Binomial Trees in Option Pricing

Peter Pavicic

Supervised by Dr. Rüdiger Frey

Bachelor's thesis for the degree of
Business and Economics

Institute for Statistics and Mathematics
Wirtschaftsuniversität Wien
Vienna, Austria
31. August 2023

Contents

List of Figures

Abstract

This bachelor thesis examines the seminal Cox-Ross-Rubinstein binomial option pricing model, and compares it to the Black-Scholes-Merton model in the context of stock options pricing. Both models are analysed from the aspect of their assumptions, pricing formulas, later extensions, potential limitations, as well as their contributions to the academic body of knowledge of option pricing. Two case studies provide an illustration to the usefulness of the binomial option pricing model. The first regards the approximation of the Black-Scholes-Merton formula for a European call option. The second examines the effectiveness of a hedging strategy against a short position in an American call option leading up to an ex-dividend date.

1 Introduction

1.1 Option definition

An options contract is a financial contract between the owner (the holder) of an option and its issuer (the writer of the option) which gives the owner the right but not the obligation to buy or sell a fixed amount of an underlying asset at an agreed upon price, named the strike price on (or leading up to) an agreed upon date, known as the expiration date). [David Hillier & Titman, 2012] The holder of the option is said to be in a long position, while the writer is said to be short the option. Since, an option presents the right but not the obligation to perform a trade, its value is always non-negative, which means that a short position's value is always non-positive . Due to this investors who write options need to be compensated for their short position. This compensation is referred to as the option premium, and is paid out by the holder of the option to its writer. [Smith, 1976]
As options contracts are legal contracts, the writer of the option is obligated to act as a counter-party to the holder of the option if and when the latter decides to exercise their right to buy (or sell) the underlying asset. Options fall under the category of financial derivatives, meaning that their value is derived from the value of an underlying asset. This can theoretically be anything but is most commonly the prices of other traded assets, such as a fixed amount of a company's stock, corporate bonds, or more exotic financial instruments, e.g. exchange traded funds, foreign currencies, another derivative, a stock index, etc. [David Hillier & Titman, 2012]
Options can be classified based on whether the right to exercise exists only on a specified date or leading up to that date, the former being referred to as a European option and the latter an American option. Another distinction in options can be made in whether the option conveys the right to buy, or to sell the underlying, the former referred to as a call option, the latter as a put option.
Options can be traded OTC (Over The Counter), in which case they can be fully tailored to the wants needs of the counter-parties entering the options contract. Exchange-traded, also known as listed options, on the other hand, are highly standardised and available to the general public.
As explained earlier, options contracts can take on a wide variety of forms depending on what the underlying asset is. This thesis will focus on standardised stock options. [David Hillier & Titman, 2012]

1.2 Option payoffs

To calculate the value of an option at the time it is written, it is best to first consider what its value can be upon the date on which it is exercised. This value is referred to as the option payoff.
The option payoff formula for European options is given by:

$$C_N = \max(0, S_N - K) \qquad P_N = \max(0, K - S_N) \tag{1}$$

[David Hillier & Titman, 2012] Where C_N denotes the value of a call option, and P_N the value of a put option, each equal to their respective payoffs on the day of expiration, N. S_N denotes the spot price (that is the available market price) of a single share of the underlying, S on day N. The payoff for American options follows analogously, however, since early exercise is possible, it is given by the difference between the prevailing spot price of the underlying and the strike price on the day of exercise.
Equations (1) can best be explained through the following examples:
Let C denote a European call option, on 100 shares of company A with a strike price of K. Let P denote a European put option with the same strike price and underlying asset as the call option. If on day N, $S_N > K$, the holder of the option can make money by exercising, which means buying the 100 shares for K each, and selling them for S_N right away. In doing this, the holder makes a profit of $(S_N - K)$ per share, in this case $100 \times (S_N - K)$. If on the other hand, on day N, $S_N < K$, the holder of the option would lose money if they were to buy at the strike price, as doing so is more expensive than buying the underlying at the prevailing spot price. Therefore a rational investor would never exercise in this case, and would keep waiting at least until $S_N \geq K$ or if N is the day of expiration, this would leave them with 0 made from the option.
In the case of the put option, P, it is easy to see that the owner who buys the underlying at the spot price S_N makes money when they can sell the asset at an expensive strike price, which is the case when $S_N < K(\iff 0 < K - S_N)$, otherwise the payoff is 0 for $S_N > K$. [Hull, 2010]
Equations (1) can be visualised by looking at the potential payoffs on day N.

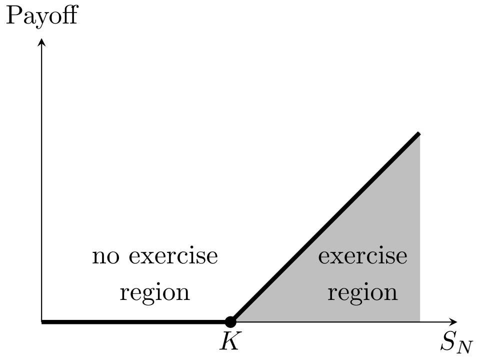

Figure 1: Call option payoff

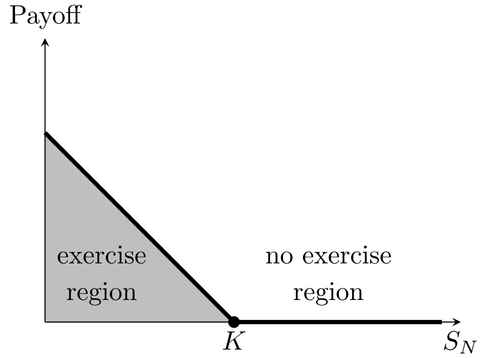

Figure 2: Put option payoff

Based on the relation between S_n and K at any timepoint n, a call or put option can be said to be:

1. *In the money* if immediate exercise would result in a positive cash flow.
2. *At the money* if immediate exercise would not bring a positive nor a negative cash flow.
3. *Out of the money* if immediate exercise would effect a negative cash flow

2 The Binomial Option Pricing Model

The binomial option pricing model built upon the famous Black-Scholes-Merton model to derive a simplified approach to pricing options which can be extended to arrive at the latter model. [Cox et al., 1979]

2.1 Notation

The following notation will be applied when describing the binomial option pricing model:

- S will denote the underlying asset of the option.
- N will denote the number of discrete time-steps of a model.
- $n \leq N$ will denote the discrete time for the model. Used as a subscript for S, S_n will denote the price of the underlying at time n.
- z_n will denote the discrete price movement of the underlying S in time-step n, which is either U for an up-movement or D for a down-movement. Z_n will denote the price movements leading up to n, e.g. $Z_3 = z_1 z_2 z_3 = UDU$.
- u, and d will denote the value of the price movements U, resp. D, which is one plus the rate of return of the price movements.
- r will denote one plus the risk-free interest rate, that is the initial investment plus the rate of return on the risk-free asset. This asset is explained in Section 2.2.
- F will denote the value of a stock option. Subscripted with n, F_n will denote the value of an option in the discrete time-step n. Subscripted with U or D, it will denote the value of the option after the corresponding price movements of the underlying. C will denote a call option, P a put option.

2.2 Assumptions

The assumptions made by the binomial option pricing model are:

1. Markets are efficient. This entails that there are no information asymmetries, all investors are rational which means they prefer more wealth to less wealth, and financial instrument prices factor in all relevant information. Further, all securities are divisible, so a fraction of any share or bond can be bought or sold at any time.

2. Existence of a risk-free asset. This implies that there is a financial asset completely free of risk, which has a fixed, positive one time-step interest rate, $r - 1$, which remains constant over time. Due to this, rational investors will never hold cash, instead investing any cash positions x, receiving xr for them after one time-step. As such all positions x are worth xr^n after n steps in the future. Equivalently, all future cash flows need to be discounted by the risk-free interest rate.

3. Principle of no arbitrage. Following the previous two assumptions, if all investors are rational and have all information available to them, all risk-less investment portfolios have the return equal to the return of the risk-free asset.

4. Absence of transaction costs. This means there are no taxes, duties, commission costs, costs of borrowing or shorting an asset. There is also no limit on the amount which can be borrowed or shorted.

5. Time discreteness. As opposed to the Black-Scholes-Merton model which assumes that trading is continuous [Black & Scholes, 1973], asset prices in the binomial model are determined on a step-by-step basis.

6. Stock price movements follow a binomial distribution. Continuing the last assumption, each stock price movement is regarded as a Bernoulli trial with a fixed up or down movement u or d with $u > r > d$, with probability $1 > q > 0$ of an up-movement occurring.

[Cox et al., 1979]

2.3 One-step model

2.3.1 Pricing European options for $N = 1$

Before turning our attention to the binomial model applicable to any number of time steps N, and more option types, let us consider the model for European options only, with $N = 1$.
As explained in Section 2.2, the binomial model assumes that each discrete time step follows a Bernoulli experiment, with the price of the underlying S going up by u with a probability of q and going down by d with a probability of $1 - q$. The prices of the underlying are then given by S_U and S_D respectively as illustrated in Figure 3.

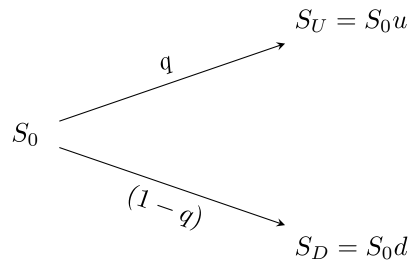

Figure 3: The price of the underlying S in a one-step binomial model

As established in Section 1.2, the values of options on the day of expiration are given by the respective payoff functions. Due to the existence of the risk-free asset, it is possible to discount any future cash flows by the risk-free interest rate. It therefore makes sense that the price of options one day before the expiration be given by discounting the expected value the possible future values, with $F_0 = [qF_U + (1 - q)F_D] / r$, where F_U, and F_D denote the values of the option in the future after one up-move resp. one down-move of the underlying S. This result would then seem to depend on the probability q, which would indicate that the value of an option depends on investors' assumptions about the price movements of the underlying. However, this is not the case, as this thesis will show in the following section. [Cox et al., 1979]

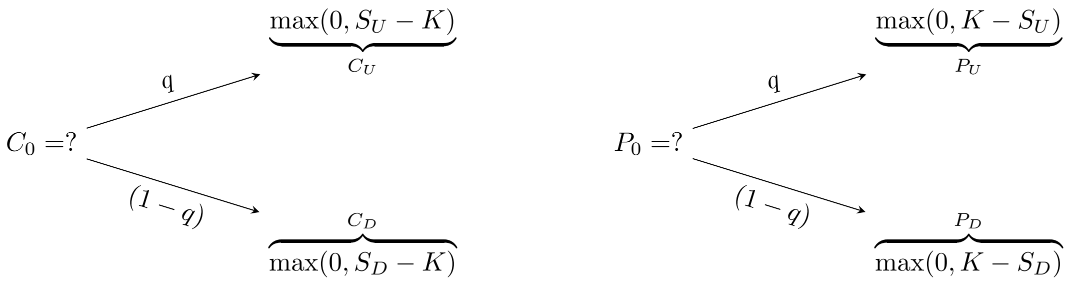

Figure 4: The price of a European call C and put option P in a one-step binomial model

2.3.2 Replicating portfolio

In order to arrive at the fair price of an option, F_0, the notion of a replicating portfolio, $\mathbf{Pf}_{\text{replicating}}(\Delta, B) = \Delta S + B$ is introduced. This portfolio consists of Δ shares of the underlying share, S, and B invested into the risk-free asset which returns r after one time step. The aim of this replicating portfolio is to replicate the value of the option for the next time-step, which in case of this model means replicating the option payoffs at $n = 1$. If this portfolio can be constructed, by the principle of no arbitrage, the value of this portfolio should equal the value of the option, and thereby a fair option premium should equal the price of constructing such a portfolio. In this case, it is possible to buy and hold the replicating portfolio from the proceeds of writing a fairly priced call or put option. Holding this portfolio of the short option and the long replicating portfolio results in net 0 cash flows at both $n = 0$ and $n = 1$ completely free of risk. This would define a perfect hedge, i.e. a position where the position is completely risk-less and no money is lost at $n = 1$. [Cox et al., 1979]

	$n = 0$	$n = 1$	
State of S	S_0	S_D	S_U
Writing an option	$-F_0$	-F_D	-F_U
Buying $\mathbf{Pf}_{\text{replicating}}$	$+(\Delta S_0 + B)$	$+\ \Delta S_0 d + Br$	$+\Delta S_0 u + Br$
Value of positions	$(\Delta S_0 + B) - F_0$	$\Delta S_0 d + Br - F_D$	$\Delta S_0 u + Br - F_U$

Table 1: Value of being short a European option and long its replicating portfolio

As can be seen from Table 1 this perfect hedge can be achieved under the conditions:

$$\begin{aligned} F_0 &= \Delta S_0 + B \\ F_D &= \Delta S_0 d + Br \\ F_U &= \Delta S_0 u + Br \end{aligned}$$

With S_0, K, r, u, d given, (F_U and F_D are also known since they equal their payoff functions in this case, which are functions of the given variables), it is possible to solve for Δ and B:

$$\Delta = \frac{F_U - F_D}{S_0(u - d)}, \quad B = \frac{uF_D - dF_U}{r(u - d)} \tag{2}$$

Since $F_0 = \Delta S_0 + B$, as discussed earlier, it is now possible to compute the fair probability π which results in the equation $F_0 = [\pi F_U + (1 - \pi)F_D]/r$:

$$\begin{aligned} F_0 &= \Delta S_0 + B \\ &= \frac{F_U - F_D}{u - d} + \frac{uF_D - dF_U}{(u - d)r} \\ &= \left[\left(\frac{r - d}{u - d}\right) F_U + \left(\frac{u - r}{u - d}\right) F_D\right] / r \end{aligned}$$

Defining

$$\pi := \left(\frac{r-d}{u-d}\right) \quad \implies \quad 1-\pi = \left(\frac{u-r}{u-d}\right) \tag{3}$$

We arrive at

$$F_0 = \frac{\pi F_U + (1-\pi)F_D}{r}$$

which defines a fair value for European call and put options.
What is important to consider is that the value π has the properties of a probability, since it is always greater than 0 and less than 1. This is because if $|\pi| > 1$ and $u > d$ (which is true by definition) $\implies r > u$, and there is a possibility of arbitrage in the market involving shorting S and investing the proceeds into the risk-free asset. π is in the context of option pricing referred to as a risk-neutral probability- The reason for this name will be elaborated on in Section 2.5.1. It can then be said that the value of an option is equal to its discounted expected value 1 time-step from now in a risk-neutral world, denoted by $\mathbb{E}^*$, whereby it is important to note that this holds with all kinds of investors, regardless of their risk-attitude.

$$F_0 = \mathbb{E}^*[F_1]/r \tag{4}$$

[Cox et al., 1979]

2.4 Pricing formula

2.4.1 European Options

As we have already established, European options are options which only allow exercise at the expiration date. Hence, their value at $n = N$ is given by the respective payoff functions. At the end of Section 2.3.1, we determined the value of a European option upon creation for a one-step model $N = 1$. Note that in the derivation of this formula, starting from Table 1 we did not stipulate where we get the values for F_U and F_D from, we simply assumed we know their value (which in the case of $N = 1$ was the respective payoff functions). Due to this, the derivation of (4) can be generalised and we have a recursive formula for pricing European options with any number of time-steps N:

$$F_n = \begin{cases} F_N & \text{if } n = N, \\ \mathbb{E}^*[F_{n+1}|F_n]/r & \text{if } n < N \end{cases} \tag{5}$$

where $\mathbb{E}^*[F_{n+1}|F_n]$ denotes the risk-neutral expected value of F_{n+1} given the current value of the option, F_n.

[Cox et al., 1979]
As is evident from Equation (5), the price of European options can be derived by first computing the values at the end-nodes and then solving the tree backwards from their values. This process can be visualised as a binomial tree illustrated in Figure 5.

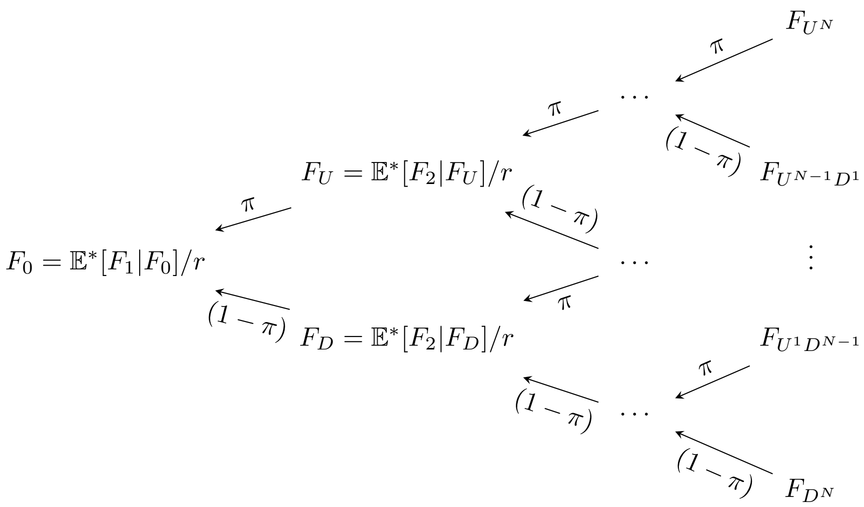

Figure 5: Solving an N-step binomial tree of a European call option backwards

Since this process of determining F_0 is dependent on the payoff formulas at the end-nodes of the binomial tree, it is possible to write a general valuation formula for a European option:

$$F_0 = \frac{1}{r^N}\left[\sum_{j=0}^{N}\binom{N}{j}\pi^j(1-\pi)^{N-j}F_{U^jD^{N-j}}\right] \tag{6}$$

In this formula, j indicates the number of up-movements of the underlying. $\binom{N}{j}$ then gives the number of paths which lead from $n = 0$ to the payoff event which occurs after j up-movements. [Cox et al., 1979]

2.4.2 Put-call parity

European put and call option prices can be related to one another using the put-call parity formula. This stipulates that European puts and calls with the same underlying and the same strike price under no arbitrage must adhere Formula (6).

$$C_0 - P_0 = S_0 - K/r^n \tag{7}$$

This can be proven using a replicating portfolio, which in this case consists of a long position in the underlying and a short position in the risk-free asset. The payoffs of the two portfolios are identical at $n = N$, as can be seen in Table 2, ignoring the trivial case of $S_N = K$, where both portfolios have a payoff of 0. In the absence of arbitrage opportunities, the relationship between C_0 and P_0 hold according to Equation (7). (S_0, K, r are assumed to be exogenous). [David Hillier & Titman, 2012]

	$n = 0$	$n = N$	
Scenario		$S_N < K$	$S_N > K$
Buying a call option	$+C_0$	0	$S_N - K$
Selling a put option	$-P_0$	$-(K - S_N)$	0
Buying S	$+S_0$	S_N	S_N
Shorting (Borrowing) K/r^N	$-K/r^N$	$-K$	$-K$

Table 2: Proof of put-call parity using a replicating portfolio

It is important to note that in the proof using the tracking portfolio, there were no assumptions made about stock price movements, only about the absence of transaction costs, existence of a risk-free asset and the availability of limitless borrowing. As such, the put-call parity holds in any model which also maintains these assumptions, including the Black-Scholes-Merton model. [Hull, 2010]

2.4.3 American Options

As established at the beginning of this thesis, American options carry the additional benefit of allowing exercise before the expiration date. This benefit adds value to a European option if there is any scenario in which early exercise is optimal. This can be the case for American put options, however, for American call options there is no instance in which exercising before expiration is beneficial. [David Hillier & Titman, 2012] This can be shown using the put-call parity formula from the previous section:

$$\begin{aligned} & C_0 - P_0 = S_0 - K/r^n \qquad (P_0 \geq 0, r > 1) \\ & \iff \\ & C_0 \geq S_0 - K/r^n > S_0 - K \qquad \text{if } n > 0 \\ \implies & C_0 > S_0 - K \end{aligned}$$

This means that the value of a European call option on a non-dividend paying stock prior to expiration is greater than its payoff if exercised immediately.
This holds equivalently for all sub-trees of an n-step binomial tree, that is for all $n < N : C_n > S_n - K$. Since a European call option is worth more than immediate exercise at any point before expiration, then this means it is always worth waiting until the expiration date until exercising the option. Therefore, the value of an American call on a non-dividend paying stock is given by the value of an equivalent European call. [Hull, 2010]
The intuitive explanation for this is that the price of the underlying can still keep increasing after any number of up-movements, even when it is deep in the money. Moreover, since for the purpose of pricing options, we assume a risk-neutral world, where the expected return on a stock is the risk-free rate, the stock price is expected to increase by exactly the risk-free rate. Additionally, paying the strike price is a negative cash flow, which is smaller the later the option is exercised due to discounting, so its payment should be deferred as much as possible. [Hull, 2010]
In the case of an American put option, the situation is not so easy. Trying to apply the put-call parity formula from Equation (7) leads to: $P_0 \geq K/r^n - S_0$, and no definitive statement can be made about the relation between $K - S_0$ and P_0. [Hull, 2010]
As such, the only way to find out if early exercise is optimal for a put option is solving the tree backwards, at each time-step also considering the value of early exercise and comparing it to the value of waiting. A modified version of the recursive formula for determining European options from Equation (5) can be derived, where the value of the American option is given by either the payoff at that time-step or the risk-neutral expected future value of the option, whichever is greater. [Cox et al., 1979]
The intuitive explanation for early exercise possibly being optimal is composed of 2 things: First, the stock price once close to 0, cannot go much lower as negative stock prices are not possible. As such, the investor can mostly lose money by waiting for a possible upturn in the stock movements, and only stands to gain very little from a further decline in the stock price. More importantly, the strike price is received in the case of a put option, which may incentivize early exercise as the same K amount n time-steps later is worth only K/r^n in today's terms. [Cox et al., 1979]

2.4.4 Dividends

Having considered the most common stock options on non-dividend paying stocks up until this point, it is possible to modify the formulas to include dividend payments.
In this section the following assumptions are made about dividend policy: Dividends are paid in cash only. They are paid immediately on the ex-dividend date, the investors who own the stock on that day receive the dividend, so there is no discounting required. This is contrary to the real world, where the dividend payout date is typically some time later, as for example in the real-life case examined in Section 4.2.
The ex-dividend date can be defined as the day on or after which a holder of the stock is no longer entitled to the dividend. Whoever is short the stock, that is owes the stock to another investor leading up to the ex-dividend date also owes the amount of the dividend. [Hull, 2010]
Following these assumptions and the assumption of no arbitrage, the price of the underlying on the ex-dividend date must drop by the value of the dividend that is paid out. Intuitively this makes sense, as leading up to the ex-dividend date, the share price contains the discounted value of the future dividend

payment, and investors would like to receive that dividend. After the ex-dividend date however, the dividend has been paid out, and owners of the stock are not entitled to a known dividend payment. [Hull, 2010]
If a stock pays a fixed dividend amount, this drop in price at the ex-dividend date can be subtracted in the payoff formulas. However, this causes a problem with binomial trees, as following the drop induced by the ex-dividend date, they do not reconnect as the standard binomial trees do. This causes a great increase in computational effort, as the number of subsequent nodes in each time-step increases exponentially. [Hull, 2010]
This is illustrated in Figure 6.

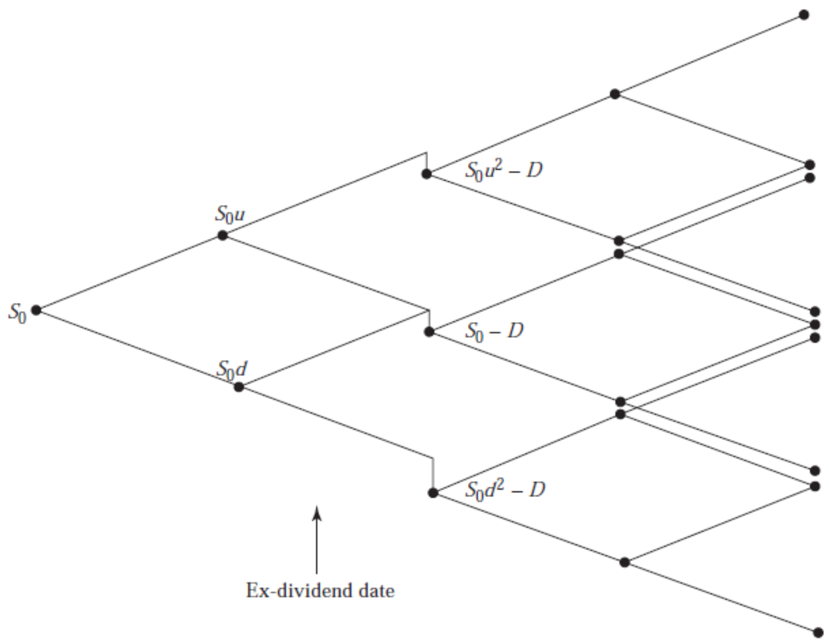

Figure 6: Non-recombining tree after fixed cash ex-dividend date [Hull, 2010]

The literature offers multiple solutions to this problem, most of which are beyond the scope of this thesis. [Vellekoop & Nieuwenhuis, 2006] [Hull, 2010]. A special case, however, is worth being mentioned: If it is assumed that the nominal dividend amount is not fixed but rather the percentage of the share price which is paid out as a dividend, i.e. the dividend yield is fixed, this means that recombining binomial trees can be constructed with any number of dividends in the model. [Cox et al., 1979]
Otherwise, another simple case presents itself when the ex-dividend date matches the expiration date for an option. It makes sense that the payoff formulas for such a European option is:

$$C_N = \max(0, S_N - D - K) \qquad P_N = \max(0, K - S_N + D)$$

With the call and put option's payoff formulas for American options following analogously, in the case when exercise is optimal on the expiration date. [Hull, 2010]
What is interesting with American call options is that the expected price drop can have an influence on the optimal exercise date. As a matter of fact, the only time that premature exercise may be is right before the ex-dividend date. In contrast, it is never optimal to exercise between ex-dividend dates. [Hull, 2010]

2.5 Risk-neutral valuation

2.5.1 No arbitrage

π from Equation (3) is called a risk-neutral probability in the context of option pricing. The only assumption used for deducting π in that section was that of no arbitrage. This, though it may seem counter-intuitive at first, means that even if the real-life probability of an up-movement occurring is not π, Equation (4) still holds and should be used to arrive at the fair value for an option. This is the case

because it is always possible to create a replicating portfolio, $\mathbf{Pf}_{\text{replicating}}(\Delta, B)$ for the cost of C_0 when using Equation (2).
If we consider a call option ignoring this fact, $C_0' \neq C_0 = \mathbf{Pf}_{\text{replicating}}$:

$$C_0' > C_0 \implies C_0' - \mathbf{Pf}(\Delta, B) > 0 \text{ at } n = 0$$

In which case the writer of the option enjoys arbitrage profits.

$$C_0' < C_0 \implies C_0' - \mathbf{Pf}(\Delta, B) < 0 \text{ at } n = 0$$

In which case the writer of the option is the source of arbitrage profits.

2.5.2 Risk-attitude

A key problem with models for pricing financial derivatives prior to the revolutionary work of Black, Scholes, and Merton was that they all included some form of risk-perception of investors, most typically in the form of the expected stock price. [Black & Scholes, 1973]
Most people in the real world are risk-averse, which presents itself in investors in that their portfolios will consist mostly of bonds, and blue chip, i.e. established stocks with low volatility. Less risk-averse investors on the other hand would be willing to invest more into riskier stocks, or other financial assets with high volatility in hopes of a greater return. In both cases however, a risk averse individual will prefer to accept a given amount of money paid out with certainty vis-à-vis some other type of investment with potentially higher expected payoff but paid out with some uncertainty. The amount of money is then referred to as that person's certainty equivalent. Risk-neutral investors therefore must be compensated for taking on risky investments, which takes shape in the stock market in the form of a risk premium. [Robert S. Pyndick, 2018]. Risk-seeking individuals, on the other hand display an affinity for risk over expected returns. Risk-seeking behaviours include most types of gambling, such as in casinos, or sports betting. These individuals pursue a bet with unfair odds even when presented with the option of receiving a certain amount greater than the expected payoff from the bet, i.e. they are even willing to "pay" a risk premium to be allowed to participate in an event with risky payoff. [Hull, 2010] Risk-neutral investors are indifferent between receiving a certain amount of cash, or taking a bet, so long as this bet has the exact expected payoff of the certain payment. This means that their certainty equivalent is exactly equal to the expected payoff. [Robert S. Pyndick, 2018].
A world where all investors are risk-neutral is much simpler than our risk-averse world. Because investors do not require compensation for taking on risk, they will invest into anything where the mean return is higher than the risk-less interest rate. Due to this, in a risk-neutral world with no arbitrage, the expected return on all financial assets is equal to the risk-less interest rate, since if it were otherwise arbitrage would be possible in either long or shorting one of the assets. [Hull, 2010]
Another great benefit of the risk-neutral world is that all cash-flows, even if uncertain need not be adjusted for the risk-premium of investors. Therefore all future cash flows are discounted by the risk-free interest rate. [Hull, 2010]
The argument for using risk-neutral probabilities stems from the fact that in the binomial option pricing model, as well as the Black-Scholes-Merton differential equation (14), there is no variable which measures the risk-attitude or expected risk of investors. This is due to the fact that in both, the derivation of the fair price of options can be determined by constructing a risk-less hedging portfolio. If, then, risk attitude does not play a role in the price of options, it is possible to assume without loss of generality that options can be valued according to their risk-neutral equivalent, which is simpler than any model with risk attitude factored in. [Hull, 2010]

2.5.3 Hedging portfolio

In the binomial option pricing model, as shown in Section 2.3.2, at each possible time-step it is possible to write an option and remain hedged until the next time-step. What is even more interesting in this, is that because the position is hedged at $n + 1$ for any n, it is theoretically possible to sell the tracking portfolio, and buy a new one, whereby both of these tracking portfolios are going to equal in value an option written on that day. Due to this, it can be proven that in the binomial model, it is possible to remain hedged all the way until an option's expiration date. [Hull, 2010] This hedging strategy is referred to as delta hedging, stemming from the symbol used for the amount of underlying to buy to remain hedged.

2.6 Limitations and extensions

Extensions The application of the binomial model is especially useful when it comes to American options, for many of which a closed-form solution is yet to be found. In this case, numerical procedures, such as recombining binomial, also known as lattice models are particularly helpful, since optimal exercise dates can be easily identified in such models. [Broadie & Detemple, 1996]
According to Rubinstein, another suitable application presents itself in an extension to the standard binomial model. He proposes assigning a risk-neutral probability distribution to the end-nodes of the binomial lattice derived from prevailing options prices' Black-Scholes implied volatilities. (Implied volatilities are explained in Section 3.5). This thereby, he argues, allows for modelling stochastic jumps and non-constant volatility throughout the model, which account for the empirically observed volatility smile, unlike the Black-Scholes formula.

Limitations One of the greatest limitations of the Cox-Ross-Rubinstein model is its quadratic computational order of growth. This means that in an algorithmic implementation, as the number of time-steps is doubled, the runtime approximately increases 4-fold. There have been improved lattice models which speed up the computations by a constant, however not the order of growth. In other cases, such as approximations which have fixed runtime, e.g. the Lower-Upper Bound Approximation, the derived values have fixed error which cannot be decreased by increasing the number of time-steps. [Broadie & Detemple, 1996].
The binomial model can be used to approximate the Black-Scholes-Merton price. [Cox et al., 1979] This is explained in detail in Section 3.4.2, and a case study is examined in Section 4.1. In this regard, the binomial option pricing model can again be improved in the aspects of speed and convergence rate, however, the order of convergence remains the same in all cases. [Leisen & Reimer, 1995].
In conclusion, it seems that though binomial option pricing models seems to be a great alternative to extended Black-Scholes-Merton models in some applications, it remains limited to these applications. This could be explained by the computational complexity of the model, vis-á-vis a closed-form solution for extended BSM models. It should be mentioned however, that in applications where it performs strongly, it can indeed be a useful tool. Such a case is examined in Section 4.2, where the standard binomial option pricing model is applied to determine a replication portfolio which hedges against a short American call option leading up to an ex-dividend date.

3 Black-Scholes-Merton model

The Black-Scholes-Merton model, originally published in 1973 by Black and Scholes revolutionized the world of finance by providing a theoretical framework to value European-style options. Their seminal paper, titled The Pricing of Options and Corporate Liabilities, as the name implies, did not only "change the game" for the options market, but also had an incredible impact on corporate finance. As the first paper to price derivatives without arbitrary assumptions of risk preferences or expected returns, it argued for a singular, fair price based on the principle of no arbitrage, derived from the construction of a portfolio, which is instantaneously perfectly risk-less, given the model's assumptions. The paper also popularised the idea that corporate liabilities, such as equity and debt can be valued similarly to stock options which is why the pricing formulas based on it could be applied to these securities as well. [Black & Scholes, 1973] For their work, Myron Scholes and Robert Merton received a Nobel Prize in Economics in 1997, which Fischer Black did not live long enough to witness. [MacKenzie, 2006]

3.1 Assumptions

The Black and Scholes clearly outline the assumptions under which their equation works, which are:

1. The short term interest rate is known and constant throughout time.
2. Stock prices follow a random walk in continuous time. Subsequently the distribution of stock prices at the end of a finite interval follow a log-normal distribution. Further, the variance rate of return on the stock is constant.
3. The stock pays no dividends.
4. The options under consideration are European.

5. Absence of transaction costs and taxes for both the stock and the options.
6. There is no limit on the amount of borrowing any fraction of any security, with the cost of borrowing equal to the short-term interest rate
7. There are no penalties for short selling.

[Black & Scholes, 1973]

3.1.1 Distribution of stock prices and returns

While a full derivation of the pricing formulas employed in the Black-Scholes-Merton model is beyond the scope of this thesis, it is worthwhile trying to understand its assumptions. While most of them are clear and similar to assumptions outlined in the context of the binomial model, there is a difference in the assumption about the development of stock prices. [Hull, 2010]
An example of a random walk in continuous time can for the case of stock price movements be understood as the succession of stock price movements on a price chart, drawn without lifting the pen. This is contrary to the real world, where successive price movements are actually given by discrete trades taking place between agreeing counter parties. A stochastic process which fits this description of a random walk is a generalised Wiener process, which will be examined closer. [Hull, 2010]
Let us define a basic Wiener process as a stochastic Markov process which has a mean of 0 and variance of 1 per year. A Markov process is a stochastic process in which only the current value of a variable is relevant to determining its future value, and not on past values or past changes in its value. The future value is then determined by a probability distribution. Stock prices are generally assumed to have this Markov property, implying that the past performance of a stock is not indicative of its future price movements. [Hull, 2010]
Formally, W is said to follow a basic Wiener process if:

1. The change ΔW during a small period of time Δt is $\Delta W = \epsilon\sqrt{\Delta t}$, where $\epsilon \sim \mathcal{N}(0, 1)$
2. The values of ΔW are independent for any two different Δt.

In this case, $W \sim \mathcal{N}(0, \Delta t)$. [Hull, 2010]
This allows us to define a generalised Wiener process, which for a variable x can be defined in terms of a Wiener process dW and constants a and b:

$$dx = adt + bdW$$

where this denotes

$$\Delta x = a\Delta t + b\Delta W \qquad \text{as } \Delta t \to 0$$

a is said to be the drift rate of x, i.e. the mean change per unit of time, While b is referred to as the variance rate, that is the variance per unit of time. This also means that

$$\Delta x \sim \mathcal{N}(a\Delta t, b^2\Delta t) \tag{8}$$

per the definition of ΔW.
A generalised Wiener process is called an Itô process if the drift and variance rates, a, and b are defined as functions of the variable x and time t. Thus, in an Itô process the drift and variance rates are no longer set to be constant, as their value changes as x changes over different time periods t. [Hull, 2010]
An Itô process can be written as:

$$dx = a(x, t)dt + b(x, t)dW$$

This can be applied to stock returns at time t as:

$$dS = \mu Sdt + \sigma SdW \tag{9}$$

or

$$\frac{dS}{S} = \mu dt + \sigma dW$$

Where we see that stock price returns follow an Itô process with the drift rate μ denoting a per-period expected rate of return and σ^2 as the variance of the stock. [Hull, 2010]
The reason why S on its own does not follow an Itô process with drift μ and variance rate σ^2 is because investors do not form expectations of nominal price changes, but rather the expected returns and variance of a stock. These formulas formalise the instantaneous price changes which take place in infinitesimally small time-steps as having a drift of the expected return μ and a variance rate of the stock's variance rate σ^2. [Hull, 2010]
Itô's lemma stipulates that if S (or more generally, any variable which follows an Itô process) follows an Itô process, then for a function $G(S,t)$:

$$dG(S,t) = \left(\frac{\partial G}{\partial S}\mu S + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial S^2}\sigma^2 S^2 \right) dt + \frac{\partial G}{\partial S}\sigma S dW \tag{10}$$

Therefore $G(S,t)$ follows an Itô process as well.
Subsequently, setting $G := \ln(S)$ results in:

$$\begin{aligned} d\ln(S) &= \left(\frac{1}{S}\mu S + 0 + \frac{1}{2}\frac{-1}{S^2}\sigma^2 S^2 \right) dt + \frac{1}{S}\sigma S dW \\ &= \left(\mu - \frac{\sigma^2}{2} \right) dt + \sigma dW \end{aligned}$$

Thereby, a change in the value of $\ln S$ from time 0 to a future value at time T is normally distributed:

$$\ln\left(\frac{S_T}{S_0}\right) = \ln S_T - \ln S_0 \sim \mathcal{N}\left[\left(\mu - \frac{\sigma^2}{2}T, \sigma^2 T\right)\right] \tag{11}$$

and

$$\ln S_T \sim \mathcal{N}\left[\ln S_0 + \left(\mu - \frac{\sigma^2}{2}T, \sigma^2 T\right)\right]$$

Showing why the assumption holds that at the end of a finite time interval stock prices are log-normally distributed when stock price movements follow a random walk. [Hull, 2010]
This formula can also be transformed to get the probability distribution for the rate of continuous compounding:

$$\begin{aligned} S_T &= S_0 e^{xT} \\ x &= \frac{1}{T}\ln\frac{S_T}{S_0} \\ x &\sim \mathcal{N}\left(\mu - \frac{\sigma^2}{2}, \frac{\sigma^2}{T}\right) \end{aligned}$$

Therefore, the continuously compounded rate of return is normally distributed with mean $\mu - \sigma^2/2$ and variance σ^2/T. This also means that the expected value of the continuously compounded rate of return is $\mu - \sigma^2/2$. [Hull, 2010]

3.2 Derivation of the pricing formulas

The full derivation of the pricing formulas are beyond the scope of this thesis, as mentioned before. The main idea of the derivation however, is important as it laid the foundation to the concept of risk-neutral valuation, and the use of hedging portfolios to arrive at the fair price of financial derivatives (the same approach used for the fair valuation in the Cox-Ross-Rubinstein model). [Hull, 2010]
First, it is argued that since derivatives derive their value from the underlying stock, their value depends on the underlying and the time, i.e. the price of a derivative f with the underlying S follows an Itô process, $f(S,t)$. Itô's lemma from Equation (10) is applied to this, and the stock price's (9) and the derivatives' Itô processes are considered for a short time interval Δt:

$$\Delta S = \mu S \Delta t + \sigma S \Delta W \tag{12}$$

$$\Delta f(S,t) = \left(\frac{\partial f}{\partial S}\mu S + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2 \right) \Delta t + \frac{\partial f}{\partial S}\sigma S \Delta W \tag{13}$$

A hedging portfolio is then constructed such that the Wiener process, ΔW is eliminated: this consists of writing one derivative, f and $-\frac{\partial f}{\partial S}$ of the underlying S:

$$\Pi = -f + \frac{\partial f}{\partial S}$$

The change in value of this portfolio $\Delta\Pi$ is then considered for a small period of time Δt, and the resulting equations when transformed still do not contain the Wiener process ΔW. The conclusion is drawn that the portfolio remains risk-less in the duration Δt. Due to the principle of no arbitrage, the return on this risk-less portfolio Π during this time must equal the return of the short term interest rate (which is assumed to be risk-less) for this time. After further calculations, the Black-Scholes-Merton differential equation for pricing derivatives is derived. The prices of different derivatives at given time points are calculated based on the boundary conditions which arise from the nature of the respective derivatives. For a European call option for example, the boundary condition is $f = \max(S - K, 0)$ at the expiration date, stipulating the relationship between S, K, and f. [Hull, 2010]

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf \tag{14}$$

As hinted at earlier, the Black-Scholes-Merton differential equation (14) was the one that lead to the concept of risk neutral valuation. This is because the equation contains only the variables for the current stock price, S, the time t, stock price volatility, σ, and the short-term interest rate, r. However, there are no variables which would depend on investors' expectations, such as the expected return of the stock. It can then be said that the fair price of derivatives is the same for any risk-attitude. This means that in the context of derivatives pricing, the simplifying assumptions of risk-neutral investors can be used for any world, not just one where investors are risk-neutral. [Hull, 2010]

3.3 Pricing formulas

Applying the boundary conditions for European call and put options, $f_T = \max(S - K, 0)$, resp. $f_T = \max(K - S, 0)$ at the expiration date, T, to the differential equation yields the fair price for European options:

$$c = S_0 N(d_1) - Ke^{-rT} N(d_2) \tag{15}$$

$$p = Ke^{-rT} N(-d_2) - S_0 N(-d_1) \tag{16}$$

where

$$d_1 = \frac{\ln(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

in which $N(x)$ is the cumulative distribution function of a standard normal distribution, $\mathcal{N}(0,1)$, and T is the time to maturity of the options.

To understand the components of the call option formula more easily, it is best to start with the expected payoff formula in a risk-neutral world:

$$S_0 N(d_1) e^{rT} - KN(d_2)$$

$N(d_2)$ in both formulas denotes the probability in a risk-neutral world that the call will be exercised. The expression $S_0 N(d_1) e^{rT}$ corresponds to the expected stock price at time T in a risk-neutral world where stock prices less than the strike price are counted as 0. Discounting this expression results in the Black-Scholes formula for European calls (16). [Hull, 2010]

3.4 Comparison with the binomial option pricing model

3.4.1 General comparison

Since the binomial option pricing model is derived from the Black-Scholes-Merton model, many of their characteristics are shared. As a matter of fact, the stochastic process used in the binomial model is itself a random walk, in which, much like in the Black-Scholes-Merton model the stock prices at the end of a fixed period length approach a log-normal distribution. [Hull, 2010]
The main characteristic which distinguishes the two models is the aspect of time in the models. While the CRR model assumes discrete time, the Black-Scholes-Merton model assumes continuous trading. While they both share principles, an obvious difference appears in the way delta-hedging can be executed. If one is to hedge using the binomial model, portfolio weights are re-balanced at each discrete time-step. However, with the Black-Scholes-Merton model, the hedging portfolio remains risk-less only in the instantaneous moment in which it is created. The hedging portfolio needs to be re-balanced continuously to remain perfectly risk-less, because the delta ($\partial f/\partial S$) calculated in Equation (13) constantly changes, as time changes. [Hull, 2010]
When adding transaction costs to the problem, it is immediately clear that due to this nature of continuous re-balancing the Black-Scholes model does not fare well in real life, as trading costs go up exponentially. With the CRR model, the problem is also immediately clear, as the optimal replicating portfolio changes very quickly in liquid markets, which translates to imperfect hedging positions in the real world [Leland, 1985]. A case study of applying a hedging strategy for the CRR model is examined later in Section 4.2.

3.4.2 Limiting case of the CRR model

The binomial option pricing model contains as its limiting case the Black-Scholes-Merton model. While the full proof of this fact is beyond the scope of this thesis, the beginning of the proof will be used in approximating the Black-Scholes-Merton value of a European call option in Section 4.1. [Cox et al., 1979].
In order to arrive at the Black-Scholes-Merton model, first the aspect of continuous trading must be approximated. This can be done by dividing the number of steps of the CRR model, N into n smaller time-steps, each with a length of h. Then $h := N/n$.
This also implies that the risk-free interest rate e^r can be transformed in terms of n. Since e^{rN} is the return over N periods of time, it is possible to derive $\hat{e}^{rn}$ such that $\hat{e}^{rn} = e^{rN}$. This is possible with $\hat{e}^r = e^{rN/n}$.
For U and D, the following properties are used:

$$\begin{aligned}
\frac{S_N}{S_0} &= U^j \cdot D^{N-j} \\
\ln\left(\frac{S_N}{S_0}\right) &= j\ln(U) + (n-j)\ln(D) \\
&= j\ln(U/D) + n\ln(D)
\end{aligned}$$

where j is the number of up-movements required to arrive at S_N. Then the expected value and variance of $\ln(S_N/S_0)$, which depend on j can be be given by:

$$\begin{aligned}
\mathbb{E}\left[\ln\left(\frac{S_N}{S_0}\right)\right] &= \ln(U/D)\mathbb{E}[j] + n\ln(D) \\
\sigma^2\left[\ln\left(\frac{S_N}{S_0}\right)\right] &= \ln^2(U/D)\sigma^2(j)
\end{aligned}$$

Let us denote the probability of an up-move to be q. Then, $\mathbb{E}[j] = nq$. Also from the fact that the binomial distribution assumes that each time-step is a Bernoulli experiment, $\sigma^2(j) = nq(1-q)$. Given this information, we can come up with estimates for the total mean return and variance over N periods:

$$\hat{\mu}n := \mathbb{E}\left[\ln\left(\frac{S_N}{S_0}\right)\right] = [q\ln(U/D) + \ln(D)]n$$
$$\hat{\sigma}^2 n := \sigma^2\left[\ln\left(\frac{S_N}{S_0}\right)\right] = q(1-q)[\ln^2(U/D)]n$$

The goal of these estimators is then to arrive at the correct empirical values for μ and σ^2, such that: $\hat{\mu}n \to \mu N$ and $\hat{\sigma}^2 n \to \sigma^2\ N$ as $n \to \infty$.
This can be shown to be the case when:

$$U = e^{\sigma\sqrt{N/n}}, \quad D = U = e^{-\sigma\sqrt{N/n}}, q = \frac{1}{2} + \frac{1}{2}(\mu/\sigma)\sqrt{N/n}.$$

The full proof goes on to show that when these conditions are satisfied and n approaches infinity, stock prices at the end of a given period N are log-normally distributed, and the Black-Scholes-Merton formula for pricing options can be achieved. In Section 4.1, this approximation of the Black-Scholes-Merton model through the formulas derived in this section is examined in the case of a European binary call option. [Cox et al., 1979]

3.5 Limitations and extensions

Extensions The original Black-Scholes model has been extended to overcome many of its shortcomings and include a number of different derivatives. Adjustments were made rather quickly after publication, in which the violation of a number of assumptions were shown to not invalidate the model, such as restrictions on the usage of proceeds from short sales, differential taxes on capital gains, and interest payments. [Smith, 1976] The model was extended by Merton to include stochastic interest rates, dividend-paying European stocks, and perpetual American puts. [Merton, 1973]
Merton also extended the model for discontinuous sample paths, such as which can arise due to firm-specific information which he argues reach the market at discrete i.i.d. time-points. He models these changes by a adding to the Brownian motion a Poisson-driven process, where he argues that even though this extended model contains the expected return in its formula, setting it equal to the risk-free interest rate is consistent with the Black-Scholes model. [Merton, 1976]
The model has further been extended by Hull and White to include pricing options with stochastic volatilities and the authors find that the Black-Scholes model overprices near-the-money options and underprices deep in-the-money and out-of-the-money options when the volatility is uncorrelated with the stock price. Further, they extend the put-call parity for European options and American calls on non-dividend paying stocks. [Hull & White, 1987]
The model, though very robust, still relies on factors such as time-to-maturity and the volatility of the underlying during the existence of the option. The latter becomes of particular interest to traders, as they begin to use the Black-Scholes-Merton model to arrive at the volatility implied by prevailing option prices, with option traders quoting prevailing options by this implied volatility measure instead of their prices. [MacKenzie, 2006]

Volatility smile and limitations In an empirical study, Rubinstein found that deep in-the-money and out-of-the-money options were regularly overpriced historically according to the Black-Scholes model in empirical option prices trading on the Chicago Board of Exchange between 1976 and 1978. [Rubinstein, 1985] This is upheld in later academic literature, with this effect, called the volatility smile, found to increase following the stock market crash of 1987. The smile seems to indicate that the implied volatility derived from the Black-Scholes-Merton model increases as options range deeper in- or out-of-the-money. When these two variables are plotted against one another, with moneyness on the x-axis, the datapoints connect in a "smile", hence the name. Rubinstein in his paper argues that though this could be the result of market inefficiencies, he claims it is unlikely due to traders' potential of greatly profiting off arbitrage opportunities. [Rubinstein, 1994]
Further empirical research using extensions of the Black-Scholes-Merton model seems to confirm that historically, the standard Black-Scholes model lacks consistency in its variables and misprices options due to the lack of inclusion of the smile. It is further argued that stock return distributions in real life are negatively skewed with higher kurtosis as opposed to that of the normal distribution presumed

by the standard Black-Scholes model. However, extensions for stochastic volatility and random-jump processes seem to model real-life price developments better. [Bakshi et al., 1997]
More recent papers also argue for the possibility of modelling for the volatility smile [Kou, 2002].
In conclusion, it seems that though the historic relevance of the seminal Black-Scholes-Merton formula cannot be overestimated, the volatility smile prevalent in the real world requires a great amount of extensions, and a perfect model for this is yet to be found.

4 Empiricism

4.1 Application I: European call on a binary option

4.1.1 Binary options

A cash-or-nothing binary option is a type of exotic option that provides a fixed payment at expiration if the option is in-the-money and pays nothing otherwise. [Hull, 2010] In this application case, the pricing of a European cash-or-nothing binary call option will be examined.
The code used in this section can be found in Appendices A.1 and A.2

4.1.2 Cox-Ross-Rubinstein model

Let a binomial model where the price is $SZ_N = S_0 z_1 z_2 \dots z_{N-1} z_N$ with $S_0 = 100, P(z_n = U) = 0.75, P(z_n = D) = 0.25, U = 1.155, D = 0.9975, e^r = 1.05$ be given. n corresponds to the discrete time. Our goal consists in using R to:

1. Compute the risk-neutral probabilities for the price to go up or down, $\pi = P(Z_k = U)$ and $1 - \pi = P(Z_k = D)$.
2. Use the binomial model to price a cash-or-nothing American binary call option F_N. F_N pays 1, if $S_N \geq K$, where $N = 10$ (the number of time steps), $n \leq N$ and $K = 200$ (strike price).
3. Use the binomial model to price a European binary call option with otherwise identical variables.
4. Use the Black-Scholes-Merton formula to value the same European binary call.
5. Approximate the Black-Scholes-Merton price derived in 4. using the binomial model, as described in Section 3.4.2.

Risk-Neutral probability Finding the risk-neutral probability of the defined binomial model is rather simple: applying the formula from Equation (3), we find:

$$\pi = \left(\frac{e^r - D}{U - D} \right) = 0.3333$$

Binomial Trees The binomial tree in Figure 7 illustrates the price movements of the underlying S proportional to their extent. As U is far greater than D in the model parameters, the tree branches out quite far up but remains relatively flat on the lower side.

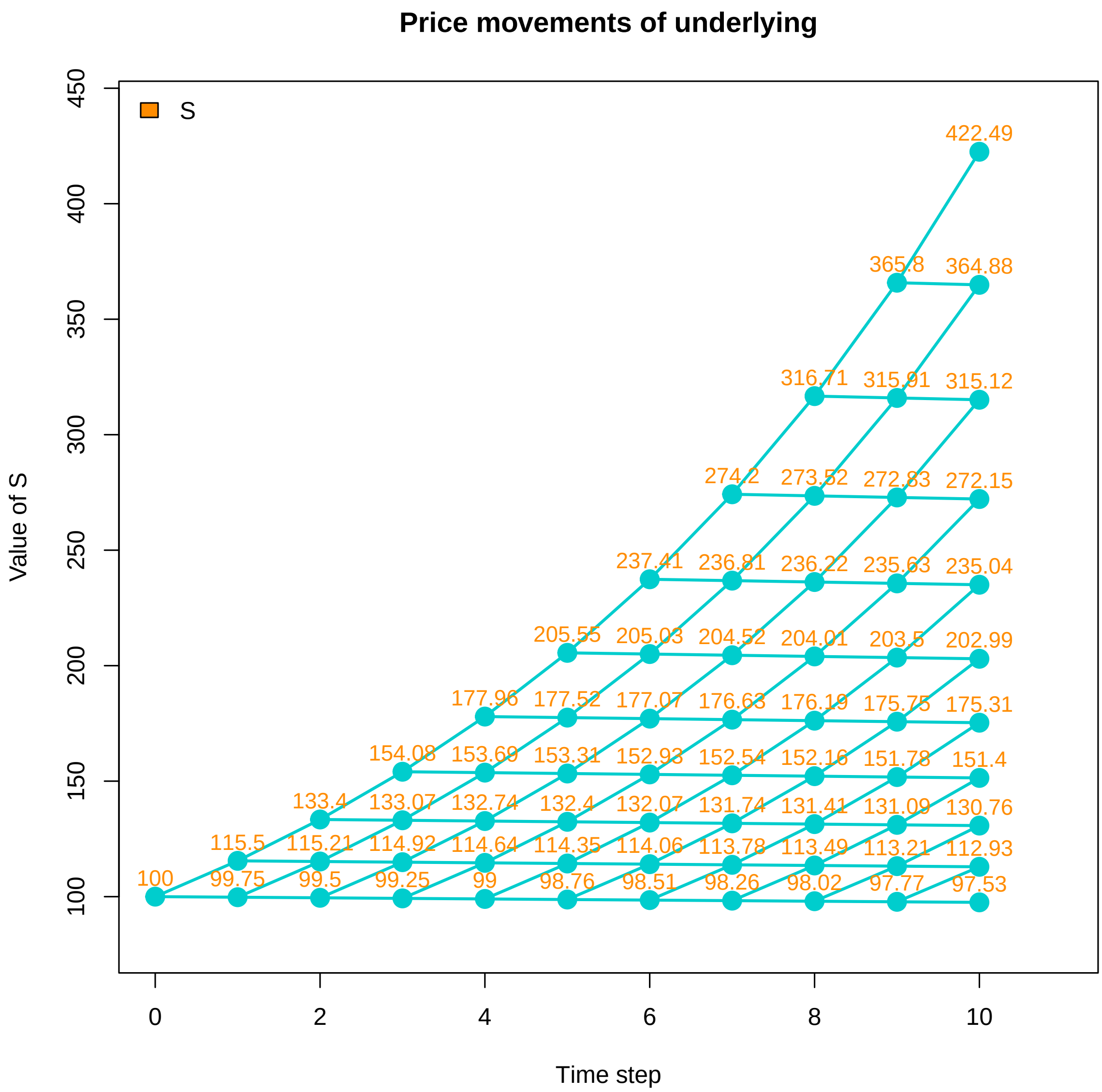

Figure 7: Price movements of the underlying S

The value of the binary American call option is 0.1404. The prices in each possible state of the underlying are given by Figure 8. As we can see, exercise becomes optimal as soon as the option finishes in the money. This makes it clear that in case of binary options, early exercise is optimal before the expiration date, as the higher strike price is not paid by the holder of the option while the payoff is higher compared to if the option were only exercised at expiration due to discounting.

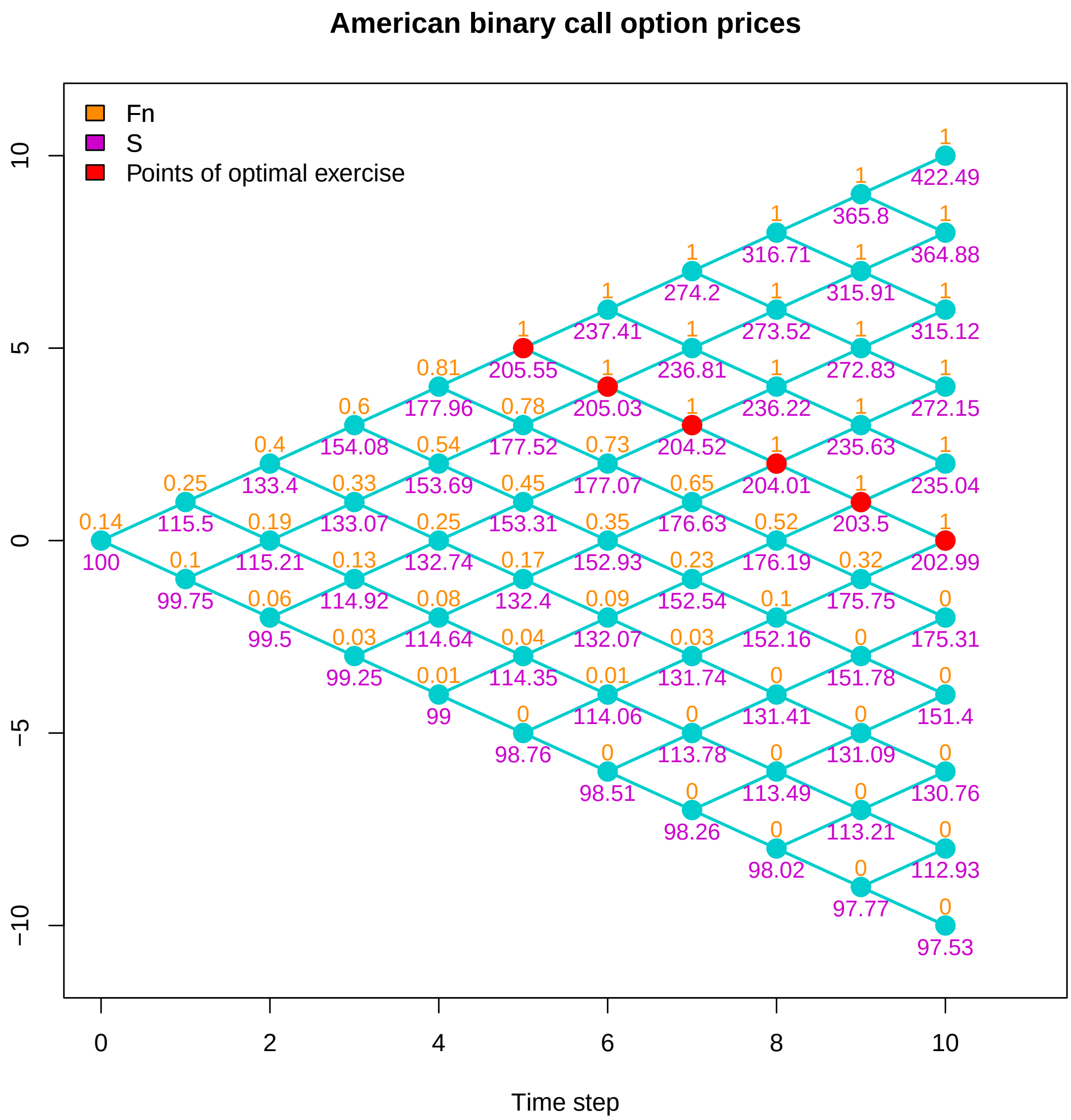

Figure 8: Value of American binary call options at each time-step

The value of the European option is 0.1308, determined by pricing through the binomial tree in Figure 9. Since exercise is only possible at the endnodes, the present value of the payoffs are smaller. The American call is worth more by 0.0096.

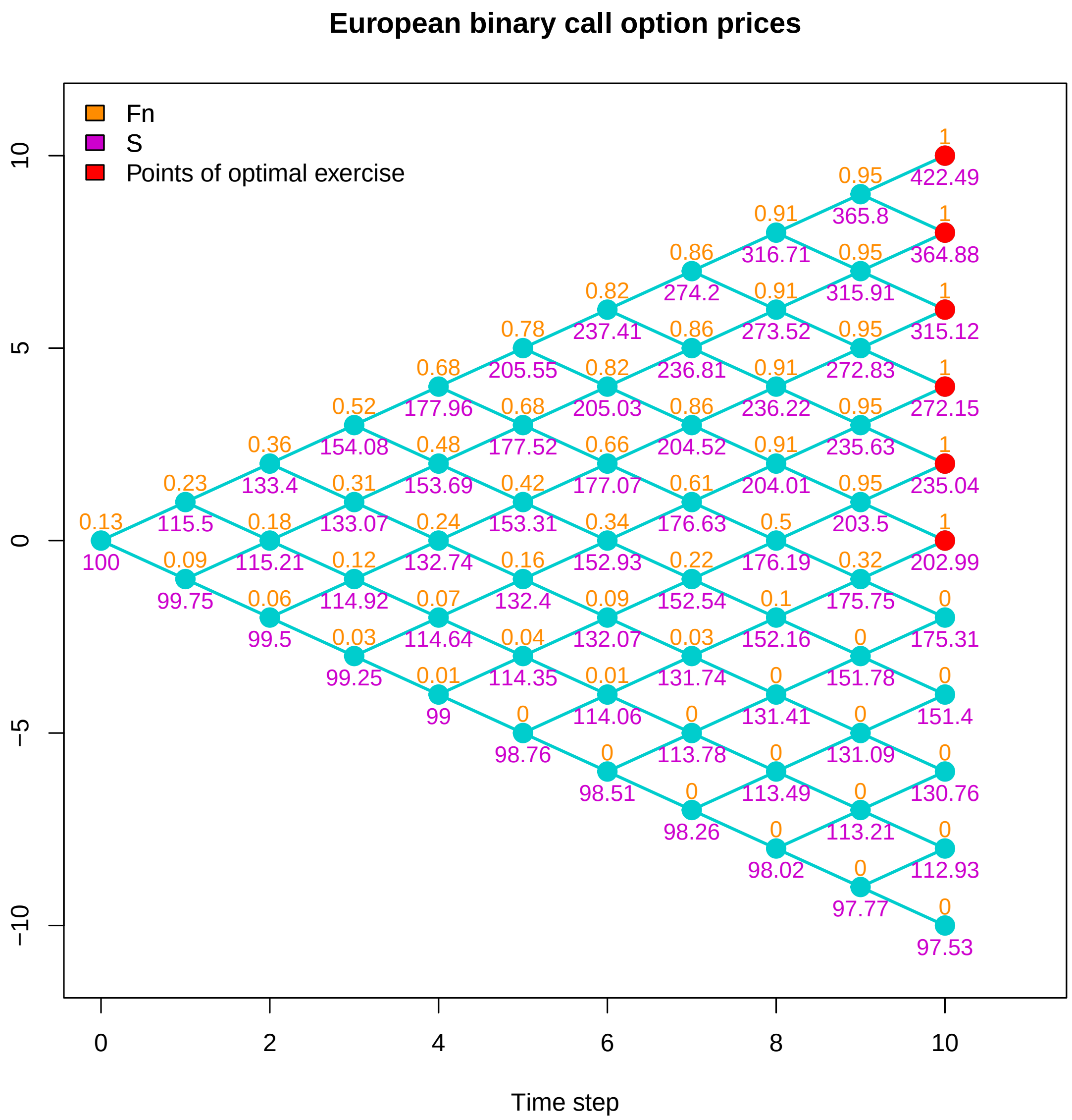

Figure 9: Value of European binary call options at each time-step

This value of 0.1308 is now going to be compared to the price derived from the Black-Scholes-Merton pricing formula.

4.1.3 Pricing using the Black-Scholes-Merton model

Before applying Equation (16) to the input parameters $S_0 = 100, K = 200, e^r = 1.05$, we need to find the per-period standard deviation of the stock price returns, σ for the underlying S. While in real life the true standard deviation of a stock is unknown [Hull, 2010], given the input parameters for the binomial tree model, both the mean return and the standard deviation of a stock are certain, and can be computed. Note that the mean return μ is equal to the risk-free rate since we assume a risk-neutral world.
Formally, σ can then be derived in the following way:

$$
\begin{aligned}
R &:= \frac{SZ_{n+1}}{SZ_n} \\
\bar{U} &:= U - 1 \\
\bar{D} &:= D - 1 \\
\mu = \mathbb{E}\left[R\right] &= \pi\bar{U} + (1-\pi)\bar{D} = 0.05 \\
\sigma^2 = EV[(R-\mu)^2] &= \pi\left(\bar{U}-\mu\right)^2 + (1-\pi)\left(\bar{D}-\mu\right)^2 = 0.0055 \\
\sigma &= \sqrt{\sigma^2} = 0.0742
\end{aligned}
$$

Where R denotes the random variable for the stock price return, $\mathbb{E}$ the expected value, $\bar{U}$ and $\bar{D}$ the returns after an up- resp. down-movement.

Now it is possible to substitute for the relevant parameters:

$$
c = S_0 N(d_1) - Ke^{-rT} N(d_2) = 0.0987
$$

It is interesting to consider why this value does not equal the price derived in the binomial tree. This can be explained by the fact that trading in the Black-Scholes-Merton model is continuous, while in the binomial tree model it is discrete, with in this case, a very low number of time-steps, $N = 10$. More importantly, the values of S_N do not perfectly follow a log-distribution. To visualise this, a binomial tree was created for $N = 2000$ with otherwise the same input parameters. The distribution of the natural logarithm of S_N, while approximately normal, displays high kurtosis, as visible in Figure 10:

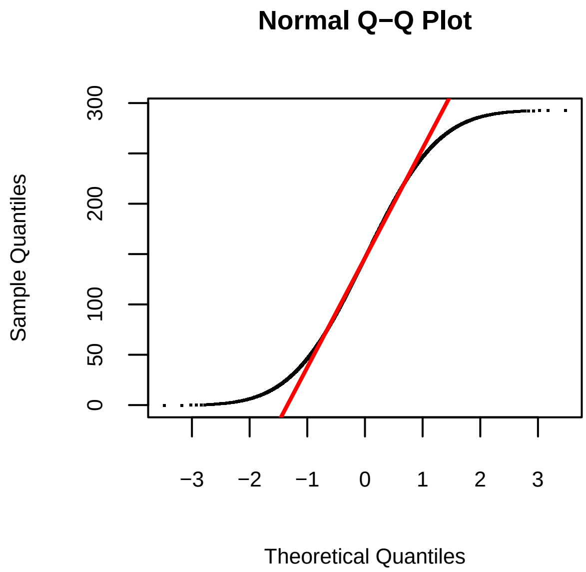

Figure 10: Distribution of $\log(S_N)$ for $N = 2000$

Nevertheless, what is interesting to consider is how this value can be approximated using the CRR model. As explained in Section 3.4.2, the CRR model can be used to arrive at the Black-Scholes-Merton model. Let us then consider these approximations, for a number of different time-steps $n = 50, 100, 150, 250, 500, 750, 1000, \ldots 9500, 9750, 10000$, but for the same model with $N = 10$. We will compare these with the value given by the BSM model in the aspects of: mean total return, $\mu \cdot N$, the total variance, $\sigma^2 \cdot N$, and the fair price.

$\mu \cdot N$	0.5
$\sigma^2 \cdot N$	0.0551
F_N	0.0987

Table 3: Black-Scholes-Model parameters

Illustrated in the top-left corner of Figure 11 though the approximations of the total mean return μ have a very narrow range, it is still interesting that the larger the number of time-steps becomes, the less accurate they seem to be. While further research would be required to confirm this, it could be possibly caused by the rounding floating point numbers undergo in computers. As the number of time-steps increase, the values of U and D in these models decrease, and so does the error introduced by the rounding. The approximations for the variance, in the top-right corner behave as expected, with a steady decrease in inaccuracy. As for the approximations of the fair price, in the bottom-left corner, while they get increasingly accurate as the time-steps become smaller and smaller, there is a considerable amount of noise even towards larger number of time-steps around 10000. After removing outliers at small number of time-steps, we can see in the bottom-right that the distribution of the approximate prices are approximately normal, but with higher kurtosis than that of the normal distribution.

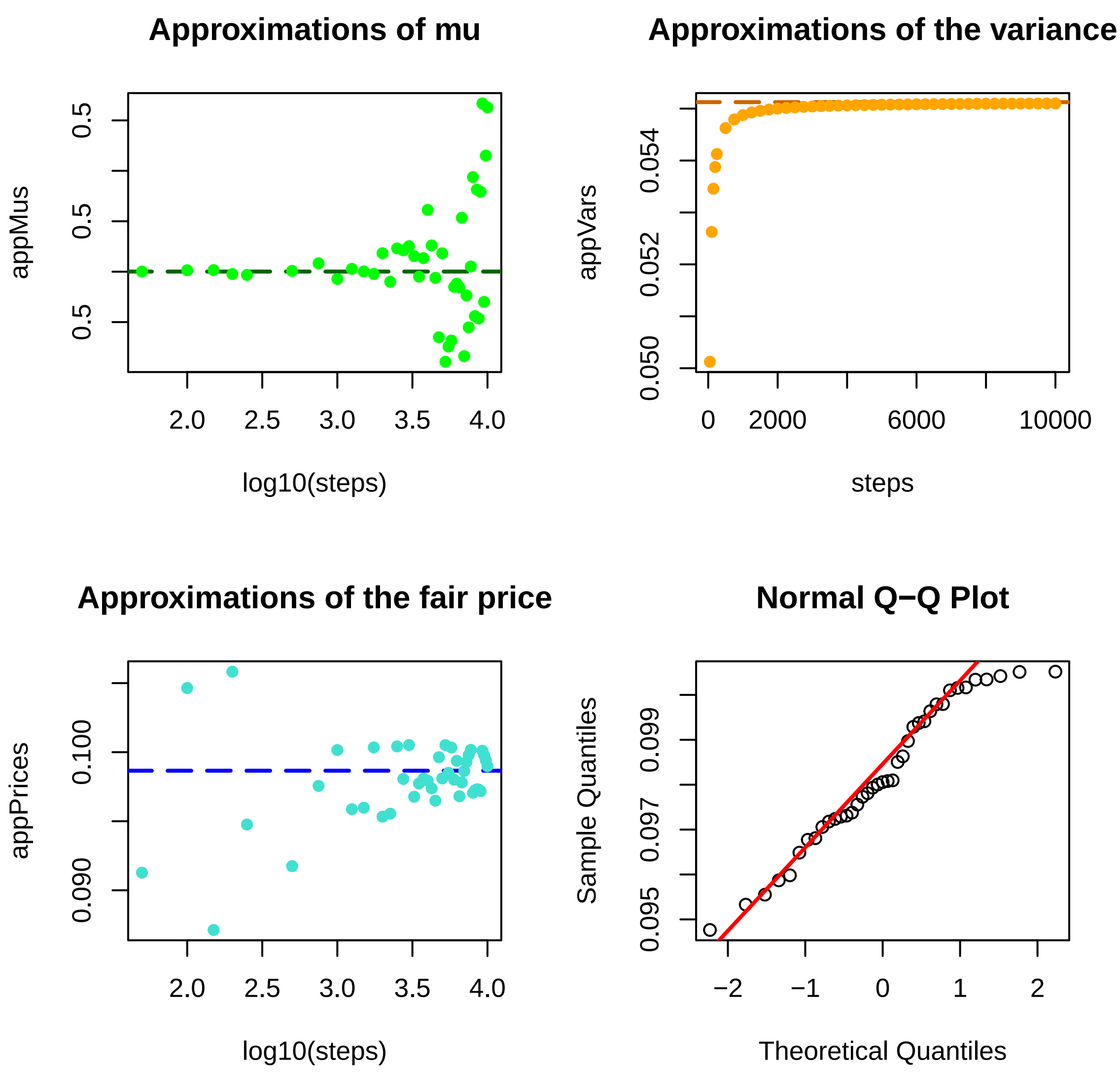

Figure 11: Approximations of the Black-Scholes-Model parameters

4.2 Application II: American call option with dividend

In this second case study, a real-life example of a call option on a single share of Morgan Stanley with a strike price of $K = 91$ is examined in the 2 week period leading up to the expiration date. The stock was announced to pay a dividend of \$0.85 [MS, 2023], to be paid on August 15^{th}, 12 business days later. The expiration date of the examined call option was to fall on the same day as the ex-dividend date, July 28^{th} 2023. In the case study, a hedging strategy using the replicating portfolio approach from Section 2.3.2 is used to hedge against a short position in a call option on day 0. The strategy involves pricing the call option using closing prices from each day, and rebalancing the replicating portfolio using the obtained measures of δ and B from Equation (2), where cash is added to purchase the replicating portfolio when this is needed.
The code used in this section can be found in Appendices A.1 and A.2
The historical stock prices used are displayed in Figure 12. Though it contains the stock movements of the 2020 stock market crash, and the following rally in 2021, the 5 year stock price history is used for the sake of simplicity.

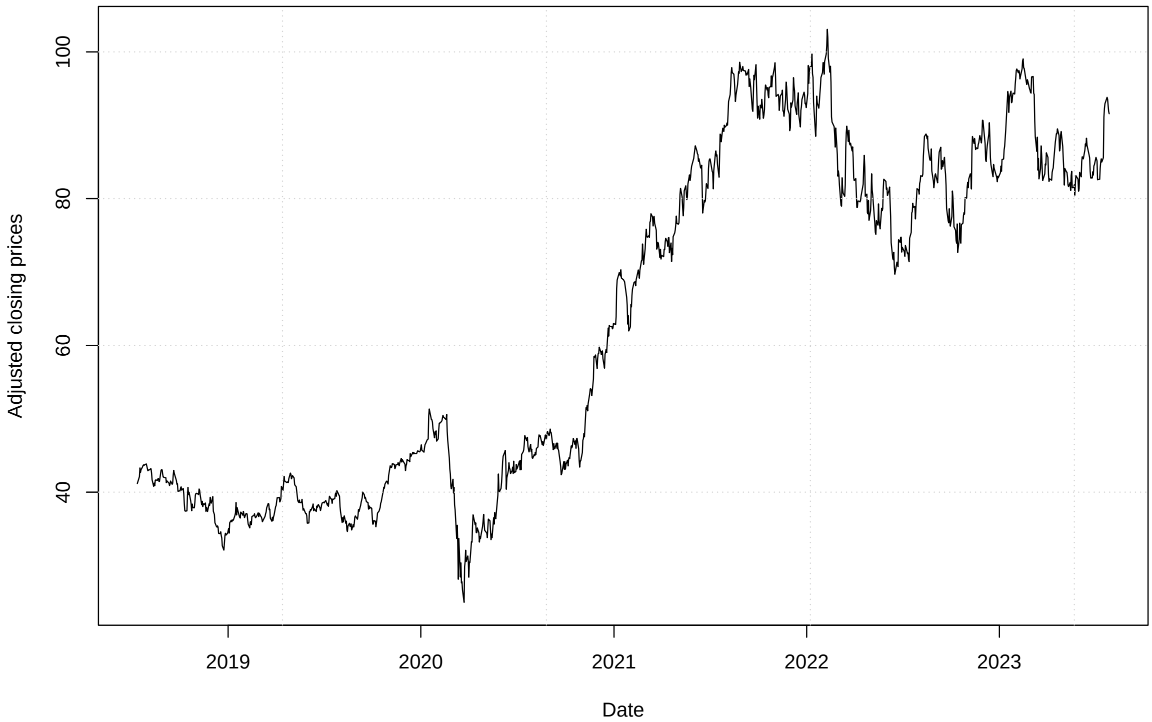

Figure 12: 5 year stock price development of Morgan Stanley

4.2.1 Pricing using the binomial model

In order to price the call option, C_0, the model parameters must first be estimated. $N = 10, K = 91$ have been chosen arbitrarily. Historical data provided by Yahoo Finance [YahooFinance, 2023] is used in finding U and D, the United States 1 year Treasury Bill yields from July 14^{th} (day 0) are used as a proxy for the risk-free interest rate, e^r. Data for the Treasury Yield on this date is sourced from MarketWatch [MarketWatch, 2023]. The results are compared to data provided by Optionistics [Optionistics, 2023], which uses data provided by Delta Neutral LLC.

The model parameters used are:

- $N = 10$
- $K = 91$
- $e^r = 1.0002$, the 1 year Treasury Bill yield on July 14$^{\text{th}}$ to the power of 1/252, to get the daily compounding rate (assuming there are 252 business days in a year).
- dividend $= 0.85e^{r-12}$
- $S_0 = 85.78$ which was the stock price on July 14$^{\text{th}}$
- $U = e^{\sigma}$
- $D = U^{-1}$
- σ is estimated as the square root of the 5-year daily stock return variance.

Using the binomial option pricing model, an option price of $C_0 = 0.6845$ is found. From these proceeds, $\Delta = 0.2061$ shares are purchased at the end of day 0, and additional $B = -16.9934$ are borrowed at the riskless interest rate e^r to finance this position until the next day, at the end of which the replicating portfolio is rebalanced. Figure 13 illustrates the binomial tree generated on day 0, with the theoretical optimal points of exercise colored red. The figure is then overlayed with a green line indicating the actual stock development during this time, and the purple line indicating the path nearest to the actual stock price development. As we can see, the actual stock movement is not that far off from the nearest path indicated on the tree, and as will be shown later, the value of call options is predicted quite well using the replicating portfolio.

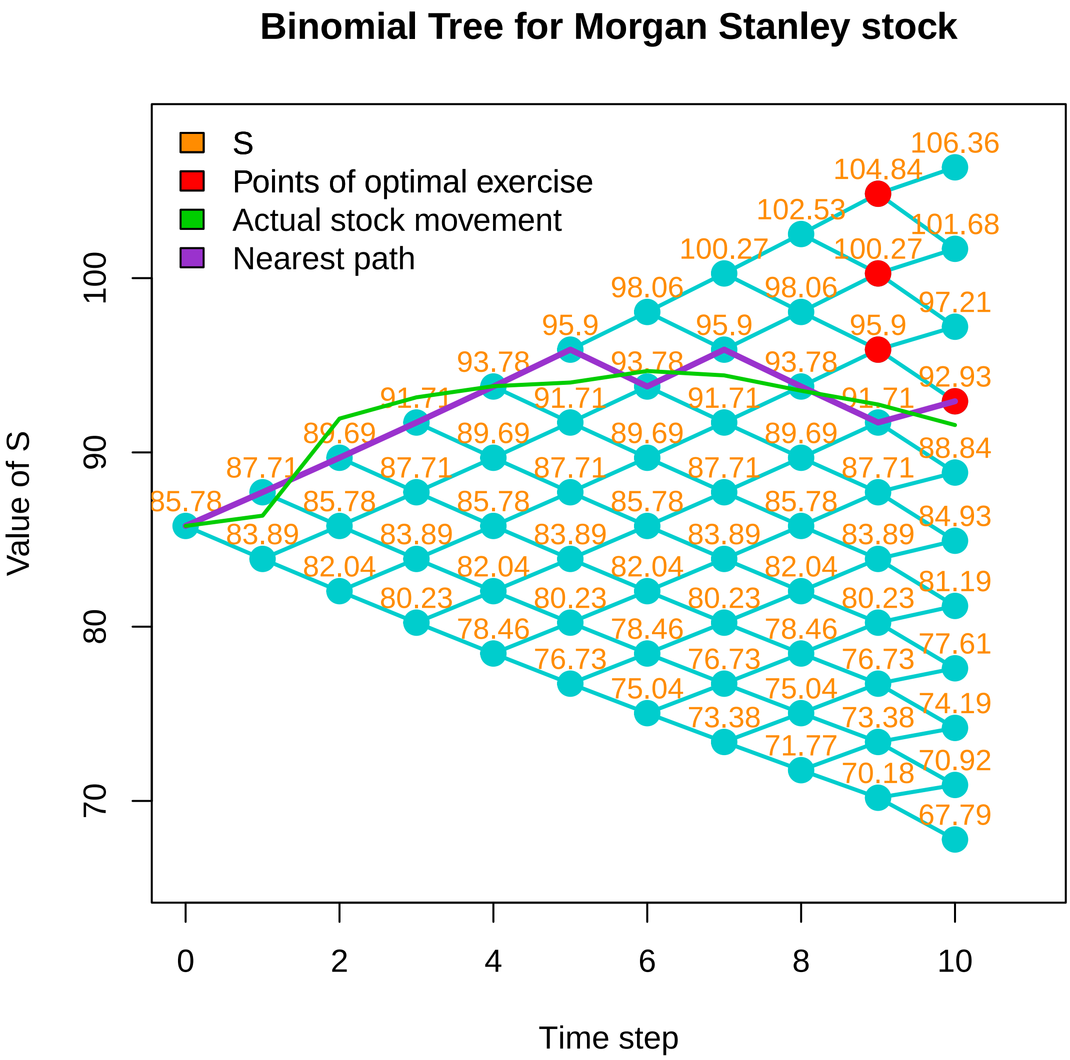

Figure 13: Binomial tree for Morgan Stanley's stock

Table 4 provides the hedging tables constructed at the end of every given day prior to exercise. These are provided until day 9, as at the end of that day exercising the option became optimal, selling just before the price drop induced by the dividend payment on the following day. The table also displays the P&L resulting from hedging on each day. As we can see in this case, the option turned out to be undervalued by the binomial model.

Day n	0	1	2	3	4	5	6	7	8	9
S	85.78	86.37	91.94	93.16	93.80	94.01	94.67	94.42	93.54	92.75
Δ	0.21	0.22	0.59	0.68	0.74	0.78	0.85	0.89	0.91	0
B	-16.99	-18.42	-51.60	-60.09	-65.64	-69.85	-76.82	-80.14	-82.57	0
Pf $= C_n$	0.68	0.71	2.79	3.42	3.71	3.68	4.03	3.69	2.74	1.75
Pf$_{-1}$		0.80	1.94	3.50	3.84	3.85	4.19	3.81	2.90	2.00
P&L	PV=0.29	0.09	-0.85	0.08	0.13	0.16	0.15	0.11	0.15	0.25

Table 4: Performance of hedging portfolio when option is exercised optimally

If the option is not exercised optimally, there is an even higher inflow at day 9, and the present value of the profit gained from the strategy is 0.607, instead of 0.295 in case of optimal exercise. The last few days of this case are seen in Table 4.2.1

Day n	8	9	10
S	93.54	92.75	91.57
Pf	2.74	1.49	0.00
$\mathbf{Pf}_{-1}$	2.90	2.00	0.63
P&L	0.15	0.51	0.06

Table 5: Performance of hedging portfolio when option is exercised at expiration

This profit and loss, in Tables 4 and 4.2.1 can be visualised as seen in Figure 14. The total hedging error when the option is exercised optimally is 1.9889, while the total hedging error when the option is exercised on day 10 is 2.3015

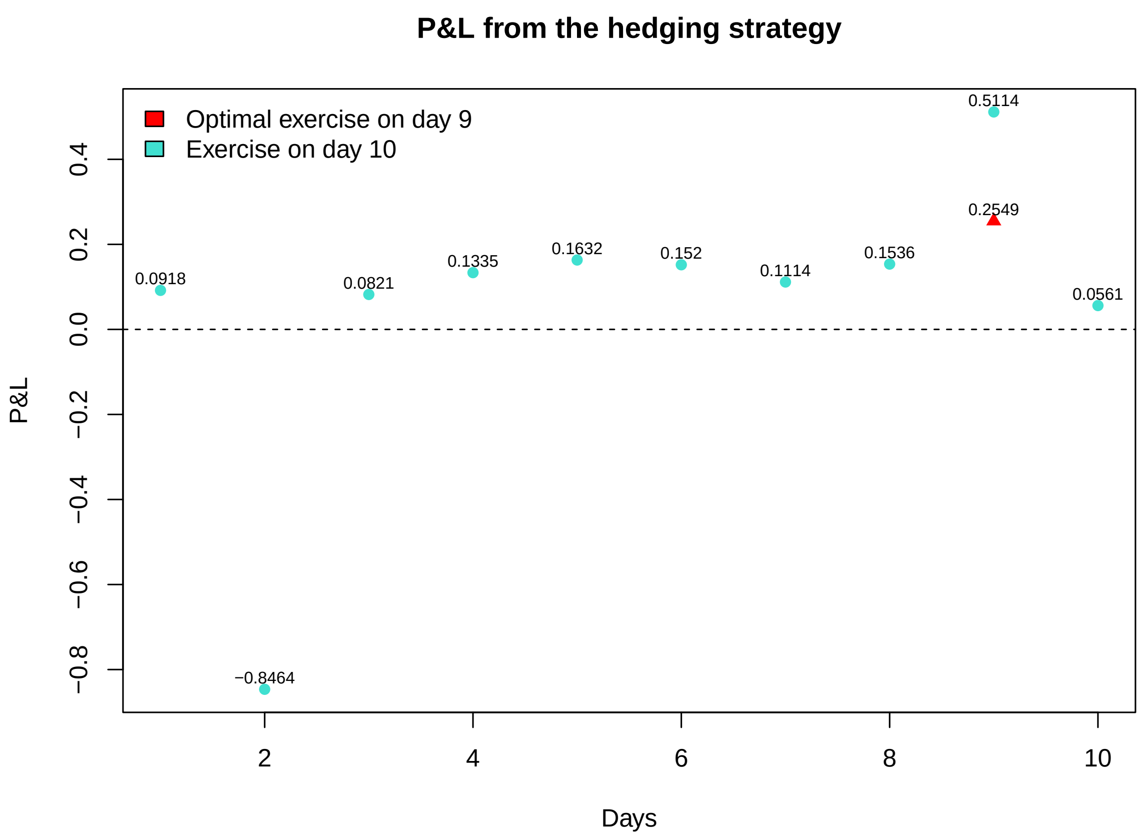

Figure 14: P&L from CRR hedging strategy

4.2.2 Comparison with real-life price

Comparing the value of the tracking portfolio created in the previous section to the actual last trade that took place each day on the stock market yields some interesting results. The proportionally drawn Figure 15 illustrates, how the value of the replicating portfolio comes very close to the historical values. However, the replicating portfolio consisted of Δ of the closing price of the stock every day, and as such, these results have little predictive power, but do show that the market value for such options comes very close to that determined by the CRR model.

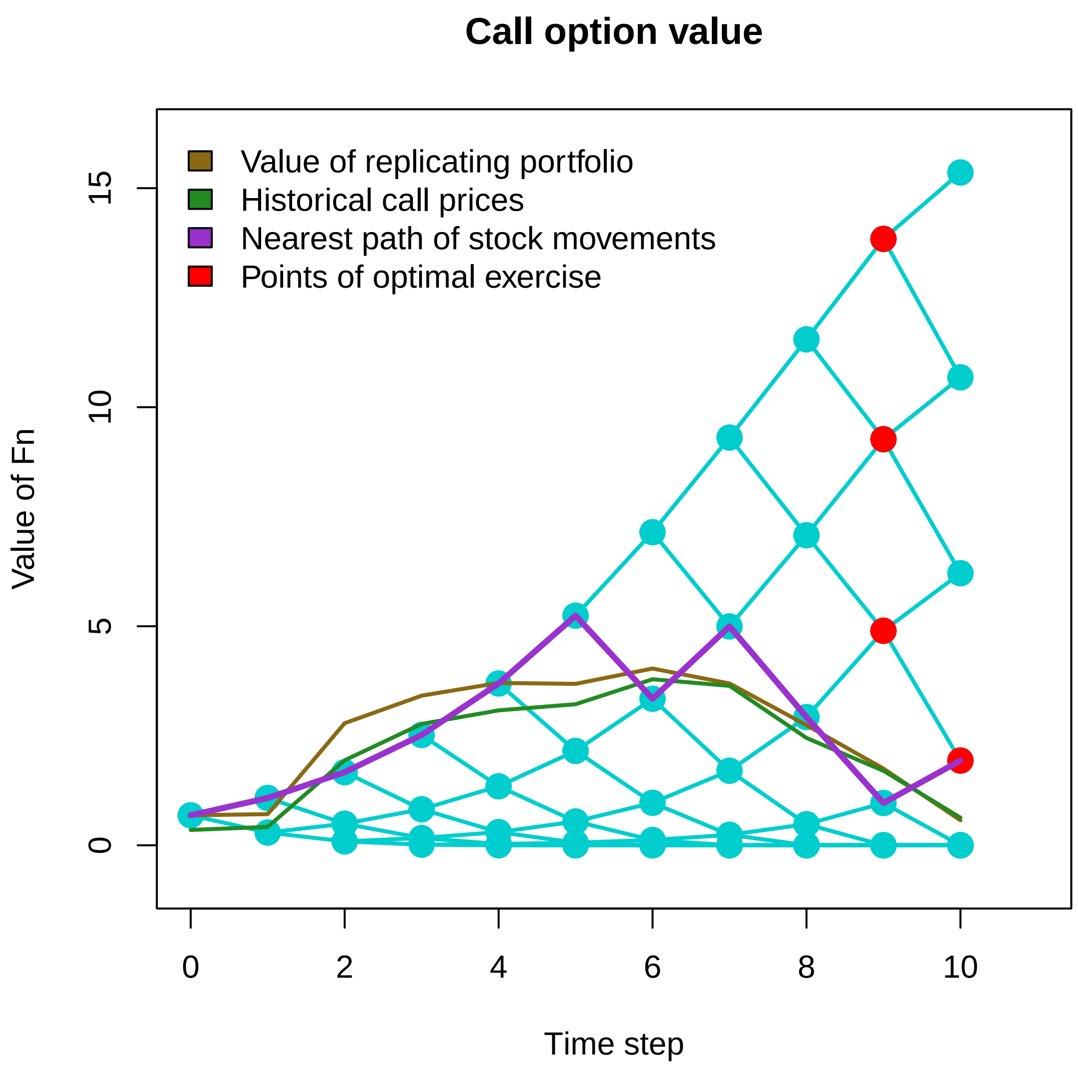

Figure 15: Comparison of replicating portfolio value with historical option prices

5 Conclusion

The Cox-Ross-Rubinstein binomial option model provides a standardised framework for the valuation of any kind of vanilla stock option. It arrives at this valuation by the principle of risk-neutral valuation, which is derived from the original Black-Scholes model of 1973. Interestingly, the former is derived from the latter, however, it is still preferable to use the binomial option pricing model when valuing American options. This is because the binomial model in each time-step examines the possibility of early exercise, whereas with a closed-form solution such as would be provided by the Black-Scholes-Merton formulas, the payoff is required to be generally known in advance. The accuracy of the binomial model, however, comes at a cost of computational efficiency. Even though new models have been found that improve this fact, they are either approximations, which have fixed accuracy, or they do not speed up the calculations significantly, with the number of computations required growing polynomially while the number of time-steps increases linearly.

The Black-Scholes-Merton model, originally described by Fischer Black and Myron Scholes in their seminal 1973 paper, changed forever how the financial world from traders to academics, and regulators treated stock options. As the first ever formula to derive the value of an option without making any assumptions about investors' risk preferences or expectations, it gave way to the principle of risk-neutral valuation, which was expanded to be used in the valuation of a number of financial derivatives, as well as corporate liabilities. While the standard model on its own employs strict and to the real world often unrealistic assumptions, countless extension have been added throughout the years to improve on its shortcomings, including its failure to account for the volatility smile exhibited by empirical option prices. Nonetheless, the Black-Scholes-Merton differential equations still serve an essential purpose in modern academic literature.

For applications I & II, binomial tree pricing and drawing algorithms were developed, with both displaying the previously mentioned computational efficiency problem. Application I serves as a simple demonstration for this algorithm, for the simple cases of an American and European cash-or-nothing binary call option. Application I goes on to serve as a proof-of-concept of the fact that the binomial model contains the Black-Scholes-Merton model as its limiting case. In this case, as the number of steps in the 10-period binomial tree gets larger and larger, the fair price is approximated more and more closely, with small, but normally distributed errors. Application II provides a case study in the effectiveness of using the lattice model to hedge a hypothetical short position in a listed American call option on 1 share of Morgan Stanley, leading up to its ex-dividend date. It is found that, although the hedging position is not perfect due to the assumption of constant volatility throughout the model, historical option prices at the end of each day very strongly followed the fair replicating portfolio as determined by the binomial option pricing model.

6 Bibliography

References

[Bakshi et al., 1997] Bakshi, G., Cao, C., & Chen, Z. (1997). Empirical performance of alternative option pricing models. *The Journal of Finance*, 52(5), 2003–2049.

[Black & Scholes, 1973] Black, F. & Scholes, M. (1973). The pricing of options and corporate liabilities. *Journal of Political Economy*, 81(3), 637–654.

[Broadie & Detemple, 1996] Broadie, M. & Detemple, J. (1996). American option valuation: New bounds, approximations, and a comparison of existing methods. *The Review of Financial Studies*, 9(4), 1211–1250.

[Cox et al., 1979] Cox, J. C., Ross, S. A., & Rubinstein, M. (1979). Option pricing: A simplified approach. *Journal of Financial Economics*, 7(3), 229–263.

[David Hillier & Titman, 2012] David Hillier, M. G. & Titman, S. (2012). *Financial Markets and Corporate Strategy Second European Edition.* McGraw-Hill Education.

[Hull & White, 1987] Hull, J. & White, A. (1987). The pricing of options on assets with stochastic volatilities. *The Journal of Finance*, 42(2), 281–300.

[Hull, 2010] Hull, J. C. (2010). *Options, Futures, and Other Derivatives.* Pearson.

[Kou, 2002] Kou, S. G. (2002). A jump-diffusion model for option pricing. *Management Science*, 48(8), 1086–1101.

[Leisen & Reimer, 1995] Leisen, D. & Reimer, M. (1995). Binomial models for option valuation—examining and improving convergence. *Applied Mathematical Finance*, 3, 319–346.

[Leland, 1985] Leland, H. E. (1985). Option pricing and replication with transactions costs. *The Journal of Finance*, 40(5), 1283–1301.

[MacKenzie, 2006] MacKenzie, D. (2006). *An Engine, Not a Camera: How Financial Models Shape Markets.* The MIT Press.

[MarketWatch, 2023] MarketWatch (Accessed in August 2023). U.S. 1 Year Treasury Bill yield. Retrieved from: `https://www.marketwatch.com/investing/bond/tmubmusd01y/download-data?countrycode=bx&mod=mw_quote_tab`.

[Merton, 1973] Merton, R. C. (1973). Theory of rational option pricing. *The Bell Journal of Economics and Management Science*, 4(1), 141–183.

[Merton, 1976] Merton, R. C. (1976). Option pricing when underlying stock returns are discontinuous. *Journal of Financial Economics*, 3(1), 125–144.

[MS, 2023] MS (2023). Morgan stanley second quarter 2023 earnings results. Retrieved from: `https://www.morganstanley.com/about-us-ir/shareholder/2q2023.pdf`.

[Optionistics, 2023] Optionistics (Accessed in August 2023). Morgan stanley historical option chain. Retrieved from: `optionistics.com`.

[Robert S. Pyndick, 2018] Robert S. Pyndick, D. L. R. (2018). *Microeconomics, Global Edition, 9/E.* PEARSON EDUCATION LIMITED.

[Rubinstein, 1985] Rubinstein, M. (1985). Nonparametric tests of alternative option pricing models using all reported trades and quotes on the 30 most active cboe option classes from august 23, 1976 through august 31, 1978. *The Journal of Finance*, 40(2), 455–480.

[Rubinstein, 1994] Rubinstein, M. (1994). Implied binomial trees. *The Journal of Finance*, 49(3), 771–818.

[Smith, 1976] Smith, C. W. (1976). Option pricing: A review. *Journal of Financial Economics*, 3(1), 3–51.

[Vellekoop & Nieuwenhuis, 2006] Vellekoop, M. H. & Nieuwenhuis, J. W. (2006). Efficient pricing of derivatives on assets with discrete dividends. *Applied Mathematical Finance*, 13(3), 265–284.

[YahooFinance, 2023] YahooFinance (Accessed in August 2023). Morgan stanley historical price data. Retrieved from: `https://finance.yahoo.com/quote/MS/history?p=MS`.

A Appendix

A.1 R functions used in Sections 4.1 and 4.2

```
evaluateCallPayoff <- function(S, K) pmax(S - K, 0)
evaluatePutPayoff <- function(S, K) pmax(K - S, 0)

.parseOptionType <- function(optionType) {
  # Parses optionType (string length 1) and
  # raises possible errors in the input
  # otherwise always returns the correct type
  # in a length 4 vector in the order:
  # binary/vanilla, long/short, american/european, call/put
  # when missing values vanilla, long, american are presumed
  # typos or additional words are not checked

  input <- strsplit(tolower(optionType), " ")[[1]]
  isLong <- FALSE
  isVanilla <- FALSE
  isEuropean <- FALSE
  isShort <- FALSE
  isBinary <- FALSE
  isAmerican <- FALSE
  isCall <- FALSE
  isPut <- FALSE

  if ("short" %in% input) isShort <- TRUE
  else isLong <- TRUE

  if ("binary" %in% input) isBinary <- TRUE
  else isVanilla <- TRUE

  if ("european" %in% input) isEuropean <- TRUE
  else if ("american" %in% input)isAmerican <- TRUE
  else {cat("American option assumed\n"); isAmerican <- TRUE}

  if ("call" %in% input)        isCall <- TRUE
  if ("put" %in% input)         isPut <- TRUE
  if ("long" %in% input)        isLong <- TRUE
  if ("vanilla" %in% input)     isVanilla <- TRUE
  if ("american" %in% input)    isAmerican <- TRUE
  if (isLong && isShort)
    stop("optionType cannot be both long and short")
  if (isVanilla && isBinary)
    stop("optionType cannot be both vanilla and binary")
  if (isEuropean && isAmerican)
    stop("optionType cannot be both European and American")
  if (isCall && isPut)
    stop("optionType cannot be both call and put")
  if (!(isCall || isPut))
    stop("call/put not specified")

  res <- c("long", "short", "vanilla", "binary", "european",
           "american", "call",  "put")[c(isLong, isShort,
                                         isVanilla, isBinary,
                                         isEuropean, isAmerican,
```

```
                                            isCall, isPut)]

  return(res)
}

priceCRR <- function(N, S0, U, D, er, K, optionType,
                     calculateExerciseDates=TRUE, dividend=0) {
  N <- N + 1 # Adding time-step 0
  rnProb <- (er - D) / (U - D)

  type <- if (length(optionType) == 4) optionType
  else .parseOptionType(optionType)

  numberOfNodes <- (N * (N + 1) / 2)
  parentNodes <- 1:(N * (N - 1) / 2)
  endNodes <- (N * (N - 1) / 2 + 1):numberOfNodes
  connections <- {
    cbind(parentNodes,
          parentNodes + rep(1:(N-1), 1:(N-1)),
          parentNodes + rep(1:(N-1), 1:(N-1)) + 1)
  }

  ########## PRICING THE TREE #########
  priceTree <- function() {

    evaluatePayoff <- {
      if (type[2] == "binary" && type[4] == "call")
        function(S, K) as.numeric(S >= K)
      else if (type[2] == "binary")
        function(S, K) as.numeric(S <= K)
      else if (type[4] == "call") evaluateCallPayoff
      else evaluatePutPayoff
    }

    valueEuropean <- function(nodeNumber) {
      if (nodeNumber >= endNodes[1]) {
        return(evaluatePayoff(resultVector["S", nodeNumber], K))
      } else {
        childNodes <- connections[nodeNumber, 2:3]
        return(sum(c(rnProb, 1 - rnProb) *
                     resultVector["Fn", childNodes]) / er)
      }
    }

    valueAmerican <- function(nodeNumber) {
      if (nodeNumber >= endNodes[1]) {
        evaluatePayoff(resultVector["S", nodeNumber], K)
      } else {
        noExercise <- valueEuropean(nodeNumber)
        exercise <-
          evaluatePayoff(resultVector["S", nodeNumber], K)

        max(exercise, noExercise)
      }
    }
```

```
# checks if premature exercise could be optimal
# (american but not vanilla call on stock with no dividends)
isPrematureOptimal <- type[3] == "american" &&
  !(type[2] == "vanilla" &&
      type[4] == "call" && dividend == 0)
valueNode <- if (isPrematureOptimal) valueAmerican
else valueEuropean

if (calculateExerciseDates) {
  resultVector <- matrix(c(S0, rep(NA, numberOfNodes*4 - 1)),
                         ncol = numberOfNodes, nrow = 4)
  rownames(resultVector) <- c("S","Fn","Exercise","timeStep")
} else {
  resultVector <- matrix(c(S0, rep(NA, numberOfNodes*3 - 1)),
                         ncol = numberOfNodes, nrow = 3)
  rownames(resultVector) <- c("S", "Fn", "timeStep")
}

resultVector["timeStep", ] <- steps <- rep(1:N, 1:N) - 1

# evaluating values of S
downMoves <- sequence(1:(N)) - 1
resultVector["S",] <- S0 * (U^(steps - downMoves) *
                              D^downMoves)
if (dividend != 0) {
  resultVector["S", endNodes] <-
    resultVector["S", endNodes] - dividend
}

# evaluating values of Fn
for (node in numberOfNodes:1) {
  resultVector["Fn", node] <- valueNode(node)
}


if (calculateExerciseDates) {
  if (!isPrematureOptimal) {
    resultVector["Exercise", parentNodes] <- FALSE
    resultVector["Exercise", endNodes] <-
      resultVector["Fn", endNodes] > 0
  } else {
    Svalues <- resultVector["S",]
    optionValues <- resultVector["Fn",]

    resultVector["Exercise",resultVector["Fn",] == 0] <- FALSE

    exerciseNotOptimal <-
      evaluatePayoff(Svalues, K) != optionValues
    # case where early exercise is optimal at first node
    if ((!exerciseNotOptimal)[1] == TRUE) {
      resultVector["Exercise", 1] <- TRUE
      resultVector["Exercise", -1] <- FALSE
    } else if (sum(exerciseNotOptimal) == length(parentNodes)
               && which(exerciseNotOptimal) == parentNodes) {
      # if exercise is only optimal at the end nodes
      resultVector["Exercise", endNodes] <-
```

```
            resultVector["Fn", endNodes] > 0
        }

        resultVector["Exercise", exerciseNotOptimal] <- FALSE

        # all nodes where premature exercise is optimal
        toConsider <- is.na(resultVector["Exercise",])
        while (any(toConsider)) {
          nodeNumber <- which(toConsider)[1]

          # if all under consideration are parent nodes
          if (nodeNumber >= endNodes[1]) {
            resultVector["Exercise", toConsider] <-
              resultVector["Fn", toConsider] > 0
            break
          }
          node <- resultVector[,nodeNumber]

          stepsRemaining <- N - 1 - node["timeStep"] # at least 1

          resultVector["Exercise", nodeNumber] <- TRUE

          # setting exercise value to false for
          # nodes only accessible from this node
          isPut <- type[4] == "put"
          for (i in 1:stepsRemaining) {
            # if call upNode, else downNode
            nextNode <- connections[nodeNumber, 2 + isPut]
            resultVector["Exercise", nextNode] <- FALSE
            nodeNumber <- nextNode
          }
          toConsider <- is.na(resultVector["Exercise",])
        }
      }
    }

    if (type[1] == "short") {
      resultVector["Fn", node] <- (-1) * resultVector["Fn", node]
    }

    return(resultVector)
  }

  ######## MAIN BODY ########
  values <- priceTree()

  # Write what the function returns
  result <- list(optionType = paste(type, collapse = " "),
                 RNProb = rnProb, value = values,
                 fairPrice = values["Fn", 1])
  return(result)
}

drawTreeGraph <- function(resultMatrix, values, title,
                          exerciseData = FALSE, toScale = FALSE,
                          showValues = TRUE) {
```

```
N <- resultMatrix["timeStep", ncol(resultMatrix)] + 1

numberOfNodes <- (N * (N + 1) / 2)
parentNodes <- 1:(N * (N - 1) / 2)
endNodes <- (N * (N - 1) / 2 + 1):numberOfNodes
connections <- {
  cbind(parentNodes,
        parentNodes + rep(1:(N-1), 1:(N-1)),
        parentNodes + rep(1:(N-1), 1:(N-1)) + 1)
}

u <- 1
d <- -1

downMoves <- sequence(1:N) - 1
steps <- rep(1:N, 1:N) - 1

# If toScale, values of first given value is used for y position
yPos <- if (toScale) resultMatrix[values[1],]
        else u*(steps - downMoves) + d*downMoves
xPos <- resultMatrix["timeStep",]

endNodeRange <- yPos[endNodes][c(length(endNodes), 1)]
endSize <- max(endNodeRange) - min(endNodeRange)
endMiddle <- mean(endNodeRange)
newSize <- endSize*1.1
newNodeRange <- endMiddle + c(-newSize / 2, newSize / 2)

plot(1, 1, col="white", main = title,
     xlab = "Time step",
     ylab = ifelse(toScale, paste("Value of", values), ""),
     xlim = c(0, N),
     ylim = newNodeRange)


upNodes <- connections[,2]
downNodes <- connections[,3]
segments(xPos[parentNodes], yPos[parentNodes],
         xPos[upNodes], yPos[upNodes],
         col="cyan3", lwd=2)

segments(xPos[parentNodes], yPos[parentNodes],
         xPos[downNodes], yPos[downNodes],
         col="cyan3", lwd=2)

points(xPos, yPos, pch = 19, lwd = 6, col = "cyan3")


if (exerciseData) {
  toDraw <- as.logical(resultMatrix["Exercise",])
  points(xPos[toDraw], yPos[toDraw],
         pch = 19, lwd = 6, col = "red")
}

colorsUsed <- c("darkorange",
                "magenta3",
```

```
                  "turquoise4",
                  "green3",
                  "darkorange3",
                  "purple",
                  "brown",
                  "deeppink",
                  "darkturquoise",
                  "darkolivegreen",
                  "midnightblue"
  )[1:length(values)]

  fontSize <- 0.9
  textXPos <- 0.5
  textYPos <- -0.7

  if (showValues) {
    for (i in seq_along(values)) {
      text(xPos, yPos, round(resultMatrix[values[i],], 2),
           adj = c(textXPos, textYPos), col=colorsUsed[i],
           cex = fontSize)
      textYPos <- textYPos + 2.5
    }
  }

  legend("topleft", legend = values, fill = colorsUsed,
         bg = "transparent", box.lty = 0)
  if (exerciseData) {
    legend("topleft",
           legend = c(values, "Points of optimal exercise"),
           fill = c(colorsUsed, "red"),
           box.lty = 0, bg = "transparent")
  }
}

# Only relevant to Application I
valueBSM <- function(S0, K, mu, sigma, er, N, optionType) {
  type <- if (length(optionType) == 4) optionType
          else .parseOptionType(optionType)
  if (type[3] == "american")
    stop("This function can only value european options")

  d1 <- (log(S0/K) + (log(er) + sigma^2 / 2)*N) / (sigma*sqrt(N))
  d2 <- d1 - sigma * sqrt(N)

  multiplier <- ifelse(type[1] == "short", -1, 1)

  if (type[2] == "binary" && type[4] == "call")
    (1 / er^N)  * pnorm(d2) * multiplier
  else if (type[2] == "binary")
    (1 / er^N)  * pnorm(-d2) * multiplier
  else if (type[4] == "call")
    (pnorm(d1) * S0 - (1 / er^N) * K * pnorm(d2)) * multiplier
  else
    ((1 / er^N) * K * pnorm(-d2) - pnorm(-d1) * S0 ) * multiplier
}
```

```r
# Only relevant to Application I
CRRapproximateBSM <- function(N, totalSteps, optionType,
                              prob, S0, K, er, U, D) {
  mu <- prob * (U-1) + (1 - prob) * (D-1)
  sigma <- sqrt(prob * ((U-1) - mu)^2 + (1-prob) * ((D-1) - mu)^2)

  Unew <- exp(sigma * sqrt(N / totalSteps))
  Dnew <- Unew^(-1)
  erNew <- er^(N / totalSteps)
  q <- 0.5 + 0.5 * (mu / sigma) * sqrt(N/totalSteps)
  if (q > 1) {
    stop("q greater than 1, q=", q,
         "\nPlease choose higher number of steps")
  }

  mu_hat <- q * log(Unew/Dnew) + log(Dnew)
  sigma_hat <- sqrt(q * (1-q) * log(Unew/Dnew)^2)

  fairPrice <- priceCRR(totalSteps, S0, Unew, Dnew, erNew, K,
                        optionType,
                        calculateExerciseDates = FALSE)$fairPrice
  res <- c(fairPrice, mu_hat, sigma_hat, totalSteps)
  names(res) <- c("fairPrice", "mu_hat",
                  "sigma_hat", "totalSteps")

  return(res)
}

# Only relevant to Application II
generateTrackingPortfolio <- function(resultsMatrix, u, d, er) {
  N <- resultsMatrix["timeStep", ncol(resultsMatrix)]
  N <- N + 1
  numberOfNodes <- (N * (N + 1) / 2)
  parentNodes <- 1:(N * (N - 1) / 2)
  endNodes <- (N * (N - 1) / 2 + 1):numberOfNodes
  connections <- {
    cbind(parentNodes,
          parentNodes + rep(1:(N-1), 1:(N-1)),
          parentNodes + rep(1:(N-1), 1:(N-1)) + 1)
  }

  delta <- B <- c(rep(NA, length(parentNodes)),
                  rep(0, length(endNodes)))
  result <- rbind(resultsMatrix, delta, B)


  uNodesFn <- resultsMatrix["Fn", connections[, 2]]
  dNodesFn <- resultsMatrix["Fn", connections[, 3]]

  result["delta", parentNodes] <- (uNodesFn - dNodesFn) /
    (resultsMatrix["S", parentNodes] * (u - d))
  result["B", parentNodes] <-
    (u * dNodesFn - d * uNodesFn) / (er * (u - d))

  return(result)
}
```

A.2 Code exclusive to Section 4.1

```
{
  N <- 10
  totalSteps <- 1000
  S0 <- 100
  U <- 1.155
  D <- 0.9975
  er <- 1.05
  K <- 200
  prob <- 0.75
  dividend <- 0
  optionType <- "american binary call"
}

myTree <- priceCRR(N, S0, U, D, er, K, optionType, dividend = 0)
riskNeutralProb <- myTree$RNProb

fPamerican <- myTree$fairPrice
drawTreeGraph(myTree$value, "S",
                "Price movements of underlying", exerciseData = F, toScale = T)

drawTreeGraph(myTree$value, c("Fn", "S"), "American binary call option prices",
                exerciseData = T, toScale = F)

optionType <- "binary european call"
myTree <- priceCRR(N, S0, U, D, er, K, optionType, dividend = 0)
fPeuropean <- myTree$fairPrice

drawTreeGraph(myTree$value, c("Fn", "S"), "European binary call option prices",
                exerciseData = T, toScale = F)

mu <- riskNeutralProb * (U-1) + (1 - riskNeutralProb) * (D-1)
sigma <- sqrt(riskNeutralProb * ((U-1) - mu)^2 + (1-riskNeutralProb) * ((D-1) - mu)^2)
BSMPrice <- valueBSM(S0, K, sigma, er, N, optionType)

steps <- c(seq(50, 200, 50), seq(250, 10000, 250))

apps <- matrix(rep(NA, length(steps) * 4), nrow = 4)
rownames(apps) <- c("fairPrice", "mu_hat", "sigma_hat", "totalSteps")

# This step takes very long
for (i in seq_along(steps)) {
  apps[,i] <-
    CRRapproximateBSM(N, steps[i], optionType, riskNeutralProb, S0, K, er, U, D)
  gc(verbose = FALSE, reset = FALSE, full = TRUE)
}


CRR2000 <- priceCRR(2000, 100, U, D, er, K, "european call", F, 0)
qqnorm(log(CRR2000$value["S", CRR2000$value["timeStep",] == 2000]))
qqline(log(CRR2000$value["S", CRR2000$value["timeStep",] == 2000]))


cat(paste("Expected total return", round(mu*N, 7),
          "\nTotal variance", round(sigma^2*N, 7),
```

```r
          "\nFair price", round(BSMPrice, 7), "\n\n"))

appMus <- appVars <- appPrices <- rep(NA, length(steps))

for (i in 1:ncol(apps)) {
  appPrices[i] <- appPrice <- apps["fairPrice", i]
  appMus[i] <- appMu <- apps["mu_hat", i] * steps[i]
  appVars[i] <- appVariance <- apps["sigma_hat", i]^2 * steps[i]

  priceDiff <- appPrice - BSMPrice
  muDiff <- appMu - mu*N
  varDiff <- appVariance - sigma^2*N

  cat(paste("CRR approximation with", steps[i], "steps",
            "\nExpected total return", round(appMu, 7),
            "\tOff by", round(muDiff, 7), "from actual value",
            "\nTotal variance", round(appVariance, 7),
            "\tOff by", round(varDiff, 7), "from actual value",
            "\nFair price", round(appPrice, 7),
            "\t\tOff by", round(priceDiff, 7), "from BSM value", "\n\n"))
}

rm(appPrice, muDiff, varDiff)


par(mfrow = c(2, 2))
plot(log10(steps), appMus, pch=16, col="green", main = "Approximations of mu")
abline(h=mu*N, col="darkgreen", lty = "longdash", lwd=2)
points(log10(steps), appMus, pch=16, col="green")

plot(steps, appVars, pch=16, col="orange", main = "Approximations of the variance")
abline(h=sigma^2*N, col="darkorange3", lty = "longdash", lwd = 2)
points(steps, appVars, pch=16, col="orange")

plot(log10(steps), appPrices, pch=16, col="turquoise",
        main = "Approximations of the fair price")
abline(h = BSMPrice, col="blue", lty="longdash", lwd=2)
points(log10(steps), appPrices, pch=16, col="turquoise")

asd <- boxplot((appPrices), horizontal = T, plot = FALSE)
appNoOutliers <- appPrices[appPrices != asd$out[1] & appPrices != asd$out[2]
               & appPrices != asd$out[3] & appPrices != asd$out[4]
               & appPrices != asd$out[5]]
qqnorm(appNoOutliers)
qqline(appNoOutliers)


# time complexity is O(n^2)
system.time(CRRapproximateBSM(N, 1000, optionType,
                              riskNeutralProb, S0, K, er, U, D))
gc()
system.time(CRRapproximateBSM(N, 2000, optionType,
                              riskNeutralProb, S0, K, er, U, D))
gc()
system.time(CRRapproximateBSM(N, 4000, optionType,
                              riskNeutralProb, S0, K, er, U, D))
```

```
gc()
system.time(CRRapproximateBSM(N, 8000, optionType,
                              riskNeutralProb, S0, K, er, U, D))
```

A.3 Code exclusive to Section 4.2

```
# MS announced a dividend increase to 85 cents per share in their
# Q2 earnings report published at the end of June 2023.
# The ex-dividend date would be on July 28th, which was
# also the expiration date of some options on the stock
# The dividend would be paid out on august 15th,
# 12 business days after the ex-dividend date
# The binomial tree will be constructed for the 2 week period
# (10 days) leading up to the ex-dividend date

# rf-rate
# Source: Yahoo Finance
YTBillData <- read.csv("1YTBillData.csv", sep = ",")
YTBillData[,1] <- as.Date(YTBillData[,1])
correctRow <- which(YTBillData[,1] == "2023-07-14")
rfRate <- YTBillData[correctRow, "Close"]
rfRate
rfRate <- as.numeric(substring(rfRate,0,nchar(rfRate)-1))/100 + 1
# This is erT for T = 1
# In the model, one time step = 1 / 252, of a year,
# since there are on average 252 business days in a year
# hence we have
oneDayRfRate <- rfRate^(1 / 252)
# for


# Source: optionistics.com
optionPriceHistory <- read.csv("MS options K=91.csv", sep = "\t")
optionPriceHistory
callPriceHistory <- optionPriceHistory[,"callPrice"]


# Stock price data from 13th of July 2018 to 28th of August 2023
stockData <- read.csv("MS.csv", sep = ",")
stockData[,1] <- as.Date(stockData[,1])

head(stockData)
tail(stockData)

plot(stockData$Date, stockData$Adj.Close, type='l',
     ylab="Adjusted closing prices", xlab="Date")
grid()

fiveYears <- which(as.logical(stockData[,1] <
                                as.Date("2023-07-14")))
fiveYearPriceHistory <- stockData[fiveYears, "Adj.Close"]

historicNumberOfDays <- length(fiveYearPriceHistory)

historicReturns <- fiveYearPriceHistory[2:historicNumberOfDays] /
```

```r
  fiveYearPriceHistory[1:(historicNumberOfDays-1)] - 1
historicMu <- mean(historicReturns)
historicSigma <- sqrt(var(historicReturns))

historicMu
historicSigma


daysToObserve <- which(as.logical(stockData[,1] >=
                                    as.Date("2023-07-14")))
toObserve <- stockData[daysToObserve,c("Date", "Close")]

nrow(toObserve)

stockValues <- toObserve[,"Close"]
dailyStockReturns <- stockValues[2:11] / stockValues[1:10] - 1

{
  N <- 10 # trading days between 14th and 28th
  S <- S0 <- stockValues[1]
  U <- exp(historicSigma)
  D <- U^(-1)
  er <- oneDayRfRate
  K <- 91
  dividend <- 0.85*er^(-12) # discounted value of dividend
  optionType <- "american call"
}

tree1 <- priceCRR(N, S0, U, D, er, K, optionType,
                  dividend = dividend)
tpf1 <- generateTrackingPortfolio(tree1$value, U, D, er)
drawTreeGraph(tree1$value, c("S"),
              "Binomial Tree for Morgan Stanley stock",
              exerciseData = TRUE, toScale = T)


# On day 1 we write a call option & buy the tracking portfolio

deltas <- Bs <- rep(NA, 10)
fairPrices <- rep(NA, 11)
exerciseData <- rep(NA, 10)


# From day 0 to 9 (1 day before expiration)
for (day in 1:10) {
  tree <- priceCRR(N, S, U, D, er, K, optionType,
                   dividend = dividend)
  fairPrices[day] <- tree$fairPrice
  exerciseData[day] <- tree$value["Exercise", 1]

  trackingPf <-
    generateTrackingPortfolio(tree$value,
                              U, D, er)[c("delta", "B"),1]
  deltas[day] <- delta <- trackingPf["delta"]
  Bs[day] <- B <- trackingPf["B"]
```

```r
  N <- N - 1
  S <- S * (1 + dailyStockReturns[day])
}

which(as.logical(exerciseData))

# Optimal tracking portfolio value for each day = fair price
# of option on that day
trackingPortfolios <- deltas * stockValues[1:10] + Bs
trackingPortfolios



# What they were worth the next day (n=1:10)
dayAfterTpf <- deltas * stockValues[2:11] + Bs * er


exerciseDay9 <- evaluateCallPayoff(stockValues[10], K)
exerciseDay10 <- evaluateCallPayoff(stockValues[11], K)

fairPrices[11] <- exerciseDay10

# hedging errors for last days:
hedgingErrorDay9 <- dayAfterTpf[9] - exerciseDay9
hedgingErrorDay10 <- dayAfterTpf[10] - exerciseDay10


PnLOfHedgeExerciseDay9 <- c(dayAfterTpf[1:8] -
                              trackingPortfolios[2:9],
                            hedgingErrorDay9)
PnLOfHedgeExerciseDay10 <- c(dayAfterTpf[1:9] -
                               trackingPortfolios[2:10],
                             hedgingErrorDay10)


# PnL value as of day 0
totalPnLExerciseDay9 <- sum(PnLOfHedgeExerciseDay9*er^(-(1:9)))
totalPnLExerciseDay10 <- sum(PnLOfHedgeExerciseDay10*er^(-(1:10)))

totalHedgingErrorExerciseDay9 <-
  sum(abs((PnLOfHedgeExerciseDay9)))
totalHedgingErrorExerciseDay10 <-
  sum(abs((PnLOfHedgeExerciseDay10)))



nearestPath <- c(1, 2, 4, 7, 11, 16, 23, 30, 39, 49, 59)
tree1$value["S",nearestPath]
# Tree from day 1 vs actual stock
drawTreeGraph(tree1$value, c("S"),
              "Binomial Tree for Morgan Stanley stock",
              exerciseData = TRUE, toScale = T)
lines(0:10, tree1$value["S", nearestPath], lwd=3, col="darkorchid3")
lines(0:10, stockValues[1:11], lwd=2, col="green3")
legend("topleft", legend = c("S", "Points of optimal exercise",
                             "Actual stock movement",
```

```
                              "Nearest path"),
       fill = c("turquoise4", "red", "green3", "darkorchid3"),
       box.lty = 0, bg = "white")



# Value of option
drawTreeGraph(tree1$value, c("Fn"),
              "Binomial Tree for Morgan Stanley stock",
              exerciseData = TRUE, toScale = T, showValues = F)
lines(0:10, fairPrices, lwd=2, col="goldenrod4")
lines(0:10, callPriceHistory, lwd=2, col="forestgreen")
lines(0:10, tree1$value["Fn", nearestPath], lwd=3, col="darkorchid3")
legend("topleft", legend = c("Value of tracking portfolio on each day",
                  "Historical call prices on each day",
                  "Points of optimal exercise"),
       fill = c("goldenrod4", "forestgreen", "red"),
       box.lty = 0, bg = "white")



summary(lm(callPriceHistory ~ tree1$value["Fn", nearestPath]))

linReg <- lm(callPriceHistory ~ fairPrices)


linReg <- lm(callPriceHistory ~ fairPrices)

qqplot(fairPrices, callPriceHistory, pch=16)
abline(linReg$coefficients)


tree1$fairPrice
stockValues[1:10]

# Hedging error visualised
plot(1:10, PnLOfHedgeExerciseDay10, col="turquoise", pch = 16)
abline(h=0, lty="dashed")
points(9, PnLOfHedgeExerciseDay9[9], col = "red", pch=17)
text(1:10, PnLOfHedgeExerciseDay10,
     round(PnLOfHedgeExerciseDay10, 4), adj=c(0.5,-0.5), cex = 0.7)
text(9, PnLOfHedgeExerciseDay9[9],
     round(PnLOfHedgeExerciseDay9[9], 4), adj=c(0.5,-0.5), cex = 0.7)
```