# Second definition of the finite and infinite

Richard Dedekind

9th March 1889

*What are numbers, and what is their purpose?*

First published in the second edition (1893) of the text *"Was sind und was sollen die Zahlen?"*, page XVII, in the form:

> A system $S$ is called finite if it can be mapped into itself in such a way that no proper part of $S$ is mapped into itself; in the opposite case, $S$ is called an infinite system.

Pursuing this definition of a finite system $S$ *without using the natural numbers*.

Let $\varphi$ be a mapping of $S$ into itself, which maps no proper part of $S$ into itself. Small Latin letters $a, b \ldots z$ always mean *elements* of $S$, capital Latin letters $A, B \ldots Z$ mean *parts* of $S$. The images of $a, A$ generated by $\varphi$ are respectively denoted by $a', A'$.

*A 'part' may not be empty.*

That $A$ is part of $B$ is expressed by $A \subseteq B$. The system consisting of the elements $a, b, c$ is denoted by $[a, b, c \ldots]$.

This gives

(1) $$S' \subseteq S$$

and

*The definition of finiteness.*

(2) from $A' \subseteq A$ it follows that $A = S$.

**1 Theorem.** $S' = S$.

▷ Every element of $S$ is an image of (at least) one element $r$ of $S$. Because from (1) it follows $(S')' \subseteq S'$, hence by (2), our proposition.

Every system $[s]$ consisting of a single element is finite because it has no proper part and is mapped into itself by the identity function. This case is *excluded* in the following; $S$ means a finite system that does *not* consist of a single element.

**2 Theorem.** Every element $s$ is different from its image $s'$, in symbols: $s \neq s'$.

▷ Because if $s = s'$, then $[s]' = [s'] = [s] \subseteq [s]$, so according to ((2)), also $[s] = S$ in contradiction to our assumption about $S$.

**3 Definition.** If $s$ is a certain element of $S$, then $H_s$ shall denote any part of $S$ that satisfies the following two conditions:

I. $s$ is element of $H_s$, so $[s] \subseteq H_s$, also

$$[s] + H_s = H_s.$$

II. If $h$ is an element of $H_s$ different from $s$, then $h'$ is also an element of $H_s$. So if $H \subseteq H_s$, but $s$ is *not contained* in H, then $H' \subseteq H_s$.

**4 Theorem.** $S$ and $[s]$ are special systems $H_s$, and $[s]$ is the *common* of all systems $H_s$ corresponding to the element s. *'Gemeinheit'*

▷ Obvious.

**5 Theorem.** $H_s = S$ or $H_s$ is a *proper* part of $S$, depending on whether $s'$ lies in $H_s$ or not.

▷ For if $s'$ lies in $H_s$, then it follows from II in 3. that $H'_s \subseteq H_s$, therefore (by 2) that $H_s = S$. Conversely, if $H = S$, then $s'$ also lies in $H_s$.

**6 Theorem.** If $H_s$ is a *proper* part of $S$, then $s'$ is the only element of $H'_s$ that lies outside $H_s$.

▷ This is because every element $k$ of $H'_s$ is the image $h'$ of at least one element $h$ in $H$. If $k = h'$ is different from $s'$, then $h$ is also different from $s$, and consequently (by II in 3.) $k = h'$ lies in $H_s$, while the element $s'$ of $H'_s$ (by 5) lies outside $H_s$.

**7 Theorem.** Every system $H'_s$ is a system $H_{s'}$, that is (by definition 3.):

I'. $s'$ is element of $H'_s$

II'. If $k$ is an element of $H'_s$ that is different from $s'$, then $k'$ also lies in $H'_s$.

▷ The first follows from the fact that $s$ lies in $H_s$, the second from the fact that $k$ lies in $H_s$ (by 6).

**8 Theorem.** If $A$, $B$, $C$ ... are special systems $H_s$ corresponding to the same $s$, then their intersection $H$ is also a system $H_s$.

▷ Because according to 3.I. $s$ is a common element of $A, B, C, \ldots$, and therefore also an element of $H$.

Furthermore, if $h$ is an element of $H$ that is different from $s$, then, by II of (3), the image $h'$ is an element of $A$, of $B$, of $C$, ..., and therefore also of $H$. $H$ therefore fulfills the two conditions I and II in definition (3) that are characteristic of every $H_s$.

**9 Definition.** If $a$, $b$ are certain elements of $S$, then the symbol $ab$ should mean the intersection of all those systems $H_b$ which (such as $S$) contain the element $a$ (*section ab*). *'Strecke ab'*

**10 Theorem.** $a$ is an element of ab, i.e. $[a] \subseteq ab$.

▷ This is because $ab$ is the intersection of all systems $H_b$ in which $a$ lies. (So a is the *start* of ab.)

**11 Theorem.** $ab$ is a system $H_b$, i.e. $[b] \subseteq ab$, and if $s$ is an element of $ab$ different from $b$, then $[s'] \subseteq ab$.

▷ This follows from (8).

So $b$ is an element (the *end*) of $ab$. If $H \subseteq ab$ but $b$ is not contained in $H$, then $H' \subseteq ab$.

**12 Theorem.** From $[a] \subseteq H_b$, follows from $ab \subseteq H_b$.

▷ Immediate consequence of definition (9).

**13 Theorem.** $aa = [a]$.

▷ This follows from (4), because $aa$ is the *intersection* of all $H_a$ that contain the element $a$ (according to 3.I). *'Durchschnitt'*

**14 Theorem.** If $b'$ is an element of $ab$, then $ab = S$.

▷ This follows from (11) and (5).

**15 Theorem.** $b'b = S$.

▷ This follows from (14) and (10).

**16 Theorem.** If $c$ is an element of $ab$, then $cb \subseteq ab$.

▷ This follows from (12), since $ab$ is an $H$ (by 11) which contains the element $c$.

**17 Theorem.** If $A+B$ means the system composed of $A, B$, then

$$a'b + b'aS.$$

▷ Because if $s$ is an element of $ab$, then $s'$ is contained in $b'a$ or $a'b$, depending on $s = b$ or different from $b$ (according to (10) or (11) and 3. II), and likewise if $s$ is an element of $b'a$, then $s'$ is contained in $a'b$ or $b'a$; therefore $(ab + b'a)' \subseteq a'b + b'a$. This leads to the theorem according to (2).

**18 Theorem.** If $a$ is different from $b$, then $ab = [a] + a'b$.

▷ For since $a$ is an element of $ab$ different from $b$, then $a'$ is an element of $ab$ (by 10, 11), and consequently (by 16) $ab \subseteq ab$; since furthermore (by 10) we also have $[a] \subseteq ab$, therefore

$$[a] + a'b \subseteq ab.$$

Furthermore: every element $s$ of $[a] + a'b$ that is different from $b$ is either $= a$ or an element of $a'b$ that is different from $b$, in both cases $s'$ is (by 10, 11) an element of $a'b$, therefore also from $[a] + a'b$, and since (by 11) also $[b] \subseteq [a] + a'b$, then $[a] + a'b$ is a system $H_b$.

Finally, since $[a] \subseteq [a] + a'b$, so (by 12)

$$ab \subseteq [a] + a'b.$$

The theorem follows from the comparison of both results.