A Formula for the Bacher Numbers

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Our goal is to compute sequence A368207 of the OEIS, dubbed 'Bacher numbers' by Donald Knuth, who is the author of this sequence.

Bacher numbers are the number of Bacher representations of n. We call a quadruple (w, x, y, z) of nonnegative integers a *Bacher representation* of n if and only if $n = wx + yz$ and $\max(w, x) < \min(y, z)$. We call a Bacher representation *monotone* if additionally $w \le x \le y \le z$.

We further define a weight for each Bacher representation,

$$\sharp(w, x, y, z) = \max(1, 2([w < x] + [y < z])),$$

where the square brackets denote Iverson brackets.

We differentiate between two cases:

1. $w = 0$: So there is only one summand, yz. If $y = z$ then $\sharp(w, x, y, z) = 1$ and otherwise 2.

2. $w > 0$: In this case $\sharp(w, x, y, z)$ only take the three values 1, 2, and 4, depending on how many of the two pairs have the same components.

For example, the weights of the Bacher representations of $n = 17$ are:

$$\begin{aligned}
\sharp(0,0,1,17) &= 2([0<0]+[1<17]) = 2;\\
\sharp(1,1,4,\ 4) &= \max(1, 2([1<1]+[4<4])) = 1;\\
\sharp(1,1,2,\ 8) &= 2([1<1]+[2<8]) = 2;\\
\sharp(1,2,3,\ 5) &= 2([1<2]+[3<5]) = 4.
\end{aligned}$$

The number of all Bacher representations of n is the sum of the weights of the monotone representations, in this case 9. And this is no coincidence: whenever n is a prime number, then the number of Bacher representations is $(n+1)/2$. The special case $\sharp(\cdots) = 1$ signals that n can be written as the sum of two squares.

We implement the sequence in the programming language 'Julia' as a function that only requires five lines. Note that we replace x by wx and yz by $n - wx$ in the weight formula, which becomes $\max(1, 2([w^2 < wx] + [y^2 < n - wx]))$, and wx becomes the summation variable running from 1 to $n/2$.

```
function bacher_number(n)  # A368207
    t(n) = (d for d in divisors(n) if d * d <= n)
    s(n) = sum(d * d == n ? 2 * d - 1 : 4 * d - 2 for d in t(n))
    c(y, w, wx) = max(1, 2 * (Int(w * w < wx) + Int(y * y < n - wx)))

    sum(sum(sum(c(y, w, wx) for y in t(n - wx) if wx < y * w; init=0)
    for w in t(wx)) for wx in 1:div(n, 2); init=s(n))
end
```

Translated into the notation of a formula, this is written

$$b(n) = s(n) + \sum_{k\in K} \sum_{w\in W} \sum_{y\in Y} \max(1, 2([w^2 < k] + [y^2 < n - k]))$$

where the square brackets denote Iverson brackets and

$$\begin{aligned} k \in K &\equiv 1 \le k \le \lfloor n/2 \rfloor; \\ w \in W &\equiv w | k,\ w^2 \le k; \\ y \in Y &\equiv y | n - k,\ y^2 \le n - k,\ k < yw. \end{aligned}$$

The first term of the sum,

$$s(n) = \sum_{\substack{d|n \\ d^2 \le n}} (1 + [d^2 < n])(2d - 1)$$

corresponds to calculating the number of representations with $w = 0$. In the Julia function, this term is integrated into the main sum as an initial value.

Roland Bacher (2023) *A quixotic proof of Fermat's two squares theorem for prime numbers*, arXiv:2210.07657.

Donald E. Knuth (2023) *A CWEB program for the number of solutions to* $n = ab + cd$ *where* $\max(a, b) < \min(c, d)$, *OEIS* A368207.