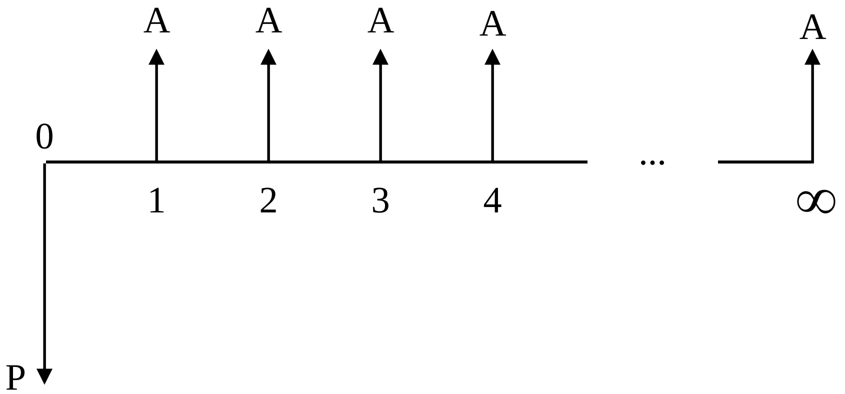
Perpetuities



- uniform series where the payments continue indefinitely
- useful approximation for projects with an estimated life of 50 years or more

$$P = \frac{A}{i}$$

- the present value P is the <u>capitalized value</u> of A

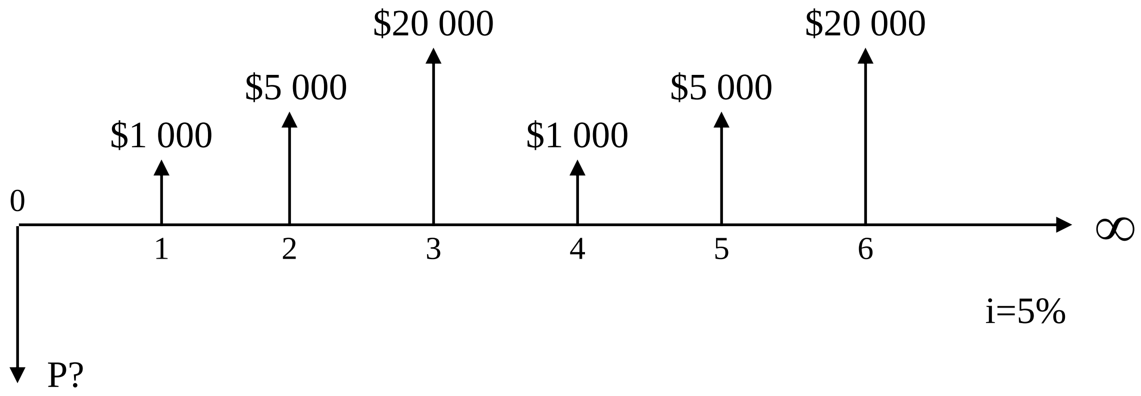
<u>Year 50 Present Value Factors</u>

(P|F 5,50) = 0.0872

(P|F 10,50) = 0.0085

(P|F 15,50) = 0.0009

Track Team Endowment



How much does the alumnus have to donate so that the resulting endowment income will continue to fund the track team competition indefinitely?

Convert each three-year segment into an annuity.

What single sum is equivalent?

$$P = 1\,000\ (P|F\ 5,1) + 5\,000\ (P|F\ 5,2) + 20\,000\ (P|F\ 5,3)$$

$$= 1\,000\ (0.9524) + 5\,000\ (0.9070) + 20\,000\ (0.8638)$$

$$= \$22\,763$$

How much of a three-year annuity will this fund?

$$A = 22\,763\ (A|P\ 5,3)$$

$$= 22\,763\ (0.3672)$$

$$= \$8\,359$$

What is the capitalized value of this three-year annuity repeating indefinitely?

$$P = \frac{A}{i} = \frac{8\,359}{0.05} = \$167\,174$$

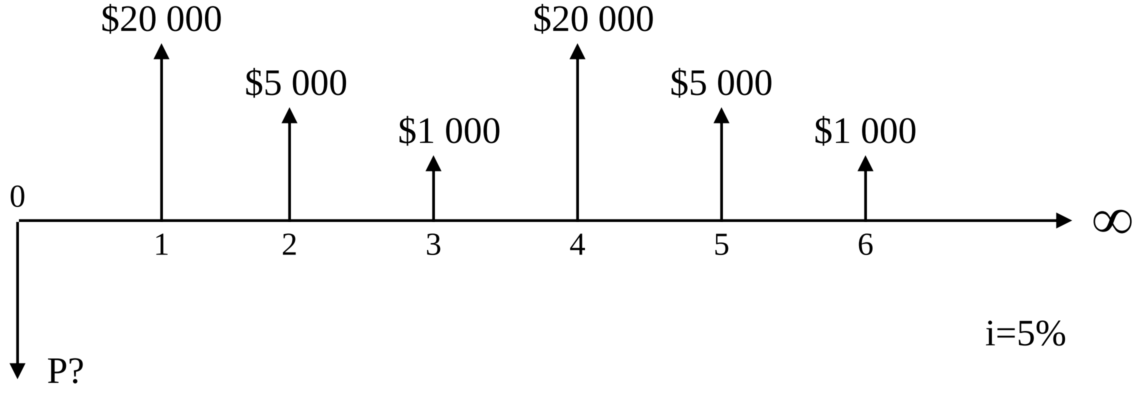
Sports Team Endowment

Year	Beginning of Year	Income (5%)	Cost	End of Year
0	--	--	--	167 174
1	167 174	8 359	1 000	174 533
2	174 533	8 727	5 000	178 260
3	178 260	8 913	20 000	167 174
4	167 174	. . .		

Note that with $167 174 in the endowment at the beginning of Year 4, the three-year cycle can repeat itself. It will again be $167 174 at the beginning of Year 7 and so on continuing indefinitely.

Sports Team Endowment

What happens if the expensive trip occurs in Year 1?



Single Sum:

P = 20 000 (P|F 5,1) + 5 000 (P|F 5,2) + 1 000 (P|F 5,3)

= 20 000 (0.9524) + 5 000 (0.9070) + 1 000 (0.8638)

= $24 447

3 Year Annuity:

A = 24 447 (A|P 5,3)

= 24 447 (0.3672)

= $8 977

Capitalized Value:

$$P = \frac{A}{i} = \frac{8\ 977}{0.05} = \$179\ 537$$

Note that the required donation has increased.

Inflation

Two approaches can be used to determine the present value of a project when the revenues and costs of the project are subject to inflationary effects.

1. Express all cash flows in terms of "constant worth" $ and use a "real" interest rate, i.e., an interest rate without an inflation component.

2. Express all cash flows in terms of actual dollar amounts and use actual interest rates.

Let j = inflation rate

C_k = "constant worth" value (end of period k)

T_k = actual value (end of period k)

i = "real" rate

$$T_k = C_k(1+j)^k$$

$$P = \sum_{k=0}^{n} C_k(1+i)^{-k}$$

$$= \sum_{k=0}^{n} T_k(1+j)^{-k}(1+i)^{-k}$$

Inflation

$$= \sum_{k=0}^{n} T_k (1 + j)^{-k} (1 + i)^{-k}$$

$$= \sum_{k=0}^{n} T_k (1 + j + i + ij)^{-k}$$

$$= \sum_{k=0}^{n} T_k (1 + d)^{-k}$$

where $d = i + j + ij$

= discount rate to account for inflationary effects

and i = “real-return” interest rate

j = anticipated inflation rate

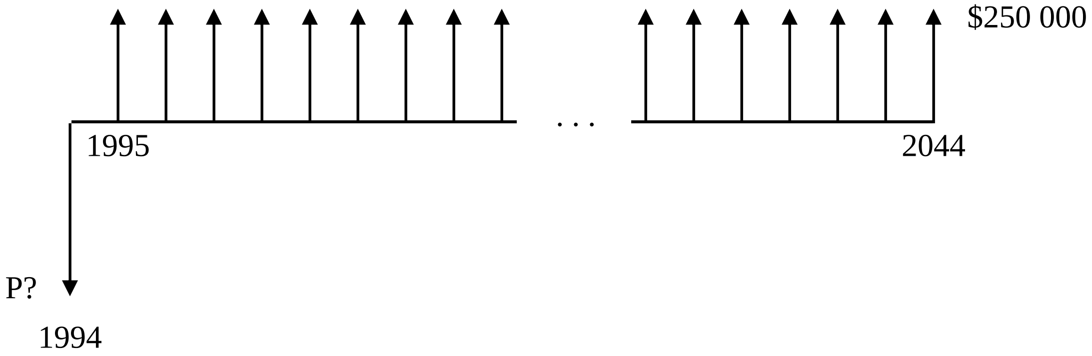
Note that d is referred to as the actual, or prevailing, interest rate. For example, bank accounts pay interest at the rate of d percent (not i).

Football Team Endowment

- Cost of the team in 1994 – $250 000 per year
- University's real return on its investment pool – 4%
- Estimated average inflation rate – 6%

How much is required to fund the team from the 1995 season through 2044?

Constant Dollar Approach



$$\begin{aligned} P &= A\ (P|A\ i,n) \\ &= 250\ 000\ (P|A\ 4\%,50) \\ &= 250\ 000\ (21.4822) \\ &= \$5.37 \text{ million} \end{aligned}$$

Football Team Endowment
Constant $ Approach

$5.37 million funds the team for the next 50 years.

How much is required to fund the team in perpetuity?

$$P = \frac{A}{i}$$

$$= \frac{250\,000}{0.04} = \$6.25 \text{ million}$$

Important Points

The endowment fund is actually earning at a rate of inflation plus real. Costs each year are increasing at the rate of inflation. But the funding requirement can be modeled on the above basis using the real rate of return and constant dollar expenses.

The symbol *i* generally refers to the interest rate in Chapter 3. However, when dealing with inflation questions, *i* represents the real interest rate. The symbol *d* represents the actual, or prevailing, rate of interest.

The textbook uses the term “then current” to refer to “actual” costs.

Football Team Endowment
Actual $ Approach

Discount Rate:

Since we are using actual costs and

$i = 4\%$

$j = 6\%$

then $\boxed{d = i + j + ij}$

$= 0.04 + 0.06 + (0.04)(0.06)$

Therefore $d = 10.24\%$ (Actual Interest Rate)

Present Value:

The cash flows form a geometric series but must be converted to a standard form to use the $(P|A_1 \; d,j,n)$ formula.

$$P_{1994} = \sum_{k=1}^{50} T_k (1+d)^{-k}$$

$$= \sum_{k=1}^{50} 250\,000(1+j)^k (1+d)^{-k}$$

Football Team Endowment
Actual $ Approach

One must state the costs of running the team in actual dollars.

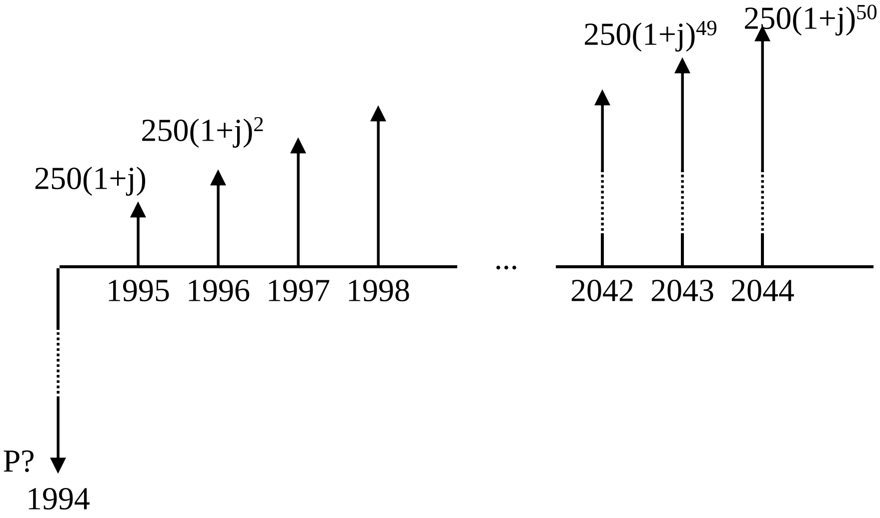
Inflation: j = 6%

T_{1994} = 250 000

T_{1995} = 250 000 (1+j) = \$265 000

T_{1994+k} = 250 000 $(1+j)^k$ k = 1,2,...,50

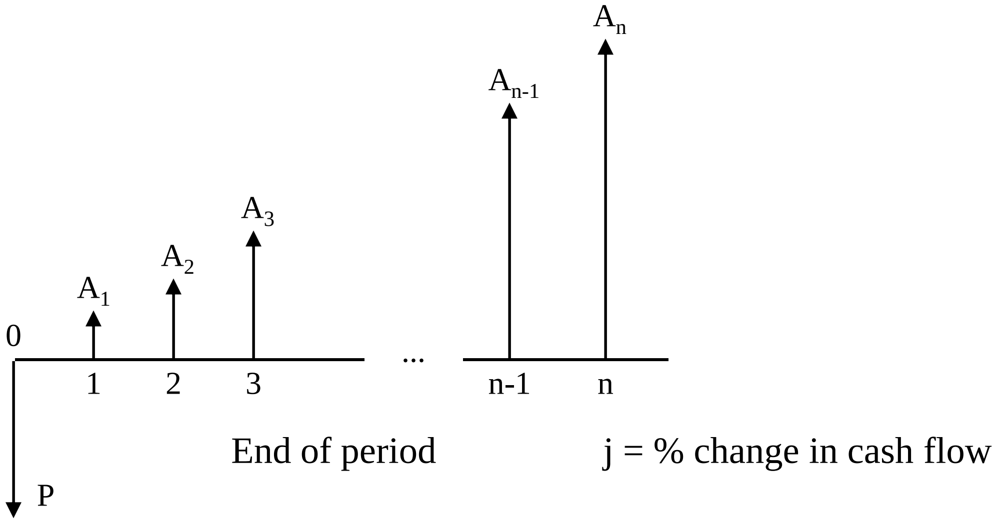
T_{2044} = 250 000 $(1.06)^{50}$ = \$4.61 million



What is the present value of these cash flows?

What is the appropriate discount rate?

Geometric Series of Cash Flows



$$A_k = A_1\,(1+j)^{k-1} \qquad k = 1,\ldots,n$$

k	Geometric Series	Football Cash Flows	Restated Football Cash Flows
1	A_1	250 000(1+j)	265 000
2	$A_1(1+j)$	$250\ 000(1+j)^2$	265 000(1+j)
3	$A_1(1+j)^2$	$250\ 000(1+j)^3$	$265\ 000(1+j)^2$
k	$A_1(1+j)^{k-1}$	$250\ 000(1+j)^k$	$265\ 000(1+j)^{k-1}$

$$\text{where } 250\ 000(1+j)\big|_{j=6\%} = 265\ 000$$

Football Team Endowment
Actual $

$$P_{1994} = \sum_{k=1}^{50} 250\,000\,(1+j)^{k}(1+d)^{-k}$$

$$= \sum_{k=1}^{50} [250\,000\,(1+j)](1+j)^{k-1}(1+d)^{-k}$$

$$= \sum_{k=1}^{50} \overbrace{265\,000}^{A_1}\,(1+j)^{k-1}(1+d)^{-k}$$

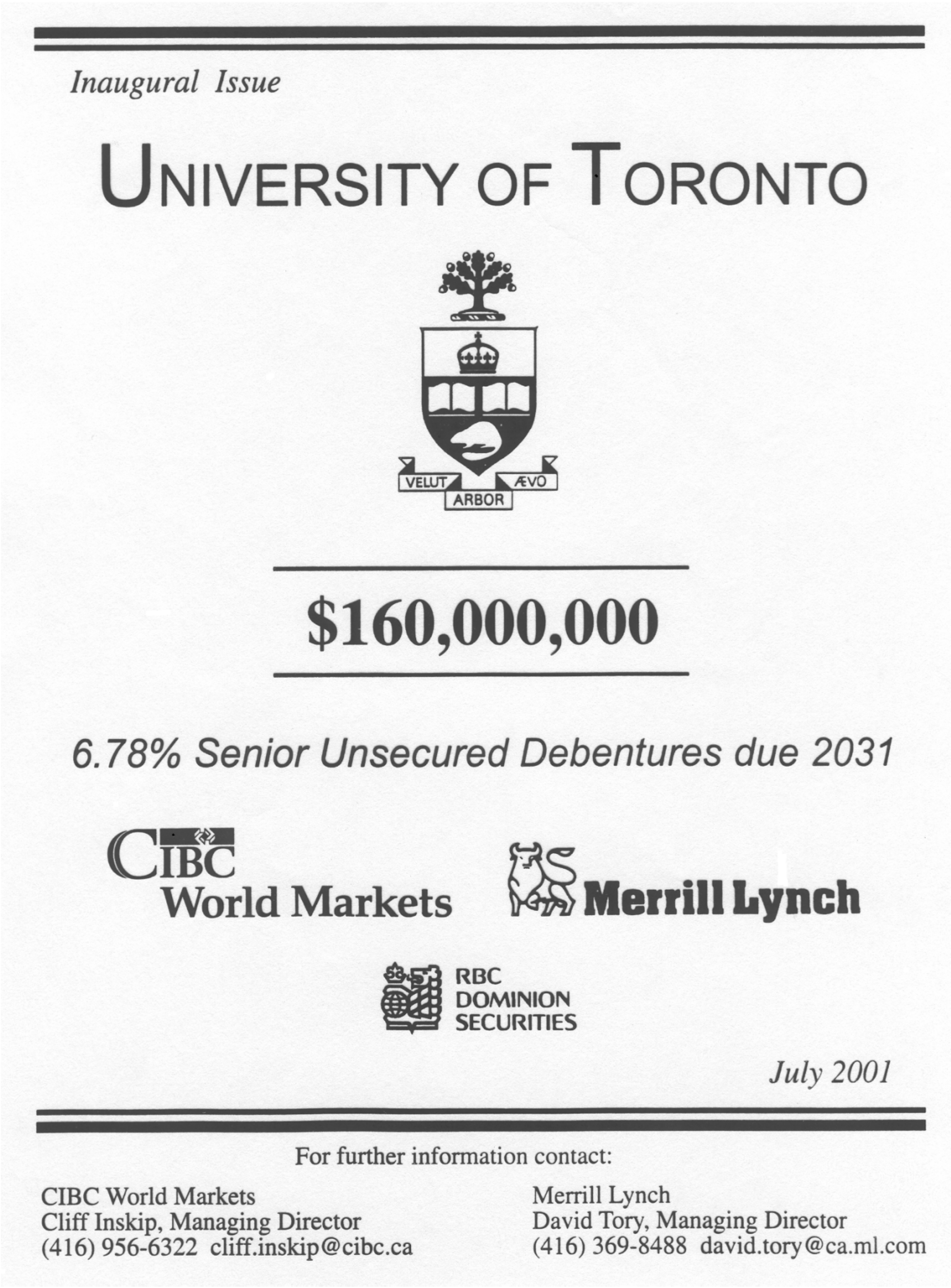
$$= 265\,000\,(P\,|\,A_1\ d,j,n)$$

$$= 265\,000\,(P|A_1\ 10.24\%,\ 6\%,\ 50)$$

$$= 265\,000\,(20.2662)$$

$$= \$5.37\ \text{million}$$

U of T Bond Issue



Inaugural Issue

UNIVERSITY OF TORONTO

VELUT ARBOR ÆVO

$160,000,000

6.78% Senior Unsecured Debentures due 2031

CIBC World Markets

Merrill Lynch

RBC DOMINION SECURITIES

July 2001

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Bond Valuation

- Bonds are one way of raising capital to finance engineering projects
- The purchase price of a bond is equivalent to the returns from the bond at an appropriate interest rate
- The yield of a bond is analogous to the calculations of the internal rate of return

Example 1: $5 000 ten year bond – 8% bond rate

Interest payment of $400 at the end of every year.

What is this payment stream equivalent to (purchase price) at an interest rate of 8% per annum?

P = 400 (P|A 8,10) + 5 000 (P|F 8,10)
= 400 (6.7101) + 5 000 (0.4632)
= $2 684 + $2 316
= $5 000 face value

Bond Valuation

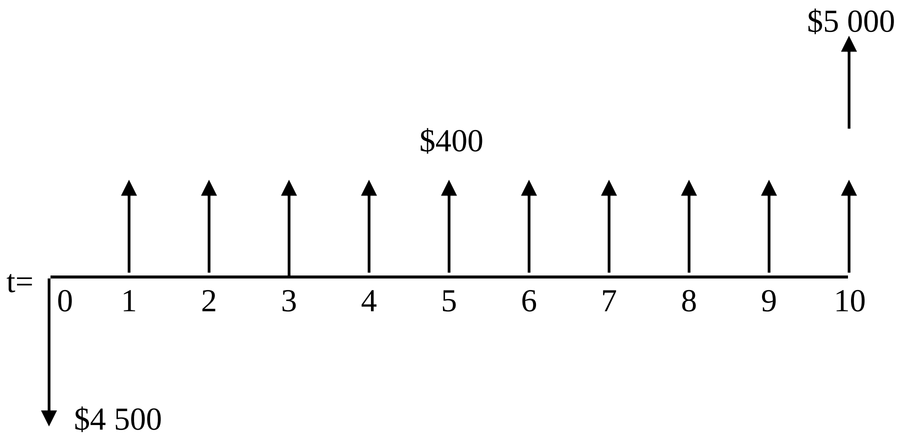
Example 2:

In trying to sell the bond, the company has discovered that no one is willing to pay $5 000 for the bond. Investors are only willing to pay $ 4 500.

Prevailing interest rates have changed.

Have they gone up or down?

What is the yield on the bond assuming a price of $4 500?



Let i equal the bond yield.

$$4\,500 \quad = \quad 400\ (P|A\ i,10) + 5\,000\ (P|F\ i,10)$$

$$4\,500 \quad = 400\left[\frac{(1+i)^{10}-1}{i(1+i)^{10}}\right]+\frac{5\,000}{(1+i)^{10}}$$

Bond Valuation

Bond Yield = Internal Rate of Return

What is the yield when the price of the bond is $4 500?

Annual Rate	PV
8	$5 000
9	$4 679
10	$4 385
11	$4 117

→ yield = 9.6%

Prevailing interest rates have gone up to 9.6%.
Note that 9.6% is k_d in the cost of capital calculation.

Also note that the bond yield of 9.6% is achievable only if the interest payments are reinvested at 9.6%.

Bond Yield Estimation

Trial & Error First Pass Calculation

	Rate (%)	PV	
	8	$5 000	
	9	4 679	
i=? →			← $4 500
	10	4 385	
	11	4 117	

Therefore $9\% < i < 10\%$

Linear Interpolation Second Pass Calculation

$$\frac{4\,679 - 4\,385}{9 - 10} = \frac{4\,500 - 4\,385}{i - 10}$$

$$i - 10 = -\frac{115}{294}$$

$$\Rightarrow i = 10 - \frac{115}{294} = \boxed{9.6\%}$$

“Open Market” Bond Sale

Example 3:

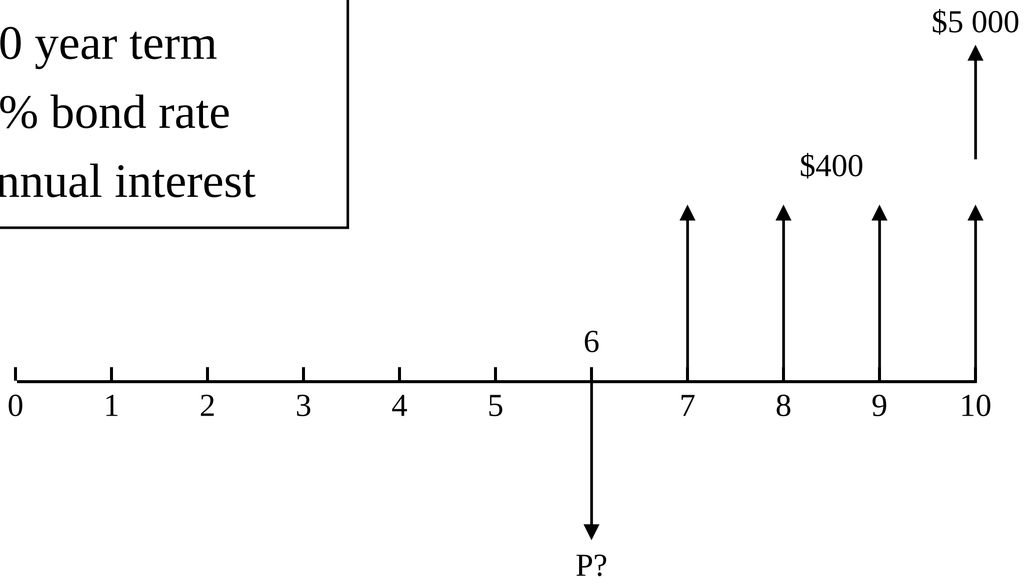
Assume interest rates have gone up to 12% and a bond holder wishes to sell the bond at the beginning of Year 7. What would the market price of the bond be at that time?

P = 400 (P|A 12,4) + 5 000 (P|F 12,4)
= $1 214.94 + $3 177.59
= $4 392.53

How does the investor achieve a 12% rate of return on this investment?

“Bearer Bond”

- $5 000 face value
- 10 year term
- 8% bond rate
- annual interest



Capital Recovery Cost

- CRC allows for the recovery of the loss of value of an asset plus the interest the funds invested would have earned if invested elsewhere.
- Annualized equivalent cost.

$$CR = P\,(A|P\ i,n) - F\,(A|F\ i,n)$$

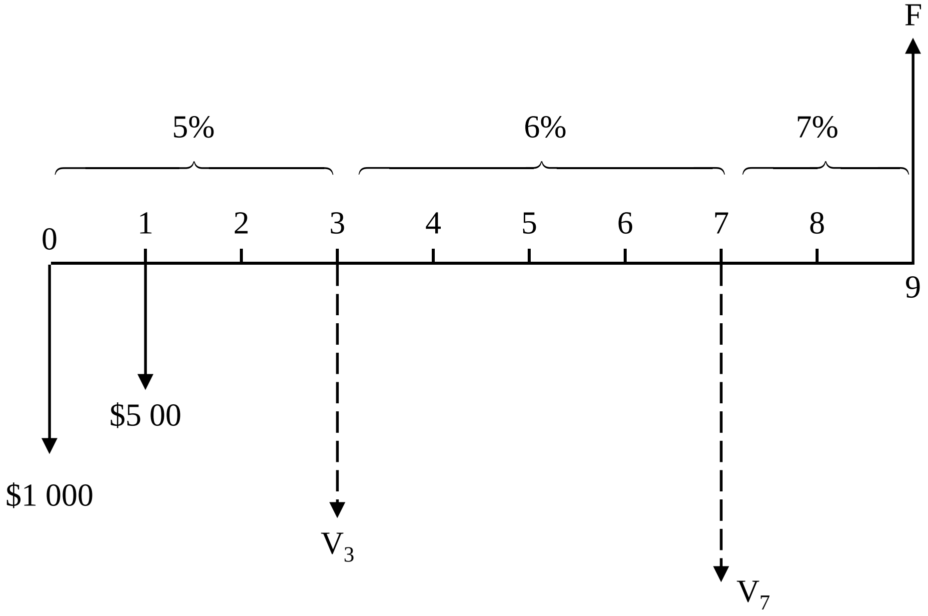
e.g. $60 000 machine with salvage value of $20 000.
Four-year project life. Interest rate of 10%

$$\begin{aligned} CR &= 60\,000\,(A|P\ 10,4) - 20\,000\,(A|F\ 10,4) \\ &= 60\,000\,(0.3155) - 20\,000\,(0.2155) \\ &= \$14\,620 \end{aligned}$$

EOY	Capital Not Recovered by EOY	Interest Due on Unrecovered Capital (10%)	Amount of Capital Recovered	Annual CR Cost
0	$60 000			
1	$51 380	6 000	8 620	$14 620
2	$41 898	5 138	9 482	$14 620
3	$31 468	4 190	10 430	$14 620
4	$19 995	3 147	11 473	$14 620
			40 005	

Changing Interest Rates

$$F = P\,(1+i_1)\,(1+i_2)\ \ldots\ (1+i_{n-1})\,(1+i_n)$$



Establish intermediate values at points where the interest rates change.

$$\begin{aligned} V_3 &= 1\ 000\ (F|P\ 5,3) + 500\ (F|P\ 5,2) \\ &= 1\ 000\ (1.1576) + 500\ (1.1025) \\ &= \$1\ 708.85 \\ V_7 &= V_3\ (F|P\ 6,4) \\ &= 1\ 709\ (1.2625) \\ &= \$2\ 157.42 \\ F &= V_7\ (F|P\ 7,2) \\ &= 2\ 157\ (1.1449) \\ &= \$2\ 470.03 \end{aligned}$$