Multiple Compounding Periods in a Year

- Not all interest rates are stipulated as annual compounding rates – often rates are compounded more frequently

$1 000 Loan – 8% Interest Compounded Quarterly

=> 2% per 3-month period for 4 interest periods

F = P (F|P 2,4)
= $1 000 (1.0824)
= $1 082.40

Equivalent to

F = P (F|P 8.24,1)
= $1 000 (1.0824)
= **$1 082.40**

Nominal Interest Rate = 8%

Per Period Interest Rate = 2%

Effective Annual Rate = 8.24%

Q 1	1 000.00	20.00	1 020.00
Q 2	1 020.00	20.40	1 040.40
Q 3	1 040.40	20.81	1 061.21
Q 4	1 061.21	21.22	**1 082.43**

Effective Interest Rates

r = nominal annual interest rate

m = number of compounding periods per year

i_{eff} = effective interest rate per year

$$i_{eff} = \left(1+\frac{r}{m}\right)^m - 1$$

$$= \left(\text{F|P } \frac{r}{m}, m\right) - 1$$

e.g. 8% Interest Compounded Quarterly

r = .08 r/m = .02

m = 4

i_{eff} = (F|P 2,4) – 1

= 1.0824 – 1 = 8.24%

UofT Tuition
"Service Charge"
1.5% per month
$i_{eff} = (1+0.015)^{12} - 1$
= (F|P 1.5,12) - 1
$\rightarrow i_{eff}$ = 19.56%
vs. 18% nominal rate

- Discrete compounding is normally used in finance
- In the limit ($m \rightarrow \infty$) implies continuous compounding
  - optional topic (Section 3.6)
  - use the discrete compounding tables in Appendix A for business purposes

Bank Account Compounding

Consider two different types of bank accounts:

1. **Savings Account:** Interest paid on minimum quarterly balance (annual rate of 3% paid on balances > $5 000)
2. **Daily Interest Account:** Interest earned from time of deposit (annual rate of 1% paid on balances > $1 000)

Each account states that the nominal rate is compounded quarterly.

Consider a customer with three monthly deposits of $10 000.

Let i_S and i_D be the monthly rates; then,

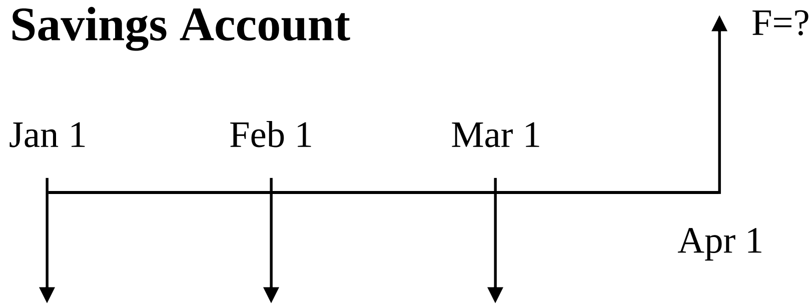
$$\begin{aligned} i_{eff} &= (1+i)^{12} - 1 \\ &= \left(1 + \frac{0.03}{4}\right)^4 - 1 \end{aligned}$$

$$\begin{aligned} \Rightarrow \quad (1+i_S)^{12} &= (1.0075)^4 \\ (1+i_S)^3 &= 1.0075 \\ \therefore \quad i_S &= (1.0075)^{1/3} - 1 = 0.00249378 \end{aligned}$$

$$\boxed{\begin{aligned} i_S &= 0.249378\% \\ i_D &= 0.083264\% \end{aligned}}$$

Interest Earned: Three Monthly Deposits of $10 000

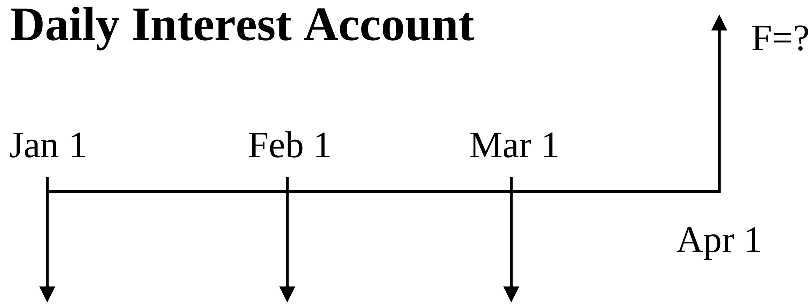
1. Savings Account



Minimum balance = 10 000   I = 75

$\boxed{\therefore \quad F = \$30\ 075.00}$

2. Daily Interest Account



- Use monthly rate of 0.083264%

January deposit	$10\ 000\ (1+.00083)^3$	$= 10\ 025.00$
February deposit	$10\ 000\ (1+.00083)^2$	$= 10\ 016.66$
March deposit	$10\ 000\ (1+.00083)$	$= 10\ 008.33$

$\boxed{\therefore \quad F = \$30\ 049.99}$

<u>OR</u>

$$F = 10\ 000\ \overset{\underline{3.00250}}{(F|A\ 0.083\%,3)}\ \overset{\underline{1.00083}}{(F|P\ 0.083\%,1)}$$
$$= 30\ 049.99$$

Interest Earned: Three Monthly Deposits of $10,000

Savings Account

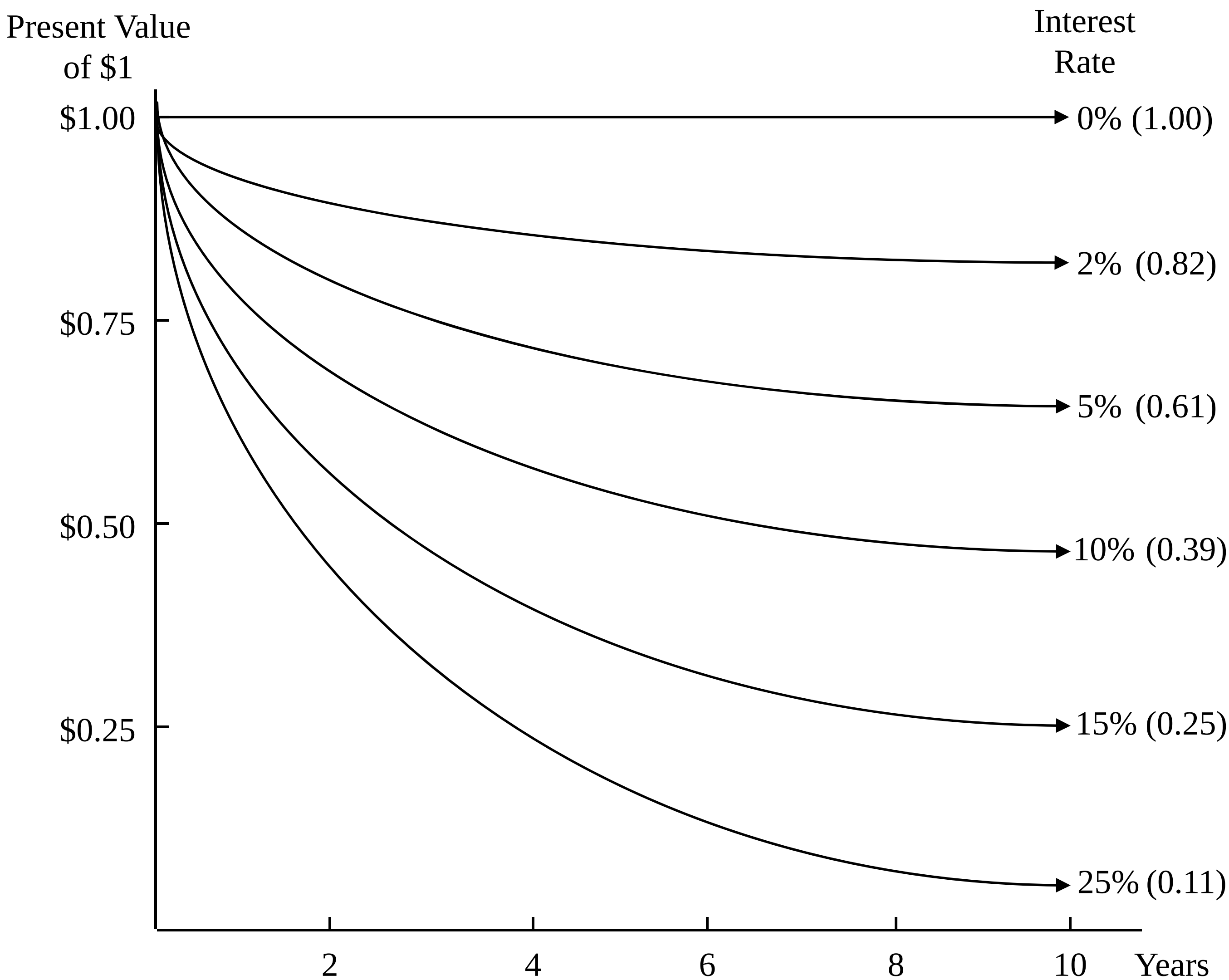
Quarterly Interest Rate:			0.7500%		
	Deposit		End-of-Quarter		End-of-Day
			Interest		Balance
January 1	10,000.00				10,000.00
January 31					10,000.00
February 1	10,000.00				20,000.00
February 28					20,000.00
March 1	10,000.00				30,000.00
March 31			75.00		**30,075.00**
Note minimum balance for the quarter is $10 000.					

Daily Interest Account

Monthly Interest Rate:			0.08326%		
	Deposit		End-of-Month		End-of-Day
			Interest		Balance
January 1	10,000.00				10,000.00
January 31			8.33		10,008.33
February 1	10,000.00				20,008.33
February 28			16.66		20,024.99
March 1	10,000.00				30,024.99
March 31			25.00		**30,049.99**

Best strategy? – Two Accounts!

Relationship Between the Present Value Factor, Inflation Rates and Time



Present Value (P|F i,n)

“Discount Factor” < 1.00

- present value factors listed are for 10 years at the stated interest rate

Series of Cash Flows - Special Cases

Three Special Cases Used Frequently in Finance

- each has a closed-form solution

1. Uniform series of cash flows (annuity)

$$A_k = A \qquad\qquad k = 1,...,n$$

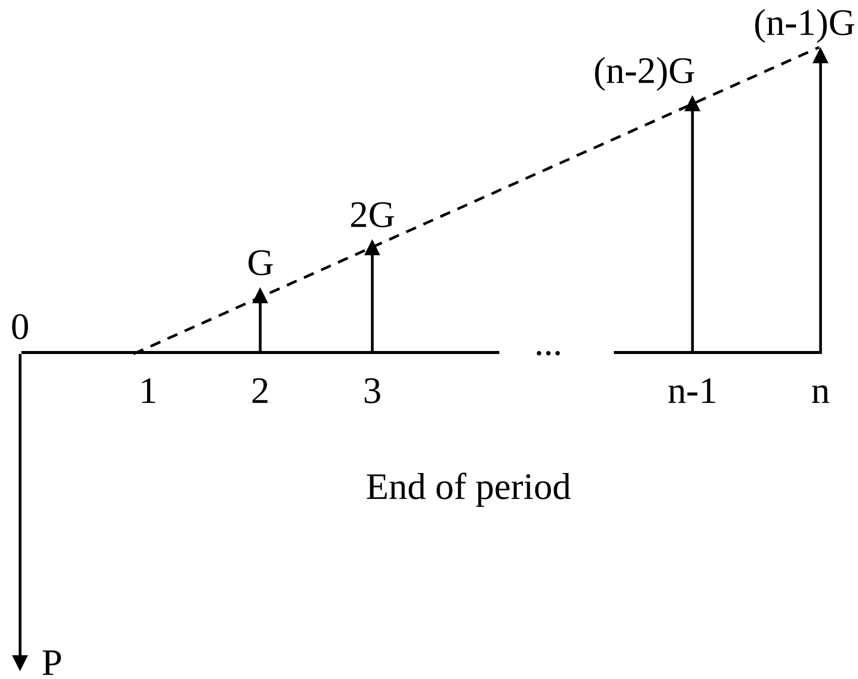
2. Gradient series of cash flows

$$A_k = \begin{cases} 0 & k = 1 \\ A_{k-1} + G & k = 2,...,n \end{cases}$$

3. Geometric series of cash flows

$$A_k = \begin{cases} A & k = 1 \\ A_{k-1}(1 + j) & k = 2,...,n \end{cases}$$

Gradient Series of Cash Flows



- Each successive cash flow increases by a fixed amount equal to G
- Note that there is <u>no</u> cash flow at end of period 1

$$A_k = (k-1)G \qquad\qquad k = 1,\ldots,n$$

$$P = \sum_{k=1}^{n} A_k (1+i)^{-k}$$

$$= G\sum_{k=1}^{n}(k-1)(1+i)^{-k}$$

$$P = G\left[\frac{1-(1+ni)(1+i)^{-n}}{i^2}\right]$$

Gradient Series, Present Value Factor

P = G (P|G i,n)

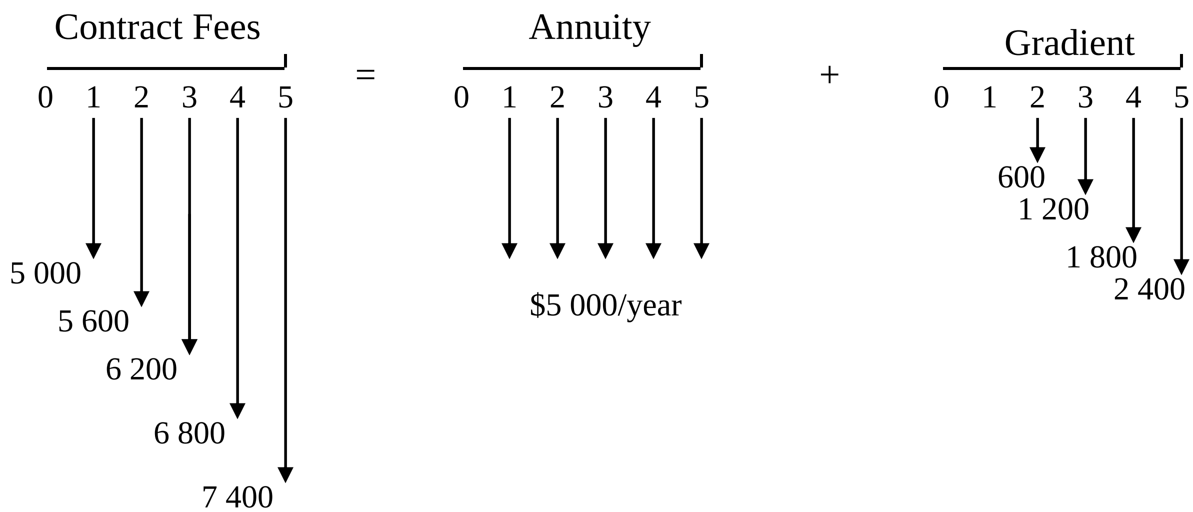
Gradient Series of Cash Flows

Computer Maintenance Service Agreement (five years)

$5 000 per year service fee

$600 annual escalation factor

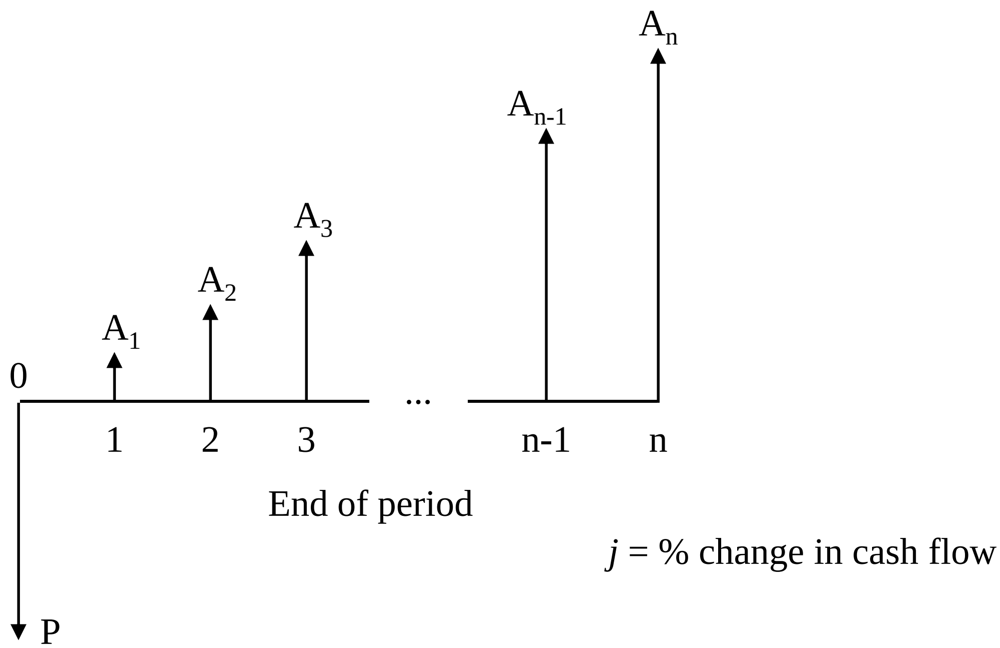
What is the Present Value Cost at a 6% Interest Rate?



i = 6%
n = 5

$$PV = A\ (P|A\ 6,5) + G\ (P|G\ 6,5)$$
$$= 5\ 000(4.2124) + 600(7.9345)$$
$$= \$25\ 822.70$$

Geometric Series of Cash Flows



- Each successive cash flow increases by a fixed percentage equal to j

$$A_k = A_{k-1}(1+j) \qquad k = 2,\ldots,n$$
$$A_k = A_1(1+j)^{k-1} \qquad k = 1,\ldots,n$$

$$P = \sum_{k=1}^{n} A_1(1+j)^{k-1}(1+i)^{-k}$$

$$P = \begin{cases} A_1\left[\dfrac{1-(1+j)^n(1+i)^{-n}}{i-j}\right] & i \neq j \\ \dfrac{nA_1}{1+i} & i = j \end{cases}$$

Geometric Series, Present Value Factor

Corporate Dividend Valuation

$$P = A_1\ (P|A_1\ i,j,n)$$

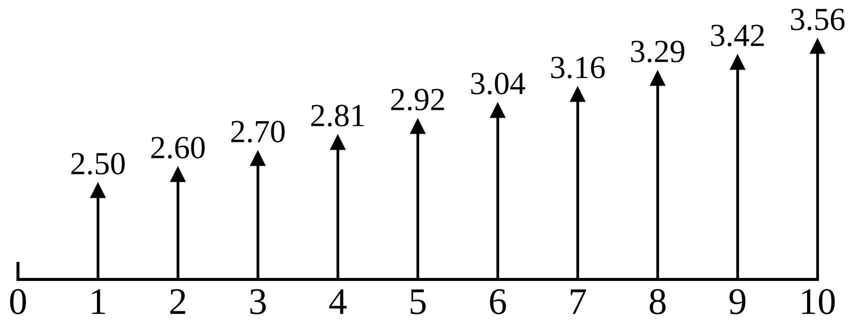
A shareholder owns 1 000 shares of XYZ Company and intends to own these shares for 10 years.

The shareholder expects next year's dividend on a share to be \$2.50. The expected annual growth in dividends is 4%.

What is the present value of the future dividend stream if money has a time value of 5%?

$$\begin{aligned} P &= A_1\ (P|A_1\ 5\%,4\%,10) \\ &= 1\ 000\ (2.50)\ (9.1258) \\ &= \$22\ 814.50 \end{aligned}$$

Geometric Series - Stock Purchase Evaluation



"Dividend Stream"

$A_1 = 2.50$
$j = 4\%$

$A_n = A_1(1+j)^{n-1}$

Present Value of Dividends per Share

$$P = A_1\,(P|A_1\ i,j,n)$$
$$= 2.50\,(P|A_1\ 5,4,10)$$
$$= 2.50\,(9.1258) = \$22.81$$

$i = 5\%$: Investor requires at least this rate of return.

Total Return: 1. from dividends

2. from value of share sold at EOY 10

Investor <u>forecasts</u> price of share to be \$50.00 at the End of Year 10.

\$22.81 \$30.70

$$PV_{\text{Future Returns}} = PV_{DIV} + 50\,(P|F\ 5,10) = \$53.51 \text{ per share}$$

Overvalued? => Sell (or "short")

Under Valued? => Buy

Stock Market Transactions

Long Position

- Shareholder believes that the share is fairly valued and intends on owning the share for a long period of time
- Return to shareholder via dividends and share price appreciation
- Usual way of making money in the stock market – “Buy Low and Sell High”

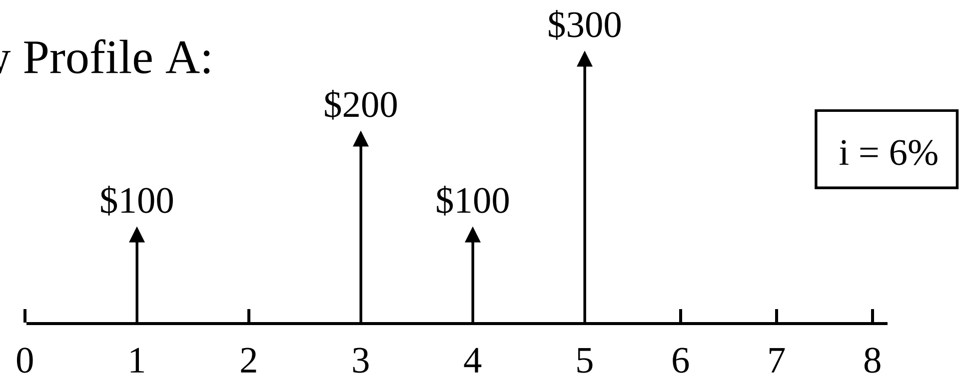
Short Position

- Requires two investors with divergent views of a company’s future
- Investor A has a long position but is willing to lend shares for a fixed period of time and receive a fee
- Investor B believes the share is over-valued and is willing to borrow shares and pay a fee
- Investor B borrows the shares, pays Investor A the fee and then sells the shares immediately
- Investor B buys back the shares before the expiry date of the short contract and returns them to Investor A
- If Investor B is correct the repurchase price will be lower than the selling price resulting in a profit on the short contract
- What are the risks to Investor A and Investor B?

Equivalence

- Two cash-flow profiles are <u>equivalent</u> at some specified interest rate, i %, if their present values are equal using that interest rate of i.

Cash Flow Profile A:



Cash Flow Profile B:

Are these two very different cash flows equivalent?

A: P = 100(P|F 6,1) + 200(P|F 6,3) + 100(P|F 6,4) + 300(P|F 6,5)
= $565.65

B: P = 91.09 (P|A 6,8)
= $565.65

Therefore, A and B are equivalent at an interest rate of 6%.

Equivalence

Web Server System

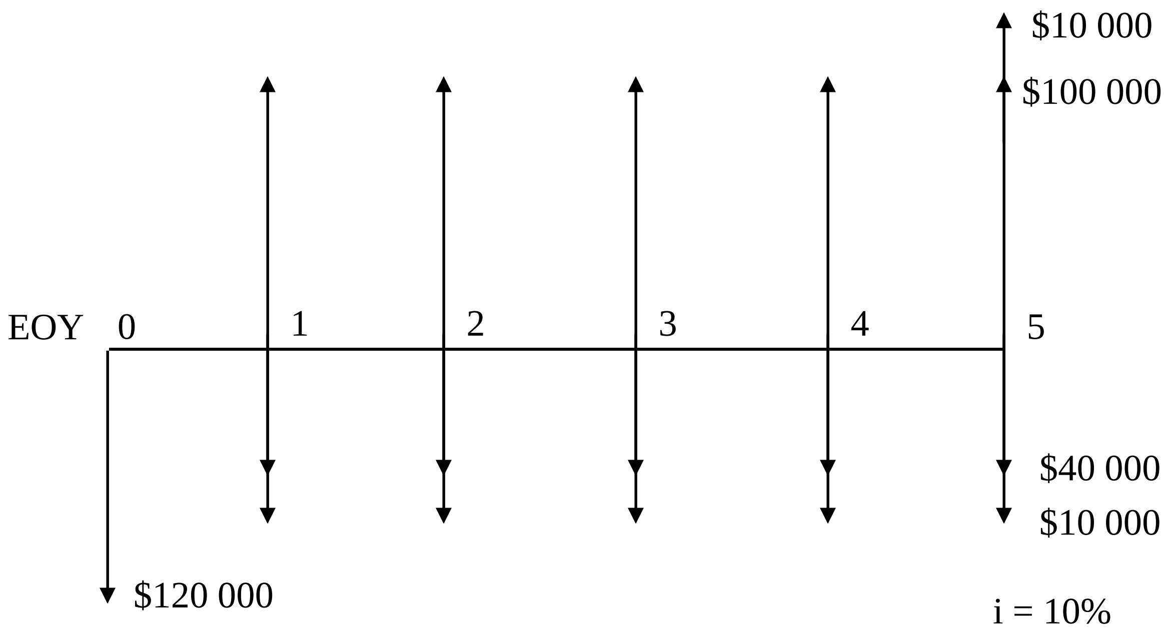
Purchase Price of server system	\$120 000
Estimated economic life of the system	5 years
Annual maintenance and Internet charges	\$ 10 000
Staff salaries	\$ 40 000
Annual estimated cash flow benefits	\$100 000
Salvage value (EOY 5)	\$ 10 000

What single sum of money at time 0 is equivalent to these cash flows using an interest rate of 10%?

Calculate the present value of the cash flows.

1. If PV = 0, then the rate of return is exactly 10%.
2. If PV > 0, then the rate of return > 10%.
3. If the PV > 0, then the FV > 0.
4. If the FV > 0, the rate of return is greater than the opportunity cost investment (10%).

Web Server System
Cash Flow Profile



EOY 0	– $120 000	(server purchase)
EOY 1 – 5	– $ 10 000	(server on-going costs)
	– $ 40 000	(staff costs)
	+ $100 000	(benefits)
	$ 50 000	
EOY 5	$ 10 000	(server salvage)

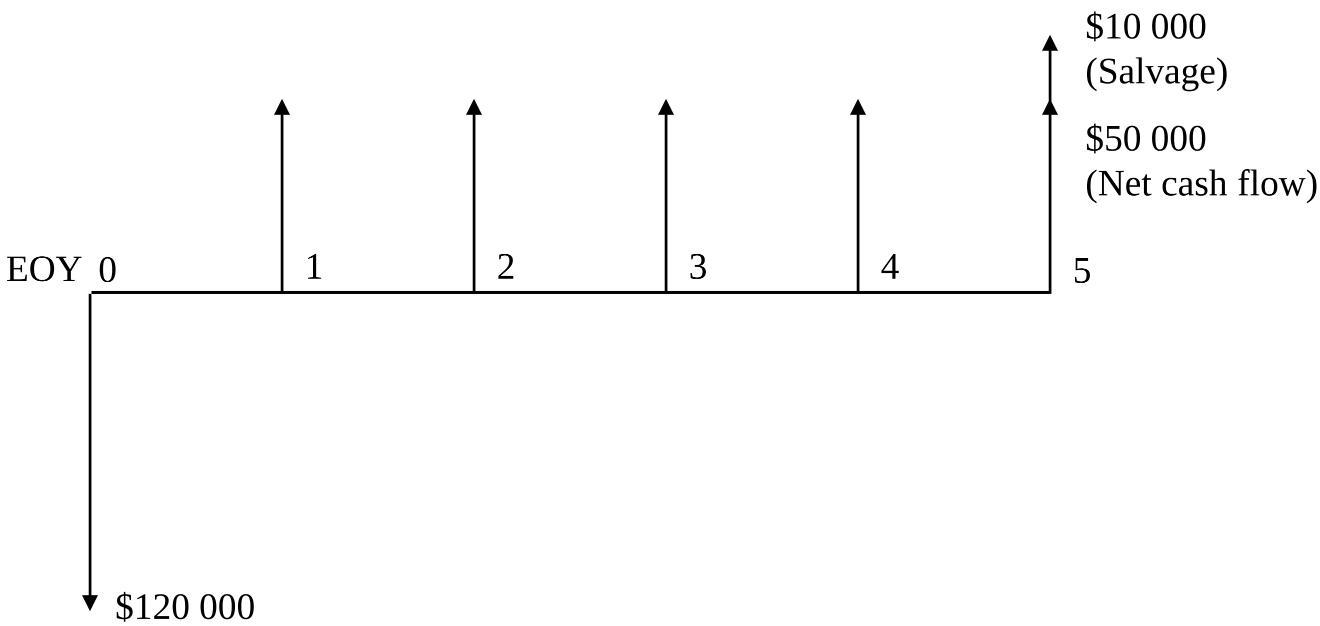
Present Value of Cash Flow Profile

$$PV|_{EOY\ 0} = -\ 120\ 000 + 50\ 000\ (P|A\ 10{,}5) + 10\ 000\ (P|F\ 10{,}5)$$

$$= -\ 120\ 000 + 50\ 000(3.7908) + 10\ 000(0.6209)$$

$$= \quad \$75\ 749$$

Web Server System
Net Cash Flow Approach



Net Cash Flow	50 000 (P|A 10,5)	189 540
Salvage	10 000 (P|F 10,5)	6 209
Total PV of Future Returns		$195 749
Less: Investment		120 000
PV of Cash Flow		$ 75 749

Web Server System - Future Value

The $120 000 can be invested in the server or at 10% (the company's time value of money & its investors' opportunity cost). What is the value of each alternative at the EOY 5?

Opportunity Cost Investment			Web Server System		
EOY	Returns	EOY Value	CF	Returns	EOY Value
0	-	120 000	-	-	0
1	12 000	132 000	50 000	-	50 000
2	13 200	145 200	50 000	5 000	105 000
3	14 520	159 720	50 000	10 500	165 500
4	15 972	175 692	50 000	16 550	232 050
5	17 569	193 261	60 000	23 205	315 255

$121 994

The Web Server Project results in being $121 994 better off than the opportunity cost investment at the EOY 5.

How does this relate to the present value?

$121 994 (P|F 10,5) = $\boxed{\$75\ 749}$

Principal & Interest In Loan Payments

- interest on money borrowed for business purposes is tax deductible (personal use borrowing is not tax deductible)
- repayment of principal is not tax deductible
- for each payment, need to know:
  - how much is interest
  - how much is repayment of principal

Principal Amount

- initial amount of money borrowed

Amortization Period

- period of time over which the indebtedness will be repaid

Mortgage Period

- period of time of the current mortgage contract

Blended Payment

- each payment contains both interest and a partial repayment of the principal

Equity Payment

- portion of each payment reducing the principal amount

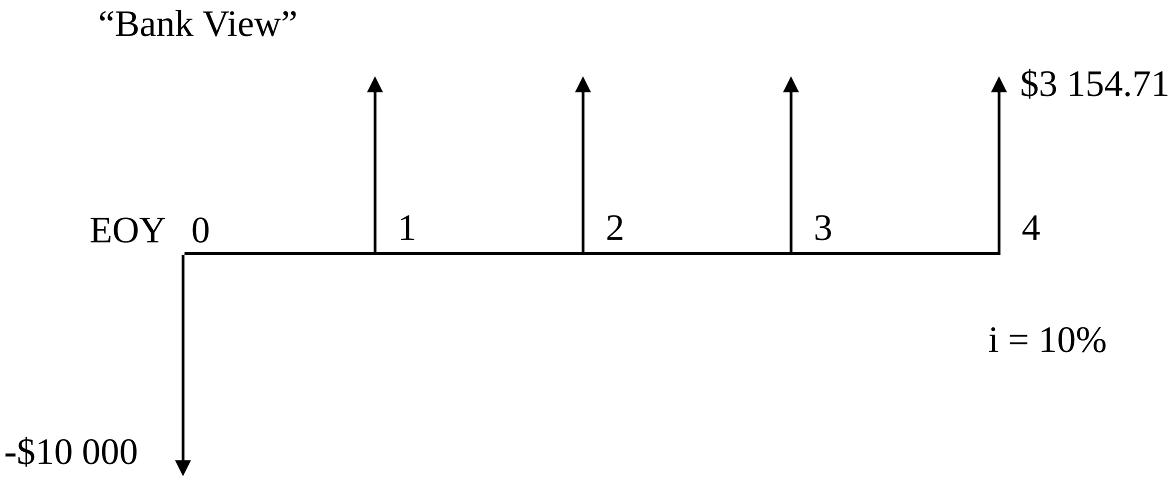
Mortgage

- loan secured by physical assets

Debenture

- loan not secured by any physical assets
- guaranteed only by the financial strength of the issuing firm

Principal & Interest in Loan Payments



4 Year Loan

Principal Amount $10 000

Interest 10% compounded annually

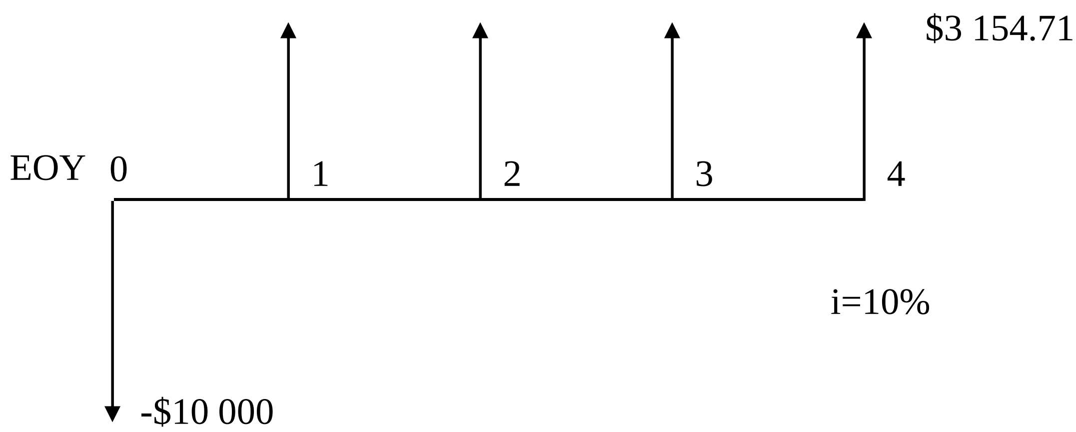
Payment Size

$$A = P\,(A|P\ 10,4)$$
$$= 10\ 000\ (0.315471)$$
$$= \$3\ 154.71$$

Future Value of Principal Plus Payments

$$F = P\,(F|P\ 10,4) + A\,(F|A\ 10,4)$$
$$= -10\ 000\ (1.4641) + 3\ 154.71\ (4.6410)$$
$$= 0$$

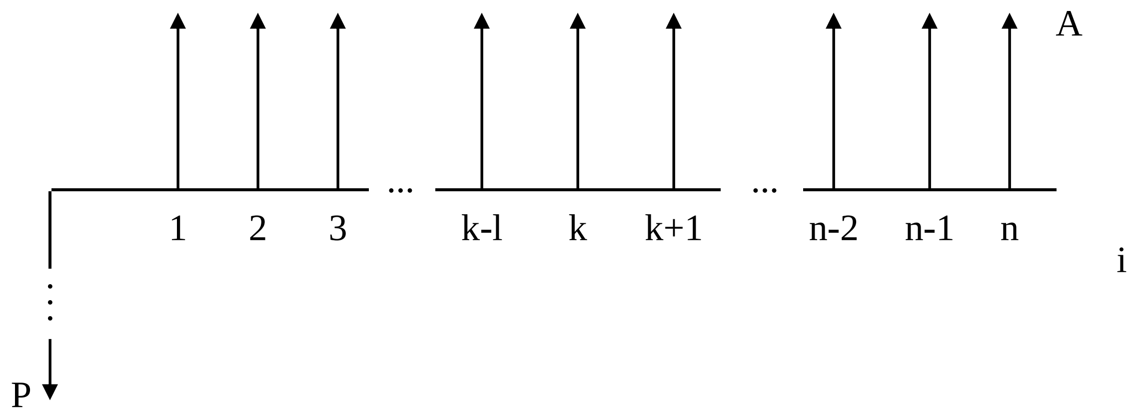
Principal and Interest in Loan Payments



EOY	Interest Calculation	Interest Payment	Equity Payment	Total Payment	Balance
0					10 000.00
1	10 000.00	1 000.00	2 154.71	3 154.71	7 845.29
2	7 845.29	784.53	2 370.18	3 154.71	5 475.11
3	5 475.11	547.51	2 607.20	3 154.71	2 867.91
4	2 867.91	286.79	2 867.91	3 154.70	0.00
Total		2 618.83	10 000.00		

Only the interest payment portion is a tax deductible expense.

Principal and Interest - Loan Payments



- What is the equity payment made in Period k?
- What is the interest payment made in Period k?

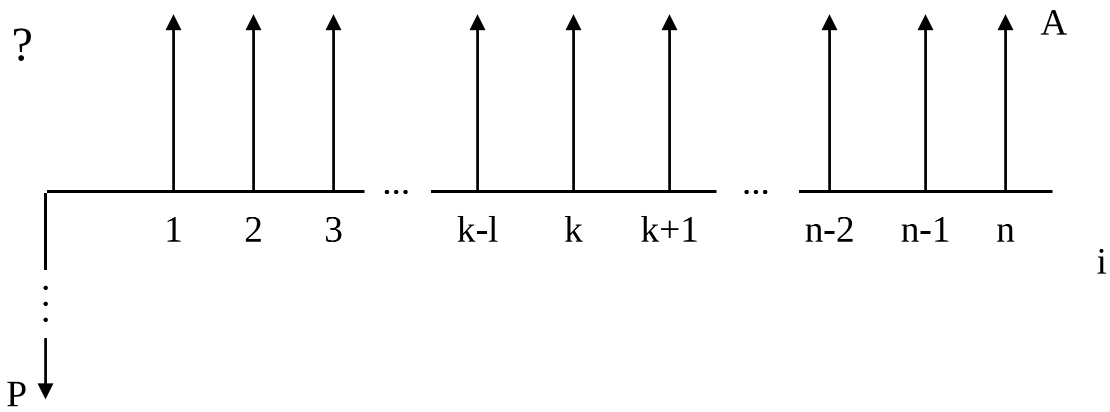
Let: P = amount borrowed (principal amount)
n = number of time periods (amortization period)
i = interest rate compounded per time period
A = payment per time period (blended payment)

Also let: E_k = equity payment in Period k
I_k = interest payment in Period k

Principal and Interest - Loan Payments

$E_k = ?$

$I_k = ?$



$$A = E_k + I_k \qquad \text{(Blended Payment)}$$

Per Period Payment: $A = P\,(A|P\ i,n)$

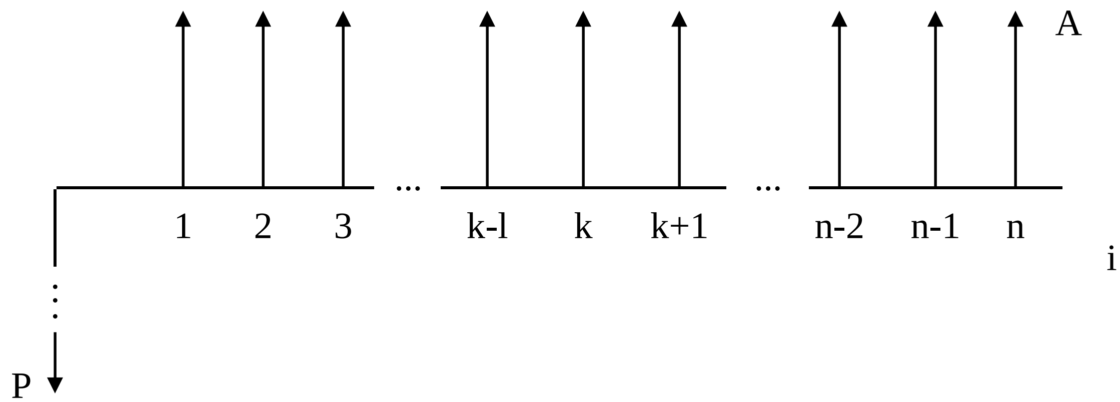
$$\Rightarrow P = A\sum_{j=1}^{n}(1+i)^{-j}$$

What is the unpaid principal remaining after making payment (k – 1)?

Number of payments left:

$$n - (k - 1) = n - k + 1$$

Principal and Interest - Loan Payments



Let U_{k-1} = unpaid principal after making payment (k – 1) then

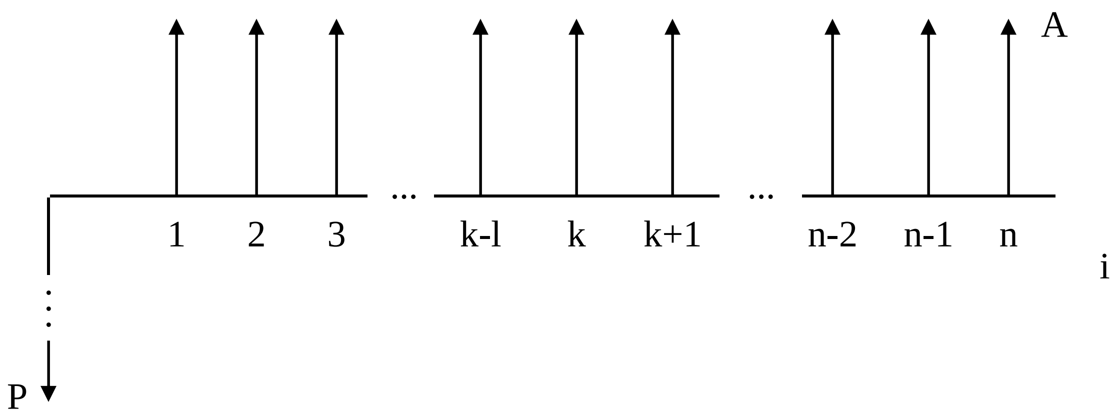
$$U_{k-1} = A\,(P|A\;i, n-k+1)$$

$$= A \sum_{j=1}^{n-k+1} (1+i)^{-j}$$

How much does payment k <u>reduce</u> the unpaid principal?

$$U_{k-1} - U_k = E_k$$

Principal and Interest - Loan Payments

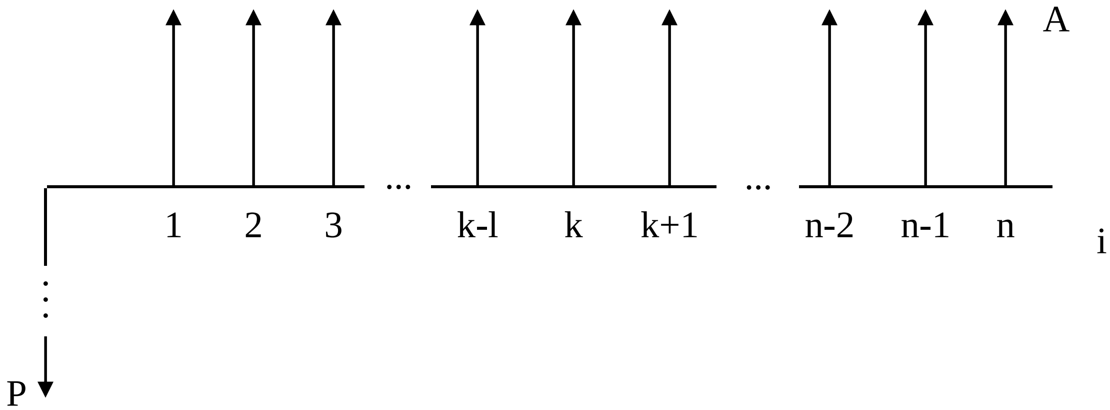


$$
\begin{aligned}
\therefore\ E_k &= A\sum_{j=1}^{n-k+1}(1+i)^{-j} - A\sum_{j=1}^{n-k}(1+i)^{-j} \\
&= A\left[\sum_{j=1}^{n-k}(1+i)^{-j} + (1+i)^{-(n-k+1)}\right] - A\sum_{j=1}^{n-k}(1+i)^{-j} \\
&= A(1+i)^{-(n-k+1)}
\end{aligned}
$$

Equity Payment in Period k

$E_k = A\ (P|F\ i,n - k + 1)$

Principal and Interest - Loan Payments



$$E_k = A\ (P|F\ i,n - k + 1)$$

and $$A = P\ (A|P\ i,n)$$

$$\boxed{E_k = P\ (A|P\ i,n)(P|F\ i,n - k + 1)}$$

Also:

$$A = E_k + I_k$$

$$\Rightarrow I_k = A - E_k$$

$$= A - A\ (P|F\ i,n - k + 1)$$

$$\boxed{I_k = A(1 - (P|F\ i,n - k\ +1))}$$

Condo Mortgage

Purchase price	\$200 000	
Down payment	\$ 25 000	(12.5% “down”)
Amount to finance	\$175 000	

Purchaser selects a 25-year amortization period.

Five-year term of initial mortgage at 7% interest compounded monthly. (Note that current 5-year rates are around 3 – 4 %.)

1. What is the split between interest and principal in the last payment?

Monthly Payment: $A = \$175\,000\ (A|P\ 7/12, 300)$

$= \$175\,000\ (0.0071)$

$= \$1\,236.86$

Equity Payment: $E_k = A\ (P|F\ i{,}n - k + 1)$

$E_{60} = A\ (P|F\ 7/12, 241)$

$= 1\,237\ (0.2462)$

$= \$304.47$

Interest Payment: $I_k = A - E_k$

$= \$932.39$

k=60 at the end of the five-year term

Condo Mortgage

2. At the end of the initial five-year term, how much of the $175 000 mortgage remains?

$$
\begin{aligned}
P|_{EOP_{60}} &= A\,(P|A\ 7/12,\ 240) \\
&= 1\ 237\ (128.93) \\
&= \$159\ 533.30
\end{aligned}
$$

This amount has to be refinanced by the next mortgage.

Over the first five-year term of the mortgage:

Total payments	(60 * $1 237)	$74 220
Mortgage reduction	($175 000 – $159 533)	<u>$15 467</u>
Interest payments!!!!		<u>$58 753</u>

The owner's equity in the condo after five years is $40 467.

The owner has invested $99 220 in the condo over the initial five-year period.

Condo Mortgage

What are the risks to the financial institution in lending money for a residential mortgage?

1. Disaster risk
   - A fire may destroy the building
2. Default risk
   - Borrower loses job and cannot afford payments
3. Housing price risk
   - Housing prices may go down
   - “mortgage under water”
4. Interest rate risk
   - Five-year term guarantees the mortgage interest rate for five years
   - Interest rates may increase requiring the bank to pay depositors a rate higher than the mortgage rate

How does the financial institution protect itself against these mortgage risks?

- Who should assume these risks?
- Who pays?