Integration

FRANCISCO CHAMERA

MAT 31201 - Trig & Introductory Calculus

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INTEGRALS - Introduction

- Given a function $f(x)$, we can differentiate it to find its derivative $f'(x)$.

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- Suppose we are given $f'(x)$ and asked to find the original function $f(x)$.
- To find $f(x)$ we apply the reverse operation to differentiation called **integration** to $f'(x)$.

- The function being integrated is called an **integrand**.

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- The result of integrating a function is an **integral** or **anti-derivative** of the original function.

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- The result of integrating a function is an **integral** or **anti-derivative** of the original function.
- The symbol for integration is $\int$.
- Thus $\int f(x)dx = F(x)$ means $F'(x) = f(x)$.

- Consider the two functions
  $f(x) = x^3 + 4x^2 + 8x + 7$ and
  $g(x) = x^3 + 4x^2 + 8x + 18.$

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- In general, the derivative of a function
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- Clearly $f(x) \neq g(x)$ but
  $f'(x) = g'(x) = 3x^2 + 8x + 8$.
- In general, the derivative of a function
  $x^3 + 4x^2 + 8x + C$, where C is any constant, is
  $3x^2 + 8x + 8$.
- This is so because the derivative of any constant is zero.

- The preceding discussion implies that

$$\int (3x^2 + 8x + 8)dx = x^3 + 4x^2 + 8x + C$$

since integration is the reverse operation to differentiation.

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- The integral $x^3 + 4x^2 + 8x + C$ is called **indefinite integral**.

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since integration is the reverse operation to differentiation.

- The integral $x^3 + 4x^2 + 8x + C$ is called **indefinite integral**.
- Hence, in general, if $\frac{dy}{dx} = f(x)$, then $y = \int f(x)dx + C$.

INTEGRALS - Constant Rule

Theorem

If k is a constant, then $\int kdx = kx + C$.

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3. $\int dx = x + C$

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For example;

1. $\int 2dx = 2x + C$
2. $\int 7dx = 7x + C$
3. $\int dx = x + C$
4. $\int -4dx = -4x + C$

INTEGRALS - Power Rule

Theorem

Let n be any real number, $n \neq -1$, then

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C.$$

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Let n be any real number, $n \neq -1$, then

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C.$$

For example;

1. $\int x^2 dx = \frac{x^{2+1}}{2+1} + C = \frac{x^3}{3} + C$

Theorem

Let n be any real number, $n \neq -1$, then

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C.$$

For example;

1. $\int x^2 dx = \dfrac{x^{2+1}}{2+1} + C = \dfrac{x^3}{3} + C$
2. $\int x^{-4} dx = \dfrac{x^{-3}}{-3} + C = -\dfrac{1}{3x^3} + C$

INTEGRALS - Properties of Indefinite integrals

Theorem

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Let $f(x)$ and $g(x)$ be two functions and let k be a constant. Then

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i. $\int kf(x)dx = k\int f(x)dx.$

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Let $f(x)$ and $g(x)$ be two functions and let k be a constant. Then

i. $\int kf(x)dx = k\int f(x)dx$.

ii. $\int [f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx$.

Theorem

Let $f(x)$ and $g(x)$ be two functions and let k be a constant. Then

i. $\int kf(x)dx = k\int f(x)dx$.

ii. $\int[f(x)+g(x)]dx = \int f(x)dx + \int g(x)dx$.

iii. $\int[f(x)-g(x)]dx = \int f(x)dx - \int g(x)dx$.

INTEGRALS - Example 1

Evaluate the following integrals;

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1. $\int(2x^3 + 3x - 4)dx$

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2. $\int (7x^6 - 3x^2 + 2x)dx$

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2. $\int (7x^6 - 3x^2 + 2x)dx$
3. $\int (19x^7 + 1)dx$

INTEGRALS - Solution to Example 1

1.

$$\int (2x^3 + 3x - 4)dx$$

$$= 2\int x^3 dx + 3\int x dx - 4\int dx$$

$$= 2\left(\frac{x^4}{4}\right) + 3\left(\frac{x^2}{2}\right) - 4x + C$$

$$= \frac{x^4}{2} + \frac{3x^2}{2} - 4x + C$$

INTEGRALS - Solution to Example 1 cont...

2.

$$\begin{aligned}
&\int (7x^6 - 3x^2 + 2x)dx \\
&= 7\int x^6 dx - 3\int x^2 dx + 2\int x dx \\
&= 7\left(\frac{x^7}{7}\right) - 3\left(\frac{x^3}{3}\right) + 2\left(\frac{x^2}{2}\right) + C \\
&= x^7 - x^3 + x^2 + C.
\end{aligned}$$

INTEGRALS - Natural Exponential functions

Since $\dfrac{d}{dx}[e^x] = e^x$, the following result is immediate;

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Let k be a constant. Then

i. $\int e^x dx = e^x + C$.

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Let k be a constant. Then

i. $\int e^x dx = e^x + C$.

ii. $\int e^{kx} dx = \dfrac{1}{k} e^{kx} + C$.

INTEGRALS - Natural Exponential functions cont...

For example,

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1. $\int e^{2x} dx = \frac{1}{2}e^{2x} + C$

For example,

1. $\int e^{2x}dx = \frac{1}{2}e^{2x} + C$
2. $\int 3e^{-4x}dx = -\frac{3}{4}e^{-4x} + C$

INTEGRALS - Trigonometric functions

$f(x)$	$\int f(x)dx$

INTEGRALS - Trigonometric functions

$f(x)$	$\int f(x)dx$
$\sin x$	$-\cos x + C$

INTEGRALS - Trigonometric functions

$f(x)$	$\int f(x)dx$
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$\csc^2 x$	$-\cot x + C$

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$\csc x \cot x$	$-\csc x + C$

INTEGRALS - Trigonometric functions

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INTEGRALS - Trigonometric functions

$f(x)$	$\int f(x)dx$
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$\cos x$	$\sin x + C$
$\csc x \cot x$	$-\csc x + C$
$\sec^2 x$	$\tan x + C$
$\sec x \tan x$	$\sec x + C$

INTEGRALS - Example 2

Evaluate the following integrals;

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1. $\int (10x^4 - 2\sec^2 x)dx$

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1. $\int (10x^4 - 2\sec^2 x)dx$
2. $\int \frac{\cos x}{2\sin^2 x}dx$

INTEGRALS - Solution to Example 2

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1.

$$\int (10x^4 - 2\sec^2 x)dx$$

$$= 10\left(\frac{x^5}{5}\right) - 2\tan x + C$$

$$= 2x^5 - 2\tan x + C$$

INTEGRALS - Solution to Example 2 cont...

2.

$$\int \frac{\cos x}{2\sin^2 x} dx$$
$$= \frac{1}{2} \int \left(\frac{1}{\sin x}\right) \left(\frac{\cos x}{\sin x}\right) dx$$
$$= \frac{1}{2} \int \csc x \cot x dx$$
$$= -\frac{1}{2} \csc x + C.$$

- Notice from above that indefinite integral of a function is another function.

- Notice from above that indefinite integral of a function is another function.
- If we are given the interval in which the integral is to be evaluated, we find **definite integral** which is a number.

INTEGRALS - Definite Integrals cont...

- The notation $\int_a^b f(x)dx$ means the integral of $f(x)$ is to be evaluated in the interval $[a, b]$.

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- The numbers a and b are called **limits of integration**.

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- The numbers a and b are called **limits of integration**.
- a is the lower limit and b is the upper limit.

Theorem

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Suppose that $\int f(x)dx = F(x)$, i.e., $F(x)$ is an anti-derivative of $f(x)$. If $f(x)$ is continuous on $[a, b]$ then

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Suppose that $\int f(x)dx = F(x)$, i.e., $F(x)$ is an anti-derivative of $f(x)$. If $f(x)$ is continuous on $[a, b]$ then

$$\int_a^b f(x)dx = F(b) - F(a).$$

INTEGRALS - Fundamental Theorem of Calculus cont...

- By the Theorem in the previous slide, the definite integral is found by first finding the integral and then substituting the limits.

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- The upper limit is substituted first.

- By the Theorem in the previous slide, the definite integral is found by first finding the integral and then substituting the limits.
- The upper limit is substituted first.
- The result of substituting the lower limit is subtracted from that of substituting the upper limit.

INTEGRALS - Example 3

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Evaluate the following integrals;

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1. $\int_2^3 (x+4)dx$

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2. $\int_{\frac{\pi}{2}}^{\pi} \frac{\sin 2x}{\sin x}dx$

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Evaluate the following integrals;

1. $\int_2^3 (x+4)dx$
2. $\int_{\frac{\pi}{2}}^{\pi} \frac{\sin 2x}{\sin x} dx$
3. $\int_2^2 4x^2 dx$
4. $\int_{-1}^2 (8x^7 + 3x^2) dx$

INTEGRALS - Solution to Example 3

1.

$$\begin{aligned}\int_2^3 (x+4)dx &= \left[\frac{x^2}{2} + 4x\right]_2^3 \\ &= \left[\frac{3^2}{2} + 4(3)\right] - \left[\frac{2^2}{2} + 4(2)\right] \\ &= \frac{9}{2} + 12 - 2 - 8 \\ &= \frac{13}{2}.\end{aligned}$$

INTEGRALS - Solution to Example 3 cont...

2.

$$\int_{\frac{\pi}{2}}^{\pi} \frac{\sin 2x}{\sin x} dx = \int_{\frac{\pi}{2}}^{\pi} \frac{2 \sin x \cos x}{\sin x} dx$$
$$= \int_{\frac{\pi}{2}}^{\pi} 2 \cos x dx = 2 \left[\sin x\right]_{\frac{\pi}{2}}^{\pi}$$
$$= 2(\sin \pi - \sin \pi/2) = -2.$$

2.

$$\int_{\frac{\pi}{2}}^{\pi} \frac{\sin 2x}{\sin x} dx = \int_{\frac{\pi}{2}}^{\pi} \frac{2 \sin x \cos x}{\sin x} dx$$
$$= \int_{\frac{\pi}{2}}^{\pi} 2 \cos x dx = 2 [\sin x]_{\frac{\pi}{2}}^{\pi}$$
$$= 2(\sin \pi - \sin \pi/2) = -2.$$

3. $\int_2^2 4x^2 dx = 4 \left[\frac{x^3}{3} \right]_2^2 = \frac{4}{3}(2^3 - 2^3) = 0.$

Theorem

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Let f and g be functions and let a, b and c be constants. Then

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Let f and g be functions and let a, b and c be constants. Then

i. $\int_a^b k dx = k(b - a)$.

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Let f and g be functions and let a, b and c be constants. Then

i. $\int_a^b k dx = k(b-a)$.

ii. $\int_a^b kf(x)dx = k\int_a^b f(x)dx$.

Theorem

Let f and g be functions and let a, b and c be constants. Then

i. $\int_a^b k dx = k(b-a)$.

ii. $\int_a^b kf(x)dx = k\int_a^b f(x)dx$.

iii. $\int_a^a f(x)dx = 0$.

Theorem

Let f and g be functions and let a, b and c be constants. Then

i. $\int_a^b k dx = k(b-a)$.

ii. $\int_a^b kf(x)dx = k\int_a^b f(x)dx$.

iii. $\int_a^a f(x)dx = 0$.

iv. $\int_a^b f(x)dx = -\int_b^a f(x)dx$.

Theorem

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Let f and g be functions and let a, b and c be constants. Then

Theorem

Let f and g be functions and let a, b and c be constants. Then

v. $\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx.$

Theorem

Let f and g be functions and let a, b and c be constants. Then

v. $\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$.

vi. $\int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)dx$ provided $a < c < b$.

INTEGRALS - Exercise 1

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Verify the following;

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1. $\int_{-1}^{0}(8x^7 + 3x^2)dx = \int_{-1}^{-0.5}(8x^7 + 3x^2)dx + \int_{-0.5}^{0}(8x^7 + 3x^2)dx$

Verify the following;

1. $\int_{-1}^{0}(8x^7+3x^2)dx = \int_{-1}^{-0.5}(8x^7+3x^2)dx + \int_{-0.5}^{0}(8x^7+3x^2)dx$
2. $\int_{0}^{\pi/2}\sin x dx = -\int_{\pi/2}^{0}\sin x dx$

- Integrals for some functions are difficult to evaluate in the function's original representation.

- Integrals for some functions are difficult to evaluate in the function's original representation.
- In some cases the variable is changed before calculating the integral and changed back at the end.

- The **Substitution Rule** is the method used to find integrals of the form $\int f(g(x))g'(x)dx$.

- The **Substitution Rule** is the method used to find integrals of the form $\int f(g(x))g'(x)dx$.
- For example, in $\int 3x^2\sqrt{x^3+1}dx$ notice that $\dfrac{d}{dx}[x^3+1] = 3x^2$.

Theorem

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If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))g'(x)dx = \int f(u)du.$$

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If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))g'(x)dx = \int f(u)du.$$

The Substitution Rule for integration is proved using the Chain Rule for differentiation.

INTEGRALS - Example 4

Evaluate the following integrals;

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1. $\int \frac{2x}{\sqrt{x^2+1}} dx$

Evaluate the following integrals;

1. $\int \frac{2x}{\sqrt{x^2+1}}dx$
2. $\int x^3 \cos(x^4+2)dx$

Evaluate the following integrals;

1. $\int \frac{2x}{\sqrt{x^2+1}}dx$
2. $\int x^3\cos(x^4+2)dx$
3. $\int \frac{\ln(x+1)}{x+1}dx$

INTEGRALS - Solution to Example 4

1. Let $u = x^2 + 1$. Then

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1. Let $u = x^2 + 1$. Then $\dfrac{du}{dx} = 2x \Rightarrow du = 2xdx$.

$$\int \frac{2x}{\sqrt{x^2+1}}dx$$

$$= \int \frac{du}{\sqrt{u}}$$

$$= \int u^{-1/2}du = \frac{u^{1/2}}{1/2} + C$$

$$= 2\sqrt{u} + C = 2\sqrt{x^2+1} + C.$$

INTEGRALS - Solution to Example 4 cont..

2. Let $u = x^4 + 2$. Then

2. Let $u = x^4 + 2$. Then
$$du = 4x^3 dx \Rightarrow x^3 dx = \frac{du}{4}.$$

2. Let $u = x^4 + 2$. Then
$du = 4x^3dx \Rightarrow x^3dx = \dfrac{du}{4}$.

$$\begin{aligned}\int \cos(x^4+2)(x^3dx) &= \int \cos u\left(\frac{du}{4}\right)\\ &= \frac{1}{4}\int \cos u du\\ &= \frac{1}{4}\sin u + C\\ &= \frac{1}{4}\sin(x^4+2) + C\end{aligned}$$

INTEGRALS - Solution to Example 4 cont..

INTEGRALS - Solution to Example 4 cont..

3. Let $u = \ln(x+1)$.

3. Let $u = \ln(x+1)$. Then $du = \dfrac{dx}{x+1}$.

3. Let $u = \ln(x+1)$. Then $du = \dfrac{dx}{x+1}$.

$$\begin{aligned}
\int \frac{\ln(x+1)}{x+1}dx &= \int \ln(x+1)\frac{dx}{x+1} \\
&= \int u du \\
&= \frac{u^2}{2} + C \\
&= \frac{(\ln(x+1))^2}{2} + C.
\end{aligned}$$

INTEGRALS - Exercise 2

Evaluate the following integrals;

Evaluate the following integrals;

1. $\int (3x^2 - 12x + 2)^{-7/3}(x - 2)dx$

Evaluate the following integrals;

1. $\int (3x^2 - 12x + 2)^{-7/3}(x-2)dx$
2. $\int e^x \sec^2(e^x)dx$

Evaluate the following integrals;

1. $\int (3x^2 - 12x + 2)^{-7/3}(x-2)dx$
2. $\int e^x \sec^2(e^x)dx$
3. $\int \frac{\sin(1/x)}{x^2}dx$

Evaluate the following integrals;

1. $\int (3x^2 - 12x + 2)^{-7/3}(x-2)dx$
2. $\int e^x \sec^2(e^x)dx$
3. $\int \frac{\sin(1/x)}{x^2}dx$
4. $\int (x^2+1)(x^3+3x)^4 dx$

Evaluate the following integrals;

1. $\int (3x^2 - 12x + 2)^{-7/3}(x-2)dx$
2. $\int e^x \sec^2(e^x)dx$
3. $\int \frac{\sin(1/x)}{x^2}dx$
4. $\int (x^2+1)(x^3+3x)^4 dx$
5. $\int \frac{\cos(\pi/x)}{x^2}dx$

Theorem

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If g' is continuous on $[a, b]$ and f is continuous on the range of $u = g(x)$, then

Theorem

If g' is continuous on $[a, b]$ and f is continuous on the range of $u = g(x)$, then

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du.$$

INTEGRALS - Example 5

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Evaluate the following integrals;

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1. $\int_0^4 \sqrt{2x+1}dx$

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1. $\int_0^4 \sqrt{2x+1}dx$
2. $\int_1^2 x^2 e^{x^3} dx$

INTEGRALS - Solution to Example 5

1. Let $u = 2x + 1$. Then

1. Let $u = 2x + 1$. Then $du = 2dx \Rightarrow dx = \dfrac{du}{2}$.

1. Let $u = 2x + 1$. Then $du = 2dx \Rightarrow dx = \dfrac{du}{2}$. Now when $x = 0$, $u = 2(0) + 1 = 1$ and $u = 9$ when $x = 4$.

INTEGRALS - Solution to Example 5 cont...

INTEGRALS - Solution to Example 5 cont...

1. $$\int_0^4 \sqrt{2x+1}dx = \int_1^9 \sqrt{u}\frac{du}{2} = \frac{1}{2}\int_1^9 u^{1/2}du$$
$$= \frac{1}{2}\left[\frac{u^{3/2}}{(3/2)}\right]_1^9 = \frac{1}{3}\left[u^{3/2}\right]_1^9$$
$$= \frac{1}{3}(9^{3/2} - 1^{3/2})$$
$$= \frac{26}{3}.$$

INTEGRALS - Solution to Example 5 cont...

2. Let $u = x^3$. Then

INTEGRALS - Solution to Example 5 cont...

2. Let $u = x^3$. Then $du = 3x^2dx$ giving
$$x^2dx = \frac{du}{3}.$$

2. Let $u = x^3$. Then $du = 3x^2dx$ giving
   $x^2dx = \dfrac{du}{3}$.
   When $x = 1$, $u = 1$ and $u = 8$ when $x = 2$.

2. Let $u = x^3$. Then $du = 3x^2 dx$ giving
$x^2 dx = \dfrac{du}{3}$.
When $x = 1$, $u = 1$ and $u = 8$ when $x = 2$.

$$\begin{aligned}
\int_1^2 x^2 e^{x^3} dx &= \frac{1}{3}\int_1^8 e^u du \\
&= \frac{1}{3}\left[e^u\right]_1^8 \\
&= \frac{1}{3}(e^8 - e).
\end{aligned}$$

INTEGRALS - Exercise 3

Evaluate the following integrals;

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1. $\int_0^{\pi/4} \sin x \cos x dx$

Evaluate the following integrals;

1. $\int_0^{\pi/4} \sin x \cos x dx$
2. $\int_1^{e^\pi} \frac{\sin(\ln x)}{x} dx$

Evaluate the following integrals;

1. $\int_0^{\pi/4} \sin x \cos x dx$
2. $\int_1^{e^\pi} \frac{\sin(\ln x)}{x} dx$
3. $\int_0^1 (x+1)e^{x^2+2x} dx$

Evaluate the following integrals;

1. $\int_0^{\pi/4} \sin x \cos x dx$
2. $\int_1^{e^\pi} \frac{\sin(\ln x)}{x} dx$
3. $\int_0^1 (x+1)e^{x^2+2x} dx$
4. $\int_0^{\frac{\pi}{2}} \cos x \sin(\sin x) dx$

i.

$$\int \frac{1}{x} dx = \ln |x| + C.$$

i.

$$\int \frac{1}{x} dx = \ln |x| + C.$$

Here we insist on absolute value of x since logarithmic function is defined for positive numbers only.

This is the integral of the function in which the numerator is the derivative of the denominator.

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Let $u = g(x)$. Then $du = g'(x)dx$.

This is the integral of the function in which the numerator is the derivative of the denominator.

Let $u = g(x)$. Then $du = g'(x)dx$.
Hence

$$\begin{aligned}\int \frac{g'(x)}{g(x)}dx &= \int \frac{du}{u} \\ &= \ln|u| + C \\ &= \ln|g(x)| + C.\end{aligned}$$

INTEGRALS - Example 6

Evaluate the following integrals;

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1. $\displaystyle\int \frac{3x}{x^2+7}dx$

Evaluate the following integrals;

1. $\displaystyle\int \frac{3x}{x^2+7}dx$
2. $\displaystyle\int \tan x dx$

1. Lat $u = x^2 + 7$. Then

1. Lat $u = x^2 + 7$. Then $du = 2xdx$ giving $xdx = \dfrac{du}{2}$.

INTEGRALS - Solution to Example 6

1. Lat $u = x^2 + 7$. Then $du = 2xdx$ giving $xdx = \dfrac{du}{2}$.

$$\int \frac{3x}{x^2+7}dx = \frac{3}{2}\int \frac{du}{u}$$
$$= \frac{3}{2}\ln|u| + C$$
$$= \frac{3}{2}\ln|x^2+7| + C.$$

INTEGRALS - Solution to Example 6 cont...

INTEGRALS - Solution to Example 6 cont...

2. $\int \tan x dx = \int \frac{\sin x}{\cos x} dx.$

2. $\int \tan x dx = \int \frac{\sin x}{\cos x} dx.$
   Let $u = \cos x$. Then $du = -\sin x dx$.

2. $\int \tan x dx = \int \frac{\sin x}{\cos x} dx.$

Let $u = \cos x$. Then $du = -\sin x dx$.

$$\begin{aligned}
\int \frac{\sin x}{\cos x} dx &= -\int \frac{du}{u} = -\ln |u| + C \\
&= -\ln |\cos x| + C \\
&= \ln \left(\frac{1}{|\cos x|} \right) + C \\
&= \ln |\sec x| + C.
\end{aligned}$$

INTEGRALS - Exercise 4

Evaluate the following integrals;

Evaluate the following integrals;

1. $\int_6^7 \frac{dx}{x-3}$

Evaluate the following integrals;

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2. $\int_0^{\pi} \cot x dx$

Evaluate the following integrals;

1. $\int_6^7 \frac{dx}{x-3}$
2. $\int_0^\pi \cot x dx$
3. $\int_{-1}^2 \frac{3x^4+12x^3+6}{x^5+5x^4+10x+12} dx$

Evaluate the following integrals;

1. $$\int_6^7 \frac{dx}{x-3}$$
2. $$\int_0^{\pi} \cot x dx$$
3. $$\int_{-1}^2 \frac{3x^4+12x^3+6}{x^5+5x^4+10x+12} dx$$
4. $$\int_2^4 \frac{6x-3}{4x^2-4x+1} dx$$

- Recall that if f and g are differentiable functions, then

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- Integrating both sides we have

- Recall that if f and g are differentiable functions, then

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + g'(x)f(x).$$

- Integrating both sides we have

$$\begin{aligned} f(x)g(x) &= \int [f'(x)g(x) + g'(x)f(x)]dx \\ &= \int f'(x)g(x)dx + \int g'(x)f(x)dx. \end{aligned}$$

INTEGRALS - Integration by Parts cont...

- Rearranging, we have

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$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx.$$

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- If we put $u = f(x)$ and $v = g(x)$, we have the following form which is easy to remember;

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- The key in integration by parts lies in making right substitutions.

- The key in integration by parts lies in making right substitutions.
- In general, we let u be a function which is easy to differentiate and let v' be a function which is easy to integrate.

INTEGRALS - Example 7

Evaluate the following integrals;

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1. $\int x \sec^2 x dx$

Evaluate the following integrals;

1. $\int x \sec^2 x dx$
2. $\int \ln x dx$

Evaluate the following integrals;

1. $\int x \sec^2 x dx$
2. $\int \ln x dx$
3. $\int e^x \sin x dx$

Evaluate the following integrals;

1. $\int x \sec^2 x dx$
2. $\int \ln x dx$
3. $\int e^x \sin x dx$
4. $\int_0^1 x^2 e^{-x} dx$

INTEGRALS - Solution to Example 7

1. Let $u = x$ and $v' = \sec^2 x$.

1. Let $u = x$ and $v' = \sec^2 x$. Then $u' = 1$ and $v = \tan x$.

1. Let $u = x$ and $v' = \sec^2 x$. Then $u' = 1$ and $v = \tan x$.

$$\int uv' = uv - \int vu'$$

$$\begin{aligned} \int x \sec^2 x dx &= x \tan x - \int \tan x dx \\ &= x \tan x - \ln\left|\sec x\right| + C \end{aligned}$$

INTEGRALS - Solution to Example 7 cont...

INTEGRALS - Solution to Example 7 cont...

2. Let $u = \ln x$ and $v' = 1$.

INTEGRALS - Solution to Example 7 cont...

2. Let $u = \ln x$ and $v' = 1$. Then $u' = 1/x$ and $v = x$.

2. Let $u = \ln x$ and $v' = 1$. Then $u' = 1/x$ and $v = x$.

$$\int uv' = uv - \int vu'$$

$$\int \ln x dx = x \ln x - \int (1/x) x dx$$

$$= x \ln x - \int dx$$

$$= x \ln x - x + C.$$

INTEGRALS - Solution to Example 7 cont...

INTEGRALS - Solution to Example 7 cont...

3. Let $u = e^x$ and $v' = \sin x$.

3. Let $u = e^x$ and $v' = \sin x$. Then $u' = e^x$ and $v = -\cos x$.

3. Let $u = e^x$ and $v' = \sin x$. Then $u' = e^x$ and $v = -\cos x$.

$$\int e^x \sin x dx = -e^x \cos x - \int e^x(-\cos x) dx$$
$$= -e^x \cos x + \int e^x \cos x dx.$$

3. Let $u = e^x$ and $v' = \sin x$. Then $u' = e^x$ and $v = -\cos x$.

$$\begin{aligned}\int e^x \sin x dx &= -e^x \cos x - \int e^x(-\cos x)dx \\ &= -e^x \cos x + \int e^x \cos x dx.\end{aligned}$$

We apply integration by parts again to $\int e^x \cos x dx$.

3. Let $u = e^x$ and $v' = \cos x$.

3. Let $u = e^x$ and $v' = \cos x$. Then $u' = e^x$ and $v = \sin x$.

3. Let $u = e^x$ and $v' = \cos x$. Then $u' = e^x$ and $v = \sin x$.
$\int e^x \cos x dx = e^x \sin x - \int e^x \sin x dx$.

3. Let $u = e^x$ and $v' = \cos x$. Then $u' = e^x$ and $v = \sin x$.
$\int e^x \cos x dx = e^x \sin x - \int e^x \sin x dx$.
Therefore $\int e^x \sin x dx =$
$-e^x \cos x + e^x \sin x - \int e^x \sin x dx$.

3. Let $u = e^x$ and $v' = \cos x$. Then $u' = e^x$ and $v = \sin x$.
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Therefore $\int e^x \sin x dx =$
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Hence $2 \int e^x \sin x dx = e^x(\sin x - \cos x)$ giving

3. Let $u = e^x$ and $v' = \cos x$. Then $u' = e^x$ and $v = \sin x$.
$\int e^x \cos x dx = e^x \sin x - \int e^x \sin x dx$.
Therefore $\int e^x \sin x dx =$
$-e^x \cos x + e^x \sin x - \int e^x \sin x dx$.
Hence $2 \int e^x \sin x dx = e^x(\sin x - \cos x)$ giving
$\int e^x \sin x dx = \dfrac{1}{2} e^x(\sin x - \cos x)$.

INTEGRALS - Solution to Example 7 cont...

4. Let $u = x^2$ and $v' = e^{-x}$.

4. Let $u = x^2$ and $v' = e^{-x}$. Then $u' = 2x$ and $v = -e^{-x}$.

4. Let $u = x^2$ and $v' = e^{-x}$. Then $u' = 2x$ and $v = -e^{-x}$.

$$\begin{aligned}\int x^2 e^{-x} dx &= -x^2 e^{-x} - \int 2x(-e^{-x}) dx \\ &= -x^2 e^{-x} + 2\int x e^{-x} dx.\end{aligned}$$

INTEGRALS - Solution to Example 7 cont...

4. For $\int xe^{-x}dx$, let $u = x$ and $v' = e^{-x}$.

4. For $\int xe^{-x}dx$, let $u = x$ and $v' = e^{-x}$. Then $u' = 1$ and $v = -\mathrm{e}^{-x}$.

4. For $\int xe^{-x}dx$, let $u = x$ and $v' = e^{-x}$. Then $u' = 1$ and $v = -e^{-x}$.
So

$$\begin{aligned} \int xe^{-x}dx &= -xe^{-x} - \int (1)(-e^{-x})dx \\ &= -xe^{-x} + \int e^{-x}dx \\ &= -xe^{-x} - e^{-x}. \end{aligned}$$

INTEGRALS - Solution to Example 7 cont...

4. Hence

4. Hence

$$\int_0^1 x^2 e^{-x} dx = \left[-x^2 e^{-x} - 2xe^{-x} - 2e^{-x}\right]_0^1$$
$$= -e^{-1} - 2e^{-1} - 2e^{-1} + 2$$
$$= 2 - \frac{5}{e}.$$

- A rational funcion is sometimes integrated by expressing it as a sum of smaller fractions called **partial fractions**.

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- The integral is found by integrating each partial fraction and add the results.

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- The integral is found by integrating each partial fraction and add the results.
- We look at the process of decomposing a given rational function into partial fractions.

INTEGRALS - First Rule

- A rational function

$$f(x) = \frac{p(x)}{q(x)}$$

can be expressed directly in partial fractions if the highest power of x in $p(x)$ is at least 1 less than the highest power of x in $q(x)$.

INTEGRALS - Example 8

- By the first rule, $\dfrac{3+2x}{x^3+1}$, can be expressed directly in partial fractions while $\dfrac{x^4+1}{x^3+2x}$ can not.

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- However, by division, $\dfrac{x^4+1}{x^3+2x} = x + \dfrac{1-2x^2}{x^3+2x}$.

INTEGRALS - Example 8

- By the first rule, $\dfrac{3+2x}{x^3+1}$, can be expressed directly in partial fractions while $\dfrac{x^4+1}{x^3+2x}$ can not.
- However, by division, $\dfrac{x^4+1}{x^3+2x} = x + \dfrac{1-2x^2}{x^3+2x}$.
- The fraction $\dfrac{1-2x^2}{x^3+2x}$ can be expressed in partial fractions.

INTEGRALS - Second Rule

- For any linear factor

$$ax + b$$

in the denominator of a rational function, there will be a corresponding partial fraction of the form

$$\frac{A}{ax+b}$$

where $A \in \mathbb{R}$.

INTEGRALS - Example 9

- Express the function

$$\frac{x}{(1-x)(2+x)}$$

in partial fractions.

- Suppose that

$$\frac{x}{(1-x)(2+x)} = \frac{A_1}{1-x} + \frac{A_2}{2+x} = \frac{A_1(2+x) + A_2(1-x)}{(1-x)(2+x)}$$

, i.e., $x = A_1(2+x) + A_2(1-x)$.

INTEGRALS - Solution to Example 9 cont....

- Let $x = -2$; $-2 = 3A_2 \Rightarrow A_2 = -2/3$.

- Let $x = -2$; $-2 = 3A_2 \Rightarrow A_2 = -2/3$.
- Let $x = 1$; $1 = 3A_1 \Rightarrow A_1 = 1/3$.

- Let $x = -2$; $-2 = 3A_2 \Rightarrow A_2 = -2/3$.
- Let $x = 1$; $1 = 3A_1 \Rightarrow A_1 = 1/3$.
- Therefore

$$\begin{aligned}\frac{x}{(1-x)(2+x)} &= \frac{1/3}{1-x} + \frac{-2/3}{2+x} \\ &= \frac{1}{3}\left(\frac{1}{1-x} - \frac{2}{2+x}\right).\end{aligned}$$

INTEGRALS - Third Rule

- For any linear factor $ax + b$ repeated r times in the denominator , there will be corresponding partial fractions of the form

- For any linear factor $ax + b$ repeated r times in the denominator , there will be corresponding partial fractions of the form

$$\frac{A_1}{(ax+b)}, \frac{A_2}{(ax+b)^2}, \ldots\ldots\ldots, \frac{A_r}{(ax+b)^r}$$

where $A_1, A_2, \ldots.., A_r \in \mathbb{R}$.

- Express the function

$$\frac{2}{(x-1)^2(x+1)}$$

in partial fractions.

- Suppose that

$$\frac{2}{(x-1)^2(x+1)} = \frac{A_1}{x-1} + \frac{A_2}{(x-1)^2} + \frac{A_3}{x+1}$$

$$= \frac{A_1(x-1)(x+1) + A_2(x+1) + A_3(x-1)^2}{(x-1)^2(x+1)},$$

i.e.,

$$2 = A_1(x-1)(x+1) + A_2(x+1) + A_3(x-1)^2.$$

INTEGRALS - Solution to Example 10 cont....

- Let $x = 1$; $2 = 2A_2 \Rightarrow A_2 = 1$.

- Let $x = 1$; $2 = 2A_2 \Rightarrow A_2 = 1.$
- Let $x = -1$;
  $2 = (-2)^2 A_3 \Rightarrow A_3 = 2/4 \Rightarrow A_3 = 1/2.$

- Let $x = 1$; $2 = 2A_2 \Rightarrow A_2 = 1$.
- Let $x = -1$;
  $2 = (-2)^2 A_3 \Rightarrow A_3 = 2/4 \Rightarrow A_3 = 1/2$.
- Equating constant terms; $2 = -A_1 + A_2 + A_3$
  $-A_1 + 1 + 1/2 = 2 \Rightarrow A_1 = -1/2$.

INTEGRALS - Solution to Example 10 cont....

- Therefore $\dfrac{2}{(x-1)^2(x+1)}$

$$= \frac{-1/2}{x-1} + \frac{1}{(x-1)^2} + \frac{1/2}{x+1}$$

$$= \frac{1}{2}\left(\frac{-1}{x-1} + \frac{2}{(x-1)^2} + \frac{1}{x+1}\right).$$

- For any quadratic factor $ax^2 + bx + c$ in the denominator , there will be corresponding partial fraction of the form

- For any quadratic factor $ax^2 + bx + c$ in the denominator , there will be corresponding partial fraction of the form

$$\frac{A_1x + A_2}{(ax^2 + bx + c)}$$

where $A_1, A_2 \in \mathbb{R}$.

INTEGRALS - Example 11

- Express the function

$$\frac{x}{x^4 - 16}$$

in partial fractions.

- By the difference of two squares $x^4 - 16 = (x+2)(x-2)(x^2+4)$.

- By the difference of two squares $x^4 - 16 = (x+2)(x-2)(x^2+4)$.
- Let

$$\frac{x}{x^4 - 16} = \frac{A_1}{x+2} + \frac{A_2}{x-2} + \frac{A_3x + A_4}{x^2+4}.$$

- By the difference of two squares $x^4 - 16 = (x+2)(x-2)(x^2+4)$.
- Let

$$\frac{x}{x^4-16} = \frac{A_1}{x+2} + \frac{A_2}{x-2} + \frac{A_3x + A_4}{x^2+4}.$$

- That is $x = A_1(x-2)(x^2+4) + A_2(x+2)(x^2+4) + (A_3x + A_4)(x+2)(x-2)$.

Let $x = 2$

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$2 = (2+2)(4+4)A_2 \Rightarrow 2 = 32A_2$ giving

$A_2 = 1/16.$

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$2 = (2+2)(4+4)A_2 \Rightarrow 2 = 32A_2$ giving

$A_2 = 1/16.$

Let $x = -2$

$-2 = (-2-2)(4+4)A_1 \Rightarrow -2 = -32A_1$ giving

$A_1 = 1/16.$

INTEGRALS - Solution to Example 11 cont....

Equating coefficients of x^3

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$0 = A_1 + A_2 + A_3 \Rightarrow 0 = 1/16 + 1/16 + A_3$

giving $A_3 = -2/16 = -1/8$.

Equating coefficients of x^3

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Equating constant terms we have

$0 = -8A_1 + 8A_2 - 4A_4$

Equating coefficients of x^3

$0 = A_1 + A_2 + A_3 \Rightarrow 0 = 1/16 + 1/16 + A_3$

giving $A_3 = -2/16 = -1/8$.

Equating constant terms we have

$0 = -8A_1 + 8A_2 - 4A_4$

$\Rightarrow 0 = -8(1/16) + 8(1/16) - 4A_4$ giving $A_4 = 0$.

Hence

$$
\begin{aligned}
\frac{x}{x^4 - 16} &= \frac{1}{16(x+2)} + \frac{1}{16(x-2)} - \frac{x}{8(x^2+4)} \\
&= \frac{1}{16}\left[\frac{1}{x+2} + \frac{1}{x-2} - \frac{2x}{x^2+4}\right].
\end{aligned}
$$

- For any quadratic factor $ax^2 + bx + c$ repeated r times in the denominator , there will be corresponding partial fractions of the form
$$\frac{A_{11}x + A_{12}}{(ax^2 + bx + c)}, \frac{A_{21}x + A_{22}}{(ax^2 + bx + c)^2}, \ldots\ldots,$$
$$\frac{A_{r1}x + A_{r2}}{(ax^2 + bx + c)^r}$$ where
$A_{11}, A_{12}, A_{21}, A_{22}, \ldots., A_{r1}, A_{r2} \in \mathbb{R}$.

INTEGRALS - Exercise 5

- Show that

$$\frac{2x-1}{(x-1)^2(x^2+x+1)^3} = \frac{1}{27(x-1)^2} - \frac{1}{27(x-1)} + \frac{x+1}{27(x^2+x+1)} - \frac{1}{9(x^2+x+1)^2} - \frac{x+3}{3(x^2+x+1)^3}.$$

INTEGRALS - Example 12

Evaluate the following integrals;

INTEGRALS - Example 12

Evaluate the following integrals;

1. $$\int \frac{x}{(1-x)(2+x)}dx$$

Evaluate the following integrals;

1. $\int \frac{x}{(1-x)(2+x)} dx$
2. $\int \frac{2}{(x-1)^2(x+1)} dx$

1. By Example 9,

1. By Example 9,
$$\frac{x}{(1-x)(2+x)} = \frac{1}{3}\left[\frac{1}{1-x} - \frac{2}{2+x}\right].$$

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Therefore

1. By Example 9,
$$\frac{x}{(1-x)(2+x)} = \frac{1}{3}\left[\frac{1}{1-x} - \frac{2}{2+x}\right].$$
Therefore
$$\int \frac{x}{(1-x)(2+x)}dx$$
$$= \frac{1}{3}\int\left(\frac{1}{1-x} - \frac{2}{2+x}\right)dx$$

INTEGRALS - Solution to Example 12 cont...

INTEGRALS - Solution to Example 12 cont...

1. $$= \frac{1}{3}\left[\int \frac{dx}{1-x} - 2\int \frac{dx}{2+x}\right]$$
$$= \frac{1}{3}[-\ln|1-x| - 2\ln|2+x|] + C.$$

INTEGRALS - Solution to Example 12 cont...

2. By example 10,

2. By example 10, $\dfrac{2}{(x-1)^2(x+1)} = \dfrac{1}{2}\left[-\dfrac{1}{x-1} + \dfrac{2}{(x-1)^2} + \dfrac{1}{x+1}\right]$.

2. By example 10, $\dfrac{2}{(x-1)^2(x+1)} = \dfrac{1}{2}\left[-\dfrac{1}{x-1} + \dfrac{2}{(x-1)^2} + \dfrac{1}{x+1}\right]$.
   Therefore

2. By example 10, $\dfrac{2}{(x-1)^2(x+1)} = \dfrac{1}{2}\left[-\dfrac{1}{x-1} + \dfrac{2}{(x-1)^2} + \dfrac{1}{x+1}\right]$.

   Therefore

$$\int \frac{2}{(x-1)^2(x+1)}dx$$
$$= \frac{1}{2}\left[-\int \frac{dx}{x-1} + \int \frac{2dx}{(x-1)^2} + \int \frac{dx}{x+1}\right]$$

INTEGRALS - Solution to Example 12 cont...

2.

$$= \frac{1}{2}\left[-\ln|x-1| + \int \frac{2dx}{(x-1)^2} + \ln|x+1|\right].$$

2.

$$= \frac{1}{2}\left[-\ln|x-1| + \int \frac{2dx}{(x-1)^2} + \ln|x+1|\right].$$

For $\displaystyle\int \frac{2dx}{(x-1)^2}$

2.

$$= \frac{1}{2}\left[-\ln|x-1| + \int \frac{2dx}{(x-1)^2} + \ln|x+1|\right].$$

For $\displaystyle\int \frac{2dx}{(x-1)^2}$ let $u = x - 1$.

2.

$$= \frac{1}{2}\left[-\ln|x-1| + \int \frac{2dx}{(x-1)^2} + \ln|x+1|\right].$$

For $\int \frac{2dx}{(x-1)^2}$ let $u = x - 1$. Then $du = dx$.

2.

$$= \frac{1}{2}\left[-\ln|x-1| + \int \frac{2dx}{(x-1)^2} + \ln|x+1|\right].$$

For $\int \frac{2dx}{(x-1)^2}$ let $u = x - 1$. Then $du = dx$.

So

$$\int \frac{2dx}{(x-1)^2} = 2\int \frac{du}{u^2} = 2\int u^{-2}du$$

$$= -\frac{2}{u} = -\frac{2}{(x-1)}.$$

2. Hence

2. Hence $\displaystyle\int \frac{2dx}{(x-1)^2(x+1)} = \frac{1}{2}\left[-\ln|x-1| - \frac{2}{x-1} + \ln|x+1|\right] + C.$

The area between a curve and the x-axis, for x between a and b, is found by the following;

The area between a curve and the x-axis, for x between a and b, is found by the following;

- If the curve of $f(x)$ is entirely above the x-axis, then the area is $\int_a^b f(x)dx$.

The area between a curve and the x-axis, for x between a and b, is found by the following;

- If the curve of $f(x)$ is entirely above the x-axis, then the area is $\int_a^b f(x)dx$.
- If the curve of $f(x)$ is entirely below the x-axis, then the area is $-\int_a^b f(x)dx$.

- If parts of the curve are below the x-axis and the other parts are above the x-axis, then the area is found by subtracting the integral for those parts of the curve that are below and adding the integrals for those lying above.

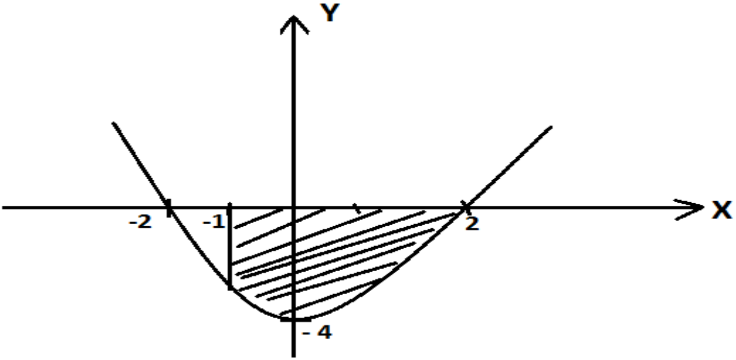
INTEGRALS - Example 13

INTEGRALS - Example 13

Find the area bounded by $y = x^2 - 4$, the x-axis and the lines $x = -1$ and $x = 2$.

INTEGRALS - Sketch for $y = x^2 - 4$ in Example 13

INTEGRALS - Sketch for $y = x^2 - 4$ in Example 13



INTEGRALS - Solution to Example 13

Since the graph is entirely below the x-axis, the area is;

Since the graph is entirely below the x-axis, the area is;

$$A = -\int_{-1}^{2} (x^2 - 4)dx = \int_{-1}^{2} (-x^2 + 4)dx$$

$$= \left[-\frac{x^3}{3} + 4x\right]_{-1}^{2} = \left(-\frac{8}{3} + 8\right) - \left(\frac{1}{3} - 4\right)$$

$$= -\frac{8}{3} + \frac{24}{3} - \frac{1}{3} + \frac{12}{3} = \frac{27}{3}$$

$$= 9 \text{ square units.}$$

INTEGRALS - Example 14

Find the area bounded by $y = x^2 + x + 4$, the x-axis and the ordinates $x = 1$ and $x = 3$.

INTEGRALS - Sketch for $y = x^2 + x + 4$ in Example 14

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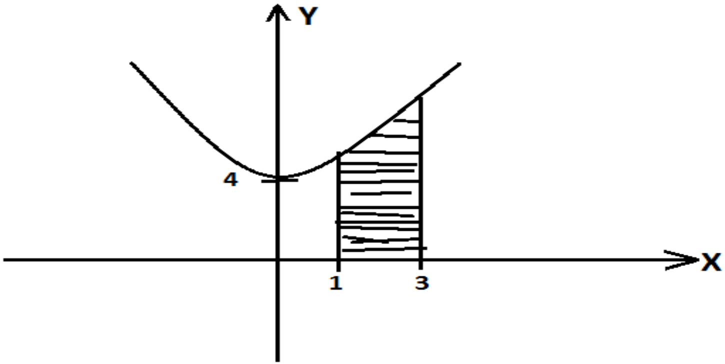


INTEGRALS -Solution to Example 14

The required area is entirely above the x-axis, so

The required area is entirely above the x-axis, so

$$A = \int_1^3 (x^2 + x + 4)dx = \left[\frac{x^3}{3} + \frac{x^2}{2} + 4x\right]_1^3$$

$$= \left(\frac{27}{3} + \frac{9}{2} + 12\right) - \left(\frac{1}{3} + \frac{1}{2} + 4\right)$$

$$= \left(\frac{54}{6} + \frac{27}{6} + \frac{72}{6}\right) - \left(\frac{2}{6} + \frac{3}{6} + \frac{24}{6}\right)$$

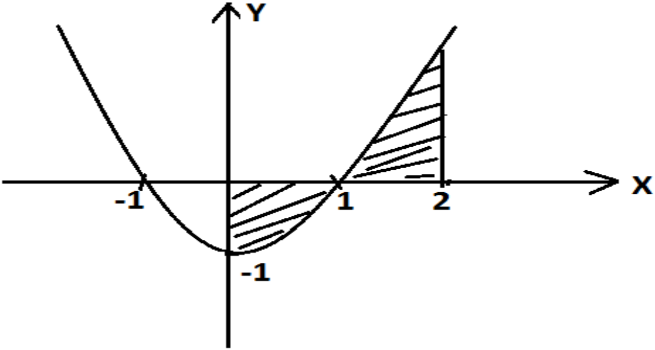
$$= \frac{153}{6} - \frac{29}{6} = \frac{124}{6} = \frac{62}{3} \text{ square units.}$$

INTEGRALS - Example 15

Find the area between the graph of $y = x^2 - 1$ and the x-axis for x between 0 and 2.

INTEGRALS - Sketch for $y = x^2 - 1$ in Example 15

INTEGRALS - Sketch for $y = x^2 - 1$ in Example 15



INTEGRALS - Solution to Example 15

The graph is below the x-axis when $0 \leq x \leq 1$.

The graph is below the x-axis when $0 \leq x \leq 1$. It is above the x-axis when $1 \leq x \leq 2$.

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The graph is below the x-axis when $0 \le x \le 1$. It is above the x-axis when $1 \le x \le 2$. Therefore

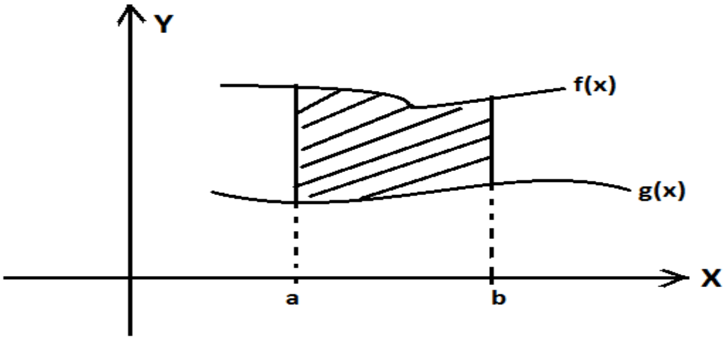
$$\begin{aligned} A &= -\int_0^1 (x^2 - 1)dx + \int_1^2 (x^2 - 1)dx \\ &= \int_0^1 (-x^2 + 1)dx + \int_1^2 (x^2 - 1)dx \\ &= \left[-\frac{x^3}{3} + x\right]_0^1 + \left[\frac{x^3}{3} - x\right]_1^2 \end{aligned}$$

INTEGRALS - Solution to Example 15 cont...

$$
\begin{aligned}
&= \left(-\frac{1}{3} + 1\right) + \left(\frac{8}{3} - 2\right) - \left(\frac{1}{3} - 1\right) \\
&= -\frac{1}{3} + \frac{3}{3} + \frac{8}{3} - \frac{6}{3} - \frac{1}{3} + \frac{3}{3} \\
&= \frac{6}{3} \\
&= 2 \quad \text{square units.}
\end{aligned}
$$

- The area between two graphs is found by subtracting the area between the lower graph and the x-axis from the area between the upper graph and the x-axis.

INTEGRALS - Area between Curves cont...



- Therefore the area of the shaded part in the diagram above is

- Therefore the area of the shaded part in the diagram above is

$$A = \int_a^b f(x)dx - \int_a^b g(x)dx$$
$$= \int_a^b [f(x) - g(x)]dx.$$

INTEGRALS - Example 16

Calculate the area of the segment cut from the curve $y = x(3 - x)$ and the line $y = x$.

INTEGRALS - Solution to Example 16

Solving for x in $x(3 - x) = x$ gives values of x at the points of intersection of the two graphs.

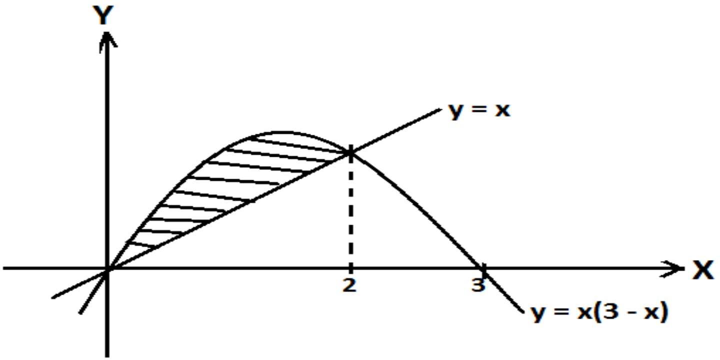
Solving for x in $x(3-x)=x$ gives values of x at the points of intersection of the two graphs. So

$$x(3-x)=x \Rightarrow 3x-x^2=x \Rightarrow x^2-2x=0$$

giving $x=0$ and $x=2$.

INTEGRALS - Sketch for Example 16

INTEGRALS - Sketch for Example 16



INTEGRALS - Solution to Example 16 cont...

$$
\begin{aligned}
A &= \int_0^2 [x(3-x)-x]dx \\
&= \int_0^2 (-x^2+2x)dx \\
&= \left[-\frac{x^3}{3}+x^2\right]_0^2 \\
&= -\frac{8}{3}+4 = \frac{4}{3} \quad \text{square units.}
\end{aligned}
$$

INTEGRALS - Example 17

Find the area of the region bounded by the curves $y = \sin x$ and $y = \cos x$, $x = 0$ and $x = \dfrac{\pi}{2}$.

INTEGRALS - Sketch for Example 17

The two graphs intersect when $\sin x = \cos x$, i.e., when $x = \dfrac{\pi}{4}$.

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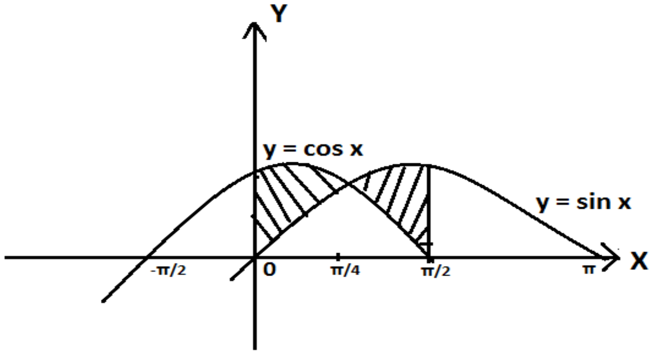

INTEGRALS - Solution to Example 17

INTEGRALS - Solution to Example 17

Note that $\cos x \geq \sin x$ when $0 \leq x \leq \dfrac{\pi}{4}$ and $\cos x \leq \sin x$ when $\dfrac{\pi}{4} \leq x \leq \dfrac{\pi}{2}$.

INTEGRALS - Solution to Example 17

Note that $\cos x \geq \sin x$ when $0 \leq x \leq \dfrac{\pi}{4}$ and $\cos x \leq \sin x$ when $\dfrac{\pi}{4} \leq x \leq \dfrac{\pi}{2}$. Therefore

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$$
\begin{aligned}
A &= \int_0^{\pi/4} (\cos x - \sin x)dx + \int_{\pi/4}^{\pi/2} (\sin x - \cos x)dx \\
&= [\sin x + \cos x]_0^{\pi/4} + [-\cos x - \sin x]_{\pi/4}^{\pi/2} \\
&= \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - 1\right) + \left(-1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) \\
&= 4/\sqrt{2} - 2 = 2\sqrt{2} - 2 \text{ square units.}
\end{aligned}
$$