# Lilongwe University of Agriculture and Natural Resources

Bunda Campus

## Basic Sciences Department


MAT 32102: Calculus I

# Differentiation

Lecture Notes


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# Contents

# 1 Limits and Continuity of Functions

## 1.1 Definition and Notation

**Definition 1.1.** *If $f(x)$ becomes arbitrarily close to a unique number $L$ as $x$ approaches $a$ from either side, then the limit of $f(x)$ as $x$ approaches $a$ is $L$. This is written*

$$\lim_{x\to a} f(x) = L.$$

In essence what this means is that the values of $f(x)$ tend to get closer and closer to the number $L$ as $x$ gets closer and closer to the number $a$ (from either side of $a$) but $a \neq a$.

Other notation for

$$\lim_{x\to a} f(x) = L$$

is

$$f(x) \to L \text{ as } x \to a.$$

In the example below, we use tables to estimate limits numerically.

**Example 1.2.**

Guess the value of

1. $\displaystyle\lim_{x\to -4} \frac{16 - x^2}{x - 1}$
2. $\displaystyle\lim_{x\to 0} \frac{1 - \cos x}{x}$
3. $\displaystyle\lim_{x\to 0} \frac{\sin x}{x}$

**Solution**

1. From Table 1, we guess that $\displaystyle\lim_{x\to -4} \frac{16 - x^2}{x - 1} = 8$.
2. From Table 2, $\displaystyle\lim_{x\to 0} \frac{1 - \cos x}{x} = 0$
3. From Table 3, $\displaystyle\lim_{x\to 0} \frac{\sin x}{x} = 1$

We can use limit laws to find the limit of a function.

| $x<-4$ | $f(x)$ | $x>-4$ | $f(x)$ |
|---|---|---|---|
| $-4.1$ | 8.1 | $-3.9$ | 7.9 |
| $-4.01$ | 8.01 | $-3.99$ | 7.99 |
| $-4.001$ | 8.001 | $-3.999$ | 7.999 |
| $-4.0001$ | 8.0001 | $-3.9999$ | 7.9999 |

Table 1: Values for $\lim_{x\to-4}\dfrac{16-x^2}{x-1}$

| $x<0$ | $f(x)$ | $x>0$ | $f(x)$ |
|---|---|---|---|
| $-1$ | $-0.4597$ | 1 | 0.4597 |
| $-0.5$ | $-0.2448$ | 0.5 | 0.2448 |
| $-0.1$ | $-0.0499$ | 0.1 | 0.0499 |
| $-0.01$ | $-0.0049$ | 0.01 | 0.0049 |
| $-0.001$ | $-0.00049$ | 0.0001 | 0.00049 |
| $-0.0001$ | $-0.000049$ | 0.0001 | 0.000049 |

Table 2: Values for $\lim_{x\to0}\dfrac{1-\cos x}{x}$

| $x<0$ | $f(x)$ | $x>0$ | $f(x)$ |
|---|---|---|---|
| $-1$ | 0.8415 | 1 | 0.8415 |
| $-0.5$ | 0.9589 | 0.5 | 0.9589 |
| $-0.1$ | 0.9983 | 0.1 | 0.9983 |
| $-0.01$ | 0.99998 | 0.01 | 0.99998 |
| $-0.001$ | 0.999999 | 0.0001 | 0.999999 |

Table 3: Values for $\lim_{x\to0}\dfrac{\sin x}{x}$

### 1.1.1 Limit Laws

Suppose that $c, L$ and $M$ are constants and that $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$. Then

1. $\lim_{x\to a} c = c$
2. $\lim_{x\to a}[cf(x)] = c\lim_{x\to a} f(x) = cL$.
   *The limit of a constant times a function is the constant times the limit of the function.*
3. $\lim_{x\to a}[f(x) + g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x) = L + M$.
   *The limit of the sum of two functions is the sum of their limits.*
4. $\lim_{x\to a}[f(x) - g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x) = L - M$.
   *The limit of the difference of two functions is the difference of their limits.*
5. $\lim_{x\to a}[f(x)g(x)] = \left[\lim_{x\to a} f(x)\right]\left[\lim_{x\to a} g(x)\right] = LM$.
   *The limit of a product of two functions is the product of their limits.*
6. $\lim_{x\to a}\dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \dfrac{L}{M}$, provided $\lim_{x\to a} g(x) = M \neq 0$.
   *The limit of a quotient of two functions is the quotient of their limits, provided the limit of the denominator is not zero.*
7. $\lim_{x\to a}[f(x)]^n = [\lim_{x\to a} f(x)]^n = L^n$
8. $\lim_{x\to a}\sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)} = \sqrt[n]{L}$ where $n$ is a positive integer.

**Example 1.3.**

Given that $\lim_{x\to 2} f(x) = 15$ and $\lim_{x\to 2} g(x) = -5$, find

1. $\lim_{x\to 2}\dfrac{f(x)}{g(x)}$
2. $\lim_{x\to 2}(f(x) - 3g(x))$
3. $\lim_{x\to 2}(2g(x) \times (f(x) + 10))$

**Solution**

We use limit laws.

1. $\lim\limits_{x \to 2} \dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x \to 2} f(x)}{\lim\limits_{x \to 2} g(x)} = \dfrac{15}{-5} = -3.$

2. $\lim\limits_{x \to 2}(f(x) - 3g(x)) = \lim\limits_{x \to 2} f(x) - 3 \lim\limits_{x \to 2} g(x) = 15 - 3(-5) = 15 + 15 = 30.$

3. $$\begin{aligned} \lim_{x \to 2}(2g(x) \times (f(x) + 10)) &= 2 \lim_{x \to 2} g(x) \times (\lim_{x \to 2} f(x) + 10) \\ &= 2(-5) \times (15 + 10) \\ &= -250 \end{aligned}$$

Some limits can be evaluated by direct substitution as in the following theorem.

**Theorem 1.4** (Direct substitution property). *If $f$ is a polynomial or a rational function and $a$ is a real number in the domain of $f$, then*

$$\lim_{x \to a} f(x) = f(a).$$

*Proof.* We prove the case when $f$ is a polynomial and leave the other part as an exercise.

Let $f(x)$ be polynomial function. Then

$$f(x) = b_n x^n + b_{n-1} x^{n-1} + ... + b_2 x^2 + b_1 x + b_0$$

where $b_0, b_1, b_2, ..., b_{n-1}, b_n \in \mathbb{R}$.

Now

$$\begin{aligned} \lim_{x \to a} f(x) &= \lim_{x \to a}(b_n x^n + b_{n-1} x^{n-1} + ... + b_2 x^2 + b_1 x + b_0) \\ &= \lim_{x \to a} b_n x^n + \lim_{x \to a} b_{n-1} x^{n-1} + ... + \lim_{x \to a} b_2 x^2 + \lim_{x \to a} b_1 x + \lim_{x \to a} b_0 \\ &= b_n a^n + b_{n-1} a^{n-1} + ... + b_2 a^2 + b_1 a + b_0 \\ &= f(a). \end{aligned}$$

□

The following results follow immediately from Theorem 1.4 ($a$ and $n$ are real numbers).

1. $\lim\limits_{x \to a} x = a$

2. $\lim\limits_{x \to a} x^n = a^n$

3. $\lim_{x \to a} \sqrt[n]{x} = \sqrt[n]{a}$

**Example 1.5.**

Evaluate the following limits:

1. $\lim_{x \to 1}(2x^4 + x^3 - 3x^2 + 4x - 20)$
2. $\lim_{x \to 2} \frac{3x-1}{3x+3}$

**Solution**

We substitute directly.

1.
$$\begin{aligned}\lim_{x \to 1}(2x^4 + x^3 - 3x^2 + 4x - 20) &= 2(1)^4 + 1^3 - 3(1)^2 + 4(1) - 20 \\ &= 2 + 1 - 3 + 4 - 20 \\ &= -16.\end{aligned}$$

2. $\lim_{x \to 2} \frac{3x-1}{3x+3} = \frac{3(2)-1}{3(2)+3} = \frac{6-1}{6+3} = \frac{5}{9}.$

Direct substitution does not work for some functions. In most cases we need to rewrite the function before making substitution. We can factorise or expand and divide out some expressions. We can also ratioalise the numerator or the denominator. These operations will give the same function but written differently.

The validity of these techniques stem from the fact that when two functions agree at all but a single number $a$ they must have identical limit behavior at $a$. So before concluding that the limit does not exist after direct substitution, it is a good idea to try and see if there is something that can be done to transform the function.

**Proposition 1.6.** *If $f(x) = g(x)$ when $x \neq a$, then*

$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x),$$

*provided the limits exist.*

**Example 1.7.**

Evaluate each limit.

1. $\lim_{x \to -3} \frac{x^2 + x - 6}{x + 3}$

2. $\lim_{x \to 1} \frac{x - 1}{x^3 - x^2 + x - 1}$

3. $\lim_{x \to \pi/4} \frac{8x \tan x - 2\pi \tan x}{4x - \pi}$

4. $\lim_{x \to 0} \frac{\sqrt{x + 1} - 1}{x}$

**Solution**

1. $\lim_{x \to -3} \frac{x^2 + x - 6}{x + 3} = \lim_{x \to -3} \frac{(x + 3)(x - 2)}{x + 3} = \lim_{x \to -3} (x - 2) = -3 - 2 = -5.$

2. $\lim_{x \to 1} \frac{x - 1}{x^3 - x^2 + x - 1} = \lim_{x \to 1} \frac{x - 1}{(x - 1)(x^2 + 1)} = \lim_{x \to 1} \frac{1}{x^2 + 1} = \frac{1}{1 + 1} = \frac{1}{2}.$

3. $$\begin{aligned} \lim_{x \to \pi/4} \frac{8x \tan x - 2\pi \tan x}{4x - \pi} &= \lim_{x \to \pi/4} \frac{(4x - \pi)2 \tan x}{4x - \pi} \\ &= \lim_{x \to \pi/4} (2 \tan x) \\ &= 2 \tan(\pi/4) = 2. \end{aligned}$$

4. $$\begin{aligned} \lim_{x \to 0} \frac{\sqrt{x + 1} - 1}{x} &= \lim_{x \to 0} \frac{\sqrt{x + 1} - 1}{x} \times \frac{\sqrt{x + 1} + 1}{\sqrt{x + 1} + 1} \\ &= \lim_{x \to 0} \frac{x + 1 - 1}{x(\sqrt{x + 1} + 1)} \\ &= \lim_{x \to 0} \frac{x}{x(\sqrt{x + 1} + 1)} \\ &= \lim_{x \to 0} \frac{1}{\sqrt{x + 1} + 1} \\ &= \frac{1}{\sqrt{0 + 1} + 1} = \frac{1}{2}. \end{aligned}$$

## 1.2 One Sided Limits

Sometimes it is necessary to consider what happens to the function $f(x)$ as $x$ approaches $a$ from one side, i.e., when $x$ is near $a$ but less than $a$ (or greater than $a$). We say that we have one sided limit.

**Definition 1.8.** *The symbol*

$$\lim_{x \to a^+} f(x)$$

*is called a* **right-hand limit.** *It refers to the limit of $f(x)$ for $x$ near to $a$ and greater than $a$. The symbol*

$$\lim_{x \to a^-} f(x)$$

*is called a* **left-hand limit.** *It refers to the limit of $f(x)$ for $x$ near to $a$ and less than $a$.*

In general we have the following theorem on the existence of the limit of a function.

**Theorem 1.9.** $\lim_{x \to a} f(x) = L$ *if and only if* $\lim_{x \to a^+} f(x) = L$ *and* $\lim_{x \to a^-} f(x) = L$.

**Example 1.10.**

1. Show that $\lim_{x \to 0} \frac{|2x|}{x}$ does not exist.

2. If $f(x) = \begin{cases} 4 - x & \text{if } x < 1 \\ 4x - x^2 & \text{if } x > 1 \end{cases}$ determine whether $\lim_{x \to 2} f(x)$ exists or not.

3. Find the value of $c$ such that $\lim_{x \to 2} f(x)$ exists if $f(x) = \begin{cases} 2x - x^2 & \text{if } x < 2 \\ x + c & \text{if } x \geq 2. \end{cases}$

**Solution**

1.
$$\lim_{x \to 0^+} \frac{|2x|}{x} = \lim_{x \to 0^+} \frac{2x}{x} = \lim_{x \to 0^+} \frac{2}{1} = 2$$

and
$$\lim_{x \to 0^-} \frac{|2x|}{x} = \lim_{x \to 0^-} \frac{-2x}{x} = \lim_{x \to 0^-} \left(-\frac{2}{1}\right) = -2.$$

Since $\lim_{x \to 0^+} \frac{|2x|}{x} \neq \lim_{x \to 0^-} \frac{|2x|}{x}$, $\lim_{x \to 0} \frac{|2x|}{x}$ does not exist.

2. $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (4x - x^2) = 4(1) - 1^2 = 4 - 1 = 3.$

   $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (4 - x) = 4 - 1 = 3.$

   Since $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^-} f(x) = 3$, $\lim_{x \to 1} f(x)$ exists and actually we have $\lim_{x \to 1} f(x) = 3$.

3. By Theorem 1.9, $\lim_{x\to 2} f(x)$ exists if and only if $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^-} f(x)$.

   Now
   $$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (x + c) = 2 + c$$
   and
   $$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} (2x - x^2) = 2(2) - 2^2 = 4 - 4 = 0.$$
   $$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^-} f(x) \Rightarrow 2 + c = 0$$
   giving $c = -2$.

Two theorems below give further properties of the limit of a function

**Theorem 1.11.** *If $f(x) \leq g(x)$ when $x$ is near $a$ (except possibly at $a$) and the limits of $f$ and $g$ both exist as $x$ approaches $a$ , then*

$$\lim_{x\to a} f(x) \leq \lim_{x\to a} g(x).$$

**Theorem 1.12** (Squeeze Theorem)**.** *If $f(x) \leq g(x) \leq h(x)$ when $x$ is near $a$ (except possibly at $a$) and*

$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$$

*then*

$$\lim_{x\to a} g(x) = L.$$

The Squeeze Theorem is also called the **Sandwich Theorem** or the **Pinching Theorem** where the function $g$ is squeezed or sandwiched between $f$ and $h$. It states that if $f$ and $h$ have a common limit as $x$ approaches $a$, then $g$ must have the same limit.

**Example 1.13.**

Prove that $\lim_{x\to 0} x \sin \dfrac{1}{x} = 0$.

**Solution**

Here we can not substitute 0 for $x$ since $\lim_{x\to 0} \sin \dfrac{1}{x}$ does not exist. No algebraic manipulation also works for this function. However since

$$-1 \leq \sin \frac{1}{x} \leq 1$$

we have

$$-x \leq x \sin \frac{1}{x} \leq x.$$

Clearly

$$\lim_{x\to 0} x = \lim_{x\to 0}(-x) = 0.$$

Hence by the Squeeze Theorem with $f(x) = -x$, $g(x) = x\sin\frac{1}{x}$ and $h(x) = x$ we have

$$\lim_{x\to 0} x\sin\frac{1}{x} = 0.$$

**Exercise 1.14.**

1. Evaluate each limit if it exists.

   (a) $\lim_{x\to 1} -450$

   (b) $\lim_{x\to 0}(x-5)\cos x$

   (c) $\lim_{x\to b}\dfrac{x^3-b^3}{x^6-b^6}$

   (d) $\lim_{x\to 0}\dfrac{3x}{\cos x}$

   (e) $\lim_{y\to 0}\dfrac{y^2}{1-\cos y}$

   (f) $\lim_{h\to 0}\dfrac{\sin 3h}{\sin 2h}$

   (g) $\lim_{x\to 0}\dfrac{3\sin 2x}{2x}$

   (h) $\lim_{x\to 3}\dfrac{x^2-2x-3}{x-3}$

   (i) $\lim_{t\to 0}\dfrac{\sqrt{t+2}-\sqrt{2}}{t}$

   (j) $\lim_{t\to 2}\dfrac{t^2-3t+2}{t-2}$

2. Find $\lim_{x\to 2^+} f(x)$ where $f(x) = \begin{cases} -x+10 & \text{if } x \le 2 \\ 2x^2 & \text{if } x > 2. \end{cases}$

3. Use the Squeeze theorem to show that

   (a) $\lim_{x\to 0} x^4\cos\left(\frac{2}{x}\right) = 0$

   (b) $\lim_{x\to 0} x^2 e^{\sin\left(\frac{1}{x}\right)} = 0$

## 1.3 Limits at Infinity and Infinite Limits

The expression

$$\lim_{x \to \infty} f(x) = L_1$$

means that $f(x)$ approaches $L_1$ as $x$ increases without bound. Similarly, the expression

$$\lim_{x \to -\infty} f(x) = L_2$$

means that $f(x)$ approaches $L_2$ as $x$ decreases without bound.

**Definition 1.15.** *If $f$ is a function and $L_1$ and $L_2$ are real numbers, then the statements*

$$\lim_{x \to \infty} f(x) = L_1$$

*and*

$$\lim_{x \to -\infty} f(x) = L_2$$

*denote the* **limits at infinity.** *The first statement is read 'the limit of $f(x)$ as $x$ approaches $\infty$ is $L_1$' and the second is read 'the limit of $f(x)$ as $x$ approaches $-\infty$ is $L_2$'.*

Below we define horizontal asymptotes.

**Definition 1.16.** *Let $k \in \mathbb{R}$. Then the line $y = k$ is called a* **horizontal asymptote** *of the curve $y = f(x)$ if either $\lim_{x \to \infty} f(x) = k$ or $\lim_{x \to -\infty} f(x) = k$.*

Most of the limit laws discussed above also hold for limits at infinity. The theorem below is an important rule when calculating limits at infinity.

**Theorem 1.17.** *Let $r$ be a rational number. If $r > 0$, then*

$$\lim_{x \to \infty} \frac{1}{x^r} = \lim_{x \to -\infty} \frac{1}{x^r} = 0.$$

The following results follow immediately from Theorem 1.17.

1. $\displaystyle\lim_{x \to \infty} \frac{1}{x^2} = 0.$
2. $\displaystyle\lim_{x \to -\infty} \frac{1}{x} = 0.$
3. $\displaystyle\lim_{x \to \infty} \frac{2}{x^3} = 2 \lim_{x \to \infty} \frac{1}{x^3} = 2 \times 0 = 0.$

**Example 1.18.**

Evaluate the following limits:

1. $\displaystyle\lim_{x\to\infty} \frac{x}{\sqrt{2x^2+1000}}$
2. $\displaystyle\lim_{x\to\infty} \frac{x^3-1}{x+1}$
3. $\displaystyle\lim_{x\to\infty} \frac{-3x^2+x+1}{x^2-2}$
4. $\displaystyle\lim_{x\to-\infty} \frac{x^2+3x}{x^3-2}$

**Solution**

1.

$$\begin{aligned}\lim_{x\to\infty} \frac{x}{\sqrt{2x^2+1000}} &= \lim_{x\to\infty} \frac{x}{x\sqrt{2+\dfrac{1000}{x^2}}}\\ &= \lim_{x\to\infty} \frac{1}{\sqrt{2+\dfrac{1000}{x^2}}}\\ &= \frac{1}{\sqrt{2+0}} = \frac{1}{\sqrt{2}}.\end{aligned}$$

Hence the line $y = \dfrac{1}{\sqrt{2}}$ is a horizontal asymptote.

2.

$$\begin{aligned}\lim_{x\to\infty} \frac{x^3-1}{x+1} &= \lim_{x\to\infty} \frac{(x^3-1)\frac{1}{x}}{(x+1)\frac{1}{x}}\\ &= \lim_{x\to\infty} \frac{x^2-\frac{1}{x}}{1+\frac{1}{x}}.\end{aligned}$$

The numerator is going to $\infty$ (because of the term $x^2$) while the denominator is going to 1.

Hence $\displaystyle\lim_{x\to\infty} \frac{x^3-1}{x+1} = \infty$.

3.

$$\lim_{x\to\infty}\frac{-3x^2+x+1}{x^2-2}\lim_{x\to\infty}\frac{(-3x^2+x+1)\frac{1}{x^2}}{(x^2-2)\frac{1}{x^2}}$$
$$=\lim_{x\to\infty}\frac{-3+\frac{1}{x}+\frac{1}{x^2}}{1-\frac{2}{x^2}}=-3.$$

Hence the line $y=-3$ is a horizontal asymptote.

4.

$$\lim_{x\to-\infty}\frac{x^2+3x}{x^3-2}=\lim_{x\to-\infty}\frac{(x^2+3x)\frac{1}{x^3}}{(x^3-2)\frac{1}{x^3}}$$
$$=\lim_{x\to-\infty}\frac{\frac{1}{x}+\frac{3}{x^2}}{1-\frac{2}{x^3}}$$
$$=\frac{0}{1}=0.$$

Hence the line $y=0$ is a horizontal asymptote.

Proposition 1.19 gives results which should be expected when calculating limits at infinity for rational functions. For a rational function $\frac{f(x)}{g(x)}$, its either the degree of $f(x)$ is less than the degree of $g(x)$, degree of $f(x)$ is greater that the degree of $g(x)$ or the two polynomials have the same degree..

**Proposition 1.19.** *Let $f(x)$ and $g(x)$ be polynomials.*

1. *If the degree of $f(x)$ is less than the degree of $g(x)$, then $\lim_{x\to\pm\infty}\frac{f(x)}{g(x)}=0$.*
2. *If the degree of $f(x)$ is greater than the degree of $g(x)$ then $\lim_{x\to\pm\infty}\frac{f(x)}{g(x)}=\infty$ or $\lim_{x\to\pm\infty}\frac{f(x)}{g(x)}=-\infty$ depending on the coefficients of leading terms.*
3. *If the degree of $f(x)$ is equal to the degree of $g(x)$ then $\lim_{x\to\pm\infty}\frac{f(x)}{g(x)}$ is the quotient of the leading coefficients.*

The expression

$$\lim_{x\to a}f(x)=\infty$$

means that the limit fails to exist because the function increases without bound as $x$ approaches $a$. Similarly, the expression

$$\lim_{x \to a} f(x) = -\infty$$

means that the limit fails to exist because the function decreases without bound as $x$ approaches $a$. A limit which is $\infty$ or $-\infty$ is called an **infinite limit**.

**Definition 1.20.**

*The line $x = a$ is a* **vertical asymptote** *of the graph of a function $y = f(x)$ if either*

$$\lim_{x \to a} f(x) = \pm\infty, \ \lim_{x \to a^+} f(x) = \pm\infty \ \text{or} \ \lim_{x \to a^-} f(x) = \pm\infty.$$

**Example 1.21.**

Evaluate the limit and identify the vertical asymptote of the graph of each function.

1. $\displaystyle\lim_{x \to 3^+} \frac{x}{x-3}$.
2. $\displaystyle\lim_{x \to 3^-} \frac{x}{x-3}$.
3. $\displaystyle\lim_{x \to 2^+} \frac{1}{2-x}$
4. $\displaystyle\lim_{x \to 0} \frac{1}{x^2}$.

**Solution**

1. If $x$ is close to 3 but larger than 3, the denominator $x - 3$ is small positive and $x$ is close to 3. So $\dfrac{x}{x-3}$ is a large positive number.

   Thus $\displaystyle\lim_{x \to 3^+} \frac{x}{x-3} = \infty$ and the line $x = 3$ is a vertical asymptote.
2. Similarly, $\displaystyle\lim_{x \to 3^-} \frac{x}{x-3} = -\infty$.
3. $\displaystyle\lim_{x \to 2^+} \frac{1}{2-x} = -\infty$ and the line $x = 2$ is a vertical asymptote.
4. $\displaystyle\lim_{x \to 0} \frac{1}{x^2} = \infty$ and the line $x = 0$ is a vertical asymptote.

**Exercise 1.22.**

1. Evaluate each limit below if it exists.

   (a) $\lim\limits_{x \to -\infty} (x^3 - 4x^2 + 5)$

   (b) $\lim\limits_{x \to \infty} \dfrac{2x^3}{3x^2 - 4}$

   (c) $\lim\limits_{x \to \infty} \dfrac{x + 1}{2x^2 + 2x + 1}$

   (d) $\lim\limits_{x \to -\infty} \dfrac{\sqrt{2x^2 + 3}}{2x + 3}$

   (e) $\lim\limits_{x \to \infty} \left( -\dfrac{\ln x}{x^4} + 1 \right)$

   (f) $\lim\limits_{x \to \infty} \left( -e^{-3x} - 1 \right)$

   (g) $\lim\limits_{x \to \infty} \cos 2x$

   (h) $\lim\limits_{x \to 2^-} \dfrac{4}{x - 2}$

   (i) $\lim\limits_{x \to 5^+} \dfrac{3x}{x - 5}$

2. Find the vertical and horizontal asymptotes for each function.

   (a) $f(x) = \dfrac{1}{x^2 - 25}$

   (b) $f(x) = \dfrac{x^3 + 5}{x}$

## 1.4 Precise Definition of a Limit

The definition of a limit given in 1.1 is inadequate since it contains vague phrases such as "$x$ is close to $a$" and "$f(x)$ gets closer and closer to $L$". Below we give the precise definition of a limit.

**Definition 1.23.** *Let $f$ be a function defined on some open interval that contains the number $a$, except possibly at $a$ itself. Then we say that the limit of $f(x)$ as $x$ approaches $a$ is $L$, and we write*

$$\lim_{x \to a} f(x) = L$$

*if for every number $\varepsilon > 0$ there is a number $\delta$ such that*

$$\text{if } 0 < |x - a| < \delta \text{ then } |f(x) - L| < \varepsilon.$$

The following equivalent statements give interpretation of Definition 1.23.

1. $\lim\limits_{x \to a} f(x) = L$ means that the distance between $f(x)$ and $L$ can be made arbitrarily small by taking the distance from $x$ to $a$ sufficiently small (but not 0).

2. $\lim\limits_{x \to a} f(x) = L$ means that the values of $f(x)$ can be made as close as we please to $L$ by taking $x$ close enough to $a$ (but not equal to $a$).

3. $\lim\limits_{x \to a} f(x) = L$ means that for every $\varepsilon > 0$ (no matter how small $\varepsilon$ is) we can find $\delta > 0$ such that if $x$ lies in the open interval $(a - \delta, a + \delta)$ and $x \neq a$, then $f(x)$ lies in the open interval $(L - \varepsilon, L + \varepsilon)$.

**Example 1.24.**

Prove that $\lim\limits_{x \to 1}(5x - 3) = 2$.

**Solution**

We need to show that for every $\varepsilon > 0$ there is $\delta > 0$ such that

$$\text{if } |x - 1| < \delta \text{ then } |(5x - 3) - 2| < \varepsilon.$$

Before the actual proof, let us analyse the problem and guess the value of $\delta$. We start with the $\varepsilon$ condition.

$|(5x - 3) - 2| < \varepsilon$ simplifies to $|5x - 5| < \varepsilon \Rightarrow 5|x - 1| < \varepsilon$.

Now (rewriting the problem) we need to show that for every $\varepsilon > 0$ there is $\delta > 0$ such that

$$\text{if } |x - 1| < \delta \text{ then } 5|x - 1| < \varepsilon.$$

Well, if $|x - 1| < \delta$ then $5|x - 1| < 5\delta$, so we just need to have $5\delta \leq \varepsilon$. Hence we choose $\delta = \dfrac{\varepsilon}{5}$.

Now that we have figured out what $\delta$ will work, we need to go back and write up an argument.

*Proof.* Given any $\varepsilon > 0$, we can define $\delta = \dfrac{\varepsilon}{5}$. Then

$$|x-1| < \delta \Rightarrow |x-1| < \frac{\varepsilon}{5} \Rightarrow 5|x-1| < \varepsilon.$$

But $5|x-1| = |(5x-3)-2|$, so we have shown that for any $\varepsilon > 0$ there is $\delta > 0$ such that

$$\text{if } |x-1| < \delta \text{ then } |(5x-3)-2| < \varepsilon.$$

$\square$

**Example 1.25.**

Prove that $\lim_{x\to 3}(4x-5) = 7$.

**Solution**

Let $\varepsilon$ be a positive number. We want to find a number $\delta$ such that

$$\text{if } |x-3| < \delta \text{ then } |(4x-5)-7| < \varepsilon.$$

But $|(4x-5)-7| = |4x-12| = 4|x-3|$. Therefore we want to show that

$$\text{if } |x-3| < \delta \text{ then } 4|x-3| < \varepsilon$$

that is

$$\text{if } |x-3| < \delta \text{ then } |x-3| < \frac{\varepsilon}{4}.$$

Hence we choose $\delta = \dfrac{\varepsilon}{4}$.

*Proof.* Given $\varepsilon > 0$ choose $\delta = \dfrac{\varepsilon}{4}$. If $|x-3| < \delta$, then

$$|(4x-5)-7| = |4x-12| = 4|x-3| < 4\delta = 4\left(\frac{\varepsilon}{4}\right) = \varepsilon.$$

Thus

$$\text{if } |x-3| < \delta \text{ then } |(4x-5)-7| < \varepsilon.$$

This proves that
$$\lim_{x \to 3}(4x - 5) = 7.$$
$\square$

**Exercise 1.26.**

Prove each statement using the precise definition of a limit.

1. $\lim\limits_{x \to 2}(14 - 5x) = 4$
2. $\lim\limits_{x \to 0} \sqrt[3]{x} = 0$
3. $\lim\limits_{x \to 2} \left(x^2 - 3x\right) = -2$
4. $\lim\limits_{x \to 4^+} \dfrac{2}{\sqrt{x - 4}} = \infty$

## 1.5 Continuous Functions

**Definition 1.27.** *A function $f$ is* **continuous** *at a number $a$ if $\lim\limits_{x \to a} f(x) = f(a)$.*

Definition 1.27 implies that for a function to be continuous at a number $a$, it has to satisfy three conditions, namely

1. $f$ is defined on an open interval containing $a$, that is $f(a)$ exists.
2. $\lim\limits_{x \to a} f(x)$ exists.
3. $\lim\limits_{x \to a} f(x) = f(a)$.

A function which is not continuous at $a$, is said to be **discontinuous** at $a$ .

**Example 1.28.**

1. Where are each of the following functions discontinuous?

   (a) $f(x) = \dfrac{x^2 - x - 6}{x - 3}$

   (b) $f(x) = \begin{cases} \dfrac{1}{x^2} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}.$

(c) $f(x) = \begin{cases} \dfrac{x^2 - x}{x^2 - 1} & \text{if } x \neq 1 \\ 1 & \text{if } x = 1. \end{cases}$

**Solution**

1. $f(3)$ is not defined, so $f$ is discontinuous at $x = 3$.

2. $f(0) = 1$ is defined but $\lim_{x \to 0} \dfrac{1}{x^2}$ does not exist. So the function is discontinuous at $x = 0$.

3. Here $f(1) = 1$ is defined.

   Now

$$\lim_{x \to 1} f(x) = \lim_{x \to 1} \frac{x^2 - x}{x^2 - 1} = \lim_{x \to 1} \frac{x(x-1)}{(x-1)(x+1)} = \lim_{x \to 1} \frac{x}{x+1} = \frac{1}{2}$$

   The function is discontinuous at $x = 1$ since $\lim_{x \to 1} f(x) \neq f(1)$.

**Example 1.29.**

Show that the function $f(x) = \begin{cases} x + 1 & \text{if } x < 2 \\ x^2 - 1 & \text{if } x = 2 \\ 2x - 1 & \text{if } x > 2 \end{cases}$ is continuous at $x = 2$.

**Solution**

$f(2) = 2^2 - 1 = 3$ is defined.

Now we consider existence of $\lim_{x \to 2} f(x)$ based on left and right limits.

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (2x - 1) = 2(2) - 1 = 3$$

and

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (x + 1) = 2 + 1 = 3.$$

It follows that $\lim_{x \to 2} f(x) = 3$.

Now since $\lim_{x \to 2} f(x) = f(2) = 3$, $f(x)$ is continuous at $x = 2$.

As seen above, continuity of functions is defined in terms of the limit of such a function. Since we have left and right limits, we can also have a function which is continuous from the left or right.

**Definition 1.30.** *A function is* **continuous from the right** *at a number a if*

$$\lim_{x \to a^+} f(x) = f(a)$$

*and is* **continuous from the left** *at a if*

$$\lim_{x \to a^-} f(x) = f(a).$$

Now we define continuity of a function on an interval.

**Definition 1.31.** *A function f is* **continuous on an interval** *if it is continuous at every number in the interval. (If f is defined only on one side of an endpoint of the interval, then* continuous *at the endpoint means* continuous *from the right or* continuous *from the left.)*

**Example 1.32.**

Show that the function $f(x) = 1 - \sqrt{1 - x^2}$ is continuous on the interval $[-1, 1]$.

**Solution**

Let $a \in [-1, 1]$. We show that $f$ is continuous at $a$.

Using limit laws we have

$$\begin{aligned}
\lim_{x \to a} f(x) &= \lim_{x \to a} \left(1 - \sqrt{1 - x^2}\right) \\
&= 1 - \lim_{x \to a} \left(\sqrt{1 - x^2}\right) \\
&= 1 - \sqrt{\lim_{x \to a}(1 - x^2)} \\
&= 1 - \sqrt{1 - a^2} \\
&= f((a).
\end{aligned}$$

Hence, by the definition of continuity, $f(x)$ is continuous at $a$ if $a \in [-1, 1]$.

**Theorem 1.33.** *Let c be a constant. If f and g are continuous at a, then the following functions are also continuous at a:*

1. $f \pm g$
2. $fg$
3. $\dfrac{f}{g}$ *if* $g(a) \neq 0$
4. $cf$

*Proof.* We prove part 2 and leave the rest as an exercise.

Since both $f$ and $g$ are continuous at $a$, we have

$$\lim_{x\to a} f(x) = f(a) \text{ and } \lim_{x\to a} g(x) = g(a).$$

Now, using limit laws

$$\begin{aligned}
\lim_{x\to a}(fg)(x) &= \lim_{x\to a} f(x)g(x) \\
&= \lim_{x\to a}\left(\lim_{x\to a} f(x)\right)\left(\lim_{x\to a} g(x)\right) \\
&= f(a)g(a) \\
&= (fg)(a).
\end{aligned}$$

Hence $fg$ is continuous at $a$. □

In the theorem below we restate the direct substitution property (Theorem 1.4) in terms of continuity of functions.

**Theorem 1.34.**
1. *Any polynomial is continuous everywhere, i.e., it is continuous on $\mathbb{R} = (-\infty, \infty)$.*
2. *Any rational function is continuous on its domain.*

*Proof.* Exercise □

**Theorem 1.35.** *The following types of functions are continuous at every number in their domains*

1. *polynomials*
2. *rational functions*
3. *root functions*
4. *trigonometric functions*
5. *inverse trigonometric functions*
6. *exponential functions*
7. *logarithmic functions*

**Example 1.36.**

1. Where is the function $f(x) = \dfrac{\ln x + \tan^{-1} x}{x^2 - 16}$ continuous?

2. Evaluate $\lim\limits_{x \to \pi} \dfrac{\cos x}{3 + \sin x}$

**Solution**

1. By Theorem 1.35, $\ln x$ is continuous for $x > 0$ and $\tan^{-1} x$ is continuous for $\mathbb{R} = (-\infty, \infty)$. Hence by Theorem 1.33, $\ln x + \tan^{-1} x$ is continuous on $(0, \infty)$. The denominator is a polynomial hence continuous everywhere (Theorem 1.35). Therefore by Theorem 1.33 the function $f$ is continuous at all positive numbers $x$ except where the denominator is equal to zero, i.e., except where $x = \pm 4$.

   Hence $f$ is continuous on the intervals $(0, 1)$ and $(1, \infty)$.

2. By Theorem 1.35, $\cos x$ is continuous. The denominator $3 + \sin x$ is the sum of two continuous functions and hence continuous by Theorem 1.33. You will notice that the denominator can not be equal to 0 since $\sin x \geq 0$.

   Hence

$$\lim_{x \to \pi} \frac{\cos x}{3 + \sin x} = \lim_{x \to \pi} f(x) = f(\pi) = \frac{\cos \pi}{3 + \sin \pi} = \frac{-1}{3 + 0} = -\frac{1}{3}.$$

**Theorem 1.37.** *If $f$ is continuous at $b$ and $\lim\limits_{x \to a} g(x) = b$, then $\lim\limits_{x \to a} f(g(x)) = f(b)$. In other words*

$$\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right).$$

**Example 1.38.**

Evaluate $\lim\limits_{x \to 1} \sin^{-1}\left(\dfrac{1 - \sqrt{x}}{1 - x}\right)$.

**Solution**

$\sin^{-1}$ is a continuous function so applying Theorem 1.37 we have

$$\begin{aligned}
\lim_{x \to 1} \sin^{-1}\left(\frac{1 - \sqrt{x}}{1 - x}\right) &= \sin^{-1}\left(\lim_{x \to 1} \frac{1 - \sqrt{x}}{1 - x}\right) \\
&= \sin^{-1}\left(\lim_{x \to 1} \frac{1 - \sqrt{x}}{(1 - \sqrt{x})(1 + \sqrt{x})}\right) \\
&= \sin^{-1}\left(\lim_{x \to 1} \frac{1}{1 + \sqrt{x}}\right) \\
&= \sin^{-1} \frac{1}{2} = \frac{\pi}{6}.
\end{aligned}$$

**Theorem 1.39.** *If $g$ is continuous at $a$ and $f$ is continuous at $g(a)$, then the composition function $f(g(x))$ is continuous at $a$.*

*Proof.* $g$ is continuous at $a$, so, by definition

$$\lim_{x \to a} g(x) = g(a).$$

$f$ is continuous at $b = g(a)$, so applying Theorem 1.37 we have

$$\lim_{x \to a} f(g(x)) = f(g(a)).$$

Hence $f(g(x))$ is continuous at $a$. $\square$

**Example 1.40.**

Where is the function $f(x) = \sin x^2$ continuous.

**Solution**

We have $h(x) = f(g(x))$ where $f(x) = \sin x$ and $g(x) = x^2$.

$g$ is continuous on $\mathbb{R}$ since it is a polynomial, and $f$ is also continuous everywhere.

Hence $h(x) = f(g(x))$ is continuous on $\mathbb{R}$.

**Theorem 1.41** (The Intermediate Value theorem)**.** *Suppose that $f$ is continuous on the closed interval $[a, b]$ and let $N$ be any number between $f(a)$ and $f(b)$ where $f(a) \neq f(b)$. Then there exists a number $c$ in $(a, b)$ such that $f(c) = N$.*

**Example 1.42.**

Show that there is a root of the equation $4x^3 - 6x^2 + 3x - 2 = 0$ between 1 and 2.

**Solution**

Let $f(x) = 4x^3 - 6x^2 + 3x - 2$. We are looking for a number $c$ between 1 and 2 such that $f(c) = 0$. So in Theorem 1.41 we take $a = 1, b = 2$ and $N = 0$.

Now $f(1) = 4 - 6 + 3 - 2 = -1$ and $f(2) = 32 - 24 + 6 - 2 = 12$.

Therefore $f(1) < 0 < f(2)$, i.e., $N = 0$ is a number between $f(1)$ and $f(2)$.

Now $f$ is continuous since it is a polynomial, so the Intermediate Value Theorem says there is a number $c$ between 1 and 2 such that $f(c) = 0$. In other words, the equation

$$4x^3 - 6x^2 + 3x - 2 = 0$$

has at least one root in the interval $(1, 2)$.

## 1.6 Tangent Lines to the Curve

A tangent line to a curve is a straight line that touches the curve at only one point (See Figure 1). We will look at how to find the equation of a tangent line to the curve at a given point. First we define the slope of the tangent line.

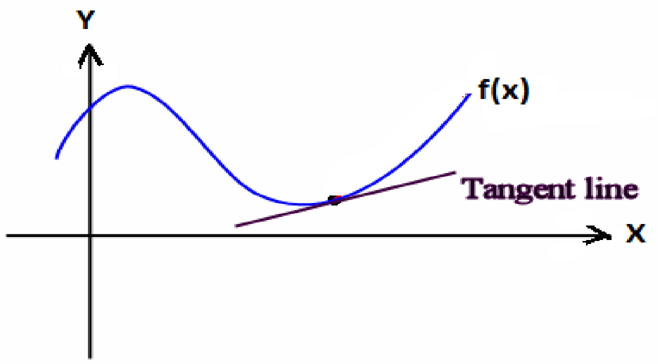


Figure 1: The tangent line

**Definition 1.43.** *The slope of the tangent line to the curve $y = f(x)$ at the point $P(a, f(a))$ is given by*

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

*provided that the limit exists.*

Once the slope of the tangent line is found, we can find the equation using **point-slope** form of the equation of straight line discussed in Module 1 of Mat 111.

**Example 1.44.**

1. Find the slope of the tangent line to the curve $y = x^3 + 1$ at the point $P(1, 2)$.
2. Find equation of the tangent line to the curve $y = x^2 + 4$ at the point $P(-1, 5)$.
3. Find the slope of the tangent line to the curve $y = 2x^2 - 4$ at $x = 2$.

**Solution**

1. $a = 1$, $f(x) = x^3 + 1$ and $f(a) = f(1) = 2$. Notice that

$$x^3 - 1 = (x - 1)(x^2 + x + 1).$$

Hence the slope is

$$m = \lim_{x \to 1} \frac{(x^3 + 1) - 2}{x - 1} = \lim_{x \to 1} \frac{(x - 1)(x^2 + x + 1)}{(x - 1)} = \lim_{x \to 1}(x^2 + x + 1) = 1 + 1 + 1 = 3..$$

2. $a = -1$, $f(x) = x^2 + 4$ and $f(-1) = 5$.

   The slope is

$$m = \lim_{x\to -1} \frac{(x^2+4)-5}{x-(-1)} = \lim_{x\to -1} \frac{x^2-1}{x+1} = \lim_{x\to 1-} \frac{(x+1)(x-1)}{(x+1)} = \lim_{x\to -1}(x-1) = -2.$$

   Using the point $(-1, 5)$ and slope $m = -2$, the equation becomes

$$y - 5 = -2(x-(-1)) \Rightarrow y = -2x - 2 + 5 \Rightarrow y = -2x + 3.$$

3. $a = 2$, $f(x) = 2x^2 - 4$ and $f(a) = f(2) = 2(2)^2 - 4 = 8 - 4 = 4$.

   The slope is

$$\begin{aligned} m &= \lim_{x\to 2} \frac{(2x^2-4)-4}{x-2} = \lim_{x\to 2} \frac{2x^2-8}{x-2} \\ &= \lim_{x\to 2} \frac{2(x^2-4)}{x-2} = \lim_{x\to 2} \frac{2(x-2)(x+2)}{x-2} \\ &= \lim_{x\to 2} 2(x+2) = 2(2+2) = 8. \end{aligned}$$

**Exercise 1.45.**

1. Let $f(x) = \begin{cases} x^2 & \text{if } -3 < x < 3 \\ x & \text{if } 3 \leq x < 5 \\ 0 & \text{if } x = 5 \\ x & \text{if } 5 < x < 7 \\ \dfrac{1}{x-10} & \text{if } x > 7. \end{cases}$

   (a) Where is $f(x)$ discontinuous?

   (b) At which of the following $x$ values 0, 3, 5, 7, and 10, is $f(x)$ continuous from the right?

   (c) At which of the following $x$ values 0, 3, 5, 7, and 10, is $f(x)$ continuous from the left?

2. Find the value of $a, b$ or $c$ such that the function is continuous on $(-\infty, \infty)$.

   (a) $f(x) = \begin{cases} \dfrac{x^3-1}{x^2-1} & \text{if } x < 1 \\ \dfrac{\sqrt{x+c}}{2} & \text{if } x \geq 1 \end{cases}$

   (b) $f(x) = \begin{cases} cx^2 + 2x & \text{if } x < 2 \\ x^3 - cx & \text{if } x \geq 2 \end{cases}$

(c) $f(x) = \begin{cases} \dfrac{x^2-4}{x-2} & \text{if } x < 2 \\ ax^2 - bx + 3 & \text{if } 2 < x < 3 \\ 2x - a + b & \text{if } x \geq 3 \end{cases}$

3. Prove that the equation has at least one real root. Then use your calculator to find an interval of length 0.01 that contains a root.

   (a) $\cos x = x^3$

   (b) $\ln x = 3 - 2x$

4. If $f(x) = x^2 + 10\sin x$ show that there is a number $c$ such that $f(c) = 1000$.

5. Use the intermediate value theorem to show that there is a root of the given equation in the specified interval.

   (a) $x^4 + x - 3 = 0$, $(1, 2)$

   (b) $\sqrt[3]{x} = 1 - x$, $(0, 1)$

   (c) $\cos x = x$, $(0, 1)$

   (d) $\ln x = e^{-x}$, $(1, 2)$

6. Use continuity to evaluate each limit.

   (a) $\displaystyle\lim_{x\to 4} \frac{5+\sqrt{x}}{\sqrt{5+x}}$

   (b) $\displaystyle\lim_{x\to \pi} \sin(x + \sin x)$

   (c) $\displaystyle\lim_{x\to 1} e^{x^2-x}$

   (d) $\displaystyle\lim_{x\to 2} \tan^{-1}\left(\frac{x^2-4}{3x^2-6x}\right)$

7. Show that the function is continuous at the given number $a$ or the given interval.

   (a) $f(x) = x^2 + \sqrt{7-x}$, $a = 4$

   (b) $f(x) = (x + 2x^3)^4$, $a = -1$

   (c) $f(x) = \dfrac{2x - 3x^2}{1 + x^3}$, $a = 1$

   (d) $f(x) = \dfrac{2x+3}{x-2}$, $(2, \infty)$

   (e) $f(x) = 2\sqrt{3-x}$, $(-\infty, 3)$

8. Prove that $f$ is continuous at $a$ if and only if $\displaystyle\lim_{h\to 0} f(a+h) = f(a)$

9. Find the slope of the tangent line to the curve $y = \sqrt{2x}$ at $x = 8$.

10. Find the equation of the tangent line to the given curve at the given point.

    (a) $y = 2x^3 - 2x + 11$ at $x = 4$

    (b) $y = \dfrac{x-3}{2-x}$ at $x = 1$.

    (c) $y = 4x^2$ at $x = -2$.

    (d) $y = x^2 + 4x + 3$ at $x = 0$.

11. Evaluate each limit.

    (a) $\displaystyle\lim_{x\to 1}\left(\frac{1}{x-1} + \frac{1}{x^2-3x+2}\right)$

    (b) $\displaystyle\lim_{x\to 0}\left(\frac{1}{x} - \frac{1}{x^2+x}\right)$

    (c) $\displaystyle\lim_{x\to 0^+}\left(\frac{1}{x} - \frac{1}{|x|}\right)$

    (d) $\displaystyle\lim_{x\to -2}\frac{x+2}{x^3+8}$

    (e) $\displaystyle\lim_{t\to 3}(2x + |x-3|)$

    (f) $\displaystyle\lim_{x\to 3^+}\frac{x+2}{x+3}$

    (g) $\displaystyle\lim_{x\to \pi^-}\cot x$

    (h) $\displaystyle\lim_{x\to 3^+}\ln(x^2-9)$

    (i) $\displaystyle\lim_{x\to 5^-}\frac{e^x}{(x-5)^3}$

12. Use the precise definition of the limit to show that

    (a) $\displaystyle\lim_{x\to 3}\frac{x}{5} = \frac{3}{5}$

    (b) $\displaystyle\lim_{x\to 2}x^3 = 8$

    (c) $\displaystyle\lim_{x\to 6}\left(\frac{x}{4} + 3\right) = \frac{9}{2}$

    (d) $\displaystyle\lim_{x\to -2}(x^2-1) = 3$

13. Determine why the function is not continuous at the given point.

    (a) $f(x) = |x-2|$, $a = 2$

    (b) $f(x) = \begin{cases} e^x & \text{if } x < 0 \\ x^2 & \text{if } x \geq 0, \end{cases}$ $a = 0$

(c) $f(x) = \begin{cases} \dfrac{1}{x-1} & \text{if } x \neq 1 \\ 2 & \text{if } x = 1, \end{cases} \quad a = 1$

14. Use squeeze theorem to show that $\lim\limits_{x\to 0} \sqrt{x^3 + x} \sin \dfrac{\pi}{x} = 0$.

15. What is wrong with the equation $\dfrac{x^2 + x - 6}{x - 2} = x + 3$.

16. Find the vertical asymptote of the function $\dfrac{x^2 + 1}{3x - 2x^2}$.

17. Find the equation of the tangent line to the curve $y = 3x^3 - 4x$ at $(-2, -16)$.

18. Find the slope of the tangent line to the curve $y = \dfrac{2x + 1}{x}$ at $x = 2$.

# 2 Definition and Techniques of Differentiation

## 2.1 Differentiation from First Principles

The process of calculating the **derivative** of a function is called **Differentiation**. There are a number of techniques which are employed when calculating the derivative of a function. In this unit we discuss all the differentiation techniques. To begin with we look at the definition of the derivative which is a technique on its own right.

**Definition 2.1.** *The derivative of a function $f$ at a number $a$ is given by*

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

*provided this limit exists.*

The technique in Definition 2.1 is called **differentiation from first principles**.

It can be shown that

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = m = \lim_{x\to a} \frac{f(x) - f(a)}{x - a},$$

so the slope of the tangent line to the curve $y = f(x)$ at the point $(a, f(a))$ is actually equal to the derivative of $f$ at $a$.

**Example 2.2.**

Find

1. $f'(3)$ given that $f(x) = 3x^2$.
2. $f'(1)$ if $f(x) = x^3$.
3. $f'(4)$ if $f(x) = \sqrt{x}$.

**Solution**

1.

$$\begin{aligned} f'(3) &= \lim_{h\to 0} \frac{f(3+h) - f(3)}{h} \\ &= \lim_{h\to 0} \frac{3(3+h)^2 - 3(3)^2}{h} \\ &= \lim_{h\to 0} \frac{3(9+6h+h^2) - 27}{h} \\ &= \lim_{h\to 0} \frac{27+18h+3h^2 - 27}{h} \\ &= \lim_{h\to 0}(18+3h) \ = 18. \end{aligned}$$

2.

$$\begin{aligned} f'(1) &= \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} \\ &= \lim_{h\to 0} \frac{(1+h)^3 - 1^3}{h} \\ &= \lim_{h\to 0} \frac{(1+2h+h^2)(1+h) - 1}{h} \\ &= \lim_{h\to 0} \frac{1+3h+3h^2+h^3 - 1}{h} \\ &= \lim_{h\to 0}(3+3h+h^2) = 3. \end{aligned}$$

3.

$$\begin{aligned} f'(4) &= \lim_{h\to 0} \frac{f(4+h) - f(4)}{h} \\ &= \lim_{h\to 0} \frac{\sqrt{4+h} - \sqrt{4}}{h} \\ &= \lim_{h\to 0} \frac{\sqrt{4+h} - \sqrt{4}}{h} \times \frac{\sqrt{4+h} + \sqrt{4}}{\sqrt{4+h} + \sqrt{4}} \\ &= \lim_{h\to 0} \frac{4+h-4}{h(\sqrt{4+h} + \sqrt{4})} \\ &= \lim_{h\to 0} \frac{1}{\sqrt{4+h} + \sqrt{4}} = \frac{1}{2\sqrt{4}} = \frac{1}{4}. \end{aligned}$$

### 2.1.1 The Derivative Function

If we replace $a$ with $x$ in Definition 2.1, we get the derivative as a function.

**Definition 2.3.** *The derivative function of $f(x)$, denoted $f'(x)$, is given by*

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

*provided the limit exits.*

**Example 2.4.**

Find the derivative of each of the following functions;

1. $f(x) = 3x^2 - 5x$.
2. $f(x) = x^2 - 4x + 3$

**Solution**

1.

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{3(x+h)^2 - 5(x+h) - (3x^2 - 5x)}{h} \\
&= \lim_{h\to 0} \frac{3x^2 + 6xh + 3h^2 - 5x - 5h - 3x^2 + 5x}{h} \\
&= \lim_{h\to 0} \frac{6xh + 3h^2 - 5h}{h} \\
&= \lim_{h\to 0} (6x + 3h - 5) \\
&= 6x - 5.
\end{aligned}$$

2.

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{[(x+h)^2 - 4(x+h) + 3] - (x^2 - 4x + 3)}{h} \\
&= \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - 4x - 4h + 3 - x^2 + 4x - 3}{h} \\
&= \lim_{h\to 0} \frac{2xh + h^2 - 4h}{h} \\
&= \lim_{h\to 0} (2x + h - 4) \\
&= 2x - 4.
\end{aligned}$$

Apart from $f'(x)$, other alternative notations for the derivative of the function $y = f(x)$ with respect to $x$ are; $y'$, $\frac{dy}{dx}$, $\frac{df}{dx}$, $\frac{d}{dx}[f(x)]$, $Df(x)$ and $D_x f(x)$.

**Exercise 2.5.**

1. Find the derivative of each function at the given number.
   (a) $f(x) = 4x^2 + x - 22$ at $x = -4$
   (b) $f(x) = x^3 - 4x + 3$ at $x = -2$
   (c) $g(x) = 22x^4 - 4x^3 + 3x - 7$ at $x = 2$
   (d) $h(x) = 23x^2 + 2x + 1$ at $x = 0$
   (e) $f(x) = 9x^2 - 7x + 15$ at $x = 1$
2. Find the derivative (from first principles) of each function.
   (a) $y = \sqrt[3]{x}$
   (b) $y = \sqrt{2x^2 - 5x}$
   (c) $y = 4x^3 - 7x + 20$
   (d) $y = x^2 - 4x^3$

### 2.1.2 Differentiability

A function is differentiable at a number if its derivative at that number exists.

**Definition 2.6.**

A function $f$ is differentiable at $x = a$ if $f'(a)$ exists. It is differentiable on an open interval $(a, b)$ [or $(a, \infty)$ or $(-\infty, a)$ or $(-\infty, \infty)$] if it is differentiable at every number in the interval.

**Example 2.7.**

Where is the function $f(x) = |x|$ differentiable?

**Solution**

For $x > 0$, $|x| = x$. Suppose the value of $h$ is such that $x+h > 0$. Then $|x+h| = x+h$.

Now

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{|x+h|-|x|}{h} \\ &= \lim_{h\to 0} \frac{(x+h)-x}{h} \\ &= \lim_{h\to 0} \frac{h}{h} = 1. \end{aligned}$$

Hence $f$ is differentiable for any $x > 0$.

For $x < 0$, $|x| = -x$ and we can choose the value of $h$ such that $|x+h| = (x+h)$.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{|x+h|-|x|}{h} \\ &= \lim_{h\to 0} \frac{-(x+h)-(-x)}{h} \\ &= \lim_{h\to 0} \frac{-h}{h} = -1. \end{aligned}$$

Hence $f$ is differentiable for any $x < 0$.

For $x = 0$ $f'(0) = \lim_{h\to 0} \dfrac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \dfrac{|h|}{h}$ if it exists. If we compute left and right limits separately we get

$$\lim_{h\to 0^+} \frac{|h|}{h} = 1 \text{ and } \lim_{h\to 0^-} \frac{|h|}{h} = -1.$$

Since we are getting different results, $f'(0)$ does not exist. It follows that $f(x) = |x|$ is differentiable at all $x$ except 0.

**Theorem 2.8.** *If f(x) is differentiable at $x = a$, then $f(x)$ is also continuous at $x = a$.*

*Proof.* Since $f$ is differentiable at $a$,

$$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$$

exits. Now

$$\lim_{x\to a}(f(x)-f(a)) = \lim_{x\to a}(x-a)\frac{f(x)-f(a)}{x-a} = \lim_{x\to a}(x-a)\lim_{x\to a}\frac{f(x)-f(a)}{x-a} = 0.$$

Thus $f'(a)=0$. Hence $\lim_{x\to a}(f(x)-f(a))=0$. Adding $f(a)$ both sides we have

$$\lim_{x\to a}(f(x)-f(a))+f(a)=f(a) \Rightarrow \lim_{x\to a}f(x)-\lim_{x\to a}f(a)+f(a)=f(a)$$

$$\lim_{x\to a}f(x)-f(a)+f(a)=f(a).$$

Hence

$$\lim_{x\to a}f(x)=f(a)$$

which is the definition for continuity of $f$ at $x=a$. $\square$

The converse of Theorem 2.8 is false since there are functions that are continuous but not differentiable. For example, the function $f(x)=|x|$ is continuous yet it is not differentiable at $x=0$ (Example 2.7).

There are three ways in which a function can fail to be differentiable.

1. By Theorem 2.8, if a function is not continuous, then it is not differentiable. So a function is not differentiable at the point where it has a discontinuity.
2. If the graph of a function has a sharp corner at the given point, then the given function is not differentiable at that point. For example $f(x)=|x|$ is not differentiable at $x=0$ since its graph has a sharp corner at this point (Example 2.7 and Figure 2).
3. If the curve of $f$ has a vertical tangent line when $x=a$ ; that is, $f$ is continuous at $a$ and
$$\lim_{x\to a}|f'(x)|=\infty,$$
then $f$ is not differentiable at $x=a$. In this case the tangent lines become steeper and steeper as $x\to a$.

## 2.2 Formulae and Techniques of Differentiation

### 2.2.1 The Derivative of a Constant Function

Differentiating a constant function gives zero.

**Theorem 2.9.** *The derivative of $f(x)=c$ with respect to $x$ is zero, i.e., $f'(x)=0$.*

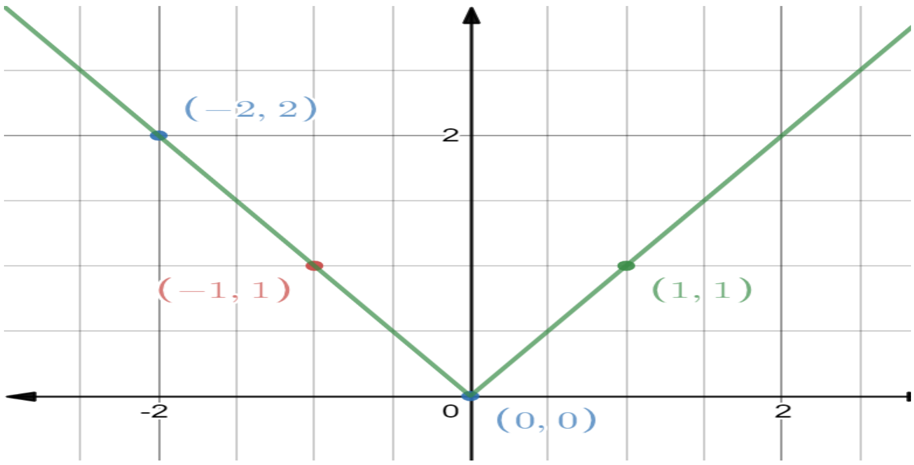


Figure 2: The Graph of $f(x) = |x|$

*Proof.* Using the formula for differentiation from first principles we have

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{c - c}{h} = 0.$$

$\square$

**Example 2.10.**

By Theorem 2.9, $\dfrac{d}{dx}(120) = 0$ and $\dfrac{d}{dx}(-305) = 0$.

### 2.2.2 The Power Rule

The power rule of differentiation gives the derivative of the function $f(x) = x^n$ where $n$ is a real number. The proof of the power rule makes use of the formula

$$x^n - a^n = (x - a)(x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \ldots + xa^{n-2} + a^{n-1}). \tag{1}$$

To verify Equation 1 you can multiply out the right-hand side or sum the second factor as a geometric series.

**Theorem 2.11.** *For any real number $n$,* $\dfrac{d}{dx}(x^n) = nx^{n-1}$.

*Proof.* We show that the derivative at $a$ is $na^{n-1}$, i.e., $f'(a) = na^{n-1}$.

$$
\begin{aligned}
f'(a) &= \lim_{x\to a} \frac{f(x)-f(a)}{x-a} \\
&= \lim_{x\to a} \frac{x^n - a^n}{x-a} \\
&= \lim_{x\to a} \frac{(x-a)(x^{n-1} + x^{n-2}a + x^{n-3}a^2 + ... + xa^{n-2} + a^{n-1})}{x-a} \\
&= \lim_{x\to a} (x^{n-1} + x^{n-2}a + x^{n-3}a^2 + ... + xa^{n-2} + a^{n-1}) \\
&= a^{n-1} + a^{n-2}a + a^{n-3}a^2 + ... + aa^{n-2} + a^{n-1} \\
&= \underbrace{a^{n-1} + a^{n-1} + a^{n-1} + ... + a^{n-1}}_{n \text{ times}} \\
&= na^{n-1}.
\end{aligned}
$$

$\square$

**Example 2.12.**

By Theorem 2.11, we have

1. $\dfrac{d}{dx}(x^5) = 5x^{5-1} = 5x^4.$
2. $\dfrac{d}{dx}(x^{3/4}) = \dfrac{3}{4}x^{3/4-1} = \dfrac{3}{4}x^{-1/4}.$
3. $\dfrac{d}{dx}(x) = 1 \times x^{1-1} = 1.$

### 2.2.3 The Constant Multiple Rule

This rule given the derivative of a function which is the product of two functions of which one of them is a constant.

**Theorem 2.13.** *If $c$ is a constant and $f$ is a differentiable function, then* $\dfrac{d}{dx}[cf(x)] = c\dfrac{d}{dx}[f(x)].$

*Proof.* Let $g(x) = cf(x)$. Then

$$\begin{aligned} g'(x) &= \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} \\ &= \lim_{h\to 0} \frac{cf(x+h)-cf(x)}{h} \\ &= \lim_{h\to 0} c\left[\frac{f(x+h)-f(x)}{h}\right] \\ &= c\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= cf'(x). \end{aligned}$$

□

**Example 2.14.**

By Theorem 2.13, we have the following:

1. If $f(x) = -4x^3$, then $\frac{d}{dx}(f(x)) = -4\frac{d}{dx}(x^3) = -4(3x^{3-1}) = -12x^2$.
2. If $f(x) = 12x^{-3}$, then $f'(x) = -36x^{-4}$.

### 2.2.4 The Sum and Difference Rule

The sum and difference rule is a combination of two rules which give the derivative of a function which is the sum or difference of some functions.

**Theorem 2.15.** *Let $f$ and $g$ be differentiable functions. Then* $\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}[f(x)] \pm \frac{d}{dx}[g(x)]$.

*Proof.* We prove the sum rule and leave the difference rule as an exercise.

Let $h(x) = f(x) + g(x)$. Then

$$\begin{aligned} h'(x) &= \lim_{h\to 0} \frac{h(x+h)-h(x)}{h} \\ &= \lim_{h\to 0} \frac{f(x+h)+g(x+h)-[f(x)+g(x)]}{h} \\ &= \lim_{h\to 0} \left[\frac{f(x+h)-f(x)}{h} + \frac{g(x+h)-g(x)}{h}\right] \\ &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} + \lim_{h\to 0} \frac{g(x+h)}{h} \\ &= f'(x) + g'(x). \end{aligned}$$

□

The sum can be extended to the sum of any number of functions. For example

$$\frac{d}{dx}[f(x)+g(x)+h(x)] = \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)] + \frac{d}{dx}[h(x)].$$

**Example 2.16.**

Find the derivative of each function:

1. $f(x) = 24x^2 + 4x - 19.$
2. $f(x) = 18x^4 - 8x^3 + 2x^2 - 17x.$
3. $f(x) = 9x^{-6} + 6x^6 - 12x$

**Solution**

1. $f'(x) = 48x + 4.$
2. $f'(x) = 72x^3 - 24x^2 + 4x - 17.$
3. $f'(x) = -54x^{-7} + 36x^5 - 12$

### 2.2.5 The Product Rule

The theorem below gives the product rule, a rule stating how to find the derivative of a function which is the product of two functions.

**Theorem 2.17.** *Let $f$ and $g$ be differentiable functions. Then*

$$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)].$$

*Proof.*

$$\begin{aligned}
(f(x)g(x))' &= \lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x)g(x)}{h}\\
&= \lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h}\\
&= \lim_{h\to 0}\left[\frac{f(x+h)g(x+h) - f(x+h)g(x)}{h} + \frac{f(x+h)g(x) - f(x)g(x)}{h}\right]\\
&= \lim_{h\to 0}\left[f(x+h)\left(\frac{g(x+h) - g(x)}{h}\right) + g(x)\left(\frac{f(x+h) - f(x)}{h}\right)\right]\\
&= \lim_{h\to 0} f(x+h)\lim_{h\to 0}\left(\frac{g(x+h) - g(x)}{h}\right) + \lim_{h\to 0} g(x)\lim_{h\to 0}\left(\frac{f(x+h) - f(x)}{h}\right).
\end{aligned}$$

Clearly $\lim_{h\to 0} g(x) = g(x)$ and $\lim_{h\to 0} f(x+h) = f(x)$. Also, by definition, $\lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = f'(x)$ and $\lim_{h\to 0} \dfrac{g(x+h)-g(x)}{h} = g'(x)$.

Hence $(f(x)g(x))' = f(x)g'(x) + g(x)f'(x)$. □

In words, the product rule is, *The derivative of a product of two functions is the first function times the derivative of the second function plus the second function times the derivative of the first function.*

**Example 2.18.**

1. Differentiate $f(x) = (2x^4 + 3x^2)(5x^3 - 4x)$.
2. If $f(x) = \sqrt{x}(cx + d)$, find $f'(4)$ in terms of $c$ and $d$.
3. If $f(x) = \sqrt{x}g(x)$ where $g(3) = 3$ and $g'(3) = 4$, find $f'(3)$.

**Solution**

1. By the product rule,

$$\begin{aligned} f'(x) &= (2x^4+3x^2)\frac{d}{dx}[5x^3-4x] + (5x^3-4x)\frac{d}{dx}[2x^4+3x^2] \\ &= (2x^4+3x^2)(15x^2-4) + (5x^3-4x)(8x^3+6x). \end{aligned}$$

2.

$$\begin{aligned} f'(x) &= \sqrt{x}\frac{d}{dx}[c+dx] + (c+dx)\frac{d}{dx}[\sqrt{x}] \\ &= d\sqrt{x} + (c+dx)\frac{1}{2}x^{-1/2} \\ &= d\sqrt{x} + \frac{c+dx}{2\sqrt{x}} \\ &= \frac{c+3dx}{2\sqrt{x}}. \end{aligned}$$

Hence $f'(4) = \dfrac{c+3d(4)}{2\sqrt{4}} = \dfrac{c+12d}{4}$.

3.

$$f'(x) = \frac{d}{dx}[\sqrt{x}g(x)] = \sqrt{x}g'(x) + g(x)\frac{1}{2}x^{-1/2} = \sqrt{x}g'(x) + \frac{g(x)}{2\sqrt{x}} = \frac{2xg'(x)+g(x)}{2\sqrt{x}}.$$

Hence $f'(3) = \dfrac{2(3)g'(3)+g(3)}{2\sqrt{3}} = \dfrac{2\times 3\times 4+3}{2\sqrt{3}} = \dfrac{27}{2\sqrt{3}}$.

### 2.2.6 The Quotient Rule

The quotient rule gives the derivative of the function which is the quotient. To prove this rule we need the result below which is called the **reciprocal rule**.

**Theorem 2.19** (Reciprocal Rule)**.** *If the function $g(x) \neq 0$ is differentiable at $x$, then* $\left(\frac{1}{g(x)}\right)' = \lim_{h\to 0} \frac{-g'(x)}{(g(x))^2}$.

*Proof.*

$$\begin{aligned}
\left(\frac{1}{g(x)}\right)' &= \lim_{h\to 0} \frac{\frac{1}{g(x+h)} - \frac{1}{g(x)}}{h} \\
&= \lim_{h\to 0} \frac{\frac{g(x)-g(x+h)}{g(x+h)g(x)}}{h} \\
&= \lim_{h\to 0} \frac{g(x)-g(x+h)}{g(x+h)g(x)h} \\
&= \left(\lim_{h\to 0} \frac{g(x)-g(x+h)}{h}\right)\left(\lim_{h\to 0} \frac{1}{g(x+h)}\right)\frac{1}{g(x)}.
\end{aligned}$$

Note that $\lim_{h\to 0} \frac{g(x)-g(x+h)}{h} = -g'(x)$ and $\lim_{h\to 0} \frac{1}{g(x+h)} = \frac{1}{g(x)}$.

Hence $\left(\frac{1}{g(x)}\right)' = \lim_{h\to 0} \frac{-g'(x)}{(g(x))^2}$. □

ow we are ready to state and prove the quotient rule. The proof makes use of both the reciprocal rule and the product rule.

**Theorem 2.20.** *If $f$ and $g$ are differentiable functions and $g(x) \neq 0$, then*

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f(x)'g(x) - f(x)g'(x)}{(g(x))^2}.$$

*Proof.* First note that

$$\left(\frac{f(x)}{g(x)}\right)' = \left(f(x).\frac{1}{g(x)}\right)'.$$

By the Product rule we have

$$\begin{aligned}\left(f(x).\frac{1}{g(x)}\right)' &= f'(x)\times\frac{1}{g(x)}+f(x)\times\left(\frac{1}{g(x)}\right)' \\ &= f'(x)\times\frac{1}{g(x)}+f(x)\times\frac{-g(x)}{(g(x))^2} \\ &= \frac{f'(x)g(x)-f(x)g'(x)}{(g(x))^2}.\end{aligned}$$

□

**Example 2.21.**

1. Find $f'(x)$ for $f(x)=\dfrac{3x-1}{2x+1}$.
2. Differentiate $y=\dfrac{2x}{4+x^2}$ with respect to $x$.
3. Find $y'$ given that $y=\dfrac{x^3}{1-x^2}$.

**Solution**

1.

$$\begin{aligned}f'(x) &= \frac{(2x+1)\frac{d}{dx}[3x-1]-(3x-1)\frac{d}{dx}[2x+1]}{(2x+1)^2} \\ &= \frac{3(2x+1)-2(3x-1)}{(2x+1)^2} \\ &= \frac{6x+3-6x+2}{(2x+1)^2} \\ &= \frac{5}{(2x+1)^2}.\end{aligned}$$

2.

$$\begin{aligned}y' &= \frac{(4+x^2)\frac{d}{dx}[2x]-(2x)\frac{d}{dx}[4+x^2]}{(4+x^2)^2} \\ &= \frac{2(4+x^2)-2x(2x)}{(4+x^2)^2} \\ &= \frac{8+2x^2-4x^2}{(4+x^2)^2} \\ &= \frac{8-2x^2}{(4+x^2)^2}.\end{aligned}$$

3.

$$\begin{aligned}
y' &= \frac{(1-x^2)\frac{d}{dx}[x^3] - x^3\frac{d}{dx}[1-x^2]}{(1-x^2)^2} \\
&= \frac{3x^2(1-x^2)+2x(x^3)}{(1-x^2)^2} \\
&= \frac{3x^2-3x^4+2x^4}{(1-x^2)^2} \\
&= \frac{3x^2-x^4}{(1-x^2)^2}.
\end{aligned}$$

### 2.2.7 The Chain Rule

Basically the Chain rule is applied to determine the derivative of a composite function.

**Theorem 2.22.** *If $y = f(u)$, $u = g(x)$, and the derivatives $\frac{dy}{du}$ and $\frac{du}{dx}$ both exist, then the derivative of the composite function defined by $y = f(g(x))$ is given by*

$$\begin{aligned}
\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} = f'(u)g'(x) \\
&= f'(g(x))g'(x).
\end{aligned}$$

The theorem below gives the chain rule combined with the power rule.

**Theorem 2.23.** *If $g$ is a differentiable function and $n$ is any real number, then*

$$\frac{d}{dx}[g(x)]^n = n[g(x)]^{n-1} \times \frac{d}{dx}[g(x)].$$

**Example 2.24.**

1. Find $f'(x)$ if $f(x) = 8(6x+21)^8$.
2. Find $\frac{dy}{dx}$ for $y = 2\sqrt{6x^2+4x+26}$.
3. Given that $y = \frac{3}{\sqrt{2x^2+2x+22}}$, find $y'$.

**Solution**

1.

$$f'(x) = 8 \times 8(6x+21)^7 \frac{d}{dx}[6x+21] = 8 \times 8 \times 6(6x+21)^7 = 384(6x+21)^7.$$

2.

$$y' = 2 \times \frac{1}{2}(6x^2+4x+26)^{-1/2} \frac{d}{dx}[6x^2+4x+26] = \frac{12x+4}{\sqrt{6x^2+4x+26}}.$$

3.

$$y' = 3 \times -\frac{1}{2}(2x^2+2x+22)^{-3/2} \frac{d}{dx}[2x^2+2x+22] = -\frac{3}{2}\frac{4x+2}{\sqrt{(2x^2+2x+22)^3}}.$$

**Exercise 2.25.**

1. Find $\dfrac{dy}{dx}$ for each function below.

   (a) $y = -12345$

   (b) $y = 14x^4 - 12x^2 + 16$

   (c) $y = (4x^2 - x)(x^3 - 8x^2 + 12)$

   (d) $y = (1 + 2x + 3x^2)(5x + 8x^2 - x^3)$

   (e) $y = \dfrac{6x^2}{2-x}$

   (f) $y = \dfrac{3x + x^4}{2x^2 + 1}$

   (g) $y = \dfrac{x}{x^3 + x^2 + x + 1}$

   (h) $y = \dfrac{1}{x^2 + 1}$

   (i) $y = (7x^2 + 3x + 24)^{22/3}$

   (j) $y = 7\sqrt{8x^3 - 21x^2 + 4x}$

   (k) $y = \dfrac{x - 3x\sqrt{x}}{\sqrt{x}}$

   (l) $y = \dfrac{3x + 2\sqrt{x}}{x}$

2. If $f(2) = -8$, $f'(2) = 3$, $g(2) = 17$ and $g'(2) = -4$, find the value of $(fg)'(2)$.

3. If $f(x) = x^2 g(x)$, $g(-7) = 2$, $g'(-7) = -9$, find the value of $f'(-7)$.

## 2.3 Implicit Differentiation

Given an equation containing the variables $x$ and $y$ for which you can not easily solve for $y$ in terms of $x$, you can find $\frac{dy}{dx}$ by doing the following;

1. Differentiate both sides of the equation with respect to $x$.
2. Move all terms containing $\frac{dy}{dx}$ to the left side of the equation and all other terms to the right side.
3. Factor out $\frac{dy}{dx}$ on the left side of the equation.
4. Solve for $\frac{dy}{dx}$.

**Example 2.26.**

1. Find $\frac{dy}{dx}$ given that $y^2 - 7y + x^2 - 4x = 10$.
2. Differentiate with respect to $x$, $x^2 + y^2 = 6y$.
3. Find the slope of the tangent line to the graph of $x^2 + y^2 + 19 = 2x + 12y$ at the point $(4, 3)$.

**Solution**

1.
$$2y\frac{dy}{dx} - 7\frac{dy}{dx} + 2x^2 - 4 \Rightarrow (2y-7)\frac{dy}{dx} = 4 - 2x^2 \Rightarrow \frac{dy}{dx} = \frac{4-2x^2}{2y-7}.$$

2.
$$2x + 2y\frac{dy}{dx} = 6\frac{dy}{dx} \Rightarrow (2y-6)\frac{dy}{dx} = -2x \Rightarrow \frac{dy}{dx} = -\frac{2x}{2y-6} = \frac{x}{3-y}.$$

3.
$$2x + 2y\frac{dy}{dx} = 2 + 12\frac{dy}{dx} \Rightarrow (2y-12)\frac{dy}{dx} = 2 - 2x \Rightarrow \frac{dy}{dx} = \frac{2-2x}{2y-12} = \frac{1-x}{y-6}.$$

The slope of the tangent line to the given curve at $(4, 3)$ is $\frac{1-4}{3-6} = 1$.

Hence the equation is $y - 3 = x - 4 \Rightarrow y - x + 1 = 0$.

**Exercise 2.27.**

1. Find $\dfrac{dy}{dx}$ for each function below.
   (a) $x^3 + y^3 = 3xy$
   (b) $x^2 + 2xy^2 = 3y + 4$
   (c) $1 - xy = x - y$
   (d) $9x^2 + 25y^2 = 225$
   (e) $y^2 = x^3$
   (f) $xy^2 - x^3y = 6$
   (g) $\sqrt{x+y} = 1 + x^3 + y^2$
   (h) $2y^3 + y^2 - 6y^5 = x^4 - 2x^3 + x^5$
2. Find the slope of the tangent line to the curve $y^3x + x^2y^2 = 6$ at the point $(2, 1)$.

## 2.4 Higher Order Derivatives

The idea of finding higher order derivatives is to differentiate derivatives of functions. If the derivative $f'(x)$ of a function $f(x)$ is differentiable then the derivative of $f'(x)$ is the **second derivative** of $f(x)$ represented by $f''(x)$ (read as $f$ double prime of $x$). Differentiating $f''(x)$ gives the **third derivative** and differentiating the third derivative gives the **fourth derivative**. We can continue differentiating $f(x)$ as long as there is differentiability.

Other notations for of the second derivative include:

$$f^{(2)}, \frac{d}{dx}\left(\frac{dy}{dx}\right), D_x^2(y) \text{ and } \frac{d^2y}{dx^2}.$$

Notations for the third derivatives include

$$f^{(3)}, \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right), D_x^3(y) \text{ and } \frac{d^3y}{dx^3}.$$

**Example 2.28.**

1. If $y = 5x^3 + 7x - 10$, find the first four derivatives.
2. Find the second derivative of the function $y = (3x - 5)^{10}$.

**Solution**

1. $\dfrac{dy}{dx} = 15x^2 + 7$, $\dfrac{d^2y}{dx^2} = 30x$, $\dfrac{d^3y}{dx^3} = 30$ and $\dfrac{d^4y}{dx^4} = 0$.
2. $y' = 10(3x - 5)^9 \times 3 = 30(3x - 5)^9$ and $y'' = 270 \times 3(3x - 5)^8 = 810(3x - 5)^8$.

**Exercise 2.29.**

1. Find $\dfrac{dy}{dx}$.
   (a) $(x + y)^2 - (x - y)^2 = x^5 + y^5$
   (b) $x^2 + y^3 = 10 - 5xy$
   (c) $x^4 + y^4 = 18$
   (d) $9x^2 + y^2 = 9$
   (e) $y^2 = x^3 + 3x^2$
   (f) $(xy)^2 + xy = 2$
   (g) $x^2 + 4y^2 = 36$
   (h) $xy = 5x^2$
2. Find $\dfrac{d^3y}{dx^3}$.
   (a) $y = \dfrac{5}{x^2}$
   (b) $y = 5x^4$
   (c) $y = 3x^5 - 2x$
   (d) $y = -x^2 + 2\sqrt[5]{x^2}$
   (e) $y = -2x^3 - 4x^{-3}$
   (f) $y = -5x^4$

## 2.5 Linear Approximations, Increments and Differentials

### 2.5.1 Linear Approximations

Given the function $y = f(x)$, we use the tangent line at $(a, f(a))$ as an approximation to the curve $y = f(x)$ when $x$ is near $a$. An equation of this tangent line

$$y = f(a) + f'(a)(x - a)$$

and the approximation

$$f(x) \approx f(a) + f'(a)(x - a)$$

is called the **linear approximation** or **tangent line approximation** of $f$ at $a$. The linear function whose graph is this tangent line, that is

$$L(x) = f(a) + f'(a)(x - a) \tag{2}$$

is called the **linearisation** of $f$ at $a$.

**Example 2.30.**

Find the linearization of the function $f(x) = \sqrt{x+3}$ at $a = 1$ and use it to approximate the numbers $\sqrt{3.98}$ and $\sqrt{4.05}$.

**Solution**

$f(x) = \sqrt{x+3} = (x+3)^{1/2} \Rightarrow f'(x) = \dfrac{1}{2}(x+3)^{-1/2} = \dfrac{1}{2\sqrt{x+3}}$ so we have $f(1) = 2$ and $f'(1) = \dfrac{1}{4}$. Hence the linearisation is

$$L(x) = f(1) + f'(1)(x - 1) = 2 + \frac{1}{4}(x - 1) = \frac{7}{4} + \frac{x}{4}.$$

The corresponding linear approximation is

$$\sqrt{x+3} \approx \frac{7}{4} + \frac{x}{4}$$

when $x$ is near 1. In particular we have

$$\sqrt{3.98} \approx \frac{7}{4} + \frac{0.98}{4} = 1.995 \text{ and } \sqrt{4.05} \approx \frac{7}{4} + \frac{1.05}{4} = 2.0125.$$

### 2.5.2 Increments

Consider the function $y = f(x)$. If $x$ changes from $x_1$ to $x_2$, then the amount of change

$$\Delta x = x_2 - x_1$$

is called an **increment** of $x$. Note that $x_2 = \Delta x + x_1$. The **increment** of $y$ is given by

$$\Delta y = f(x_2) - f(x_1) = f(x_1 + \Delta x) - f(x_1) \text{ (See Figure 3).}$$

**Example 2.31.**

Suppose $f(x) = 3x^2 - 5$.

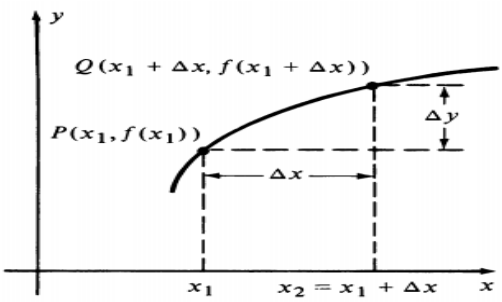


Figure 3: Increments

1. If $x$ is given an increment $\Delta x$, find $\Delta y$.
2. Use $\Delta y$ to calculate the numerical change in $y$ if $x$ changes from 2 to 2.1.

**Solution**

1.

$$\begin{aligned}\Delta y &= f(x+\Delta x) - f(x)\\ &= [3(x+(\Delta))^2 - 5] - [3x^2 - 5]\\ &= 3(x^2 + 2x(\Delta x) + (\Delta x)^2) - 5 - 3x^2 + 5\\ &= 3x^2 + 6x(\Delta x) + 3(\Delta x)^2 - 3x^2\\ &= 6x(\Delta x) + 3(\Delta x)^2.\end{aligned}$$

2. $\Delta x = 2.1 - 2 = 0.1.$

   Hence

$$\Delta y = 6(2)(0.1) + 3(0.1)^2 = 12(0.1) + 3(0.001) = 1.2 + 0.03 = 1.23.$$

### 2.5.3 Differentials

**Definition 2.32.** *Let $y = f(x)$ where $f$ is differentiable, and let $\Delta x$ be an increment of $x$.*

i. *The* **differential** *$dx$ of the independent variable $x$ is $dx = \Delta x$.*

ii. *The* **differential** *$dy$ of the dependent variable $y$ is $dy = f'(x)\Delta x = f'(x)dx$.*

**Example 2.33.**

If $y = 3x^2 - 5$, use $dy$ to approximate $\Delta y$ if $x$ changes from 2 to 2.1

**Solution**

If $y = 3x^2 - 5$, by Example 2.31 $\Delta y = 1.23$. Now

$$dy = f'(x)dx = 6xdx.$$

$$x = 2,\ \Delta x = dx = 0.1 \text{ and } dy = 6(2)(0.1) = 1.2.$$

Hence our approximation is correct to the nearest tenth.

**Exercise 2.34.**

1. Find a general formula for $\Delta y$ and calculate the change in $y$ corresponding to the stated values of $x$ and $\Delta x$.

   (a) $y = 2x^2 - 4x + 5$, $x = 2$ and $\Delta x = -0.2$

   (b) $y = 2x^3 - 4$, $x = -1$ and $\Delta x = 0.12$

   (c) $y = \dfrac{1}{x+2}$, $x = 0$ and $\Delta x = -0.03$

2. Find $\Delta y$, $dy$ and $dy - \Delta y$

   (a) $y = 3x^2 + 5x - 2$

   (b) $y = 4 - 7x = 2x^2$

   (c) $y = x^{-2}$

3. Differentiate, from first principles, each function with respect to $x$.

   (a) $f(x) = 16x^3 - 4x^2 + 3x - 11$

   (b) $f(x) = \dfrac{1-x}{2+x}$

   (c) $f(x) = \sqrt{4x}$ at $x = 4$

   (d) $f(x) = x^3 + 10x^2 + 12x - 22$.

4. Find $y'$.

   (a) $y = (x+1)^{100}$

   (b) $y = 2x - \dfrac{1}{x}$

   (c) $y = (x^2+3)(x^2+5x-7)$

   (d) $y = \dfrac{x^2+1}{x^2-x}$

(e) $y = \dfrac{x^2 + 3x - 4}{x + \sqrt{x}}$

(f) $y = \left(\dfrac{2x - 3}{7x + 1}\right)^5$

(g) $y = 3(x^2 + 2x)^{3/2}(4x - 3)^5$

5. Find $\dfrac{dy}{dx}$ for each function below.

(a) $x^2 + 2xy + 3y^2 = 12$

(b) $y^3 + xy - x^2 = 0$

(c) $2x^3 + 5xy^2 - 2y^4 = 10$

(d) $x^2y + 4xy^2 = 2y$

(e) $(3x - y)(2x + 3y) = 8$

(f) $y(x^3 + y^3) = (x + 1)(x + 4)$

(g) $\sqrt{xy} - x + y^2 = 0$

(h) $\dfrac{(x + 2y)^2}{4x - y} + y = 0$

6. Find $f^{(4)}(b)$.

(a) $f(x) = 4x^5 - 3x^3 - 2x + 3$ and $b = 1$

(b) $f(x) = x^8 + x^4 - x^3 + x^2$ and $b = -1$

(c) $f(x) = x^6 - x^4 + x^2 - 10$ and $b = 0$

# 3 Derivatives of Transcendental Functions

## 3.1 Trigonometric Functions

We look at derivatives of the six trigonometric functions. To begin with we consider derivative of $\sin x$ in the example below.

**Example 3.1.**

Find $\dfrac{d}{dx}[\sin x]$.

**Solution**

We use differentiation from first principles.

$$\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}\\
&= \lim_{h\to 0}\frac{\sin x\cos h+\cos x\sin h-\sin x}{h}\\
&= \lim_{h\to 0}\left[\sin x\left(\frac{\cos h-1}{h}\right)+\cos x\left(\frac{\sin h}{h}\right)\right]\\
&= \lim_{h\to 0}\sin x\times\lim_{h\to 0}\frac{\cos h-1}{h}+\lim_{h\to 0}\cos x\times\lim_{h\to 0}\frac{\sin h}{h}.
\end{aligned}$$

Clearly $\lim_{h\to 0}\sin x=\sin x$ and $\lim_{h\to 0}\cos x=\cos x$. Also you will recall that $\lim_{h\to 0}\frac{\sin h}{h}=1$. It can be easily shown that $\lim_{h\to 0}\frac{\cos h-1}{h}=0$. Therefore

$$\lim_{h\to 0}\sin x\lim_{h\to 0}\frac{\cos h-1}{h}+\lim_{h\to 0}\cos x\lim_{h\to 0}\frac{\sin h}{h}=\sin x\times(0)+\cos x\times(1)=\cos x.$$

Hence $\frac{d}{dx}[\sin x]=\cos x$.

Table 4 gives derivatives of all trigonometric functions. The angle $x$ is given in radians.

| $f(x)$ | $\sin x$ | $\cos x$ | $\tan x$ | $\cot x$ | $\csc x$ | $\sec x$ |
|---|---|---|---|---|---|---|
| $\frac{d}{dx}[f(x)]$ | $\cos x$ | $-\sin x$ | $\sec^2 x$ | $-\csc^2 x$ | $-\csc x\cot x$ | $\sec x\tan x$ |

Table 4: Derivatives of trigonometric functions

**Example 3.2.**

Find the derivative of each function with respect to $x$:

1. $y=x^2\cos x$
2. $y=\sin x^6$
3. $y=\tan\sqrt{x^3}$
4. $y=\sin e^x\cos x$

**Solution**

1. By the product rule,

$$\begin{aligned} y' &= 2x\cos x + x^2\frac{d}{dx}[\cos x] \\ &= 2x\cos x + x^2(-\sin x) \\ &= 2x\cos x - x^2\sin x. \end{aligned}$$

2. $y' = \cos x^6 \dfrac{d}{dx}[x^6] = 6x^5\cos x^6$

3. $y' = \sec^2\sqrt{x^3}\dfrac{d}{dx}[\sqrt{x^3}] = \sec^2\sqrt{x^3} \times \dfrac{3}{2} \times x^{\frac{1}{2}} = \dfrac{3}{2}\sqrt{x}\sec^2\sqrt{x^3}$

4. $y' = \sin e^x(-\sin x) + \cos x\cos e^x\dfrac{d}{dx}[e^x] = -\sin e^x\sin x + e^x\cos x\cos e^x$

### 3.1.1 Inverse Trigonometric Functions

Inverse trigonometric functions were covered in Year one. In Figure 4 we have graphs of inverse trigonometric functions. Here we are interested on their derivatives.

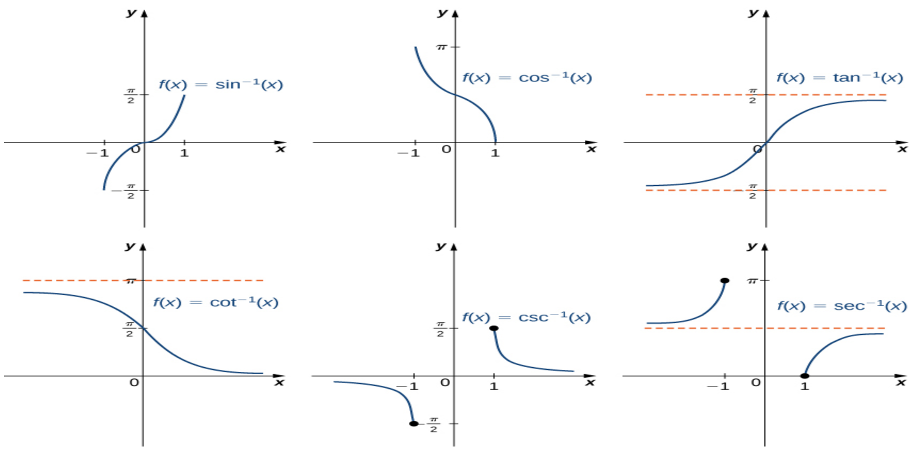


Figure 4: Graphs of Inverse Trigonometric Functions

**Example 3.3.**

Find the derivative of each function below with respect to $x$.

1. $f(x) = \sin^{-1} x$
2. $f(x) = \cos^{-1} x$
3. $f(x) = \tan^{-1} x$

**Solution**

1. Let $y = \sin^{-1} x$. Then $x = \sin y$. Differentiating both sides with respect to $x$ gives
$$1 = \cos y \frac{dy}{dx}.$$
Now from $\sin^2 y + \cos^2 y = 1$ we have $\cos y = \sqrt{1 - \sin^2 y}$.
$$\frac{dy}{dx} = \frac{1}{\sqrt{1 - \sin^2 y}} = \frac{1}{\sqrt{1 - x^2}}.$$
Hence $\dfrac{d}{dx}[\sin^{-1} x] = \dfrac{1}{\sqrt{1 - x^2}}$.

2. Let $y = \cos^{-1} x$. Then $x = \cos y$. Differentiating both sides with respect to $x$ we have
$$1 = -\sin y \frac{dy}{dx}.$$
Now
$$\frac{dy}{dx} = \frac{1}{-\sin y} = \frac{1}{-\sqrt{1 - \cos^2 x}} = -\frac{1}{\sqrt{1 - x^2}}.$$
Hence $\dfrac{d}{dx}[\cos^{-1} x] = -\dfrac{1}{\sqrt{1 - x^2}}$

3. Let $y = \tan^{-1} x$. Then $x = \tan y$. Differentiating both sides with respect to $x$ we have
$$1 = \sec^2 y \frac{dy}{dx}.$$
From $\sin^2 y + \cos^2 y = 1$ we have $\tan^2 y + 1 = \sec^2 y$. Now
$$\frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{\tan^2 y + 1} = \frac{1}{x^2 + 1}.$$

Hence $\dfrac{d}{dx}[\tan^{-1} x] = \dfrac{1}{x^2 + 1}$.

Derivatives of other inverse trigonometric functions are derived in a similar manner. We present them in Table 5.

**Example 3.4.**

Find the derivative of each function below with respect to $x$.

1. $y = e^{\tan^{-1} x}$

| $f(x)$ | $\frac{d}{dx}[f(x)]$ |
|---|---|
| $\sin^{-1} x$ | $\frac{1}{\sqrt{1-x^2}}$ |
| $\cos^{-1} x$ | $-\frac{1}{\sqrt{1-x^2}}$ |
| $\tan^{-1} x$ | $\frac{1}{1+x^2}$ |
| $\csc^{-1} x$ | $-\frac{1}{x\sqrt{x^2-1}}$ |
| $\sec^{-1} x$ | $-\frac{1}{x\sqrt{x^2-1}}$ |
| $\cot^{-1} x$ | $-\frac{1}{1+x^2}$ |

Table 5: Derivatives of Inverse Trigonometric Functions

2. $y = \tan^{-1} e^x$
3. $y = \ln \sin^{-1} x$

**Solution**

1. $y' = e^{\tan^{-1} x} \dfrac{d}{dx}[\tan^{-1} x] = e^{\tan^{-1} x} \dfrac{1}{1+x^2} = \dfrac{e^{\tan^{-1} x}}{1+x^2}.$
2. $y' = \dfrac{1}{1+(e^x)^2} \dfrac{d}{dx}[e^x] = \dfrac{e^x}{1+e^{2x}}.$
3. $y = \dfrac{1}{\sin^{-1} x} \dfrac{d}{dx}[\sin^{-1} x] = \dfrac{1}{(\sin^{-1} x)\sqrt{1-x^2}}$

**Exercise 3.5.**

1. If $y = \cos x + \sin x$, show that $y'' = -y$.
2. Use the Quotient Rule of differentiation to show that
   (a) $\dfrac{d}{dx}[\cot x] = -\csc^2 x$
   (b) $\dfrac{d}{dx}[\csc x] = -\csc x \cot x$
3. Differentiate each function with respect to $x$.
   (a) $f(x) = \cos^{-1} x^2$

(b) $f(x) = \sin^{-1}\sqrt{2x}$

(c) $f(x) = \cot^{-1}\sqrt{x-1}$

(d) $f(x) = x\sin^{-1}x + \sqrt{1-x^2}$

(e) $f(x) = \tan^2(3-x^2)$

(f) $f(x) = x^3 - x^2\sin x$

(g) $f(x) = 6 + 4\sqrt{x}\csc x$

(h) $f(x) = \sin^1(2x)$

(i) $f(x) = x\tan^{-1}\sqrt{x}$

(j) $f(x) = \tan^{-1}(\pi x)$

(k) $f(x) = \sec^{-1}x^5$

(l) $f(x) = \tan^{-1}(\ln x + \pi)$

(m) $f(x) = x\sin^{-1}(\ln x)$

(n) $f(x) = \dfrac{1}{\sin^{-1}x}$

(o) $f(x) = x\tan^{-1}\sqrt{x}$

4. Prove that

   (a) $\dfrac{d}{dx}[\sec^{-1}x] = -\dfrac{1}{x\sqrt{x^2-1}}$

   (b) $\dfrac{d}{dx}[\cot^{-1}x] = -\dfrac{1}{1+x^2}$

## 3.2 Exponential Functions

Exponential function is a function of the form

$$y = a^{f(x)}$$

where $a$ is a positive constant which is not equal to 0. Our interest is on exponential function base $e$, but before we go there let us look at the derivative of $f(x) = a^x$ for any base $a > 0$. Here is the derivative using differentiation from first principles.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{a^{x+h}-a^x}{h} \\ &= \lim_{h\to 0} \frac{a^x a^h - a^x}{h} \\ &= \lim_{h\to 0} \frac{a^x(a^h-1)}{h} \\ &= a^x \lim_{h\to 0} \frac{a^h-1}{h}. \end{aligned}$$

You will note that $\lim_{h\to 0} \frac{a^h-1}{h}$ is the derivative of $a^x$ at $x=0$, i.e., $\lim_{h\to 0} \frac{a^h-1}{h} = f'(0)$.

Hence for $f(x) = a^x$, we have

$$\frac{d}{dx}[a^x] = f'(0)a^x. \tag{3}$$

It follows that the function $f(x) = a^x$ is differentiable everywhere whenever it is differentiable at $x = 0$.

Using advanced calculus, it can be shown that $\lim_{h\to 0} \frac{a^h-1}{h} = \ln a$. Hence the derivative in Equation 3 is

$$\frac{d}{dx}[a^x] = a^x \ln a. \tag{4}$$

Alternative proof of 4, using derivatives of natural logarithm function, is provided in the section below.

Now since $a^x > 0$ for all real numbers, the simplest derivative formula in Equation 3 occurs when $f'(0) = 1$. The irrational number $e$ satisfies $f'(0) = 1$.

**Definition 3.6.** *$e$ is the number such that* $\lim_{h\to 0} \frac{e^h-1}{h} = 1$.

Using Equation 3 and Definition 3.6 we have

$$\frac{d}{dx}[e^x] = e^x. \tag{5}$$

Note that 5 can also be obtained from Equation 4 by noting that $\ln e = 1$. The following theorem follows from Equation 5 and the Chain rule.

**Theorem 3.7.** *Let $y = e^{f(x)}$ where $f(x)$ is a differentiable function. Then*

$$y' = f'(x)e^{f(x)}.$$

**Example 3.8.**

Find the derivative of each function with respect to $x$.

1. $y = 5^x$
2. $y = e^{-3x}$
3. $y = e^{\sqrt{x}}$
4. $y = e^x - x$
5. $y = e^{x^2} - x^3$

**Solution**

1. $y' = 5^x \ln 5$
2. $y' = (e^{-3x})\dfrac{d}{dx}(-3x) = -3e^{-3x}$.
3. $y' = e^{\sqrt{x}}\dfrac{d}{dx}[\sqrt{x}] = \dfrac{1}{2\sqrt{x}}e^{\sqrt{x}}$.
4. $y' = e^x - 1$
5. $y' = e^{x^2}\dfrac{d}{dx}[x^2] - 3x^2 = 2xe^{x^2} - 3x^2$

**Exercise 3.9.**

Differentiate each function with respect to $x$.

1. $y = 2^{(4x^2+3x-1)}$
2. $y = 7^{x^2-x}$
3. $y = 2e^x - 8^x$
4. $y = 5^x$
5. $y = 3^{-2x^3}$
6. $y = -6e^{3x}$
7. $y = -4x^3e^{2x^2}$

## 3.3 Logarithmic Functions

In this section we find the derivatives of the logarithmic functions $\log_a f(x)$ and, in particular, the natural logarithmic function $y = \ln f(x)$. First we look at the derivative of $y = \log_a x$.

**Proposition 3.10.** $\dfrac{d}{dx}[\log_a x] = \dfrac{1}{x \ln a}$.

*Proof.* Let $y = \log_a x$. Then $a^y = x$. Differentiating both sides implicitly with respect to $x$ we have

$$a^y \ln a \frac{dy}{dx} = 1$$

giving

$$\frac{dy}{dx} = \frac{1}{a^y \ln a} = \frac{1}{x \ln a}.$$

□

Proposition 3.10 and the chain rule give the following theorem.

**Theorem 3.11.** *If $f(x)$ is a differentiable function, then*

$$\frac{d}{dx}[\log_a f(x)] = f'(x) \times \frac{1}{f(x) \ln a}.$$

The following results follow immediately from Theorem 3.11:

$$\frac{d}{dx}[\ln x] = \frac{1}{x} \tag{6}$$

and

$$\frac{d}{dx}[\ln f(x)] = \frac{f'(x)}{f(x)}. \tag{7}$$

Now we provide proof for $\dfrac{d}{dx}[a^x] = a^x \ln a$ in Equation 4.

Let $y = a^x$. Then $\ln y = \ln a^x = x \ln a$. Differentiating both sides with respect to $x$ we have

$$\frac{1}{y}\frac{dy}{dx} = \ln a \Rightarrow \frac{dy}{dx} = y \ln a = a^x \ln a.$$

**Example 3.12.**

Find $y'$ for each function below.

1. $y = \log_4(x^2 + 7)$
2. $y = x \ln x$
3. $y = \log_2(\cos^2 x)$
4. $y = \log 4x^2$

**Solution**

1. $y' = \dfrac{1}{(x^2+7)\ln 4}\dfrac{d}{dx}[x^2+7] = \dfrac{2x}{(x^2+7)\ln 4}.$
2. $y' = \ln x + x\dfrac{d}{dx}[\ln x] = \ln x + x\left(\dfrac{1}{x}\right) = \ln x + 1.$
3. $$\begin{aligned} y' &= \frac{1}{\ln 2\cos^2 x}\frac{d}{dx}[\cos^2 x] = \frac{1}{\ln 2\cos^2 x}2\cos x\frac{d}{dx}[\cos x] \\ &= \frac{-2\cos x\sin x}{\ln 2\cos^2 x} = -\frac{\sin 2x}{\ln 2\cos^2 x}. \end{aligned}$$
4. $y' = \dfrac{1}{4x^2\ln 10}\dfrac{d}{dx}[4x^2] = \dfrac{8x}{4x^2\ln 10} = \dfrac{2}{x\ln 10}.$

## 3.4 Logarithmic Differentiation

This is a technique we apply to differentiate particularly complicated functions involving products, quotients or powers. Given a function $y = f(x)$ where $f(x)$ is a combination of products, quotients or powers, we follow these steps to find the derivative:

1. Take natural logarithm (ln) both sides: $\ln y = \ln(f(x))$.
2. Use the laws of logarithms to simplify the right hand side as much as possible.
3. Differentiate implicitly with respect to $x$.
4. Solve the resulting equation for $y'$.

**Example 3.13.**

Find $y'$.

1. $y = x^{\frac{1}{x}}$
2. $y = (\sin x)^x$
3. $y = \dfrac{(2x+1)^{\frac{1}{3}}(1+x^2)}{(2+x^4)^{\frac{2}{5}}}$

**Solution**

1. $y = x^{\frac{1}{x}} \Rightarrow \ln y = \ln x^{\frac{1}{x}} = \dfrac{1}{x}\ln x.$
   Differentiating both sides we have
   $$\frac{y'}{y} = -\frac{1}{x^2}\ln x + \frac{1}{x}.\frac{1}{x} = -\frac{\ln x}{x^2} + \frac{1}{x^2} \Rightarrow y' = y\left(\frac{1}{x^2} - \frac{\ln x}{x^2}\right) = x^{\frac{1}{x}}\left(\frac{1}{x^2} - \frac{\ln x}{x^2}\right).$$
2. $y = (\sin x)^x \Rightarrow \ln y = \ln(\sin x)^x = x \ln \sin x.$
   Differentiating both sides we have
   $$\frac{y'}{y} = \ln \sin x + x\frac{\cos x}{\sin x} = \ln \sin x + x \cot x \Rightarrow y' = y(\ln \sin x + x \cot x) = (\sin x)^x(\ln \sin x +$$
3. Taking ln both sides and simplifying the right hand side gives
   $$\ln y = \frac{1}{3}\ln(2x+1) + \ln(1+x^2) - \frac{2}{5}\ln(2+x^4).$$
   Differentiating both sides we have
   $$\frac{y'}{y} = \frac{2}{3(2x+1)} + \frac{2x}{1+x^2} - \frac{2}{5(2+x^4)}.$$
   Hence
   $$\begin{aligned} y' &= y\left(\frac{2}{3(2x+1)} + \frac{2x}{1+x^2} - \frac{2}{5(2+x^4)}\right) \\ &= \frac{(2x+1)^{\frac{1}{3}}(1+x^2)}{(2+x^4)^{\frac{2}{5}}}\left(\frac{2}{3(2x+1)} + \frac{2x}{1x^2} - \frac{2}{5(2+x^4)}\right). \end{aligned}$$

Be careful in distinguishing the power rule $\left(\dfrac{d}{dx}[x^n] = nx^{n-1}\right)$ and the rule for differentiating exponential functions $\left(\dfrac{d}{dx}[a^x] = a^x \ln a\right)$. In the first case the base is variable and the exponent is constant whereas in the second case the base is constant and the exponent is a variable. The following are four cases for exponents and bases.

1. Both base and exponent are constants: Here the derivative is zero. i.e.,
$$\frac{d}{dx}[a^b] = 0.$$
2. The base is a variable while the exponent is a constant: Here we use the power rule and, where necessary, the chain rule.
$$\frac{d}{dx}[f(x)]^b = b[f(x)]^{b-1}f'(x).$$
3. The base is a constant while the exponent is a variable: Here we use Equation 4 and the chain rule.
$$\frac{d}{dx}[a^{f(x)}] = a^{f(x)}\ln a f'(x).$$
4. Both base and exponents are variables: Here we use logarithmic differentiation as in examples above.

### 3.4.1 The Number $e$ as a limit

Let $f(x) = \ln x$. Then, from discussion above, $f'(x) = \frac{1}{x}$. It follows that $f'(1) = 1$.

$$\begin{aligned}
f'(1) &= \lim_{h\to 0}\frac{f(1+h)-f(1)}{h}\\
&= \lim_{h\to 0}\frac{f(1+x)-f(x)}{x}\\
&= \lim_{x\to 0}\frac{\ln(1+x)-\ln 1}{x}\\
&= \lim_{x\to 0}\frac{1}{x}\ln(1+x) = \lim_{x\to 0}\ln(1+x)^{\frac{1}{x}}.
\end{aligned}$$

Since $f'(1) = 1$, we have $\lim_{x\to 0}\ln(1+x)^{\frac{1}{x}} = 1$.

Now
$$e = e^1 = e^{\lim_{x\to 0}\ln(1+x)^{\frac{1}{x}}} = \lim_{x\to 0} e^{\ln(1+x)^{\frac{1}{x}}} = \lim_{x\to 0}(1+x)^{\frac{1}{x}}.$$

Hence
$$e = \lim_{x\to 0}(1+x)^{\frac{1}{x}}. \tag{8}$$

In equation 8 if we put $n = \frac{1}{x}$ we see that $n \to \infty$ as $x \to 0^+$. Hence another way of writing 8 is
$$e = \lim_{x\to 0}\left(1+\frac{1}{n}\right)^n. \tag{9}$$

**Exercise 3.14.**

1. If $x^y = y^x$ use implicit and logarithmic differentiation to find $\dfrac{dy}{dx}$.

2. Find the derivative of each function below with respect to $x$.

   (a) $y = \ln\left[x\ln(x^2 1)\right]$

   (b) $y = \dfrac{x^{\frac{3}{4}}\sqrt{x^2+1}}{(3x+2)^5}$

   (c) $y = x^{\sqrt{x}}$

   (d) $y = \ln(x^4 \sin^2 x)$

   (e) $y = \sqrt{x}e^{x^2}(x^2+1)^{10}$

   (f) $y = \ln(x^4 + 5x)$

   (g) $y = \log_\pi \dfrac{x^2+1}{x+1}$

   (h) $y = xe^{x^2} + \dfrac{x}{\ln x}$

   (i) $y = \dfrac{x}{e^{x^2}} + \dfrac{\ln x}{x}$

## 3.5 Hyperbolic Functions

The hyperbolic functions are analogs of the circular function or the trigonometric functions. Generally, the hyperbolic function takes place in the real argument called the hyperbolic angle. The basic hyperbolic functions are hyperbolic sine (sinh), hyperbolic cosine (cosh) and hyperbolic tangent (tanh). From these three basic functions, the other functions such as hyperbolic cosecant (csch), hyperbolic secant (sech) and hyperbolic cotangent (coth) functions are derived.

### 3.5.1 Definitions and Identities

**Definition 3.15.** *The* **hyperbolic cosine function**, *denoted* $\cosh x$, *is defined, for all real values of* $x$, *by the relation*

$$\cosh x = \frac{1}{2}\left(e^x + e^{-x}\right). \tag{10}$$

**Definition 3.16.** *The* **hyperbolic sine function**, *denoted* $\sinh x$, *is defined, for all real values of* $x$, *by the relation*

$$\sinh x = \frac{1}{2}\left(e^x - e^{-x}\right). \tag{11}$$

| Function | Domain | Range |
|---|---|---|
| $\sinh x$ | $\mathbb{R}$ | $\mathbb{R}$ |
| $\cosh x$ | $\mathbb{R}$ | $(1, \infty)$ |
| $\tanh x$ | $\mathbb{R}$ | $(-1, 1)$ |
| $\coth x$ | $(-\infty, 0) \cup (0, \infty)$ | $(-\infty, -1) \cup (1, \infty)$ |
| $\operatorname{sech} x$ | $\mathbb{R}$ | $(0, 1)$ |
| $\operatorname{csch} x$ | $(-\infty, 0) \cup (0, \infty)$ | $(-\infty, 0) \cup (0, \infty)$ |

Table 6: Domain and Range of a Hyperbolic Function

The two definitions 3.15 and 3.16 define the very basic hyperbolic functions. The rest of the functions are defined in terms of these two as follows;

$$\tanh x = \frac{\sinh x}{\cosh x}, \ \coth x = \frac{1}{\tanh x} = \frac{\cosh x}{\sinh x}, \ \operatorname{sech} = \frac{1}{\cosh x} \text{ and } \operatorname{csch} = \frac{1}{\sinh x}.$$

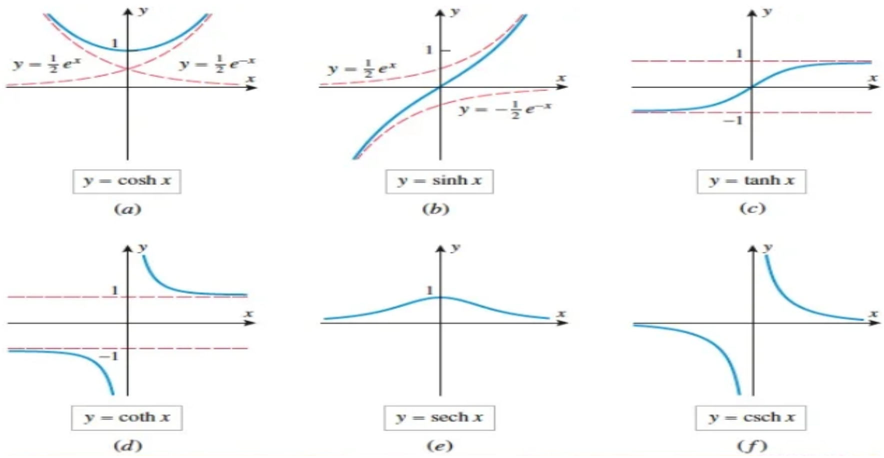


Figure 5: Graphs of Hyperbolic Functions

Graphs of the six hyperbolic functions are shown in Figure 5. Table 6 gives domain and range for each function.

The hyperbolic functions satisfy a number of identities that are similar to well-known trigonometric identities. For example if we substitute $-x$ for $x$ in $\sinh x$ in Equation 11 we obtain

$$\sinh(-x) = \frac{1}{2}\left(e^{-x} - e^{x}\right) = -\frac{1}{2}\left(e^{x} - e^{-x}\right) = -\sinh x.$$

Similarly $\cosh(-x) = \cosh x$. It follows that the functions $\cosh x$ and $\sinh x$ are even and odd respectively (just like $\cos x$ and $\sin x$ functions).

**Example 3.17.**

Prove that $\cosh^2 x - \sinh^2 x = 1$.

**Solution**

$$\begin{aligned} L.H.S &= \left[\frac{1}{2}\left(e^x + e^{-x}\right)\right]^2 - \left[\frac{1}{2}\left(e^x - e^{-x}\right)\right]^2 \\ &= \frac{1}{4}\left(e^{2x} + 2 + e^{-2x}\right) - \frac{1}{4}\left(e^{2x} - 2 + e^{-2x}\right) \\ &= \frac{1}{4}\left(e^{2x} + 2 + e^{-2x} - e^{2x} + 2 - e^{-2x}\right) \\ &= \frac{4}{4} = 1. \end{aligned}$$

It is easy to show that $\cosh x + \sinh x = e^x$ and that $\cosh x - \sinh x = e^{-x}$. Now

$$(\cosh x + \sinh x)(\cosh x - \sinh x) = \cosh^2 x - \sinh^2 x = e^x \times e^{-x} = 1.$$

This gives another way of verifying the identity in Example 3.17.

If we divide by $\cosh^2 x$ both sides of the identity in Example 3.17, we obtain

$$1 - \frac{\sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x} \Rightarrow 1 - \tanh^2 x = \operatorname{sech}^2 x.$$

Similarly, dividing both sides by $\sinh^2 x$ gives the identity

$$\coth^2 - 1 = \operatorname{csch}^2 x.$$

**Example 3.18.**

Prove that $\cosh 2x = 2\cosh^2 x - 1$.

**Solution**

$$\begin{aligned} 2\cosh^2 x - 1 &= 2\left(\frac{1}{2}\left(e^x + e^{-x}\right)\right)^2 - 1 \\ &= 2\left(\frac{1}{4}\left(e^{2x} + 2 + e^{-2x}\right)\right) - 1 \\ &= \frac{1}{2}\left(e^{2x} + e^{-2x}\right) + 1 - 1 \\ &= \frac{1}{2}\left(e^{2x} + e^{-2x}\right) \\ &= \cosh 2x. \end{aligned}$$

**Example 3.19.**

Prove that $\cosh 2x = 1 + 2\sinh^2 x$.

**Solution**

$$\begin{aligned}
1 + 2\sinh^2 x &= 1 + 2\left(\frac{1}{2}\left(e^x - e^{-x}\right)\right)^2 \\
&= 1 + 2\left(\frac{1}{4}\left(e^{2x} - 2 + e^{-2x}\right)\right) \\
&= 1 + \frac{1}{2}\left(e^{2x} + e^{-2x}\right) - 1 \\
&= \frac{1}{2}\left(e^{2x} + e^{-2x}\right) \\
&= \cosh 2x.
\end{aligned}$$

**Example 3.20.**

Prove that $\cosh(x - y) = \cosh x \cosh y - \sinh x \sinh y$.

**Solution**

$$\begin{aligned}
R.H.S &= \cosh x \cosh y - \sinh x \sinh y \\
&= \frac{1}{2}(e^x + e^{-x})\frac{1}{2}(e^y + e^{-y}) - \frac{1}{2}(e^x - e^{-x})\frac{1}{2}(e^y - e^{-y}) \\
&= \frac{1}{4}\left(e^{x+y} + e^{x-y} + e^{-(x-y)} + e^{-(x+y)}\right) - \frac{1}{4}\left(e^{x+y} - e^{x-y} - e^{-(x-y)} + e^{-(x+y)}\right) \\
&= \frac{1}{4}\left(e^{x+y} + e^{x-y} + e^{-(x-y)} + e^{-(x+y)} - e^{x+y} + e^{x-y} + e^{-(x-y)} - e^{-(x+y)}\right) \\
&= \frac{1}{4}\left(2e^{x-y} + 2e^{-(x-y)}\right) \\
&= \frac{1}{2}\left(e^{x-y} + e^{-(x-y)}\right) \\
&= \cosh(x - y) \\
&= L.H.S.
\end{aligned}$$

**Exercise 3.21.**

1. If $\tanh x = \dfrac{12}{13}$, find the values of the other hyperbolic functions at $x$.
2. If $\cosh x = \dfrac{5}{3}$ and $x > 0$, find the values of the other hyperbolic functions at $x$.

| $f(x)$ | $\frac{d}{dx}[f(x)]$ |
|---|---|
| $\sinh x$ | $\cosh x$ |
| $\cosh x$ | $\sinh x$ |
| $\tanh x$ | $\operatorname{sech}^2 x$ |
| $\operatorname{csch} x$ | $-\operatorname{csch} x \coth x$ |
| $\operatorname{sech} x$ | $-\operatorname{sech} x \tanh x$ |
| $\coth x$ | $-\operatorname{csch}^2 x$ |

Table 7: Derivatives of Hyperbolic Function

3. Prove that

   (a) $\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$

   (b) $\sinh(x-y) = \sinh x \cosh y - \cosh x \sinh y$

   (c) $\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$

   (d) $\sinh x + \sinh y = 2\sinh\left(\frac{x+y}{2}\right)\cosh\left(\frac{x-y}{2}\right)$

   (e) $\cosh x - \cosh y = 2\sinh\left(\frac{x+y}{2}\right)\sinh\left(\frac{x-y}{2}\right)$

   (f) $\sinh 2x = 2\sinh x \cosh x$

   (g) $\cosh 2x = \cosh^2 x + \sinh^2 x$

   (h) $\cosh^2 \frac{x}{2} = \frac{1+\cosh x}{2}$

   (i) $\sinh^2 \frac{x}{2} = \frac{\cosh x - 1}{2}$

   (j) $\tanh(\ln x) = \frac{x^2-1}{x^2+1}$

   (k) $\tanh(x+y) = \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y}$

### 3.5.2 Derivatives of Hyperbolic Functions

Differentiation of hyperbolic functions is similar to that of trigonometric functions. For example

$$\frac{d}{dx}[\sinh x] = \frac{d}{dx}\left[\frac{e^x - e^{-x}}{2}\right] = \frac{e^x + e^{-x}}{2} = \cosh x.$$

Table 7 shows derivatives of hyperbolic functions.

**Example 3.22.**

Find $f'(x)$ for each function below .

1. $f(x) = \sinh 3x$
2. $f(x) = 3x \cosh 2x$
3. $f(x) = 4x \sinh(3x^2 - 1)$
4. $f(x) = 3\cosh^2 2x - 13 \sinh 3x^2$
5. $f(x) = 5x^2 \cosh^2 4x$

**Solution**

1. $f'(x) = 3\cosh 3x$
2. $f'(x) = 3\cosh 2x + 3x(2\sinh 2x) = 3\cosh 2x + 6x \sinh 2x$
3. $$\begin{aligned} f'(x) &= 4\sinh(3x^2 - 1) + 4x(6x)\cosh(3x^2 - 1) \\ &= 4\sinh(3x^2 - 1) + 24x^2 \cosh(3x^2 - 1). \end{aligned}$$
4. $f'(x) = 3(2\cosh 2x)(2\sinh 2x) - 13(6x \cosh 3x^2) = 12\cosh 2x \sinh 2x - 65x \cosh 6x^2$
5. $f'(x) = 10x\cosh^2 4x + 5x^2(2\cosh 4x)(4\sinh 4x) = 10x \cosh^2 4x + 40\cosh 4x \sinh 4x$

### 3.5.3 Inverse Hyperbolic Functions and Derivatives

Recall that only one to one functions have inverses. Now you will note that both sinh and tanh functions are one to one implying that they both have inverses. cosh is not one to one. It follows that cosh has got no inverse. However, restricting the domain of cosh to $[0, \infty)$ gives a one to one and hence an invertible function. The inverse hyperbolic cosine function is defined as the inverse of this restricted function.

Using definition of an inverse function, we see that

$$\sinh^{-1} x = y \text{ whenever } \sinh y = x,$$

$$\cosh^{-1} x = y \text{ whenever } \cosh y = x \ (y \geq 0) \text{ and}$$

$$\tanh^{x-1} x = y \text{ whenever } \tanh y = x.$$

**Example 3.23.**

Find $y = \sinh^{-1} x$.

**Solution**

$y = \sinh^{-1} x \Rightarrow x = \sinh y = \dfrac{e^y - e^{-y}}{2}$. This implies that

$$e^y - 2x - e^{-y} = 0.$$

Multiplying throughout by $e^y$ gives

$$(e^y)^2 - 2xe^y - 1 = 0.$$

This is a quadratic equation in $e^y$ with $a = 1$, $b = -2x$ and $c = -1$. Using the quadratic formula we have

$$\begin{aligned} e^y &= \frac{-(-2x) \pm \sqrt{(-2x)^2 - 4(1(-1))}}{2} \\ &= x \pm \frac{\sqrt{4x^2+4}}{2} \\ &= x \pm \frac{\sqrt{4} \times \sqrt{x^2+1}}{2} \\ &= x \pm \sqrt{x^2+1}. \end{aligned}$$

Note that $e^y > 0$ but $x - \sqrt{x^2+1} < 0$ since $x < \sqrt{x^2+1}$. Therefore

$$e^y = x + \sqrt{x^2+1}.$$

It follows that

$$y = \ln\left(x + \sqrt{x^2+1}\right).$$

**Example 3.24.**

Show that $\cosh^{-1} x = \ln\left(x + \sqrt{x^2-1}\right)$.

**Solution**

Let $y = \cosh^{-1} x$. Then $x = \cosh y = \dfrac{e^y + e^{-y}}{2}$. Multiplying by $e^y$ and rearranging we have

$$(e^y)^2 - 2xe^y + 1 = 0.$$

So

$$e^y = \frac{-(-2x) \pm \sqrt{(-2x)^2 - 4(1)(1)}}{2} = x \pm \sqrt{x^2-1}.$$

Again we ignore the negative part and have

$$e^y = x + \sqrt{x^2 - 1}.$$

Hence

$$y = \ln\left(x + \sqrt{x^2 - 1}\right).$$

**Example 3.25.**

Show that $\tanh^{-1} x = \dfrac{1}{2} \ln \dfrac{x+1}{1-x}$.

**Solution**

Let $\tanh^{-1} x = y$. Then $x = \tanh y = \dfrac{e^y - e^{-y}}{e^y + e^{-y}}$. This gives

$$xe^y + xe^{-y} = e^y - e^{-y}.$$

Multiplying by $e^y$ and rearranging we have

$$x\left(e^y\right)^2 + x - \left(e^y\right)^2 + 1 = 0 \Rightarrow (x-1)\left(e^y\right)^2 + (x+1) = 0$$

$$\left(e^y\right)^2 = \frac{-(x+1)}{x-1} = \frac{x+1}{1-x} \Rightarrow e^y = \sqrt{\frac{x+1}{1-x}}.$$

Hence $y = \ln\sqrt{\dfrac{x+1}{1-x}} = \dfrac{1}{2}\ln\dfrac{x+1}{1-x}$.

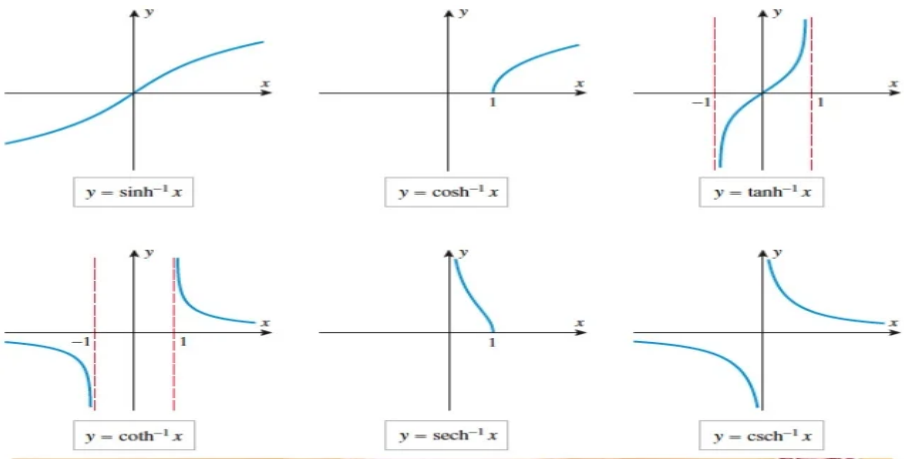


Figure 6: Graphs of Inverse Hyperbolic Functions

Inverses of the remaining three hyperbolic functions are derived in a similar manner. Graphs of inverse hyperbolic functions are shown in Figure 6.

The inverse hyperbolic functions are all differentiable because the hyperbolic functions are differentiable.

| $f(x)$ | $\frac{d}{dx}[f(x)]$ |
|---|---|
| $\sinh^{-1} x$ | $\frac{1}{\sqrt{x^2+1}}$ |
| $\cosh^{-1} x$ | $\frac{1}{\sqrt{x^2-1}}$ |
| $\tanh^{-1} x$ | $\frac{1}{1-x^2}$ |
| $\operatorname{csch}^{-1} x$ | $-\frac{1}{\lvert x\rvert\sqrt{x^2+1}}$ |
| $\operatorname{sech}^{-1} x$ | $-\frac{1}{x\sqrt{1-x^2}}$ |
| $\coth^{-1} x$ | $\frac{1}{1-x^2}$ |

Table 8: Derivatives of Inverse Hyperbolic Function

**Example 3.26.**

Show that $\frac{d}{dx}[\sinh^{-1} x] = \frac{1}{\sqrt{1+x^2}}$.

**Solution**

Let $y = \sinh^{-1} x$. Then $\sinh y = x$. Differentiating this implicitly with respect to $x$ , gives

$$\cosh y \frac{dy}{dx} = 1.$$

Since $\cosh^2 y - \sinh^2 y = 1$ and $\cosh y \geq 0$, we have $\cosh y = \sqrt{1+\sinh^2 y}$. Hence

$$\frac{dy}{dx} = \frac{1}{\cosh y} = \frac{1}{\sqrt{1+\sinh^2 y}} = \frac{1}{\sqrt{1+x^2}}.$$

Another way to show that $\frac{d}{dx}[\sinh^{-1} x] = \frac{1}{\sqrt{x^2+1}}$ is to use

$$\frac{d}{dx}[\sinh^{-1} x] = \frac{d}{dx}\left[\ln(x+\sqrt{x^2+1})\right]$$

and compute the derivative on the right hand side. Derivatives of inverse hyperbolic functions are given in Table 8.

**Example 3.27.**

Find the derivative of each function with respect to $x$.

1. $y = 5x^2 \cosh^{-1}(x^3)$

2. $y = \tanh^{-1}(\sin x)$.

3. $y = 6x^{-4} - \cosh^{-1}\left(4x^7\right)$

4. $y = -8\coth^{-1}(21x^3)$

**Solution**

1.
$$\begin{aligned} y' &= 10x(\cosh^{-1} x^2) + 5x^2 \times \frac{1}{\sqrt{(x^3)^2 - 1}} \times \frac{d}{dx}[x^3] \\ &= 10x\cosh^{-1} x^3 + \frac{5x^2}{\sqrt{x^6 - 1}} \times 3x^2 \\ &= 10x\cosh^{-1} x^3 + \frac{15x^4}{\sqrt{x^6 - 1}}. \end{aligned}$$

2.
$$\begin{aligned} y' &= \frac{1}{1 - \sin^2 x}\frac{d}{dx}[\sin x] \\ &= \frac{1}{1 - \sin^2 x} \times \cos x = \frac{\cos x}{\cos^2 x} = \frac{1}{\cos x} = \sec x. \end{aligned}$$

3. $y' = -24x^{-5} - \dfrac{1}{\sqrt{(4x^7)^2 - 1}}\dfrac{d}{dx}[4x^7] = -\dfrac{24}{x^5} - \dfrac{28x^6}{\sqrt{16x^{14} - 1}}$.

4. $y' = -\dfrac{8}{1 - (21x^3)^2}\dfrac{d}{dx}[21x^3] = -\dfrac{8 \times 63x^2}{1 - 441x^6} = -\dfrac{504x^2}{1 - 441x^6}$.

**Exercise 3.28.**

1. Find the numerical value of each function: $\sinh 0$, $\tanh 0$, $\sinh(\ln 2)$, $\sinh 2$, $\cosh(\ln 3)$, $\sinh^{-1} 1$, $\cosh 3$ and $\sinh 2$.

2. Use the definitions of the hyperbolic functions to find each of the following limits

   (a) $\lim\limits_{x \to \infty} \tanh x$.

   (b) $\lim\limits_{x \to \infty} \sinh x$.

   (c) $\lim\limits_{x \to 0^+} \coth x$.

(d) $\lim_{x \to 0^-} \coth x$.

(e) $\lim_{x \to -\infty} \operatorname{csch} x$.

3. Find the derivative of each function with respect to $x$.

(a) $y = \operatorname{sech}^2 e^x$

(b) $y = \ln \cosh x$

(c) $y = \cosh(\sinh^{-1} x)$

(d) $y = \sinh^{-1}(\cos^2 x)$

(e) $y = \ln \sinh x - \cosh^{-1}\left(e^{x^2}\right)$

(f) $y = \sin\left[\cosh x + \tanh^{-1}\left(\frac{x^2}{2}\right)\right]$

(g) $y = \sinh^{-1}(x^3 + e^x)$

(h) $y = x \coth(1 + x^2)$

(i) $y = x \sinh x - \cosh x$

(j) $y = x^2 \cosh^{-1}\left(3x^5\right)$

(k) $y = \dfrac{\sinh x}{\cosh^{-1} x}$

(l) $y = \tanh^{-1}\left(e^{3x} - \sqrt{x+4}\right)$

(m) $y = \tanh^{-1}\sqrt{x}$

(n) $y = \arctan(\tanh x)$

(o) $y = \sqrt[4]{\dfrac{1 + \tanh x}{1 - \tanh x}}$

4. If $x = \ln(\sec\theta + \tan\theta)$, show that $\sec\theta = \cosh x$.

5. Find $\dfrac{dy}{dx}$ for each function below.

(a) $y = x \sin x$

(b) $y = \cos x - 2\tan x$

(c) $y = x^3 \cos x$

(d) $y = \dfrac{\sin x}{1 + \cos x}$

(e) $y = \cos x + 6\cos^{-1} x$

(f) $y = \tanh x + x^2 \operatorname{csch} x$

(g) $y = \sin(\cos 2x)$

(h) $y = \dfrac{\tan x}{1 + \tan x}$

(i) $y = \sin^{-1} \dfrac{1}{x}$

(j) $y = xe^x - x^2e^x$

(k) $y = 3^{\sin x}$

(l) $y = (\ln x)^5$

(m) $y = 2x \ln x + x$

(n) $y = \log_5(2x + 1)$

(o) $y = \dfrac{(3x + 1)^4}{(4x - 3)^6}$

(p) $y = \dfrac{x^{\frac{2}{3}}(4x^2 - 5x)^{\frac{5}{4}}}{(x - 5)^{\frac{7}{8}}}$

(q) $y = \dfrac{\sin^2 x \tan x^4}{(x^2 + 2)^3}$

(r) $y = x^x$

(s) $y = x^{\sin x}$

(t) $y = x \tan^{-1} x + \ln \sqrt{1 - x^2}$

(u) $y = \coth^{-1} \sqrt{x^2 + 1}$

(v) $y = e^{\cosh 3x}$

(w) $y = x \coth(4 + 2x^2)$

(x) $y = (\sin x)^{\cosh x}$

(y) $y = \sqrt[4]{\dfrac{1 + \tanh x}{1 - \tanh x}}$

(z) $y = x \sin^{-1} \dfrac{x}{3} - \sqrt{9 + x^2}$

6. Prove that

(a) $\sin^{-1}(\tanh x) = \tan^{-1}(\sinh x)$

(b) $\dfrac{1 + \tanh x}{1 - \tanh x} = e^{2x}$

# 4 Applications of Derivatives

## 4.1 Rates of Change and Related Rates of Change

### 4.1.1 Rates of Change

Derivatives have applications in areas such as physics, chemistry, biology and economics. Here we look at how derivatives are applied in these and many areas as rates of change.

**Definition 4.1.** *The* **average rate of change** *of $y = f(x)$, with respect to $x$ in the interval $[x_1, x_2]$, is the quotient*

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

**Definition 4.2.** *The* **rate of change** *of $y = f(x)$, with respect to $x$ is the derivative*

$$\frac{dy}{dx} = f'(x).$$

**Example 4.3.**

The cost (in MK) of producing $x$ units of a certain commodity is $C(x) = 50 + \sqrt{x}$.

1. Find the average rate of change of $C$ with respect to $x$ when the production level is changed from $x = 100$ to $x = 169$.
2. Find the instantaneous rate of change of $C$ with respect to $x$ when $x = 100$.

**Solution**

1. Average rate of change is

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{(50 + \sqrt{169}) - (50\sqrt{100})}{169 - 100} = \frac{13 - 10}{69} = \frac{3}{69} = 0.04347.$$

2. The rate of change is the derivative:

$$C'(x) = \frac{1}{2\sqrt{x}}.$$

At $x = 100$ we have

$$C'(100) = \frac{1}{2\sqrt{100}} = \frac{1}{2 \times 10} = \frac{1}{20} = 0.05.$$

**Example 4.4.**

A balloon has a small hole and its volume $V$ (in $cm^3$) at time $t$ (sec) is given by $V = 66–10t–0.01t^2$ where $t > 0$. Find the rate of change of volume after 10 seconds.

**Solution**

$V = 66–10t–0.01t^2 \Rightarrow V'(t) = -10 - 0.02t.$

When $t = 10$, $V'(10) = -10 - 0.02(10) = -10.2cm^3/sec$

### 4.1.2 Related Rates of Change

Here we look at situations in which two or more rates are related. We say that we have related rates of change and associated problems are solved by following steps below.

1. Read the problem and where necessary draw a diagram.
2. Represent the given information and the unknowns by mathematical symbols.
3. Write an equation involving the rate of change to be determined. If the equation involves more than one variable, it may be necessary to reduce the equation to one variable.
4. Differentiate each term of the equation with respect to time.
5. Substitute all values and all known rates of change into the resulting equation.
6. Solve the resulting equation for the desired rate of change.
7. Write the answer and indicate the units of measure.

**Example 4.5.**

When the area of a square is increasing twice as fast as its diagonals, what is the length of a side of a square?

**Solution**

Let $z$ represent the diagonal of the square. The area of a square is $A = \frac{z^2}{2}$.

$$\text{Since } \frac{dA}{dt} = 2\frac{dz}{dt},\ 2\frac{dz}{dt} = z\frac{dz}{dt} \text{ giving } z = 2\ .$$

Let $s$ be a side of the square. Since the diagonal is $z = 2$, then

$$s^2 + s^2 = z^2 \Rightarrow 2s^2 = 4 \Rightarrow s^2 = 2 \Rightarrow s = \sqrt{2}.$$

**Example 4.6.**

Find the surface area of a sphere at the instant when the rate of increase of the volume of the sphere is nine times the rate of increase of the radius.

**Solution**

Volume of a sphere is $V = \frac{4}{3}\pi r^3$ and surface area is $S = 4\pi r^2$.

Differentiating volume gives $\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$.

Since

$$\frac{dV}{dt} = 9\frac{dr}{dt}$$

we have

$$9\frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt} \text{ or } 9 = 4\pi r^2.$$

Since $S = 4\pi r^2$, the surface area is $S = 9$ square units.

**Example 4.7.**

A point $P(x, y)$ moves on the graph of the equation $y = x^3 + x^2 + 1$, the $x$ coordinate changing at the rate of 2 units per second. How fast is the $y$ coordinate changing at the point $(1, 3)$.

**Solution**

$$\frac{dy}{dt} = (3x^2 + 2x)\frac{dx}{dt}.$$

Now $\frac{dx}{dt} = 2$, and at $(1, 3)$ $x = 1$.

So

$$\frac{dy}{dt} = (3(1)^2 + 2(1))2 = 10.$$

Hence $y$ changes by 10 units/sec.

**Exercise 4.8.**

1. The pressure $P$, of a given mass of gas kept at constant temperature, and its volume $V$ are connected by the equation $PV = 500$. Find $\dfrac{dP}{dV}$ when $V = 20$.

2. Water is running out of a conical funnel at the rate of $5cm^3/s$. The radius of the funnel is 10 cm and the height is 20 cm. How fast is the water level dropping when the water is 10 cm deep? $\left[V = \dfrac{1}{3}\pi r^2 h\right]$

3. The radius of a spherical balloon is increasing at a rate of 3 cm/min. At what rate is the volume increasing when the radius is 5 cm?

4. The function $n(t) = 200t - 100\sqrt{t}$ describes the spread of a virus where $t$ is the number of days since the initial infection and $n$ is the number of people infected. Find the rate at which $n$ is increasing at the instant when $t = 4$.

5. The profit P made from selling a certain item is related to the number sold $x$ by the formula $P(x) = 10000 - x^2 + 520x$. What is the rate of change of profit with respect to number sold when $x = 260$?

6. The area of a circle is increasing at the rate of 3 cm$^2/s$. Find the rate of change of the circumference when the radius is 2 cm.

7. A conical water tank with vertex down has a radius of 10 cm at the top and is 24 cm high. If water flows out of the tank at the rate of 20 cm$^3/min$, how fast is the depth of the water decreasing when the water is 16 cm deep?

8. Two sides of a triangle have lengths of 12 m and 15 m. The angle between them is increasing at the rate of $2°/min$. How fast is the area of a triangle increasing when the angle between the sides of fixed length is $60°$?

9. The radius of a sphere is increasing at a constant rate of 2 cm/min. In terms of the surface area, what is the rate of change of the volume of the sphere?

10. A spherical balloon is being inflated. Find the volume of the balloon at the instant when the rate of increase the surface area is eight times that rate of increase of the radius of the sphere.

11. Two cars leave an intersection at the same time. The first car is going due east at the rate of 40 km/h and the second is going due south at the rate of 30 km/h. How fast is the distance between the cars increasing when the first car is 120 km from the intersection?

## 4.2 Maximum and Minimum Values

The maximum and minimum values of a function can be identified by looking at its graph. However this is not possible for a function with a finite domain. In this section we use derivatives to find the maximum and minimum values of a given function.

**Definition 4.9.** *Let $c$ be a number in the domain $D$ of a function $f$. Then $f(c)$ is the*

i. **absolute maximum** *value of $f$ on $D$ if $f(c) \geq f(x)$ for all $x \in D$.*

ii. **absolute minimum** *value of $f$ on $D$ if $f(c) \leq f(x)$ for all $x \in D$.*

An absolute maximum or minimum is sometimes called a **global maximum** or **minimum**. The maximum and minimum values of $f$ are called **extreme values** or **extrema** of $f$. Extreme values for a small interval are called **local maximum** or **local minimum** values.

**Definition 4.10.** *The number $f(c)$ is a*

i. **local maximum** *value of $f$ if there exists an open interval $(a, b)$ containing $c$ such that $f(c) \geq f(x)$ for all $x$ in $(a, b)$.*

ii. **local minimum** *value of $f$ if there exists an open interval $(a, b)$ containing $c$ such that $f(c) \leq f(x)$ for all $x$ in $(a, b)$.*

The word 'local' in local maximum (or minimum) is used to show that $f(c) > f(x)$ (or $f(c) < f(x)$) where $x$ is near $a$. In other words we are interested in small intervals only.

Some functions have extreme values while others do not have. The theorem below states conditond each function must satisfy in order to have extreme values.

**Theorem 4.11** (The Extreme Value Theorem)**.** *If $f$ is continuous on a closed interval $[a, b]$, then $f$ attains an absolute maximum value $f(a)$ and an absolute minimum value $f(d)$ at some numbers $c$ and $d$ in $[a, b]$.*

The following theorem states that if $f(c)$ is the maximum or minimum value of $f$, then the derivative of $f$ at $c$, if it exists, is equal to 0.

**Theorem 4.12** (The Fermat's Theorem)**.** *If $f$ has a local maximum or minimum at $c$, and if $f'(c)$ exists, then $f'(c) = 0$.*

*Proof.* We prove the case for the local maximum and leave the case for the local minimum as an exercise.

Suppose $f(c)$ is the local maximum of $f$ at $c$. Then $f(c) > f(x)$ for all $x$ near $c$. Now if $h$ is sufficiently close to 0, where $h$ is positive or negative, we have

$$f(c) \geq f(c+h) \Rightarrow f(c+h) - f(c) \leq 0.$$

Dividing both sides by $h$ where $h > 0$ we have

$$\frac{f(c+h) - f(c)}{h} \leq 0.$$

Taking the right jhand limit of both sides of the inequality gives

$$\lim_{h \to 0^+} \frac{f(c+h) - f(c)}{h} \leq \lim_{h \to 0^+} 0 = 0.$$

Since $f'(c)$ exists, we have

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h} = \lim_{h \to 0^+} \frac{f(c+h) - f(c)}{h}$$

so we have shown that $f'(c) \leq 0$.

Following similar steps for $h < 0$ gives $f'(c) \geq 0$. Now both $f'(c) \leq 0$ and $f'(c) \geq 0$ must be true. It follows that $f'(c) = 0$. □

By Fermat's Theorem extreme values of a function $f$ occur at a number $c$ where $f'(c) = 0$ or $f'(c)$ does not exist. Such numbers are called **critical numbers**.

**Definition 4.13.** *A number $c$ in the domain of $f$ is a* **critical number** *of $f$ if either $f'(c) = 0$ or $f'(c)$ does not exist.*

**Example 4.14.**

Find the critical numbers of the function $f(x) = 5x^{-1/2} + 6x^{3/2}$.

**Solution**

$$f'(x) = 5\left(-\frac{1}{2}\right) x^{-1/2-1} + 6\left(\frac{3}{2}\right) x^{3/2-1} = -\frac{5}{2}x^{-3/2} + 9x^{1/2} = \frac{-5 + 18x^2}{2x^{3/2}}.$$

$f'(x) = 0$ if $18x^2 - 5 = 0$, that is $x = \pm\dfrac{5}{18}$ and $f'(x)$ does not exist when the denominator is equal to 0. i.e., when $x = 0$.

Hence the three critical numbers are $\pm\dfrac{5}{18}$ and 0.

To find absolute extrema of a function on a closed interval $[a, b]$ we follow the following steps:

i. Find all the critical numbers of $f$.

ii. Calculate $f(c)$ for each critical number $c$.

iii. Calculate $f(a)$ and $f(b)$.

iv. The absolute maximum of $f$ on $[a, b]$ is the largest of the values obtained in (ii) and (iii). The absolute minimum of $f$ on $[a, b]$ is the smallest of the values obtained in (ii) and (iii).

**Example 4.15.**

Find the absolute maximum and minimum values of the function
$f(x) = x^2 - 12x + 5$ in the interval $[0, 3]$.

**Solution**

$f'(x) = 6x - 12.$

i. Solving $f'(x) = 0$ gives one critical number 2.

ii. $f(2) = 3(2)^2 - 12(2) + 5 = 12 - 24 + 5 = -7.$

iii. $f(0) = 5$ and $f(3) = 3(3)^2 - 12(3) + 5 = 27 - 36 + 5 = -4.$

iv. Therefore the absolute maximum value of $f(x) = x^2 - 12x + 5$ on $[0, 3]$ is $f(0) = 5$ and the absolute minimum value is $f(2) = -7$.

**Example 4.16.**

Find the absolute maximum and minimum values of the function
$f(x) = x^4 - 2x^2 + 3$ in the interval $[-2, 3]$.

**Solution**

$f'(x) = 4x^3 - 4x = 4x(x^2 - 1).$

i. Solving $f'(x) = 0$ gives three critical numbers 0, 1 and $-1$.

ii. $f(0) = 3$ and $f(1) = f(-1) = 2.$

iii. $f(-2) = (-2)^4 - 2(-2)^2 + 3 = 16 - 8 + 3 = 11$ and $f(3) = 3^4 - 2(3)^2 + 3 = 81 - 18 + 3 = 66.$

iv. Therefore the absolute maximum value of $f(x) = x^4 - 2x^2 + 3$ on $[-2, 3]$ is $f(3) = 66$ and the absolute minimum value is $f(1) = f(-1) = 2$.

**Exercise 4.17.**

Find the absolute maximum and minimum values of the given function on the given interval

1. $f(x) = x - \ln x$, $[1/2, 2]$
2. $f(x) = x^3 - 3x + 1$, $[0, 3]$
3. $f(x) = (x^2 - 1)^3$, $[-1, 2]$
4. $f(x) = 2x^3 - 3x^2 - 12x + 1$, $[-2, 3]$
5. $f(x) = \dfrac{x^2 - 4}{x^2 + 4}$, $[-4, 4]$
6. $f(x) = \dfrac{x}{x^2 + 1}$, $[0, 2]$
7. $f(x) = \ln(x^2 + x + 1)$, $[-1, 1]$
8. $f(x) = e^{-x} - e^{-2x}$, $[0, 1]$
9. $f(x) = xe^{-x^2/8}$, $[-1, 4]$
10. $f(x) = 2\cos x + \sin 2x$, $[0, \pi/2]$
11. $f(x) = \sqrt[3]{x}(8 - x)$, $[0, 8]$

## 4.3 The Mean Value Theorem

The mean value theorem gives a very important result in calculus especially in this unit. We need the following result in order to effectively discuss the mean value theorem.

**Theorem 4.18** (Rolle's Theorem)**.** *Let $f$ be a function such that*

1. *$f$ is continuous on the closed interval $[a, b]$.*
2. *$f$ is differentiable on the open interval $(a, b)$.*
3. *$f(a) = f(b)$*

*Then there is a number $c$ in $(a, b)$ such that $f'(c) = 0$.*

*Proof.* There are three cases:

**Case I** $f(x) = k$ where $k$ is a constant

Here $f'(x) = 0$ so the number $c$ can be taken to be any number in the interval $(a, b)$.

**Case II** $f(x) > f(a)$ for some $x$ in $(a, b)$

By the Extreme Value Theorem, $f$ has a maximum value somewhere in $[a, b]$. Since $f(a) = f(b)$, it must attain this maximum value at a number $c$ in the open interval $(a, b)$. Then $f$ has a local maximum at $c$ and hence $f$ is differentiable at $c$. Hence, by Fermat's theorem, $f'(c) = 0$.

**Case III** $f(x) < f(a)$ for some $x$ in $(a, b)$

By the Extreme Value Theorem, $f$ has a minimum value in $[a, b]$. Since $f(a) = f(b)$, $f$ attains this minimum value at a number $c$ in the interval $(a, b)$. Hence $f'(c) = 0$ by Fermat's theorem.

□

**Example 4.19.**

Prove that the equation $x^3 + x - 1 = 0$ has exactly one real root.

**Solution**

First we use Intermediate Value Theorem (IVT) to show existence of the root.

Let $f(x) = x^3 + x - 1$. Then $f(0) = -1 < 0$ and $f(1) = 1 > 0$. $f$ is continuous since it is a polynomial, so the IVT states that there is a number $c$ between 0 and 1 such that $f(c) = 0$. Hence the given equation has a root.

To prove uniqueness of the root we use Rolle's Theorem and argue by contradiction. Suppose on the contrary that there are two roots $a$ and $b$. Then $f(a) = 0 = f(b)$, and since $f$ is a polynomial, it is differentiable on $(a, b)$ and continuous on $[a, b]$. Therefore, by Rolle's theorem, there is a number $c$ between $a$ and $b$ such that $f'(c) = 0$. But

$$f'(x) = 3x^2 + 1 \geq 1 \text{ for all } x$$

so $f'(x)$ can not be equal to zero. This is a contradiction.

Hence the equation has exactly one real root.

Rolle's theorem is used to prove the mean value theorem.

**Theorem 4.20** (The Mean Value Theorem)**.** *Let $f$ be a function which is continuous on the closed interval $[a, b]$ and differentiable on the open interval $(a, b)$.*

*Then there is a number $c$ in $(a, b)$ such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

*or equivalently*

$$f(b) - f(a) = f'(c)(b - a).$$

*Proof.* Let $g(x)$ be the secant line to $f(x)$ passing through $(a, f(a))$ and $(b, f(b))$. We know that the equation of the secant line is $y - y_1 = m(x - x_1)$.
$g(x) - f(a) = \dfrac{f(b) - f(a)}{b - a}(x - a)$ giving

$$g(x) = \frac{f(b) - f(a)}{b - a}(x - a) + f(a). \tag{12}$$

Let $h(x) = f(x) - g(x)$. Then by Equation 12,

$$h(x) = f(x) - \left(\frac{f(b) - f(a)}{b - a}(x - a) + f(a)\right).$$

$h(a) = h(b) = 0$ and $h(x)$ is continuous on $[a, b]$ and differentiable on $(a, b)$.

Thus applying the Rolles theorem, there is some $x = c$ in $(a, b)$ such that $h'(c) = 0$. Now

$$h'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}.$$

For some $c$ in $(a, b)$, $h'(c) = 0$. Thus

$$h'(c) = f'(c) - \frac{f(b) - f(a)}{(b - a)} = 0$$

giving

$$f'(c) = \frac{f(b) - f(a)}{(b - a)}.$$

□

**Example 4.21.**

1. Let $f(x) = x^3 + 2x^2 - x - 1$, find all numbers $c$ that satisfy the conditions of the Mean Value Theorem in the interval $[-1, 2]$.
2. Find the value of $c$ by using Mean Value Theorem: $f(x) = x(x - 1)(x - 2)$, $[0, 1/2]$.

**Solution**

1. $f(x) = x^3 + 2x^2 - x - 1 \Rightarrow f'(x) = 3x^2 + 4x - 1 \Rightarrow f'(c) = 3c^2 + 4c - 1$.
   Now $f(2) = 2^3 + 2(2)^2 - 2 - 1 = 8 + 8 - 2 - 1 = 13$ and $f(-1) = 1$.
   Now by the mean value theorem
   $$f'(c) = 3c^2 + 4c - 1 = \frac{f(2) - f(-1)}{2 - (-1)} = \frac{13 - 1}{2 + 1} = 4.$$
   So $3c^2 + 4c - 5 = 0$ giving $c = \dfrac{-4 \pm \sqrt{4^2 - 4(3)(-5)}}{6} = \dfrac{-4 \pm \sqrt{76}}{6}$.
   Hence $c = \dfrac{-4 + \sqrt{76}}{6} \in [-1, 2]$.
2. The mean value theorem applies to $f$ since $f$ is continuous in $[0, 1/2]$ and differentiable on $(0, 1/2)$.
   Now $f'(x) = 3x^2 - 6x + 2$. Using the mean value theorem we have
   $$f'(c) = 3c^2 - 6c + 2 = \frac{f(1/2) - f(0)}{1/2 - 0} = \frac{3}{4} \Rightarrow 12c^2 - 24c + 5 = 0 \Rightarrow c = 1 \pm \frac{\sqrt{21}}{6}$$
   Hence $c = 1 - \dfrac{\sqrt{21}}{6} \in \left(0, \dfrac{1}{2}\right)$.

The mean value theorem has a number of applications in differential calculus. The following theorem gives one such application.

**Theorem 4.22.** *If $f'(x) = 0$ for all $x$ in the interval $(a, b)$, then $f$ is constant on $(a, b)$.*

*Proof.* Let $x_1$ and $x_2$ be numbers in the interval $(a, b)$ with $x_1 < x_2$. Since $f$ is differentiable on $(a, b)$, it must be differentiable on $(x_1, x_2)$ and continuous on $[x_1, x_2]$. By applying the Mean Value Theorem to $f$ on the interval $[a, b]$ we get a number $c$ such that $x_1 < c < x_2$ and

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1). \tag{13}$$

Since $f'(x) = 0$ for all $x$ we have $f'(c) = 0$, and so Equation 13 becomes

$$f(x_2) - f(x_1) = 0 \text{ or } f(x_2) = f(x_1).$$

Therefore $f$ has the same value at any two numbers $x_1$ and $x_2$ in $(a, b)$. This means that $f$ is constant on $(a, b)$. □

**Corollary 4.23.** *If $f'(x) = g'(x)$ for all $x$ in an interval $(a, b)$, then $f - g$ is constant on $(a, b)$; that is $f(x) = g(x) + c$ where $c$ is a constant.*

*Proof.* Let $F(x) = f(x) - g(x)$. Then

$$F'(x) = f'(x) - g'(x) = 0$$

for all $x$ in $(a, b)$. Thus by Theorem 4.22, $F$ is constant, i.e., $f - g$ is constant. □

**Example 4.24.**

Prove the identity $\tan^{-1} x + \cot^{-1} x = \dfrac{\pi}{2}$.

**Solution**

This identity can be proved without using calculus. However, the proof is very easy when calculus is involved.

If $f(x) = \tan^{-1} x + \cot^{-1} x$ then

$$f'(x) = \frac{1}{1 + x^2} - \frac{1}{1 + x^2} = 0$$

for all values of $x$. Therefore $f(x) = C$, a constant. To determine the value of $C$ , we put $x = 1$ (since we can find the exact value of $f(1)$). Now

$$C = f(1) = \tan^{-1} 1 + \cot^{-1} 1 = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}.$$

**Exercise 4.25.**

1. Find the value of $c$ which satisfies the Rolle's theorem for the function
   (a) $f(x) = (x - 1)^2(x - 2)^3$ on the interval $[1, 2]$.
   (b) $f(x) = x^3 - x^2 - 6x + 2$ on $[0, 3]$
2. Find the values of $c$ that satisfy the Mean Value Theorem.
   (a) $f(x) = -x^2 + 8x - 17$ on $[3, 6]$
   (b) $f(x) = x^3 - 9x^2 + 24x - 18$ on $[2, 4]$
   (c) $f(x) = -\dfrac{x^2}{2} + x - \dfrac{1}{2}$ on $[-2, 1]$
   (d) $f(x) = \dfrac{x^2}{2} - 2x - 1$ on $[-1, 1]$
   (e) $f(x) = x^3 + 3x^2 - 2$ on $[-2, 0]$

(f) $f(x) = -x^3 + 4x^2 - 3$ on $[0, 4]$

3. Determine if the Mean Value Theorem can be applied. If it can, find all values of $c$ that satisfy the theorem. If it cannot, explain why not.

   (a) $f(x) = -\dfrac{x^2}{4x+8}$ on $[-3, -1]$

   (b) $f(x) = -\dfrac{-x^2+9}{4x}$ on $[1, 3]$

   (c) $f(x) = (x-3)^{2/3}$ on $[1, 4]$

4. Use the Mean Value Theorem to prove that $|\sin a - \sin b| \leq |a - b|$ for all real values $a$ and $b$ where $a \neq b$.

5. Let $f(x) = \dfrac{x^3 - x^2}{x - 1}$ on $[0, 2]$. Show that there is no value of $c$ such that $f'(c) = \dfrac{f(2) - f(0)}{2 - 0}$. Is this a counterexample to the mean value theorem? Why or why not?

6. Show that the equation $x^4 + 4x + c = 0$ has at most two roots.

## 4.4 Derivatives and the Shape of a Graph

### 4.4.1 Increasing and Decreasing Functions

Derivatives are used to find intervals in which a function is increasing or decreasing. Before we look at how that is done, we define increasing and decreasing functions.

**Definition 4.26.** i. *A function $f$ is increasing on an interval $I$ if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$.*

ii. *A function $f$ is decreasing on an interval $I$ if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$.*

iii. *A function $f$ is constant on an interval $I$ if $f(x_1) = f(x_2)$ for every $x_1$ and $x_2$ on $I$.*

The increasing and decreasing test is given in the theorem below.

**Theorem 4.27** (Increasing/decreasing Test)**.** *Let $f(x)$ be a function defined on an interval $I$. Then*

*(a) If $f'(x) > 0$ on $I$, then $f$ is increasing on $I$.*

*(b) If $f'(x) < 0$ on $I$, then $f$ is decreasing on $I$.*

*Proof.* (a) Let $x_1$ and $x_2$ be any two numbers in the interval with $x_1 < x_2$. We need to show that $f(x_1) < f(x_2)$.

Since $f'(x) > 0$ (given) $f$ is differentiable on $[x_1, x_2]$. So by the mean value theorem there is a number $c$ between $x_1$ and $x_2$ such that

$$f(x_2) - f(x_1) = f'(x)(x_2 - x_1). \tag{14}$$

Now $f'(c) > 0$ by our assumption and $x_2 - x_1 > 0$ since $x_1 < x_2$. Thus the right hand side of Equation 14 is positive, and so

$$f(x_2) - f(x_1) > 0 \text{ or } f(x_1) < f(x_2).$$

Hence $f$ is increasing.

(b.) Exercise

□

**Example 4.28.**

Find intervals in which the function $f(x) = x^2 - 4x + 7$ is increasing.

**Solution**

$$f'(x) = 2x - 4$$

Now $f'(x) > 0$ when $2x - 4 > 0 \Rightarrow x > 2$. Similarly $f'(x) < 0$ when $x < 2$.

Hence $f$ is increasing in the interval $(2, \infty)$ and decreasing in the interval $(-\infty, 2)$.

**Example 4.29.**

Find intervals on which $f(x) = (x^2 - 9)^{2/3}$ is increasing or decreasing.

**Solution**

$$f'(x) = \frac{2}{3}(x^2 - 9)^{-1/3}(2x) = \frac{4x}{3(x^2 - 9)^{1/3}}$$

Setting $f'(x) = 0$ gives $x = 0$. Additionally $f'(x)$ does not exist when the denominator is equal to 0, i.e., when $x^2 - 9 = 0 \Rightarrow x = 3$ or $-3$.

Hence the critical numbers are $-3$, 0 and 3.

| **Interval** | $(-\infty, -3)$ | $(-3, 0)$ | $(0, 3)$ | $(3, \infty)$ |
|---|---|---|---|---|
| **Test Number** | $-5$ | $-1$ | $1$ | $5$ |
| $f'(x)$ | $-$ | $+$ | $-$ | $+$ |
| $f(x)$ | Decreasing | Increasing | Decreasing | Increasing |

Table 9: Test table for $f(x) = (x^2 - 9)^{2/3}$

The four intervals are $(-\infty, -3)$, $(-3, 0)$, $(0, 3)$ and $(3, \infty)$. In Table 10 we test in order to identify intervals in which the function is increasing or decreasing.

From Table 10 $f$ is increasing on $[-3, 0]$ and $[3, \infty)$ and decreasing on $(-\infty, -3]$ and $[0, 3]$.

### 4.4.2 Local Extrema: The First Derivative Test

By the increasing/decreasing test, if a function $f$ is increasing in the interval $[x_1, x_2]$, then $f'(x) > 0$ in this interval. Suppose $f$ is then decreasing in the interval $[x_2, x_3]$ where $x_1 < x_2 < x_3$. Then the derivative of $f$ changes from positive to negative at $x = x_2$. This means the graph of $f$ has a peak at $x_2$. The value $f(x_2)$ is clearly a local maximum value. Similarly $f(x_2)$ will be a local minimum value if $f'(x)$ changes from negative to positive at $x = x_2$.

**Theorem 4.30** (The first derivative test)**.** *Suppose $c$ is a critical number of a continuous function $f$.*

i. *If $f'(x)$ changes from positive to negative at $c$, that is $f'(x) > 0$ for $a < x < c$ and $f'(x) < 0$ for $c < x < b$, then $f(c)$ is a local maximum value of $f$.*

ii. *If $f'(x)$ changes from negative to positive at $c$, then $f(c)$ is a local minimum value of $f$.*

iii. *If $f'(x)$ does not change sign at $c$, that is $f'(x) > 0$ or $f'(x) < 0$ on both sides of $c$, then $f(c)$ is not a local extremum of $f$.*

**Example 4.31.**

Find the local maximum and minimum values of the function $f(x) = (x^2 - 9)^{2/3}$.

**Solution**

By Table 10 in Example 4.29, $f'(x)$ changes from negative to positive at both 3 and $-3$. It changes from positive to negative at 0.

Hence

$$f(3) = f(-3) = 0$$

is the local minimum value and

$$f(0) = (-9)^{2/3} = \sqrt[3]{(-9)^2} = \sqrt[3]{81} \approx 4.3267$$

is the local maximum value.

**Example 4.32.**

Find the relative extrema of the function $f(x) = (x^2 - 1)^{2/3}$.

**Solution**

$$f'(x) = \frac{2}{3}(x^2 - 1)^{-1/3}(2x) = \frac{4x}{3(x^2 - 1)^{1/3}}$$

Equating the numerator to 0 gives $x = 0$ and equating the denominator to 0 gives $x = \pm 1$. This gives three critical numbers, namely $-1$, 0 and 1.

The four intervals are $(-\infty, -1)$, $(-1, 0)$, $(0, 1)$ and $(1, \infty)$.

| **Interval** | $(-\infty, -1)$ | $(-1, 0)$ | $(0, 1)$ | $(1, \infty)$ |
|---|---|---|---|---|
| **Test Number** | $-2$ | $-1/2$ | $1/2$ | 2 |
| $f'(x)$ | $-$ | $+$ | $-$ | $+$ |

Table 10: Test table for $f(x) = (x^2 - 1)^{2/3}$

$f(x)$ has relative maximum at $x = 0$ and relative minimum at $x = -1$ and $x = 1$.

Hence

$$f(0) = 1$$

is the relative maximum value and

$$f(-1) = f(1) = 0$$

is the relative minimum value of $f$.

### 4.4.3 Concavity Test

The concavity test uses the second derivative of a function to describe the shape of its graph.

**Definition 4.33.** *If the graph of $f$ lies above all of its tangents on an interval $I$, then it is called* **concave upward** *on $I$. If the graph of $f$ lies below all of its tangents on $I$, it is called* **concave downward** *on $I$.*

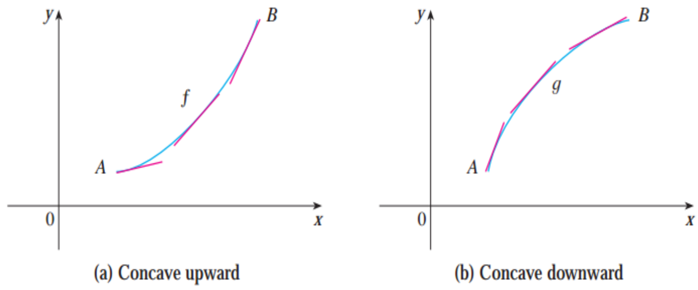


Figure 7: concavity

Figure 7 shows the graph of $f$ which is concave upward and the graph of another function $g$ which is concave downward.

**Theorem 4.34** (Concavity Test). *Let $f$ be a twice differentiable function.*

i. *If $f''(x) > 0$ for all $x$ in the interval $I$, then the graph of $f$ is concave upward on $I$.*

ii. *If $f''(x) < 0$ for all $x$ in $I$ then the graph of $f$ is concave downward.*

Below we define a point where concavity changes from upward to downward or vice versa.

**Definition 4.35.** *A point $P$ on a curve is called* **a point of inflection** *if $f$ is continuous at $P$ and the graph of $f$ changes from concave upward to concave downward or from concave downward to concave upward.*

**Example 4.36.**

Find the inflection point of the function $f(x) = x^3 - 3x^2 - 144x$.

**Solution**

$$f'(x) = 3x^2 - 6x - 144 \text{ and } f''(x) = 6x - 6.$$

$f''(x) > 0$ when $x > 0$ and $f''(x) < 0$ when $x < 0$.

Hence the curve changes from concave upward to concave downward at $x = 1$. Hence the point of inflection is $(1, f(1)) = (1, -146)$.

**Example 4.37.**

Find intervals in which the graph of $f(x) = x^4 + 28x^3 + 10x$ is concave upward or concave downward.

**Solution**

$$f'(x) = 4x^3 + 84x^2 + 10 \text{ and } f''(x) = 12x^2 + 168x = 12x(x + 14).$$

Solving the inequality $f''(x) > 0$ we find two solution intervals, namely $(-\infty, -14)$ and $(0, \infty)$. Similarly $f''(x) < 0$ in the interval $(-14, 0)$.

Hence the graph of $f$ is concave upward in $(-\infty, -14)$ and $(0, \infty)$, and concave downward in $(-14, 0)$.

### 4.4.4 Local Extrema: The Second Derivative Test

Another application of the second derivative is the following test for maximum and minimum values. It is a consequence of the Concavity Test.

**Theorem 4.38** (The Second Derivative Test)**.** *Suppose $f$ is continuous near $c$.*

i. *If $f'(c) = 0$ and $f''(c) > 0$, then $f(c)$ is a local minimum.*

ii. *If $f'(c) = 0$ and $f''(c) < 0$, then $f(c)$ is a local maximum.*

The second derivative test does not work when $f''(c) = 0$ or when $f''(c)$ does not exist since it gives no information in these cases. The first derivative test should be used whenever such situations are encountered.

**Example 4.39.**

Find the relative extrema for the function $f(x) = \dfrac{x^3}{3} - x^2 - 3x$.

**Solution**

$$f'(x) = x^2 - 2x - 3$$

$f'(x) = 0 \Rightarrow x^2 - 2x - 2 = 0 \Rightarrow (x - 3)(x + 1) = 0$ giving two critical numbers $x = -3$ and $x = 1$.

$$f''(x) = 2x - 2$$

Now

$$f''(3) = 2(3) - 2 = 4 > 0$$

and

$$f''(-1) = 2(-1) - 2 = -4 < 0.$$

Hence, by the second derivative test, $f(3) = \dfrac{3^3}{3} - 3^2 - 3(3) = -9$ is the relative minimum value of $f$ and $f(-1) = \dfrac{5}{3}$.

**Example 4.40.**

If $f(x) = 2x^3 + 2x^2 - 36x$, find the local maximum and minimum values of $f$ by the second derivative test.

**Solution**

$$f'(x) = 6x^2 + 6x - 36 = 6(x^2 + x - 6) = 6(x+3)(x-2)$$

$f'(x) = 0$ gives two critical numbers $x = -3$ and $x = 2$.

$$f''(x) = 12x + 6$$

So

$$f''(2) = 12(2) + 6 = 30 > 0$$

and

$$f''(-3) = 12(-3) + 6 = -36 + 6 = -30 < 0.$$

Hence

$$f(2) = 2(2)^3 + 3(2)^2 - 36(2) = 16 + 12 - 72 = -44$$

is the local minimum value of $f$ and

$$f(-3) = 2(-3)^3 + 3(-3)^2 - 36(-3) = -54 + 27 + 108 = 81$$

is the local maximum value of $f$.

**Exercise 4.41.**

1. Find critical numbers for each function below.
   (a) $f(x) = \dfrac{1-x}{x^2 + 2x - 15}$
   (b) $f(x) = 15 - (3-x)(x^2 - 8x + 7)^{1/3}$
   (c) $f(x) = \sqrt[5]{x^2 - 6x}$
   (d) $f(x) = 8x^3 + 81x^2 - 42x - 8$
2. Find the intervals where each function is increasing or decreasing;
   (a) $f(x) = x^3 + x^2 - 8x + 12$

(b) $f(x) = e^x + e^{-2x}$

(c) $f(x) = e^{-x} + x$

(d) $f(x) = 4x^3 - 18x^2 + 48x - 290$

3. Find the local extrema [*compare the values obtained by the first and second derivative tests*].

(a) $f(x) = x^2(x-4)$

(b) $f(x) = 3x^4 + 8x^3 - 174x^2 - 360x$

(c) $f(x) = e^{-x} + x$

(d) $f(x) = x^3 + x^2 - 8x + 12$

(e) $f(x) = x^4 - 8x^2$

(f) $f(x) = 3x^4 - 44x^3 + 144x^2$

4. Determine interval(s) where the graph of the function $f$ is concave upward and interval(s) where it is concave downward, then find point(s) of inflection.

(a) $f(x) = x^4 - 4x^3$

(b) $f(x) = x^{2/3}(6-x)^{1/3}$

(c) $f(x) = -x^{5/3} + 3x^{2/3}$

(d) $f(x) = \dfrac{x^2-4}{x^2-25}$

(e) $f(x) = 3x^4 - 44xx^3 + 144x^2$

## 4.5 Indeterminate Forms and l'Hospital's Rule

The limit of some functions can not be evaluated using limit laws. For example limit laws can not be used to evaluate

$$\lim_{x\to 0} \frac{\sin x}{x}$$

since

$$\lim_{x\to 0} \sin x = \lim_{x\to 0} x = 0.$$

In general, the limit of the form

$$\lim_{x\to a} \frac{f(x)}{g(x)}$$

where $f(x) \to 0$ and $g(x) \to 0$ as $x \to a$ is called an **indeterminate form of type** $\dfrac{0}{0}$. If the limit exists, it will not be found by using limit laws. For rational functions we can cancel common factors:

$$\lim_{x\to 2} \frac{x^2-2x}{x^2-4} = \lim_{x\to 2} \frac{x(x-2)}{(x-2)(x+2)} = \lim_{x\to 2} \frac{x}{x+2} = \frac{2}{4} = \frac{1}{2}.$$

The limit of the form

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

where $f(x) \to \infty$ (or $-\infty$) and $g(x) \to \infty$ (or $-\infty$) as $x \to a$ is called an **indeterminate form of type** $\frac{\infty}{\infty}$. This type of the limit exists only for some type of functions. For rational functions they are evaluated by dividing numerator and denominator by the highest power of $x$ that occurs in the denominator:

$$\lim_{x \to \infty} \frac{2x^2 - 3}{x^3 - 2x} = \lim_{x \to \infty} \frac{\frac{2}{x} - \frac{3}{x^3}}{1 - \frac{2}{x^2}} = 0.$$

L'Hospital's rule applies to indeterminate form of types $\frac{0}{0}$ and $\frac{\infty}{\infty}$.

**Theorem 4.42** (L'Hospital's Rule)**.** *Suppose $f$ and $g$ are differentiable and $g'(x) \neq 0$ on an open interval $I$ that contains $a$ (except possibly at $a$). Suppose that*

$$\lim_{x \to a} f(x) = 0 \textit{ and } \lim_{x \to a} g(x) = 0$$

*or that*

$$\lim_{x \to a} f(x) = \pm\infty \textit{ and } \lim_{x \to a} g(x) = \pm\infty$$

*then*

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$

By the Theorem 4.42, the limit of a quotient of functions is equal to the limit of the quotient of their derivatives, provided that the given conditions are satisfied. It is especially important to verify the conditions regarding the limits of $f$ and $g$ before using l'Hospital's theorem. L'Hospital's Rule is also valid for one-sided limits and for limits at infinity or negative infinity; that is $x \to a$ can be replaced by $x \to a^+$, $x \to a^-$, $x \to \infty$ or $x \to -\infty$.

**Example 4.43.**

Evaluate

1. $\displaystyle\lim_{x \to 1} \frac{x - 1}{\sqrt{x} - 1}$

2. $\displaystyle\lim_{x \to 0} \frac{\sin mx}{\sin nx}$

3. $\lim\limits_{x\to 0} \dfrac{\tan^{-1}(2x)}{3x}$

4. $\lim\limits_{x\to \infty} \dfrac{\ln(\ln x)}{\sqrt{x}}$

5. $\lim\limits_{x\to \pi} \dfrac{\sin x}{1-\cos x}$

**Solution**

1. $\lim\limits_{x\to 1} \dfrac{x-1}{\sqrt{x}-1} = \lim\limits_{x\to 1} \dfrac{1}{\left(\dfrac{1}{2\sqrt{x}}\right)} = \lim\limits_{x\to 1} 2\sqrt{x} = 2.$

2. $\lim\limits_{x\to 0} \dfrac{\sin mx}{\sin nx} = \lim\limits_{x\to 0} \dfrac{m\cos mx}{n\cos nx} = \dfrac{m}{n}.$

3. $\lim\limits_{x\to 0} \dfrac{\tan^{-1}(2x)}{3x} = \lim\limits_{x\to 0} \dfrac{\left(\dfrac{2}{1+4x^2}\right)}{3} = \dfrac{2}{3}.$

4. 
$$\begin{aligned}
\lim_{x\to\infty} \frac{\ln(\ln x)}{\sqrt{x}} &= \lim_{x\to\infty} \frac{\left(\dfrac{1}{x\ln x}\right)}{\left(\dfrac{1}{2\sqrt{x}}\right)} \\
&= \lim_{x\to\infty} \frac{2\sqrt{x}}{x\ln x} \\
&= \lim_{x\to\infty} \frac{2\sqrt{x}}{x\ln x} \\
&= \lim_{x\to\infty} \frac{2}{\sqrt{x}\ln x} \\
&= 0.
\end{aligned}$$

5. Here $\sin x$ approaches 0 as $x \to \pi$ but $1-\cos x$ does not approach 0. Therefore the l'Hospital's rule can not be applied here. However the function is continuous at $\pi$ so the limit is evaluated as follows:
$$\lim_{x\to\pi} \frac{\sin x}{1-\cos x} = \frac{\sin\pi}{1-\cos\pi} = \frac{0}{1-(-1)} = 0.$$

### 4.5.1 Indeterminate Form of Type $0.\infty$

The value of

$$\lim_{x\to a}[f(x)g(x)]$$

where $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = \infty$ (or $-\infty$) is not clear to find especially if it exists. There is a struggle between $f$ and $g$. If $f$ wins, the answer will be 0 ; if $g$ wins, the answer will be $\infty$ (or $-\infty$). Or there may be a compromise where the answer is a finite nonzero number. We say that we have an indeterminate form of type $0.\infty$. To deal with this we write the product $fg$ as the quotient

$$fg = \frac{f}{1/g} \text{ or } fg = \frac{g}{1/f}.$$

This converts the given limit into an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$ so that we can use l'Hospital's Rule.

**Example 4.44.**

Evaluate

1. $\lim_{x\to-\infty} xe^x$
2. $\lim_{x\to\infty} e^{-x}\ln x$
3. $\lim_{x\to1^+}(x-1)\tan\frac{\pi x}{2}$

**Solution**

1. $\lim_{x\to-\infty} xe^x = \lim_{x\to-\infty}\frac{x}{e^{-x}} = \lim_{x\to-\infty}\frac{1}{-e^{-x}} = \lim_{x\to-\infty}(-e^x) = 0.$
2. $\lim_{x\to\infty} e^{-x}\ln x = \lim_{x\to\infty}\frac{\ln x}{e^x} = \lim_{x\to\infty}\frac{1/x}{e^x} = \lim_{x\to\infty}\frac{1}{xe^x} = 0.$
3. $\lim_{x\to1^+}(x-1)\tan\frac{\pi x}{2} = \lim_{x\to1^+}\frac{x-1}{\cot\frac{\pi x}{2}} = \lim_{x\to1^+}\frac{1}{-\csc^2(\pi x/2)\frac{\pi}{2}} = -\frac{2}{\pi}.$

### 4.5.2 Indeterminate Form of the Type $\infty - \infty$

If $\lim\limits_{x\to a} f(x) = \infty$ and $\lim\limits_{x\to a} g(x) = \infty$ then the limit

$$\lim_{x\to a}[f(x) - g(x)]$$

is called an indeterminate form of type $\infty - \infty$. There is contest between $f$ and $g$ where if $f$ wins the answer is $\infty$ while if $g$ wins the answer is $-\infty$ or they will compromise on a finite number. To deal with this we try to convert the difference into a quotient (for instance, by using a common denominator, or rationalization, or factoring out a common factor) so that we have an indeterminate form of type $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$.

**Example 4.45.**

Evaluate

1. $\lim\limits_{x\to\frac{\pi}{2}} (\sec x - \tan x)$

2. $\lim\limits_{x\to 0} \left(\dfrac{3}{x} - \dfrac{1}{e^x - 1}\right)$

**Solution**

1.
$$\begin{aligned}
\lim_{x\to\pi/2} (\sec x - \tan x) &= \lim_{x\to\pi/2} \left(\frac{1}{\cos x} - \frac{\sin x}{\cos x}\right) \\
&= \lim_{x\to\pi/2} \frac{1 - \sin x}{\cos x} \\
&= \lim_{x\to\pi/2} \frac{-\cos x}{-\sin x} \\
&= 0.
\end{aligned}$$

2. $\lim\limits_{x\to 0} \left(\dfrac{3}{x} - \dfrac{1}{e^x - 1}\right) = \lim\limits_{x\to 0} \dfrac{3(e^x - 1) - x}{x(e^x - 1)} = \lim\limits_{x\to 0} \dfrac{3e^x - 3 - x}{xe^x - x}.$

   Applying the l'Hospital's rule we have $\lim\limits_{x\to 0} \dfrac{3e^x - 1}{xe^x + e^x - 1}$.

   Applying the rule again we have

$$\lim_{x\to 0} \frac{3e^x}{xe^x + e^x + e^x} = \lim_{x\to 0} \frac{3e^x}{xe^x + 2e^x} = \frac{3}{2}.$$

### 4.5.3 Indeterminate Form of the Type $0^0$, $\infty^0$ or $1^\infty$

The limit

$$\lim_{x\to a}[f(x)]^{g(x)}$$

gives rise to the following indeterminate forms:

1. Type $0^0$ if $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$.
2. Type $\infty^0$ if $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = 0$.
3. Type $1^\infty$ if $\lim_{x\to a} f(x) = 1$ and $\lim_{x\to a} g(x) = \infty$.

To deal with these we can either take natural logarithms:

$$\text{let } y = [f(x)]^{g(x)} \text{ then } \ln y = g(x)\ln f(x)$$

or write the function as an exponential:

$$[f(x)]^{g(x)} = e^{g(x)\ln f(x)}.$$

**Example 4.46.**

Evaluate

1. $\lim_{x\to 0^+} (\sin x)^{\tan x}$
2. $\lim_{x\to\infty} \left(1+\dfrac{1}{x^2}\right)^x$

**Solution**

1. Let $y = (\sin x)^{\tan x}$. Then $\ln y = \tan x \ln \sin x$. So

$$\lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} \tan x \ln \sin x = \lim_{x\to 0^+} \frac{\ln \sin x}{\cot x}.$$

Applying l'Hospital's rule we have

$$\lim_{x\to 0^+} \frac{(\cos x)/\sin x}{-\csc^2 x} = \lim_{x\to 0^+} (-\sin x \cos x) = 0.$$

Hence

$$\lim_{x\to 0^+} (\sin x)^{\tan x} = \lim_{x\to 0^+} e^{\ln y} = e^0 = 1.$$

2. Let $y = \left(1 + \dfrac{1}{x^2}\right)^x$. Then $\ln y = x \ln\left(1 + \dfrac{1}{x^2}\right)$. So

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} x \ln\left(1 + \frac{1}{x^2}\right) = \lim_{x\to\infty} \frac{\ln\left(1 + \frac{1}{x^2}\right)}{1/x}.$$

Applying l'Hospital's rule we have

$$\lim_{x\to\infty} \frac{\left(-\frac{2}{x^3}\right) / \left(1 + \frac{1}{x^2}\right)}{-1/x^2} = \lim_{x\to\infty} \frac{2/x}{1 + 1/x^2} = 0.$$

Hence

$$\lim_{x\to\infty} \left(1 + \frac{1}{x^2}\right)^x = \lim_{x\to\infty} e^{\ln y} = e^0 = 1.$$

**Exercise 4.47.**

1. $\displaystyle\lim_{x\to 0} \frac{\sin^2 x}{\tan x^2}$
2. $\displaystyle\lim_{x\to\infty} \frac{\ln(1 + e^x)}{5x}$
3. $\displaystyle\lim_{x\to 0} \frac{x}{\sin^{-1} 3x}$
4. $\displaystyle\lim_{x\to 0} \frac{\sin^{10} x}{\sin(x^{10})}$
5. $\displaystyle\lim_{x\to 0} \frac{\tan x - \sin x}{x^3}$
6. $\displaystyle\lim_{x\to 0} \frac{x + \tan 2x}{x - \tan 2x}$
7. $\displaystyle\lim_{x\to 0} \frac{\tan 2x}{\tanh 2x}$
8. $\displaystyle\lim_{x\to 0} \frac{2x - \sin^{-1} x}{2x + \tan^{-1} x}$
9. $\displaystyle\lim_{x\to 0^+} (\cot x)^{\sin x}$
10. $\displaystyle\lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^{x^2}$
11. $\displaystyle\lim_{x\to\infty} (-\ln x)^x$

## 4.6 Summary of Curve Sketching

While graphing calculators and other devices can be used to produce nice graphs, the use of calculus enables us to discover the most interesting aspects of graphs and in many cases to calculate maximum and minimum points and inflection points exactly instead of approximately. The following guidelines serve as checklist when sketching the curves using calculus.

**Guidelines for Sketching a Curve**

A. **Domain:** determine the domain of the function

B. **Intercepts:** $x$ and $y$ intercepts are found by setting $y = 0$ and $x = 0$ respectively. This can be omitted if the equation is difficult to solve.

C. **Symmetry:** recall that even functions are symmetric about the $y$ axis while odd functions are symmetric about the origin. Also if $f(x+p) = f(x)$ for all $x$ in the domain of $f$ where $p$ is a positive constant, then $f$ is called **periodic function** while $p$ is called the **period**.

D. **Assymptotes:** recall that $L$ is a horizontal asymptote if either $\lim\limits_{x\to\infty} f(x) = L$ or $\lim\limits_{x\to-\infty} f(x) = L$. The line $x = a$ is a vertical asymptote if either $\lim\limits_{x\to a} = \pm\infty$, $\lim\limits_{x\to a^+} = \pm\infty$ or $\lim\limits_{x\to a^-} = \pm\infty$. For rational functions we can locate the vertical asymptotes by equating the denominator to 0 after canceling any common factors. But for other functions this method does not apply.

E. **Intervals of Increase or Decrease:** compute the derivative and see intervals in which it is positive and intervals in which it is negative.

F. **Local Maximum and Minimum Values:** use the first derivative test or the second derivative test to identify maximum or minimum values.

G. **Concavity and Points of Inflection:** compute the second derivative and find intervals where the curve is concave upward and concave downward and identify inflection points.

H. **Sketch the curve:** draw the graph using the information above. Sketch the asymptotes as dashed lines. Plot the intercepts, maximum and minimum points, and inflection points. Then make the curve pass through these points.

**Example 4.48.**

Sketch the graph of the function $f(x) = -\frac{1}{3}x^3 + x - \frac{2}{3}$.

**Solution**

Please verify all the calculations and conclusions below.

A. The domain is the set of all real numbers.

B. The $x$ intercepts are $(1, 0)$ and $(-2, 0)$. The $y$ intercept is $(0, -2/3)$.

C. There is no symmetry.

D. There are no assymptotes.

E. Increasing interval: $x \in (-1, 1)$. Decreasing interval: $x \in (-\infty, -1) \cup (1, \infty)$

F. Using the first derivative test we have,
Maximum value point: $(1, 0)$ and minimum value point: $(-1, -4/3)$

G. Concave upward interval: $x \in (-\infty, 0)$. Concave downward interval: $x \in (0, \infty)$

H. We provide the sketch in Figure 8.

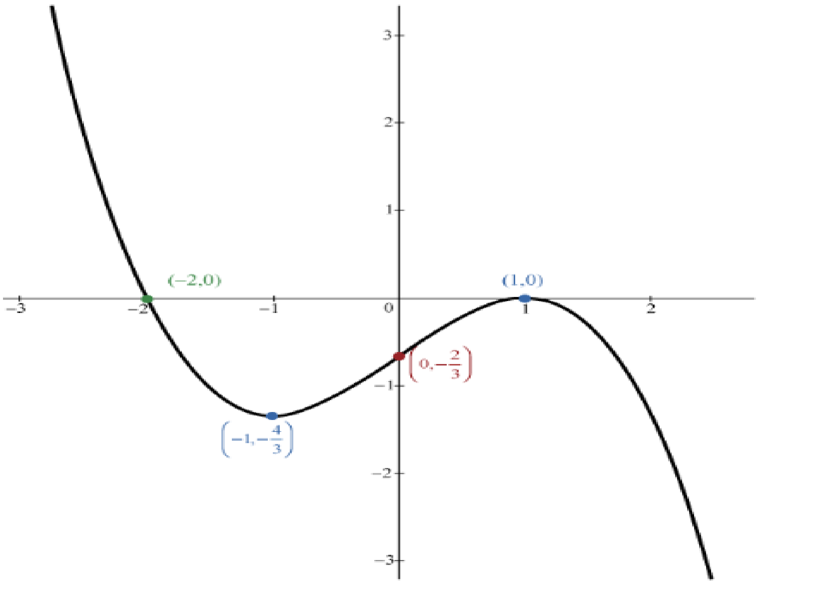


Figure 8: The graph of the function $f(x) = -\frac{1}{3}x^3 + x - \frac{2}{3}$

**Exercise 4.49.**

Use the guidelines of this section to sketch the curve.

1. $y = x^3 - 12x^2 + 36x$

2. $y = 2 + 3x^2 - x^3$

3. $y = x^4 - 8x^2 + 3$

4. $y = \dfrac{x^2 - 4}{x^2 - 2x}$

5. $y = \dfrac{x^2}{x^2 + 9}$

6. $y = 1 + \dfrac{1}{x} + \dfrac{1}{x^2}$

7. $y = \dfrac{\sqrt{1 - x^2}}{x}$

8. $y = \sqrt[3]{x^3 + 1}$

9. $y = 2\sqrt{x} - x$

## 4.7 Optimisation Problems

There are a number of applications to the methods of finding extreme values discussed in sections above. For example, the aim of every business is minimise costs while maximising profits. If you are travelling your aim may be to minimise both transportation time and costs.

An **optimization problem** is a word problem in which:

1. Two quantities are related, one of them (dependent) being a function of the other (independent).

2. The goal is to identify the value of the independent quantity that will make the dependent quantity largest or smallest within a certain acceptable range.

### Steps in Solving Optimization Problems

When solving an optimisation problem,

1. Read carefully and understand the problem.

2. For most problems it is useful to draw a diagram and identify given and required quantities.

3. Ensure that the quantity to be optimized is expressed as a function of a single independent variable.

4. If the available information leads to a relationship that involves other, auxiliary variables, find a suitable way to eliminate them by writing them in terms of the needed independent variable.

5. Ensure that the optimal values found are within the practical restrictions required by the problem, that is, within the allowable domain for the function.

6. Remember that a critical number may be a maximum or a minimum or neither and you need to check what it is before reaching a conclusion about the problem.

7. Look up any formula you need instead of guessing.

8. Use letters to denote variables, but use the actual numbers for any constants

**Example 4.50.**

We have a piece of cardboard that is 50 cm by 20 cm and we are going to cut out the corners and fold up the sides to form a box. Determine the height of the box that will give a maximum volume.

**Solution**

1. First we provide the sketch in Figure 9.

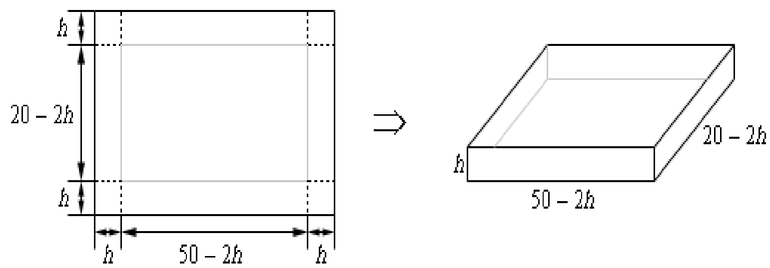


Figure 9: The sketch of the problem in Example 4.50

2. The volume equation is $V(h) = h(50 - 2h)(20 - 2h) = 4h^3 - 140h^2 + 1000h$.

3. Finding critical points: $V'(h) = 12h^2 - 280h + 1000$ giving $h = \dfrac{35 \pm 5\sqrt{19}}{3}$, $h = 4.4018$ or $18.9315$.

   From Figure 9, we see that the limits on $h$ must be $h = 0$ and $h = 10$. Note that neither of these really make physical sense but they do provide limits on $h$. So, we must have $0 \leq h \leq 10$ and this eliminates the second critical point and so the only critical point we need to worry about is $h = 4.4018$.

4. Because we have limits on $h$ we can quickly check to see if we have maximum by plugging in the volume function: $V(0) = 0$, $V(4.4018) = 2030.34$ and $V(10) = 0$.

   So, we can see then that the height of the box will have to be $h = 4.4018$ in order to get a maximum volume.

**Example 4.51.**

We want to construct a cylindrical can with a bottom but no top that will have a volume of $30cm^3$. Determine the dimensions of the can that will minimize the amount of material needed to construct the can.

**Solution**

1. We provide the sketch in Figure 10.

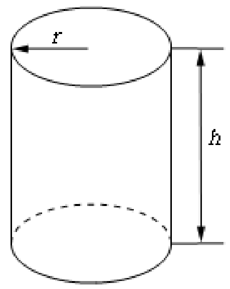


Figure 10: The sketch of the problem in Example 4.51

2. We are told that the volume of the can must be $30cm^3$ and so

$$30 = \pi r^2 h. \tag{15}$$

   We are being asked to minimize the amount of material needed to construct the can

$$A = 2\pi r h + \pi r^2. \tag{16}$$

   Recall that the can will have no top and so the second term will only be for the area of the bottom of the can.

3. Now solving Equation 15 for $h$ gives

$$h = \frac{30}{\pi r^2}.$$

   Plugging this Equation 16 gives,

$$A(r) = 2\pi r \left(\frac{30}{\pi r^2}\right) + \pi r^2 = \frac{60}{r} + \pi r^2.$$

4. Finding the critical point(s): $A'(r) = -\frac{60}{r^2} + 2\pi r = \frac{2\pi r^3 - 60}{r^2}$ giving one critical point $r = \sqrt[3]{\frac{60}{2\pi}} = 2.1216$.

   Note that $r = 0$ can not be a critical point because the function does not exist there.

5. The second derivative of the volume function is

$$A''(r) = \frac{120}{r^3} + 2\pi.$$

   From this we can see that the second derivative is always positive for positive $r$ (which is always true in this case since $r$ is the radius of a can). Therefore, provided $r$ is positive, $A(r)$ will always be concave up and so the single critical point we got must be a relative minimum and hence must be the value that gives a minimum amount of material.

6. Now the corresponding height of the can is

$$h = \frac{30}{\pi(2.1216)^2} = 2.1215.$$

   Hence the final dimensions are $r = 2.1216$ and $h = 2.1215$.

**Exercise 4.52.**

1. A carpenter is building a rectangular room with a fixed perimeter of 100 m. What are the dimensions of the largest room that can be built? What is its area?

2. From a thin piece of cardboard 30 cm by 30 cm, square corners are cut out so that the sides can be folded up to make a box. What dimensions will yield a box of maximum volume? What is the maximum volume?

3. The cost of a computer system increases with increased processor speeds. The cost $C$ of a system as a function of processor speed is estimated as $C = 5s^2 - 4s + 1000$ where $s$ is the processor speed in MHz. Find the processor speed for which cost is at a minimum.

4. Which point on the graph of $y = \sqrt{x}$ is closest to the point $(5, 0)$?

5. Two vertical poles, one 4 m high and the other 16 m high, stand 15 m apart on a flat field. A worker wants to support both poles by running rope from the ground to the top of each post. If the worker wants to stake both ropes in the ground at the same point, where should the stake be placed to use the least amount of rope?

6. A geometry student wants to draw a rectangle inscribed in a semicircle of radius 8. If one side must be on the semicircle's diameter, what is the area of the largest rectangle that the student can draw?

## 4.8 Newton's Method

Newton's method is a technique for generating numerical approximate solutions to equations of the form $f(x) = 0$. It is a process that can find roots of functions whose graphs cross or just "touch" the $x$ -axis. We start by simply making a guess for the solution. For example we could base the guess on a sketch of the graph of $f(x)$. Call the initial guess $x_1$. Next find the linear (tangent line) approximation to $f(x)$ near $x_1$. Let us call the linear approximation $F(x)$. It is

$$F(x) = f(x_1) + f'(x_1)(x - x_1).$$

Now, instead of trying to solve $f(x) = 0$, we solve the linear equation $F(x) = 0$ and call the solution $x_2$.

$$0 = F(x) = f(x_1) - f'(x_1)(x - x_1) \Rightarrow x - x_1 = -\frac{f(x_1)}{f'(x_1)}$$

giving

$$x = x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}.$$

See Figure 11.

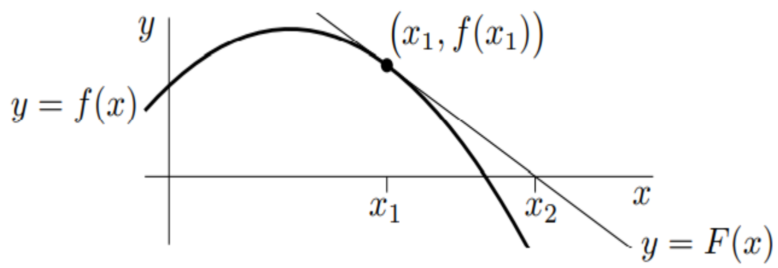


Figure 11: An illustration of Newton's Method

Now we repeat but starting with the second guess $x_2$ rather than $x_1$. This gives the third guess

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}.$$

**Algorithm for Newton's Method**

To solve an equation $f(x) = 0$,

1. Make a preliminary guess about the solution, $x_1$. The guess can be based on the sketch of the graph of $f(x)$.

2. Define $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$.

3. Iterate, i.e., for each $x_n$ $(n \geq 3)$ calculate a new estimate

$$x_{n+1} = x_n - \frac{f(n_n)}{f'(n_n)}.$$

**Example 4.53.**

Starting with $x_1 = 2$, find the third approximation $x_3$ to the root of the equation $x^3 - 2x - 5 = 0$.

**Solution**

$f(x) = x^3 - 2x - 5 \Rightarrow f'(x) = 3x^2 - 2$. So

$$x_{n+1} = x_n - \frac{x_n^3 - 2x_n - 5}{3x_n^2 - 2}.$$

When $n = 1$ we have

$$x_2 = x_1 - \frac{x_1^3 - 2x_1 - 5}{3x_1^2 - 2} = 2 - \frac{2^3 - 2(2) - 5}{3(2)^2 - 2} = 2.1.$$

When $n = 2$ we have

$$x_3 = x_2 - \frac{x_2^3 - 2x_2 - 5}{3x_2^2 - 2} = 2.1 - \frac{(2.1)^3 - 2(2.1) - 5}{3(2.1)^2 - 2} \approx 2.0946.$$

**Example 4.54.**

Use the Newton's method to solve the equation $2x - \tan x = 0$.

**Solution**

$$x_{n+1} = x_n - \frac{2x_n - \tan x_n}{2 - \sec^2 x_n}.$$

We try $x_1 = 1$. Then

$$x_2 = 1 - \frac{2 - \tan 1}{2 - \sec^2 1} = 1.31048.$$

Similar calculation yield

$$x_3 = 1.22393$$

$$x_4 = 1.17605$$

$$x_5 = 1.16593$$
$$x_6 = 1.16556$$
$$x_7 = 1.16556.$$

**Example 4.55.**

Use the Newton's method to solve the equation $2x - \cos x = 0$.

**Solution**

$$x_{n+1} = x_n - \frac{2x_n - \cos x_n}{2 + \sin x_n}.$$

Try $x_1 = 0$. Then

$$x_2 = 0 - \frac{2(0) - \cos 0}{2 + \sin 0} = \frac{1}{2} = 0.5.$$
$$x_3 = 0.45063$$
$$x_4 = 0.45018$$
$$x_5 = 0.45018.$$

**Exercise 4.56.**

1. Use Newton's method to find $\sqrt[6]{2}$ correct to eight decimal places (this is the same as solving the equation $x^6 - 2 = 0$).
2. Solve each equation and correct your solution to 5 decimal places.
   (a) $x^3 - 7 = 0$
   (b) $x^5 + 3x - 4 = 0$
   (c) $x^3 - 5 = 0$
   (d) $x^3 + 3x - 5 = 0$
   (e) $x^3 - x - 1 = 0$
   (f) $2x + x\sin(x + 3) - 5 = 0$
3. Put $x_1 = 3$ and use Newton's method to find the first two iterates $x_2$ and $x_3$ for the function $f(x) = x^3 - 3x^2 + x - 1$.
4. If $f(x) = x^3 - x^2 - 2x$ show that the hypothesis of Rolle's theorem are satisfied on the interval $[-1, 2]$ and find all values of $c$ that satisfy the theorem.
5. If $f(x) = e^x$ show that the hypothesis of the mean value theorem are satisfied on the interval $[0, 1]$ and find all values of $c$ that satisfy the theorem.

6. Determine the intervals on which the graph of the function $f(x) = \dfrac{x^2+9}{x^2-25}$ is concave upward or concave downward.

7. If $f''(x) = x^2(x+3)(x-5)$ find the values of $x$ at which the graph of $f$ has a change in concavity.

8. Evaluate $\lim\limits_{x\to 1/2} \dfrac{3-6x}{4x^2-1}$.

9. Ship A is 32 km north of ship Band is sailing due south at 16 km/h. Ship B is sailing due east at 12 km/h. At what rate is the distance between them changing at the end of an hour?

10. Water is being withdrawn from a conical tank 3 m in radius and 10 m deep at the rate of $4m^3/min$. How fast is the surface falling when the depth of the water is 6 m? How fast is the area of the surface decreasing at this instant? $[V = \frac{1}{3}\pi r^2 h$ and $A = \pi r^2]$

11. From a thin piece of cardboard 10 cm by 10 cm, square corners are cut out so that the sides can be folded up to make a box. What dimensions will yield a box of maximum volume? What is the maximum volume?

12. Find the intervals in which the function is increasing or decreasing.

    (a) $f(x) = x^2$

    (b) $f(x) = 2x^3 + 3x^2 - 12x + 1$

    (c) $f(x) = x^5 - x^3$

13. Find the relative extrema of the function $f(x) = 2x^3 + 3x^2 - 12x + 1$.