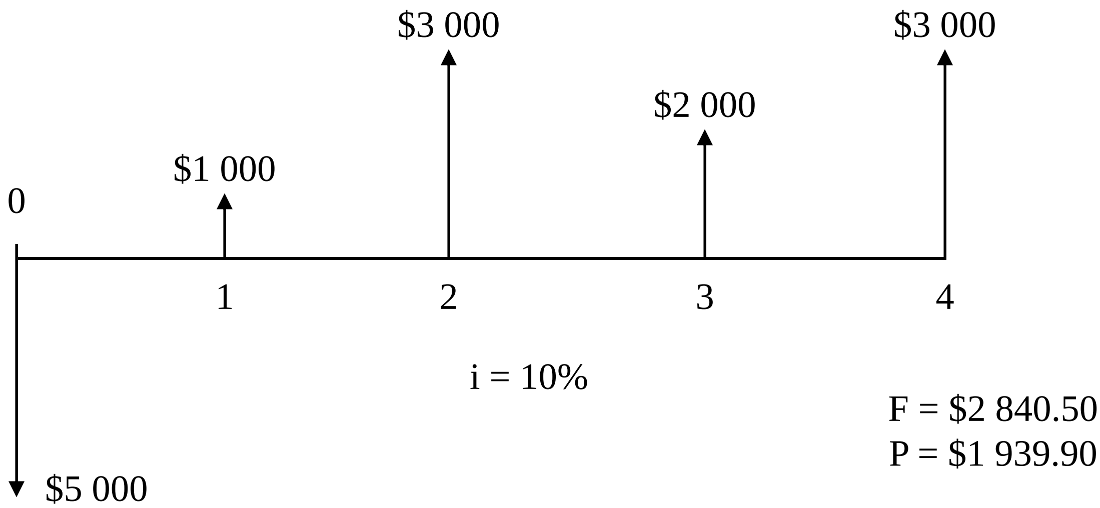
P (EOY 0) = F (EOY 4) (P|F 10,4)
= 2 840.50 (0.6830)
= $1 939.90



Future Value - Bank Account Example

Future value of a series of cash flows

$$F = A_0(1+i)^n + A_1(1+n)^{n-1} + ... + A_{n-1}(1+i) + A_n$$



- 10% interest compounded annually (positive or negative account balance)
- bank account (with overdraft privileges)

EOY	Amount "Rented" During Year	Interest 10%	Amount Deposited/ Withdrawn	Account Balance
0	--	--	– 5 000	– 5 000
1	– 5 000	– 500	1 000	– 4 500
2	– 4 500	– 450	3 000	– 1 950
3	– 1 950	– 195	2 000	– 145
4	– 145	– 14.50	3 000	**$2 840.50**

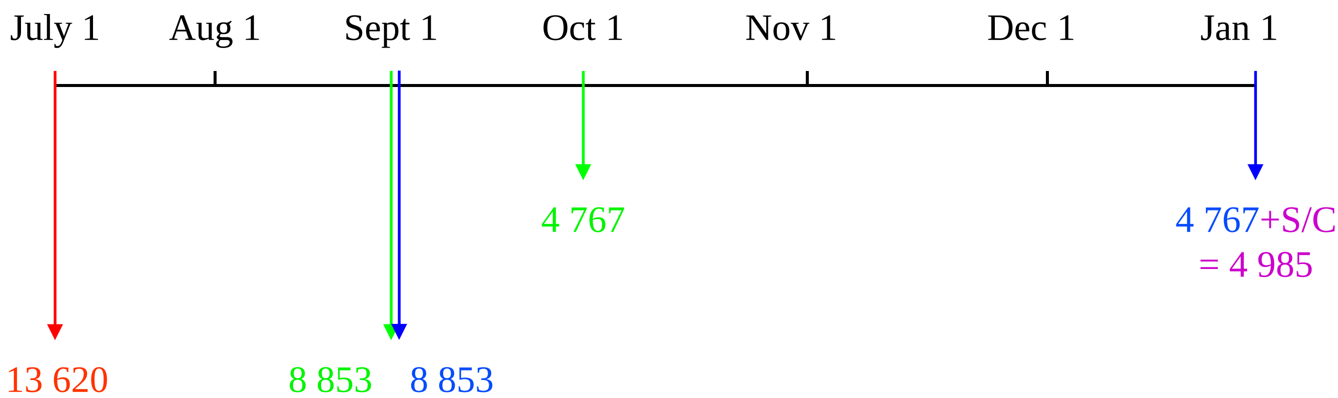
Optimal Tuition Payment Strategy

1. Fees invoice sent in July.
2. U of T recommends early payment.
   – at least first installment by August 15
3. Payment by installments allowed.
   – Total Fee $13 620
   – Minimum First Installment $8 853
   – Balance $4 767
4. First installment due before the first day of class.
5. Remainder of tuition fees due January 15.
6. Service charge (i.e. interest) charged on the balance outstanding from October 15.
   – Rate charged - 1.5% per month compounded
   => 19.56% per annum!

What is the optimal payment strategy that is consistent with money having a "time value?"

2015-2016
Fee Schedule

Tuition Payment Strategy



1 Interest on bank savings account: 0.25% per month

$\Rightarrow$ 3.04% per annum

2 U of T "service charge": 1.5% per month

$\Rightarrow$ 19.56% per annum

1 $PV|_{\text{July 1}} = \boxed{\$13\ 620}$

2 $PV|_{\text{July 1}} = 8\ 853(P|F\ 0.25\%,2) + \underbrace{(4\ 767 + S/C)}_{\text{Payment (January)}}(P|F\ 0.25\%,6)$

$\text{Payment(January)} = 4\ 767\ (F|P\ 1.5\%,3) = \$4\ 985$

$PV|_{\text{July 1}} = 8\ 853\ (P|F\ 0.25\%,2) + 4\ 767\ (F|P\ 1.5\%,3)(P|F\ 0.25\%,6)$

$= 8\ 853\ (0.9950) + 4\ 767\ (1.0457)(0.9851) = \boxed{\$13\ 720}$

3 $PV|_{\text{July 1}} = 8\ 853\ (P|F\ 0.25\%,2) + 4\ 767\ (P|F\ 0.25\%,3)$

$= 8\ 853\ (0.9950) + 4\ 767\ (0.9925)$

$= \boxed{\$13\ 540}$

Optimal Tuition Payment Strategy

Alternative 3 is the optimal payment strategy and has a Present Value (July 1) savings of:

$$PV_1|_{\text{July 1}} - PV_3|_{\text{July 1}} = 13\,620 - 13\,540 = \$80$$

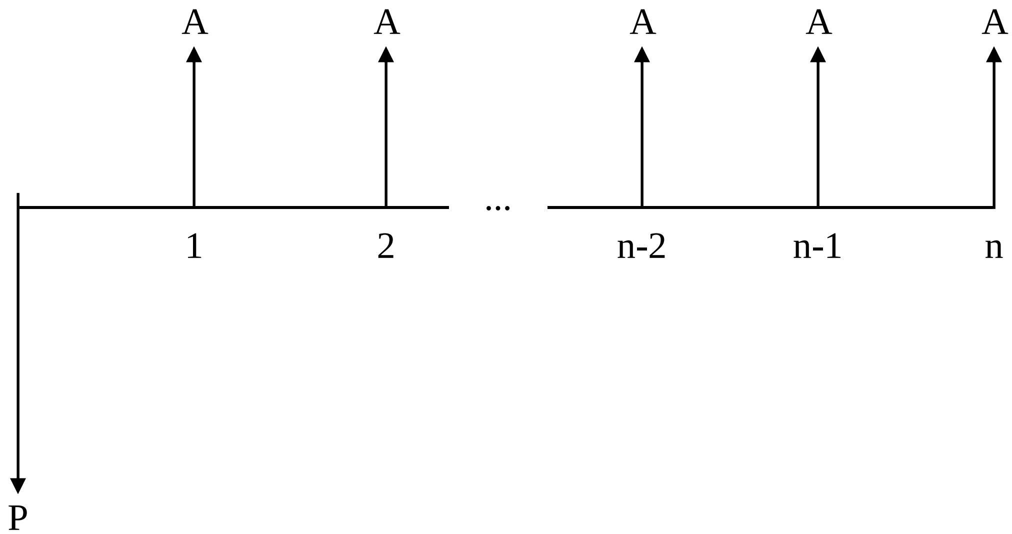
These savings are created by the time value of money at 0.25% per month.

Only $13 540 need be deposited into a bank account on July 1 to meet the required September & October payments.

	Beginning of Month Balance	Interest (0.25%)	Payment to U of T	End of Month Balance
July 1	$13 540	$34	--	$13 574
Aug 1	13 574	34	--	13 608
Sept 1	13 608	12	$8 853	4 767
Oct 1	4 767	0	4 767	0
Total Interest		$ 80		

Uniform Series of Cash Flows

- A uniform series of cash flows exists when all cash flows are equal



- Also called an annuity

$$P = A(1+i)^{-1} + A(1+i)^{-2} + \ldots + A(1+i)^{-(n-1)} + A(1+i)^{-n}$$

$$= \sum_{k=1}^{n} A(1+i)^{-k}$$

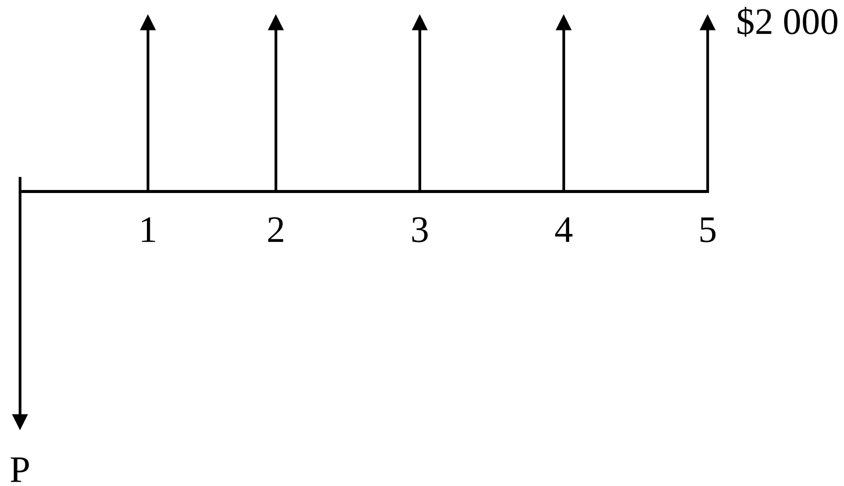
$$P = A\left[\frac{(1+i)^n - 1}{i(1+i)^n}\right]$$

Uniform series, Present Value Factor

P = A(P|A i,n)

Uniform Series of Cash Flows
- Annuity Funding

- Annuity funding



An individual wishes to deposit a single sum of money into a savings account so that five annual withdrawals of $2 000 can be made. The first withdrawal is to occur a year after the deposit and the rate of interest is 7%.

$$P = A\left[\frac{(1+i)^n - 1}{i(1+i)^n}\right] = A(P|A\ i,n)$$

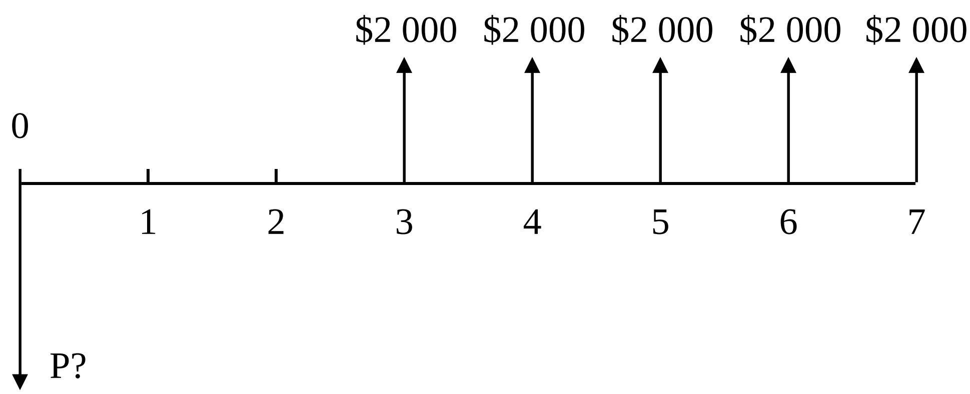
$$= 2\,000\left[\frac{(1.07)^5 - 1}{0.07(1.07)^5}\right] = 2\,000\ (P|A\ 7,5)$$

$$= 2\,000\ [4.1002] = \$8\,200.40$$

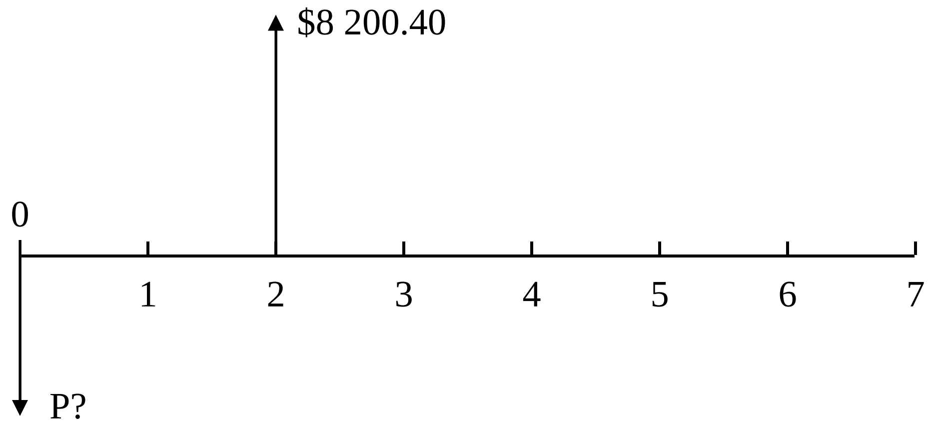
Therefore, if $8 200.40 is deposited into the fund, then five annual withdrawals can be made.

Deferred Annuity Funding

Suppose that the first withdrawal will not occur until three years after the deposit. How much must be deposited to receive the five annual $2 000 payments?



But we know that at Year 2 $8 200.40 will be sufficient to generate 5 payments of $2 000.



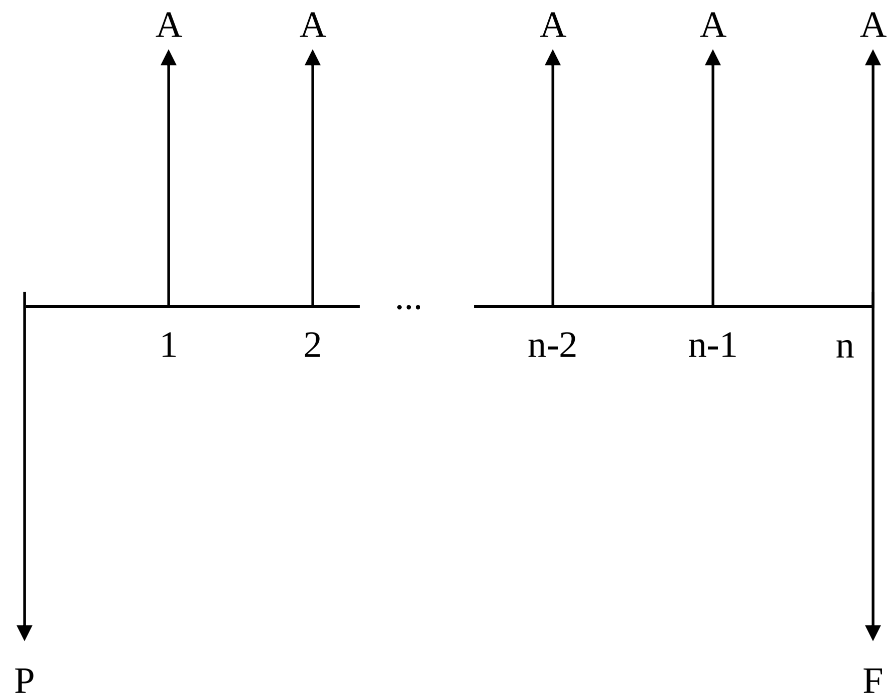
Therefore, we must discount the single sum of $8 200.40 back to Year 0.

$$P = \frac{\$8\ 200.40}{(1.07)^2} = \$8\ 200.40\ (P|F\ 7\%,2)$$

$$= \$7\ 162.55$$

Only $7 162.55 must be deposited. Deferring the withdrawal for two years reduces the deposit required by $1 037.85.

Applications of the Uniform Series of Cash Flows

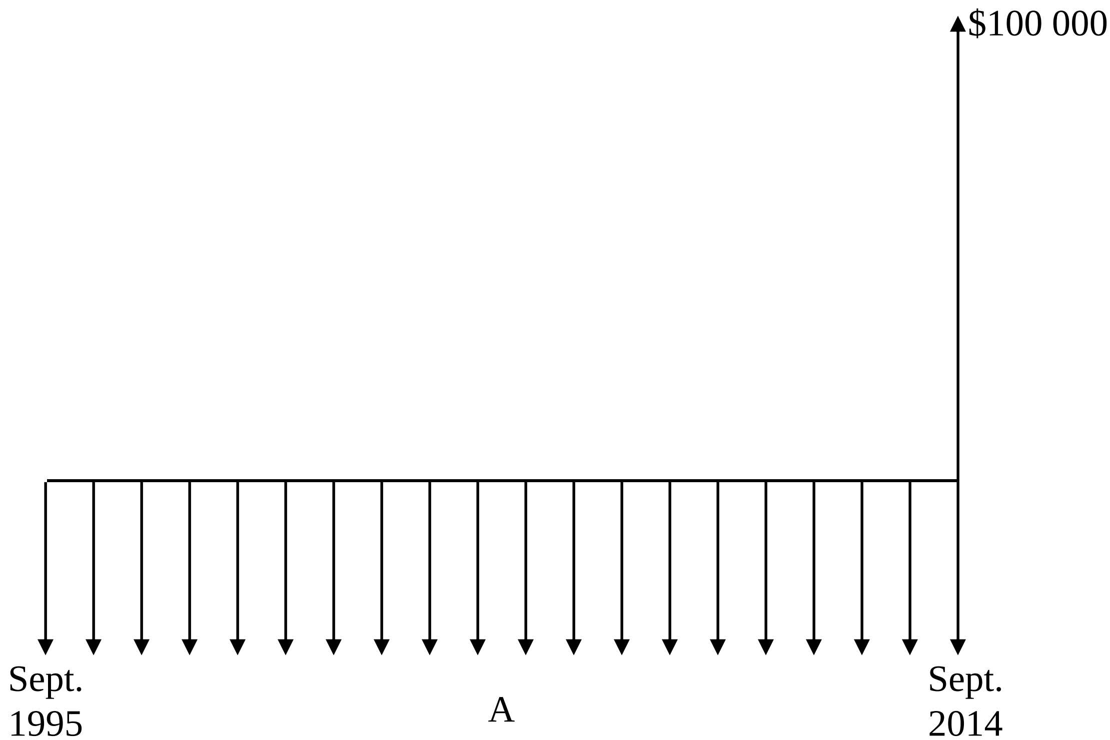


Present Value Factor, Annuity	P = A (P|A i,n)
Capital Recovery Factor	A = P (A|P i,n)
Future Value Factor, Annuity	F = A (F|A i,n)
Sinking Fund Factor	A = F (A|F i,n)

P|A – valuation occurs one time period before first cash flow

F|A – valuation occurs just after last cash flow

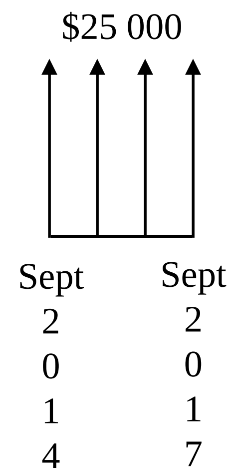
Sinking Fund - College Education Planning



- Funding a college education over 20 years with an equal annual contribution
- Need to estimate the following in September 2014:
  - Four-year cost of education ($100 000)
  - average interest rate over 20 years (7%)

$$
\begin{aligned}
A &= F\,(A|F\ i,n) \\
&= 100\,000\,(A|F\ 7,20) \\
&= 100\,000\,(0.0244) \\
&= \$2\,440 \text{ per year}
\end{aligned}
$$

College Education Planning
- Four Annual Installments



$$
\begin{aligned}
A &= \overbrace{[25\,000 + 25\,000\ (P|A\ 7{,}3)]}^{90\,608}\ (A|F\ 7{,}20) \\
&= [25\,000(1 + 2.6243)](0.0244) \\
&= \$2\,211 \text{ per year}
\end{aligned}
$$