1. Zadanie

1. Define the absolute value of a real number. Name the properties of the absolute value of a real number and explain its goniometric importance. Define the interval and give the interval operations.

?

2. Prove the regularities of combination numbers and demonstrate it in Pascal's triangle.

3. Give the equations of the all lines which pass through the point A = [1; 2] and their distance from the point B = [1; -1] is $d = \frac{3\sqrt{2}}{2}$.

2. Zadanie

1. Give the analytic expression of a line in the plane, the relation among the coefficients in the general equations of parallel and perpendicular lines.

2. Derive the sine rule.

3. The graph of a function f is the parabola with the vertex V = [3; -7] which cuts across the axis y at the point Y = [0; -25]. Write the equation of the function f.

3. Zadanie

1. Describe different types of the proof of a statement and the mathematical sentence $A \rightarrow B$.

2. Given arithmetic progression $\{\alpha_k\}_{n=1}^{\infty}$ such that $a_4+a_7+a_{10}=15$, $a_5+a_6+a_{11}=9$. Decide and prove whether there is k such that $a_k = 0$ and m such that $a_m = 1$.

3. Find the value of $m \in R$ such that the expression $a(m) = \frac{2m+1}{m-2}$ has the values:

   A) From the interval <-1; 0)
   B) Greater than the expression 2m

zadanie 4

1. Define the function over set R, state the ways of expressing the function. Define the properties of the function over set $M \subset D(f)$ and exemplify these properties graphically.

2. The square KLMN is inscribed into the equilateral triangle ABC with the side a. Prove that the length of the side of the square is $a(2\sqrt{3}-3)$.

3. Calculate the probability that a randomly chosen two-digit number:

a) is not divisible by 5 nor by 7

b) is not divisible by 5 or is not divisible by 7.

zadanie 5

1. Explain the following terms: hypothesis, axiom, statement, negation of the statement, statement operations (state).

2. Prove that:

$$\sqrt{16+\sqrt{17}} > \sqrt{16-\sqrt{17}} + 1.$$

3. Find two real ~~...~~ numbers x, y such that three numbers $3, x, y$ are the three consecutive terms of a geometric progression and numbers $x, y, 18$ are the three consecutive terms of an arithmetric progression.

√

Zadanie 6

1. Give the form of the quadratic equation and inequality and explain different solving methods.

√

2. Prove that:

$$\log\left(1+\frac{1}{2}\right)+\log\left(1+\frac{1}{3}\right)+\log\left(1+\frac{1}{4}\right)+\ldots+\log\left(1+\frac{1}{2001}\right)=\log 1\text{[illegible]}$$

√

3. Given triangle ABC such that $a = 4\,cm$, $b = 8\,cm$, $\gamma = 100^\circ$. Calculate the radius of the circle circumscribed around the triangle.

zadanie 7

√1. Give the equation of the linear function, sketch the graph and name its properties.

2. Prove that the allowable values of $x \in R$ satisfy

√ a) $$\frac{2\sin 2x - \sin 4x}{2\sin 2x + \sin 4x} = tg^2 x$$

√ b) $$\frac{\cos 2x}{cotg^2 x - tg^2 x} = \frac{1}{4} \sin^2 2x$$

√

3. Given cuboid ABCDEFGH ($|AB| = 4cm$, $|BC| = 3cm$, $|AE| = 5cm$) In the parallel projection, demonstate the angle between the line FD and the plane ACM, where M is the midpoint of the edge DH. In the properly chosen plane, demonstrate the real value of this angle and calculate it as well.

zadanie 8

1. Give the equation of the quadratic function, explain $D(f)$, $R(f)$ and the function properties using its graph.

2. Prove that:

a) $\forall n \in N: 30 / (n^5 - n)$

b) $\forall a, b, c \in N: a/b \wedge a/(b-c) \Rightarrow a/c$

3. Two $(\neq)$ groups have 26 elements in total and 160 second-class combinations without repetition. Calculate the number of elements in each group.

zadanie 9

✓ 1. Give the equation of the inverse proportion, fractional function of the first degree and explain their mutual relationship.

✓ 2. Prove that the sequence $\left\{\frac{3n-2}{n+1}\right\}_{n=1}^{\infty}$ is bounded. Find the recurrence relation of the sequence.

✓ 3. Calculate the value of $k \in R$ so that the equation $x^2+y^2+6x-4y+k=0$ is the equation of a circle. How to change k so that,

a) the circle touches the y-axis

b) the circle passes through the point $M=[2,1]$.

zadanie 10

1. Define the exponential and logarithmic function, explain their properties using graphs and explain the invertibility between these functions.

2. Prove that:

a) $\forall a,b,c \in \mathbb{N}: a|b \wedge a|c \Rightarrow a|(bx+yc)$ $x,y \in \mathbb{Z}$

b) $\forall x,y \in \mathbb{R}^+ : (x+y)\cdot\left(\frac{1}{x}+\frac{1}{y}\right) \geq 4$

3. Given obtuse triangle ABC with the obtuse angle at the vertex B ($\beta = 100°$), $|AB| = 90cm$. The angle between the median to the side c and side c is $110°$. Calculate the length of the median to the side c and the length of the side AC.

zadanie 11

1. Define the arithmetic and geometric progression and give the particular example of both progressions. Decribe the basic relations of these progressions

2. Derive Euclidean leg theorem.

3. a) Plot the following sets into the number line and rewrite them as intervals:

$A = \{x \in R; |x+2| < 4\}$

$B = \{x \in R; |x-2| \geq 6\}$

b) Name the elements of the sets X, Y such that:

a) $X \subset \{1, 2, 3, 5, 6, 9\}$
$Y \subset \{2, 3, 4, 5, 8\}$
$X \cap Y = \{5\}$

b) $X \cup Y = \{3, 5, 6, 8\}$
$X \cap \{4, 6, 8\} = \emptyset$
$Y \cap \{1, 3, 5, 7\} = \{3, 5\}$

c) $X \cap Y = \emptyset$
$X' \cap Y = \{3, 5, 7\}$
$X \cap Y' = \{2, 4, 6, 8\}$

d) $X \cap Y = \{2, 6, 7\}$
$X \subset \{5\}$

Zadanie 12

1. Define functions $\sin x$, $\cos x$, on the unit circle Name the properties of these functions using their graphs. Explain the periodicity of a function using the particular example of these functions.

2. Given point D from the base BC of the equilateral triangle ABC, given point E which is the foot of the perpendicular construed from the point D to the side AB. Prove that $|AE| = 3/4\,|AB|$ if and only if D is the midpoint of BC.

3. 1) Given true statements A, B and the false statement C. Decide which of the compound statements is true:

a) $(A \wedge B) \Rightarrow B$

b) $A \Rightarrow (B \wedge C)$

c) $(A \vee B) \Leftrightarrow C$

2) Form the negation of the following statements

a) $\forall a,b,c \in N: a | bc \Rightarrow (a|b \vee a|c)$

b) $\forall a \in N: 2|a \Leftrightarrow 2|a^2$

Zadanie 13

1. Describe the important line segments in the triangle and their properties. Give the relation between the sides and angle of a scalene triangle.

2. Prove that the sequence $\left\{\frac{5n-1}{n+2}\right\}_{n=1}^{\infty}$ is monotonous and bounded above.

3. Find $a, b \in R$ so that the graph of the function $f: y = a \cdot \cos\left(x - \frac{\pi}{3}\right) + b$ passes through the points $A = [0, -1]$, $B = [\pi, 3]$. How to change the value of b so that the graph of the function f passes through the point $O = [0, 0]$?

Zadanie 14

1. Define function tgx on the unit circle. Describe its properties using the graph.

2. Prove that the diagonals e, f and sides a, b of the parallelogram ABCD satisfy the relationship: $e^2 + f^2 = 2.(a^2+b^2)$

3. The final results of the test were the following: 15 students got 1,
35 students got 2,
30 students got 3,
15 students got 4,
5 students got 5.

Calculate the averge mark of the test, mode, median. Demonstrate the results graphically.

Zadanie 15

1. Classify polygons according to:
   a) the number and length of the sides
   b) the relation to the insribed circle of the polygon and the circle circumscrib around the polygon.

State the properties of polygans and formulae for the calculation of their area.

√

2. Derive Euclidean altitude theorem.

√

3. The snm of a two-digit number is 7. If we change the order of the digits, we get the number which, when multiplied by the original number, is 1462. Identify the original number.

✓ zadanie 16

1. Define the reflection in a line and the reflection in a point and decribe their properties. Give the examples of the shapes which are axially and centrally symmetrical. Perform the transformation of the point and the line in the axial and central symmetry and decribe the mutual position of the object and the image of the line.

2. Prove that the three numbers:

✓ a) $\sin 2x$, $\cos x$, $\frac{1}{2 tg x}$, $x \in (0; \pi)$ form three consecutive terms of a geometric progression

✓ b) log 16, log 8, log 4 form three consecutive terms of an arithmetic progression.

✓

3. Solve the system of linear equations in dependence on the parameter $a \in R$. Calculate the value of a such that both roots of the system are positive.

$$2x + ay - 2 = 0$$
$$3x + 6y + 2 = 0$$

Zadanie 17

1. Define the geometric shapes (line segment, angle, zone, triangle, circle k, circle K, parallelogram, trapezium) using the set operations or the characteristic property.

2. Given sequence of squares: the vertices of the square $A_2B_2C_2D_2$ are the midpoints of the sides of the square $A_1B_1C_1D_1$, the vertices of the square $A_3B_3C_3D$ are the midpoints of the sides of the square $A_2B_2C_2D_2$ etc. Suppose that p_i is the perimeter of the square $A_iB_iC_iD_i$, a_i is the area of the square $A_iB_iC_iD_i$. Prove that the sequences $\{p_i\}_{i=1}^{\infty}$; $\{a_i\}_{i=1}^{\infty}$ are geometric progressions and find their common ratio.

3. Solve these equations in the set R:

a) $\log x^{2\log\sqrt{x}} + \log\frac{1}{x^2} = 3$

b) $\frac{1}{3^x} = \frac{1}{\sqrt{3}}\cdot\sqrt[6]{27^{3-3x}}\cdot\left(\frac{1}{9}\right)^{x+3}$

Zadanie 18

1. Define the translation and rotation and describe their properties. Perform the transformation of the point and the line in the translation and rotation, describe the mutual position of the object and the image.

2. Given cone with the radius of the base of 4cm and height of 6 cm which is divided into two objects with the same volume by the plane parallel with the base. Prove that the circumference of the circle which is the section is $4\pi\sqrt{4}$.

3. Calculate the values of the remaining goniometric functions without solving the angle x if $tg x = -3$, $x \in \left(\frac{3}{2}\pi, 2\pi\right)$. Calculate the values of $\sin 2x$ and $\cos 2x$.

zadanie 19

✓

1. Define the sequence, state the ways of expressing the sequence and the sequence properties, graph of the sequence.

✓

2. Given the expressions:

$$V = \log 0{,}01 + 2\log 10\sqrt{10} + \log_3 \frac{1}{125} - \ln e^3$$

$$U = \log_6 \sqrt{216} + \log_7 \frac{1}{49} - \log_{0{,}5} 8 + \log \frac{1}{0{,}001}$$

Prove that $V + U = -6$.

✓

3. Given cube ABCDEFGH. Construct:

a) the line of intersection of the planes ACE and BHP, where P is the midpoint of the edge FG.

b) the section by the plane KLM, where K, L, M are the midpoints of the line segments AE, AB, EG successively.

√ Zadanie 20

1. Explain the following terms: angle of 2 lines, angle of the line and the plane, angle of two planes, perpendicularity of the lines and planes, orthoganal projection of the point to the ~~plain~~ plane and the line

√

2. Derive the formula for the calculation of the roots of the quadratic equation,

$$ax^2+bx+c=0.$$

3. Given functions:

√ a) $f: y=\frac{1}{4}(10^x+1)$. Identify the set over which there exists the inverse function to the function f and write its equation.

√ b) $g: y=\frac{2x-6}{-x+2}$. Find the asymptotes to the function g and all $x \in D(f)$ such that $g(x) \geq 0$.

Zadanie 21

1. Define the circle as the set of points of the given property and derive its central and general equation. Describe the mutual position of the line and the circle. State the ways of identifying the mutual position of the line and the circle

2. Given area of the equilateral triangle A and perimeter P. Prove that $2\sqrt[4]{27A^2} = P$

3. Find the parameter $p \in R$ so that the difference of the roots of the quadratic equation $px^2 + x + p - x^2 - 2px = 0$ is equal to $-\frac{4}{3}$.

✓ Zadanie 22

1. Explain the following terms: prime number, composite number, coprime numbers, highest common divisor, lowest common multiple. Set the criteria for divisibility of natural numbers.

✓

2. Prove that the allowable values of x satisfy

$$\frac{\sin 2x}{1-\cos 2x} + \frac{1+\cos 2x}{\sin 2x} = \frac{2}{\operatorname{tg} x}$$

✓

3. The rectangle of which the sides are in the ratio 2:1 is inscribed into the equilateral triangle ABC with the length of the side 4. Calculate the dimensions of the rectangle.

Zadanie 23

1. Explain the basic terms in probability: event, certain event, impossible event. (give the particular examples). State Laplace scheme, formula for the calculation of the probability of independent events and an opposite event.

2. Prove that any natural number n satisfies:

$$2C(n, 2n-1) = C(n, 2n)$$

3. Find the coordinates of the point C from the line $p: 2x+y=0$ so that the triangle ABC with the base AB is isosceles and $A[6,4]$, $B=[2,-2]$.

Zadanie 24

1. Explain the following terms: variations without and with repetition, permutations, combinations, formulas for their calculation and practical examples.

2. Derive the relations between the roots and the coefficients of the quadratic equation
$$ax^2+bx+c=0$$

3. Given lines
$$a: (3-p)x+12y+2-q=0$$
$$b: 2x-2y+2=0$$
Examine their mutual position in dependance on the parameters p, q.

✓ Zadanie 25

1. Explain the following terms: $n!$, combination number, state the properties of the combinati[on] numbers.

✓

2. Given leg $a = 4cm$ and median $m_a = 6$ cm in the right-angled triangle with the hypotenuse c. Prove that $m_a : m_b = \frac{\sqrt{6}}{2}$. ?

3. Given circle $k: x^2+y^2=25$ and line $p: 3x+4y+a=$
Find $a \in R$ so that:

a) the line p is the secant of the circle k

b) the line p is not the secant nor the tangent of the circle k.

√ Zadanie 26

1. Explain the following terms: distance of two points, distance of the point from the line and the point from the plane, distance of the parallel lines, distance of the line from the plane which is parallel with the line. Sketch the given situations in the optional parallel ~~proj~~ projection.

√

2. Prove that when rolling three dice of a different colour, it is more probable to roll the sum of 10 than sum of 9.

√

3. Solve:

a) in set Z: $x^2 - 3(x+1) - x < 0$

b) in set R: $2 + \frac{3}{x+1} \geq \frac{2}{x}$

✓ Zadanie 27

1. Define the following terms: statistical dataset, arithmetic mean, modus, median, standard deviation, statistic dispersion. State the ways of picturing the statistical dataset - various types of graphs (polygon, histogram).

✓

2. Prove that any positive real number a satisfies: $a^3 + 1 \geq a + a^2$.

3. Explain the construction steps to draw the triangle ABC if $\alpha = 50°$, $\beta = 100°$, radius of the inscribed circle is $\rho = 2cm$.

√ Zadanie 28

1. Explain the following terms surface area of the solid, volume of the solid. State the examples of solids and formulas for the calculation of their volumes and areas. Name the solids which are rotational.

√

2. Prove that the value of the expression

$$\left[\left(a^{0,5}+b^{0,5}\right)^2-\left(\frac{\sqrt{a}-\sqrt{b}}{a^{1,5}-b^{1,5}}\right)^{-1}\right]\cdot(ab)^{-0,5}$$ is constant

for the allowable values of $a, b \in R$.

√

3. Write the equation of the circle which touches two lines $p: 3x-4y+1=0$,

$q: 3x-4y+5=0$.

The centre of the circle lies on the line $m: 3x+2y=0$.

Zadanie 29

✓

1. Define the isometry and similarity of shapes. State the theorems about the isometry and similarity of triangles. Explain the usage of the ratio of the similarity when changing the length of the line segment.

2. Prove:

✓ a) any natural number n satisfies:
If the sum of numbers $7n$ and $4n$ is divisible by 55, then their difference is divisible by 15.

✓

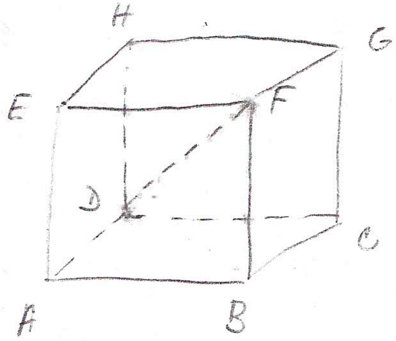
b) the sum of the cubes of three consecutive natural numbers is divisible by 3.

✓?

3. Calculate $a, b \in R$ so that the graph of the function $f: y = a \cdot \log_2 x + b$ passes through the point $A = \left[\frac{1}{2}; 1\right]$, $B = [2; 3]$. Find out whether the point $C = [32; 7]$ lies on the graph of this function. How to change the function f so that $H(f) = \langle 0, \infty)$?

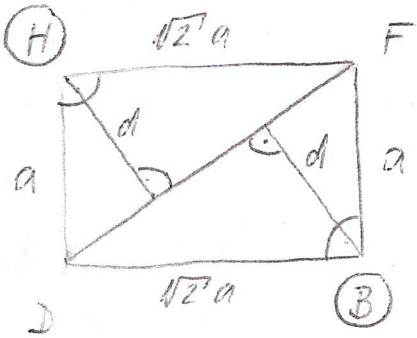
zadanie 30:

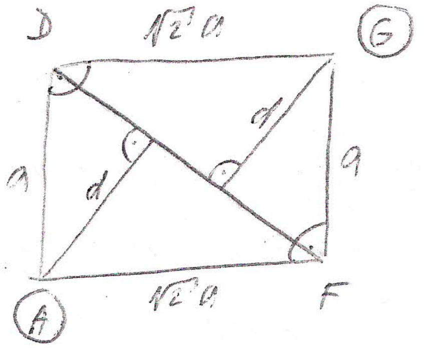
2.

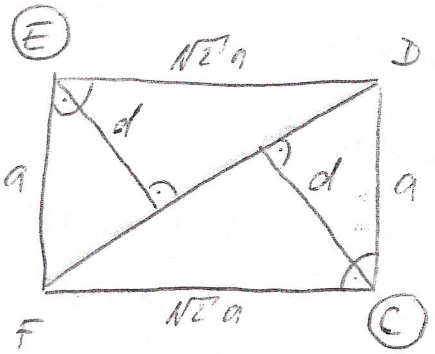


DF → space diagonal

- DF lies in planes $\overrightarrow{EFC}$; $\overrightarrow{BFH}$; $\overrightarrow{AFG}$
- we want to prove that the distances $|H;FD|;|B;FD|;|A;FD|;|G;FD|;|E;FD|$ and $|C;FD|$ are the same







$|DB|^2 = 2a^2$

$|DB| = \sqrt{2}a$

$|AF|^2 = 2a^2$

$|AF| = \sqrt{2}a$

$|FC|^2 = 2a^2$

$|FC| = \sqrt{2}a$

∴ the $\triangle HFD \cong \triangle BDF \cong \triangle AFD \cong GDF \cong CFD \cong EDF$ according to sus theory

- if we want to express the distance d we can:

$$S_{\triangle_1} = \frac{a \cdot \sqrt{2}a}{2} = \frac{\sqrt{2}a^2}{2}$$

$$S_{\triangle_2} = \frac{|DF| \cdot d}{2}$$

$$S_{\triangle_2} = \frac{\sqrt{3}a \cdot d}{2}$$

$$|DF|^2 = a^2 + (\sqrt{2}a)^2$$

$$|DF|^2 = a^2 + 2a^2$$

$$|DF|^2 = 3a^2$$

$$|DF| = \sqrt{3}a$$

$$S_{\triangle 1} = S_{\triangle 2}$$

$$\frac{\sqrt{2}a^2}{\cancel{2}} = \frac{\sqrt{3}a \cdot d}{\cancel{2}}$$

$$d = \frac{\sqrt{2}a^{\cancel{2}}}{\sqrt{3}\cancel{a}} = \frac{\sqrt{6}a}{3}$$

TRANSFORMATIONS IN THE PLANE

If each point X in the plane is associated with a unique point X' in the plane, then this correspondence is called the transformation in the plane.

X $\xrightarrow{\text{trans.}}$ X'

↳ point before trans. / original shape = OBJECT (vzor)

↳ point created after the trans. = IMAGE (obraz)

↳ it's a kind of correspondence between object and image

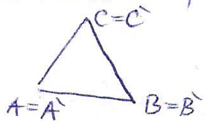
Transformations → isometries (zhodné zobrazenia)
↘ similarities (podobné zobrazenia)

1. ISOMETRY → the transformation in the plane is called isometry if any 2 points X, Y and their images X', Y' satisfy the equality that: $|XY| = |X'Y'|$ (the length of $XY = X'Y'$)

Isometry is the distance-preserving transformation.

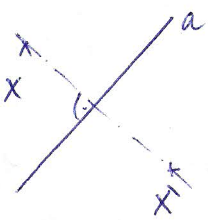
Isometries are: Identity
Axial symmetry (Reflection in a line) — osová súmernosť
Central symmetry (Reflection in a point)
Rotation (otočenie)
Translation (posunutie)

→ Identity → all the points of this transformation are invariant (samodružné) (a point is invariant when object is congruent with image)



→ Reflection in a line / Axial symm. → this transformation is given by the line, which is called the axis of symmetry / mirror line (os súmernosti)

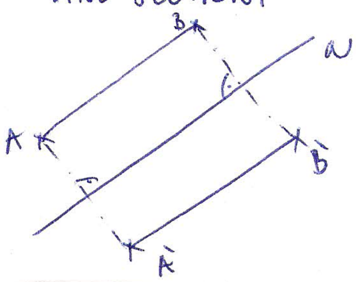
IMAGE OF: POINT



- suppose there is a line a and the point X → we construct perpendicular line to the line a passing through point X
- we plot the same distance as between point X and line a to the opposite half line and we find point X'

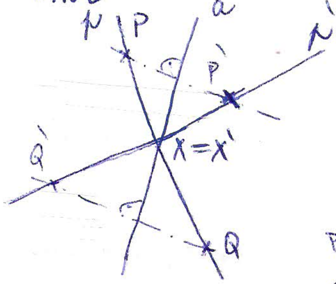
$XX' \perp a$; $|X, a| = |X', a|$

LINE SEGMENT



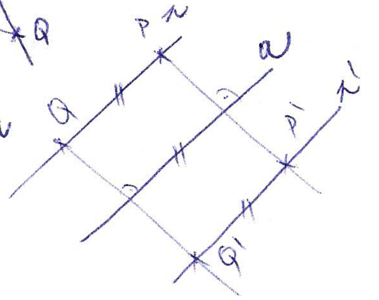
- line segment AB, mirror line a
- we find the images of the end points of the line segment

LINE



- suppose there is line p intersecting with the mirror line
- we choose 2 points on the line p - we create their images
- the image of the line passes through the image of the points and also through one invariant point

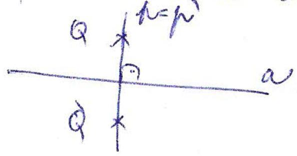
LINE // w. a



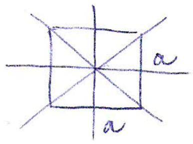
all 3 lines are parallel

$|p, a| = |p', a|$

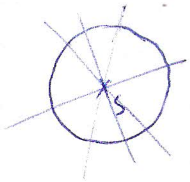
INVARIANT POINTS | LINES :
- all the points on a mirror line
- mirror line itself
- any line perpendicular to the mirror line



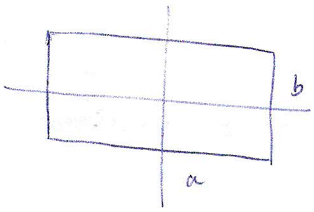
AXIALLY SYMMETRICAL SHAPES :



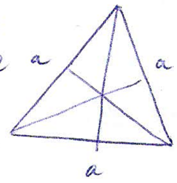
SQUARE : 4 mirror lines : diagonals + the lines passing through the midpoints of the sides



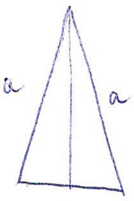
CIRCLE : infinite number of mirror lines passing through the centre.



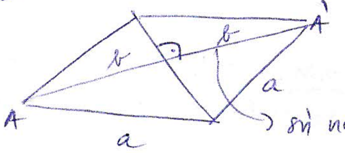
RECTANGLE : 2 mirr. lines : lines passing through the midpoints of the sides



EQUILATERAL Δ : 3 mirr. lines : angle bisectors = medians = altitudes



ISOSCELES Δ : 1 mirr. line : median = angle bisector = = altitude

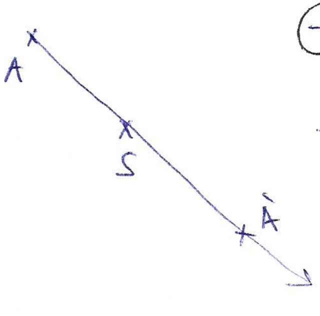


RHOMBUS : 2 mirr. lines : diagonals

→ sú na seba kolmé a navzájom sa rozpoľujú

→ Reflection in a point (central symm.) – this transformation is always given by a point = " the centre of symmetry "

DOINT:

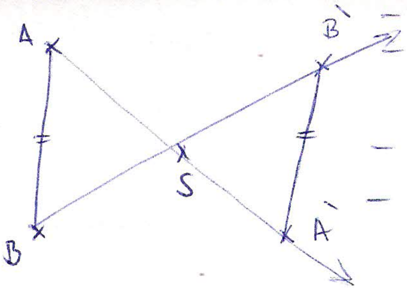


- we draw a half line AS (not SA – the direction is very important)
- → we plot the same distance and we get a point A' which is from a half line
- S is the midpoint of the line AA'

1) $\overrightarrow{AS}$

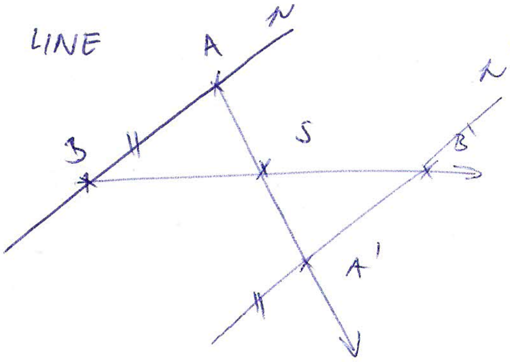
2) A'; $A' \in \overrightarrow{AS} \wedge S = A \dot{-} A'$

LINE SEGMENT:



- we find images of the end points
- we create half line $\overrightarrow{AS}$ (→)
  half line $\overrightarrow{BS}$ (→)
- line segment $A`B`$
- mutual position of AB and $A`B`$: $AB \parallel A`B`$
  and $|AB| = |A`B`|$

LINE

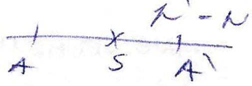


- 2 points from line p → half lines → images → we join the images

$p \parallel p`$

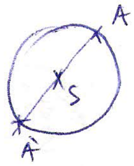
$|S;p| = |S;p`|$

INVARIANT POINTS | LINES :
- centre of symmetry (COS) is the only invariant point ($S=S`$)
- lines passing through the COS are all invariant

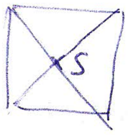


- if COS is the midpoint of line segment AB, then the line segment is invariant

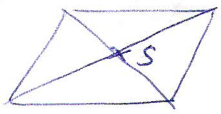
CENTRALLY SYMMETRICAL SHAPES:



CIRCLE : COS = centre of the circle



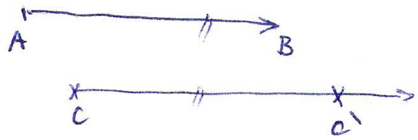
square : COS = point of intersection of 2 diagonals



RHOMBUS (same as □)

→ Translation – is given by the directed line segment ("orientovaná úsečka") (segment having length and the direction

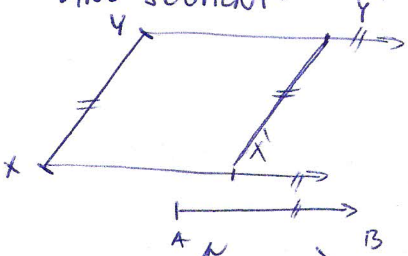
POINT:



- we draw a half line from the point C parallel with the directed line segment AB → we plot the same distance on the new half line ⇒ C`

$AB \parallel CC`$ ; $|AB| = |CC`|$

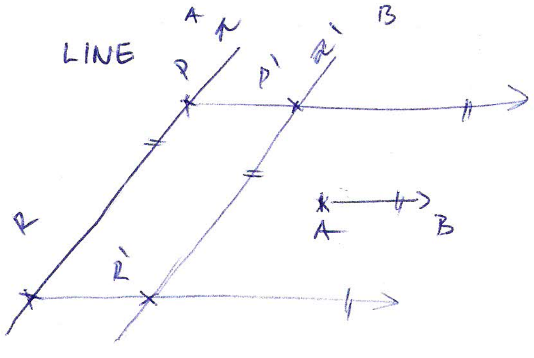
LINE SEGMENT:



↳ the same but with 2 points

$XY \parallel X`Y`$ & $|XY| = |X`Y`|$

LINE

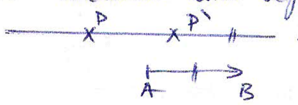


↳ 2 points from line p

$p \parallel p`$   $|p;p`| = |AB|$

INVARIANT POINTS | LINES: • no invariant points

• if line is ∥ with directed line segment AB then the line is invariant



$p = p'$

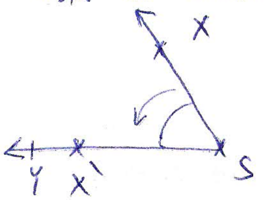
→ **Rotation** – is given by the point called the centre of rotation and the direction angle of rotation

• if the direction angle is:

- $\alpha > 0$, then it is measured in the anticlockwise direction
- $\alpha < 0$, ... clockwise direction

POINT

$R_{S,\alpha} = 60°$ (rotation is given by the centre S and the dir. angle is 60)



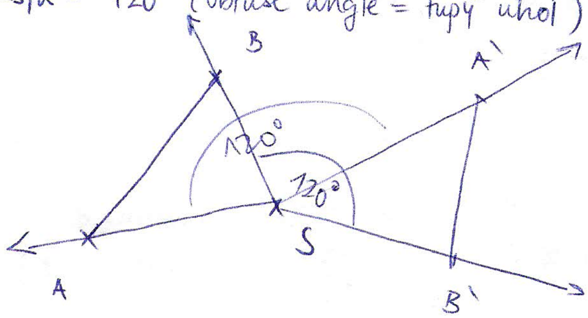
1 – half line $\overrightarrow{SX}$

2 – we measure 60° in anticlockwise direction, we draw the angle XSY; $|\measuredangle XSY| = \alpha = 60°$

3 – X'; $X' \in \overrightarrow{SY} \wedge |SX| = |SX'|$

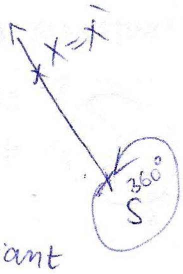
LINE SEGMENT

$R_{S,\alpha} = -120°$ (obtuse angle = tupý uhol)



$|AB| = |A'B'|$

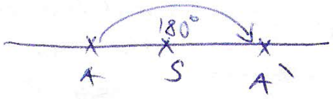
$AB \nparallel A'B'$



INVARIANT POINTS, LINES:

POINTS
- if $R_{S,\alpha} = 360°$ → then the point is invariant
- centre of rotation is invariant

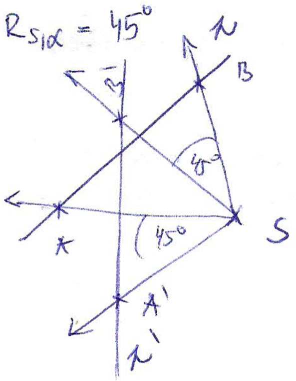
LINES
- if $\alpha = 360°$ → then the line is invariant
- if $\alpha = 180°$ and the centre is from the line, then the line is also invariant



$p = p'$

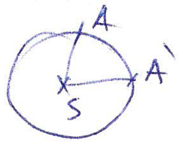
LINE

$R_{S,\alpha} = 45°$

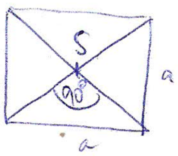


$|\measuredangle p, p'| = 45°$ as well (because these lines intersect at some point)

SHAPES:

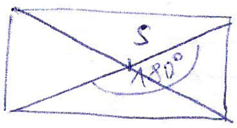


CIRCLE: centre of rotation = centre of circle; α doesn't matter



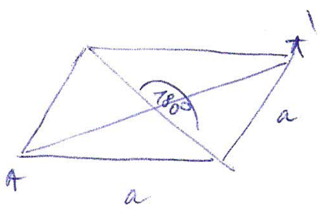
SQUARE

$\alpha = 90°$



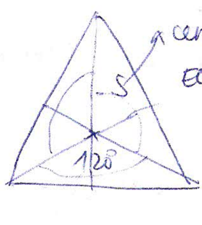
RECTANGLE

$\alpha = 180°$



RHOMBUS

$\alpha = 180°$



centroid = ťažisko

EQUILATERAL Δ

$\alpha = 120°$

2. SIMILARITY → ratio-preserving transformation

The transformation in the plane is called a simillarity if there is a positive real number k ($k \in \mathbb{R}^+; k \neq 1$) such that any two points X, Y and their images X' and Y' satisfy the equality | relationship:

$|XY| = k \cdot |X'Y'|$
↳ "scale factor"
= koeficient podobnosti

if $k > 1$ ... similarity is enlargement
$0 < k < 1$ ... reduction
($k = 1$ ... isometry)

CONGRUENCY AND SIMILARITY OF SHAPES
(= zhodnosť a podobnosť útvarov)

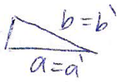
- Two objects are congruent if there is the distance-preserving transformation (= isometry) which converts one object to the other one.

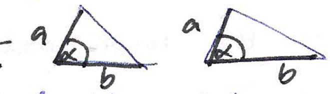
 1) 2 circles are congruent if their radii are equal
 2) 2 polygons are ~~eq~~ congruent if one of the following conditions holds: (stačí keď jedna z nich)
 - their corresponding sides and corresponding diagonals are equal
 OR
 - their corresponding sides and corresponding angles are equal

- Two objects are simmilar if there is the ratior preserving transformation (= simillarity) which ~~comes~~ converts one object to the other one.

CONGRUENCY AND SIMILARITY OF △

2 triangles are CONGRUENT if one of the conditions holds:

1) SSS - all 3 corresponding sides are equal 

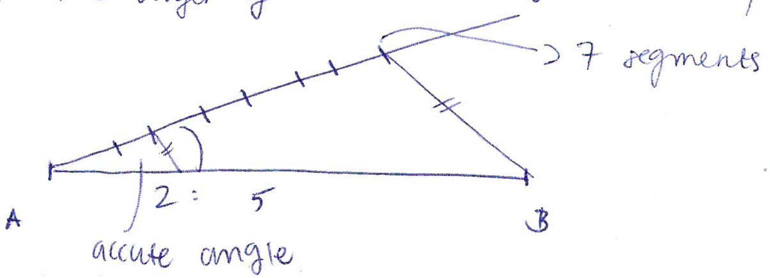


2) SAS - there are two pairs of equal sides and the angle~~s~~ enclosed by these sides are equal
3) ASA - there is one pair of equal sides and the angles of these sides are also pairwise equal ← 

4) SSA - there are two pairwise equal sides and the angle opposite the larger of these 2 sides ~~are~~ is equal

2 triangles are SIMMILAR if one of the conditions holds:

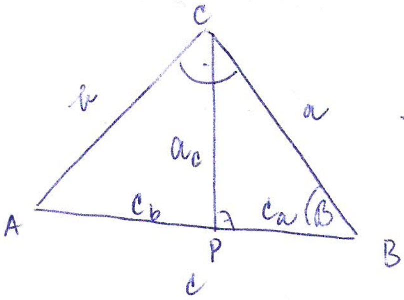
1) SSS - if the ratios of the lengths of corresponding sides are equal
2) SAS - the ratios of the lengths of 2 corresponding sides are equal and the angles enclosed (between them) by these 2 sides are the same
3) ASA - two corresponding angles are equal
4) SSA - the ratios of the lengths of two corresponding sides are equal and the angles opposite the larger of these two sides are also equal



2 : 5

EUCLIDEAN THEOREMS – the theorems can be applied only in the right Δ

1) Euclidean altitude theorem



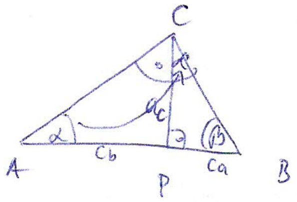
– the foot of the altitude divides the hypotenuse into c_a and c_b (because it's adjacent to the leg a / b)

– $c_a + c_b = c$

$$\boxed{\text{PROVE THAT: } a_c^2 = c_a \cdot c_b}$$

– to prove theorem we need to find two simmilar triangles:

$\Delta ABC \sim \Delta PBC$



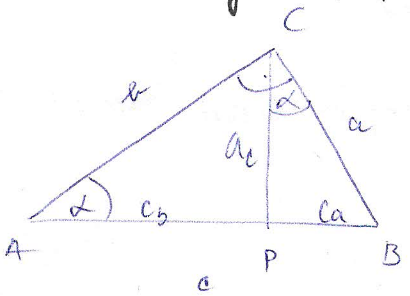
– we use the trigonometric ratio of $tg\,\alpha$:

$$\left.\begin{aligned} \Delta PBC:\ tg\,\alpha &= \frac{c_a}{a_c} \\ \Delta APC:\ tg\,\alpha &= \frac{a_c}{c_b} \end{aligned}\right\} \text{they have to be equal}$$

$$\frac{c_a}{a_c} = \frac{a_c}{c_b} \quad / \cdot a_c \cdot c_b$$

$$\underline{c_a \cdot c_b = a_c^2}$$

2) Euclidean leg theorem



"hypotenuse times the segment adjacent to the side..."

$a^2 = c \cdot c_a$

$$\boxed{\text{PROVE THAT: } b^2 = c \cdot c_b}$$

– we will use the trigonometric function sine:

$$\left.\begin{aligned} \Delta PBC:\ \sin\alpha &= \frac{c_a}{a} \\ \Delta ABC:\ \sin\alpha &= \frac{a}{c} \end{aligned}\right\}$$

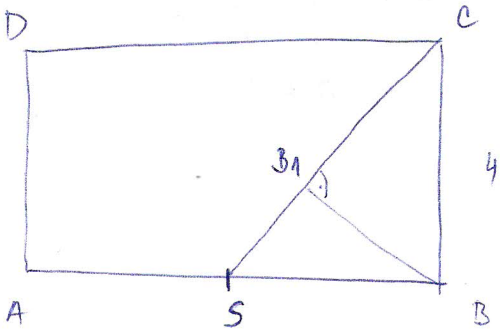
$$\frac{c_a}{a} = \frac{a}{c} \quad / \cdot a \cdot c$$

$$\underline{c_a \cdot c = a^2}$$

– when we sum up these equations we get the pythagorean theorem

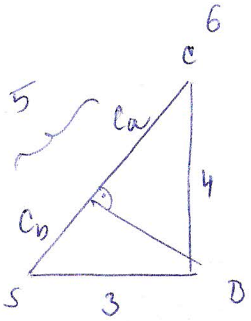
$$a^2 = c \cdot c_a$$
$$b^2 = c \cdot c_b$$
$$\overline{a^2 + b^2 = c\,(\underbrace{c_a + c_b}_{c})}$$
$$\underline{a^2 + b^2 = c^2}$$

PRÍKLADY



Prove : $|SB_1| : |B_1C| = 9 : 16$

1. we calculate $|SC|$ using pythagorean theorem
2. we can calculate the lengths of segments using euclidean leg theorems
3. we calculate the ratio between $c_b\,(SB_1)$ and $c_a\,(B_1C)$



$16 = 5 \cdot c_a$

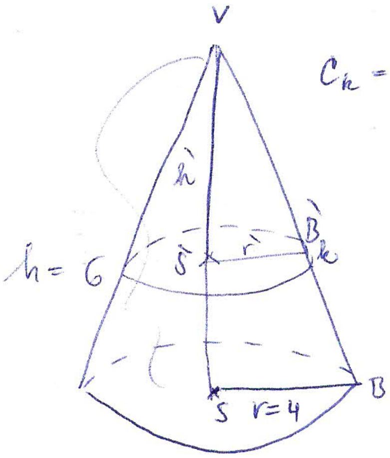
$\frac{16}{5} = c_a$

$9 = 5 \cdot c_b$

$\frac{9}{5} = c_b$

$\frac{c_b}{c_a} = \frac{\frac{9}{5}}{\frac{16}{5}} = \underline{\underline{\frac{9}{16}}}$

Given cone with the radius of the base of 4 cm and the height of 6 cm which is divided into 2 objects with the same volume by the plane parallel with the base. Prove that the circumference of the circle which is the section is $4\pi\sqrt[3]{4}$.



$C_k = 4\pi\sqrt[3]{4}$

$V = \frac{1}{3} \cdot \pi r^2 \cdot h$

$V = \frac{1}{\cancel{3}} \cdot \pi \cdot 16 \cdot \cancel{6}^{\,2}$

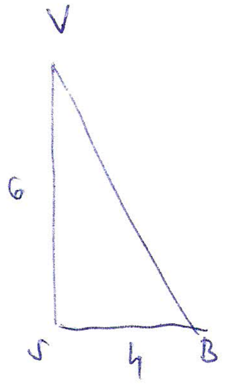
$V = 32\pi$

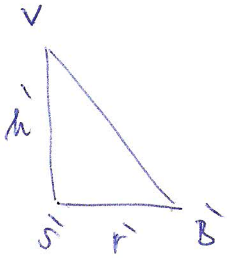
$V' = \frac{1}{2} V$

$V' = 16\pi$

$\cancel{\pi} r'^2 \cdot h' \frac{1}{3} = 16\cancel{\pi}$

$r'^2 \cdot h' = 48$





$\frac{6}{4} = \frac{h'}{r'}$

$\frac{6}{4} r' = h'$

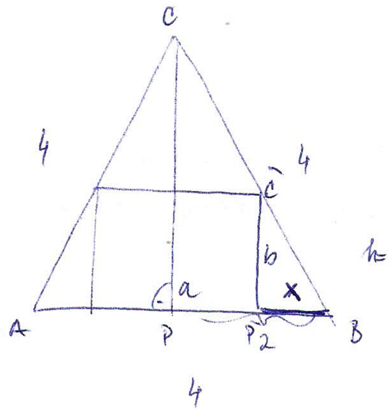
$r'^3 \cdot \frac{6}{4} = 48$

$r'^3 = 32.$

$r' = 2 \cdot \sqrt[3]{4}$

$O = 2\pi r' = 2 \cdot \pi \cdot 2 \cdot \sqrt[3]{4} = 4 \cdot \pi \cdot \sqrt[3]{4}$

Equilateral Δ and rectangle is inscribed into it. Find dimensions of the ▭.



$x = 2 - \frac{a}{2}$

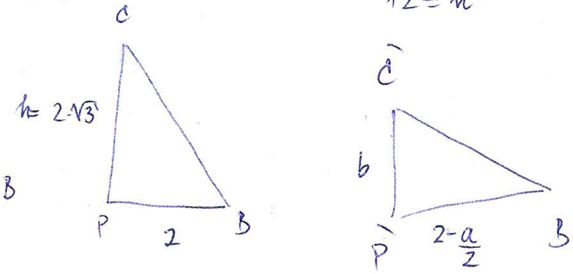
$a : b = 2 : 1 \longrightarrow a = 2b$

$\frac{a}{b} = \frac{2}{1}$

$a = 2b$

1) we find altitude $16 = 4 + h^2$

$12 = h^2$



$\underbrace{\qquad\qquad}$

similar Δs:

$$\frac{2\sqrt{3}}{2} = \frac{\frac{b}{2}}{2-\frac{a}{2}}$$

$$\frac{2\sqrt{3}}{2} = \frac{2b}{4-a}$$

$$\frac{(4-a)2\sqrt{3}}{\cancel{4}2} = \cancel{4}b$$

$$\frac{(4-2b)2\sqrt{3}}{\cancel{4}\,2} = b$$

$$4\sqrt{3} - 2b\sqrt{3} = 2b$$

$$4\sqrt{3} = 2b + 2b\sqrt{3}$$

$$4\sqrt{3} = 2b(1+\sqrt{3})$$

$$\frac{\cancel{4}^{2}\sqrt{3}}{\cancel{2}(1+\sqrt{3})} = b$$

$$b = \frac{\sqrt{3}}{1+\sqrt{3}} \cdot \frac{1-\sqrt{3}}{1-\sqrt{3}} = \frac{\sqrt{3}-3}{-2}$$

$$a = 2\cancel{2\sqrt{3}(4-a)}$$

$$\frac{2\cdot\sqrt{3}}{2} = \frac{\frac{b}{1}}{\frac{4-a}{2}}$$

$$\frac{2\sqrt{3}}{2} = \frac{2b}{4-a}$$

$$\frac{(4-a)\cancel{2}\sqrt{3}}{\cancel{2}} = \cancel{2}\cdot\frac{a}{\cancel{2}}$$

$$4\sqrt{3} = a + \sqrt{3}a$$

$$4\sqrt{3} = a(1+\sqrt{3})$$

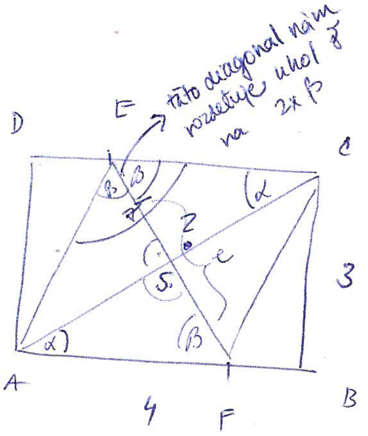
$$\frac{4\sqrt{3}}{1+\sqrt{3}} \cdot \frac{1-\sqrt{3}}{1-\sqrt{3}} = \frac{4\sqrt{3}-12}{-\cancel{4}2} = \frac{\cancel{2}(2\sqrt{3}-6)}{-\cancel{2}} = \underline{\underline{-2\sqrt{3}+6 = a}}$$

$\frac{a}{b} = \frac{2}{1}$

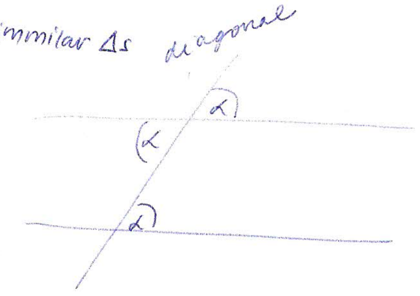
$b = \frac{a}{2}$

$$\underline{b} = \frac{-\cancel{2}\sqrt{3}+\cancel{6}^{3}}{\cancel{2}} = \underline{\underline{-\sqrt{3}+3}}$$

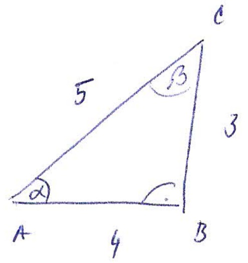
Rectangle, AFCE= Rhombus, calculate EF

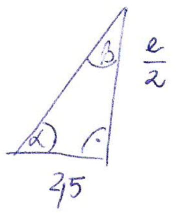


1) simmilar Δs



$\Delta ABC \sim \Delta ASE$





$$\frac{3}{4} = \frac{\frac{e}{2}}{\frac{2,5}{1}}$$

$$\frac{3}{4} = \frac{e}{5}$$

$$15 = 4e$$

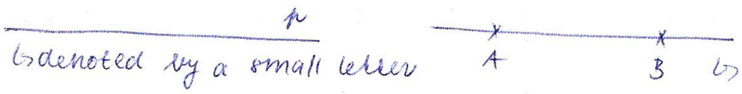
$$e = \frac{15}{4}$$

3

BASIC GEOMETRIC SHAPES

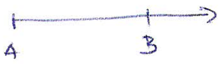
- **POINT** – it indicates the position and has no size or dimension; usually denoted by capital letters ^+A

- **LINE** – a set of infinite number of points
  - shortest distance between two points and it continues forever in both directions

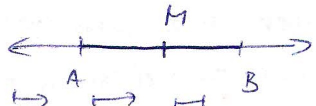


↳ denoted by a small letter ↳ line defined by 2 points

- **HALF-LINE** – is the subset of a line with one end point
  - it continues forever in just one direction
  - notation: $\overrightarrow{AB}$

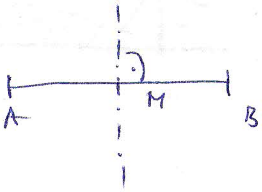


- **LINE SEGMENT** – subset of a line with two end points
  - intersection of two half-lines with opposite direction

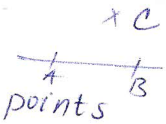


$\overrightarrow{AB} \cap \overrightarrow{BA} = \overline{AB}$

  - Midpoint of the line segment $\overline{AB}$: a point which is equidistant from both end points $M = A \doteq B$
  - Perpendicular bisector (=os úsečky) – is a line, which is perpendicular to the line segment and passes through the midpoint

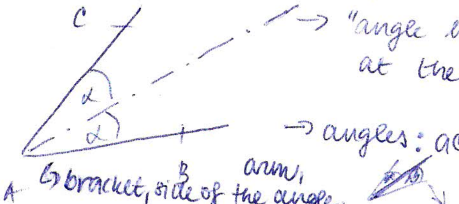


- **PLANE** – a two dimensional surface with infinite width and length
- **HALF-PLANE** – is a subset of the plane which is given by: 1) 3 non-collinear points



  2) a line and a point which doesn't belong to that line
  3) 2 intersecting lines
  4) 2 parallel lines

- **ANGLE** – the intersection of 2 half planes with intersecting boundary lines



→ "angle bisector" = os uhla – a half line with the end point at the vertex → it divides the angle into 2 equal angles

→ angles: acute, right, obtuse, straight, reflex, round (full turn)
$180 < \alpha < 360$ $360°$

↳ bracket, side of the angle, arm,

- adjacent angles – angles with 1 common arm
- vertical angles – 2 intersecting lines – the angles are opposite eachother
- complementary → $\alpha + \beta = 90°$
- suplementary → $\alpha + \beta = 180°$

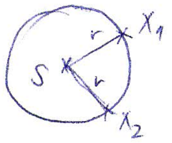
- **ZONE** (=rovinný pás) – the intersection of 2 half planes with parallel boundary lines

- **TRIANGLE** – intersection of three half-planes which are given by three non-collinear points: $\Delta ABC = \overrightarrow{ABC} \cap \overrightarrow{BAC} \cap \overrightarrow{ACB}$
  - ABC – vertices; a, b, c – 3 sides; 3 interior angles

division acc. to the: sides:
↳ equilateral – all 3 sides are equal
↳ isosceles – 2 sides are equal
↳ scalene = všeobecný

angles:
↳ acute
↳ obtuse
↳ right

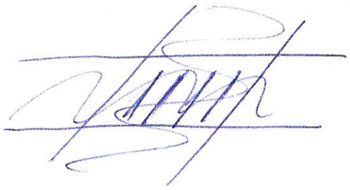
• CIRCLE – the set of points in the plane which are equidistant from a single fixed point called the centre of the circle



$$k = \{ X \in E_2 : |SX| = r \}$$

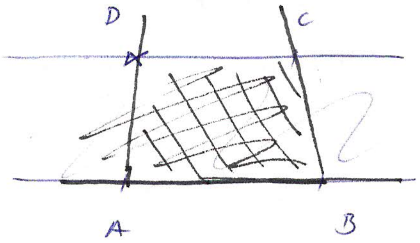
$$(K = \{ X \in E_2 : |SX| \leq r \})$$

• PARALLELOGRAM – the intersection of 2 zones with intersecting boundary lines

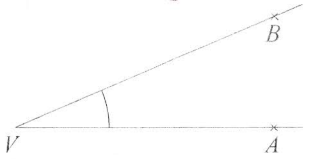


• TRAPEZIUM – given zone with boundary lines AB, CD

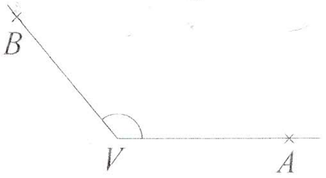
– then trapezium is the intersection of the given zone and $\measuredangle ABC$ and $\measuredangle DAB$



The acute angle is any angle whose measure is greater than $0°$ and less than $90°$.

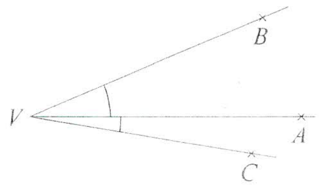


The obtuse angle is any angle whose measure is greater than $90°$ and less than $180°$.



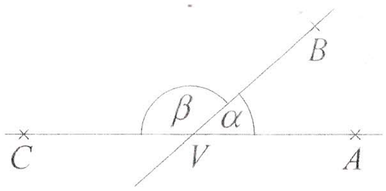
PAIRS OF ANGLES

Adjacent angles are any angles that share a common side and a common vertex.



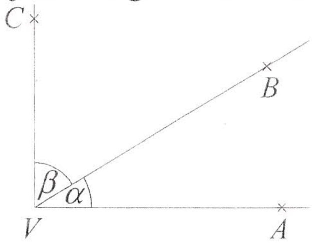
Supplementary angles are two adjacent angles whose sum is $180°$.

$$\alpha + \beta = 180°$$



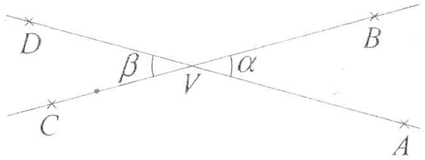
Note: We say that α is a supplement of β or β is a supplement of α or we say that α and β are supplementary.

Complementary angles are two adjacent angles which add up to $90°$.



Note: We say that α is a complement of β or β is a complement of α or we say that α and β are complementary.

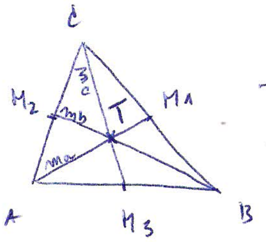
Vertical /opposite angles form a pair of non-adjacent angles formed by two intersecting lines. They are equal in measure. We say that they are congruent.



Note: The symbol for congruence is $\cong$. In the picture above, angles α and β are congruent. In mathematical sentence: $\alpha \cong \beta$.

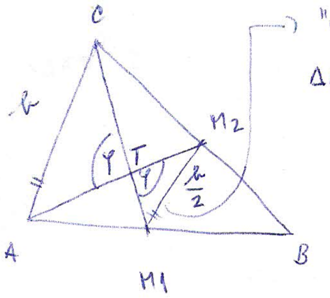
4. Important line segments in a Δ

• **MEDIAN** of the Δ (= ťažnica) – a line segment drawn (zostrojená) from the vertex of the Δ to the midpoint of the opposite side



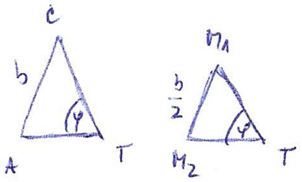
– all 3 medians intersect at one point which is called the **centroid** (ťažisko)

– centroid divides the medians in the ratio 2:1 (prove it :)



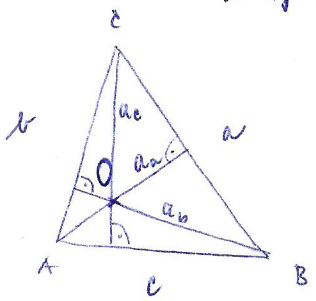
"midline" – vždy je to polovica strany, s kt. je ∥

$\Delta ATC \sim \Delta TM_1M_2$



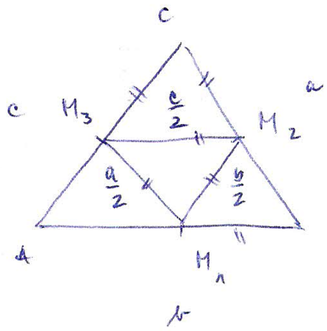
$$\frac{AT}{b} = \frac{M_2T}{\frac{b}{2}} \Rightarrow \frac{AT}{b} = \frac{2M_2T}{b} \Rightarrow |AT| = 2|M_2T|$$

• **ALTITUDE / HEIGHT** of the Δ – a line segment drawn from the vertex of the Δ perpendicular to the opposite side



– all three heights intersect at one point = "**ORTHOCENTRE**"

• **MIDLINE** of the Δ – a line segment joining 2 midpoints of 2 sides of the Δ



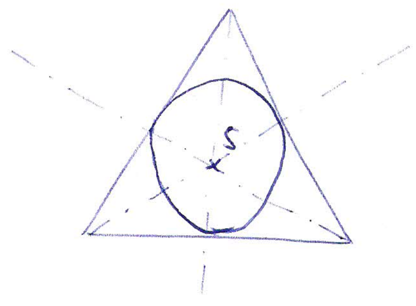
– 3 midlines divide Δ into 4 congruent triangles

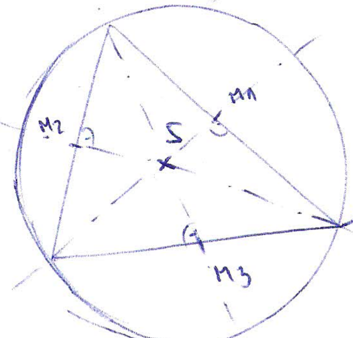
– their length is the half of the side to which they are parallel

Incircle & circumcircle of a Δ

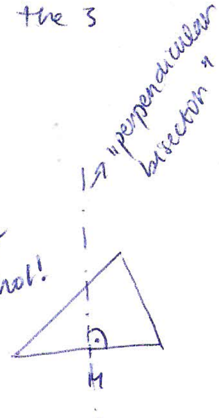
INCIRCLE of the Δ – the circle which touches all 3 sides; the centre of the circle is the point of intersection of the (interior) 3 angle bisectors

CIRCUMCIRCLE of the Δ – the circle which passes through all 3 vertices of the Δ and the centre of the circumcircle is the point of intersection of the 3 perpendicular bisectors of the sides



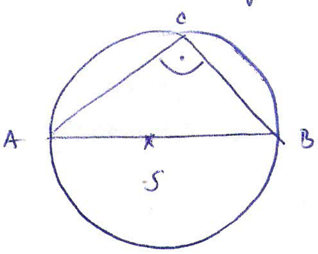


nemusí ísť cez vrchol!



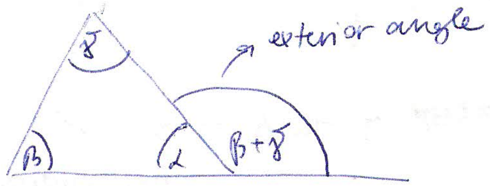
Thales circle

- a special type of circumcircle of the triangle but one side of the △ is the diameter of the ○



4 special theorems which work in any △ :

1) The sum of the lengths of any 2 sides is greater than the length of the 3rd one
2) The sum of interior angles adds up to 180°
3) The side opposite the largest ∢ is the largest one
4) Any exterior angle of the △ is equal to the sum of 2 interior angles which are not adjacent to the exterior ∢



5. Polygons

- polygon is a many-sided planimetric figure / shape

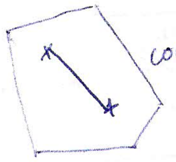
→ acc. to the length of sides and values of interior angles polygons are divided into:

a) **REGULAR** – polygons in which all sides and interior angles are equal
↳ square, pentagon, hexagon ..., equilateral Δ

b) **IRREGULAR** – polygons which are not regular

→ another division:

a) **CONVEX** – polygons in which all interior angles are less than 180°
(angles: acute / obtuse)
– if we choose 2 points from the polygon and we join them → entire line segment lies in the polygon



convex

b) **CONCAVE** – polygons with at least one reflex interior angle ($180° < \varphi < 360°$)

concave

→ part of a line segment which doesn't lie in the polygon

→ division acc. to the number of sides: triangle, quadrilateral, pentagon, hexagon, heptagon, octagon, nonagon, decagon ...

Cyclic and tangent polygons (tetivové a dotyčnicové)

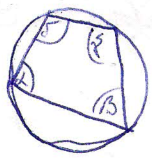
CYCLIC POLYGON – a polygon whose vertices lie on a circle
- it doesn't have to be regular
- all the sides of CP are chords (tetivy) of the circle

CP

→ **cyclic quadrilateral** – a cyclic polygon with exactly 4 sides
- in any cyclic quadrilateral the sum of opposite angles adds up to 180°

$\alpha + \xi = 180°$

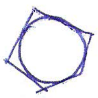
$\beta + \delta = 180°$



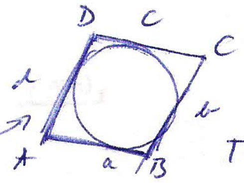
CQ

TANGENT POLYGON – a polygon whose sides are tangent to the circle

TP



→ **tangent quadrilateral** – the sums of opposite sides are equal $a + c = b + d$



TQ

- number of **diagonals** in the convex polygons → we use combinations without repetition

$$C(2, \overset{\text{number of vertices}}{n}) - n = \frac{n!}{(n-2)!\,2!} - n = \frac{n(n-1)\cancel{(n-2)!}}{\cancel{(n-2)!}\,2} - \overset{\text{common denominator}}{n} = \frac{n^2 - n - 2n}{2} = \frac{n^2 - 3n}{2} = \boxed{\frac{n(n-3)}{2}}$$

} FORMULA FOR CALCULATION THE NUMBER OF DIAGONALS

↳ factor

2 poins of n-gon

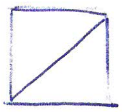
↳ we need to substract number of sides bcs they are not diagonals

- the sum of interior angles in the convex polygons:

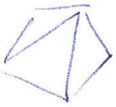
$\boxed{(n-2) \cdot 180°}$

↳ number of sides

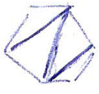
Grafický dôkaz: → vidíme, že nám to dáva aritmetickú postupnosť



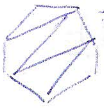
4-uholník
2.180



5-uholník
3.180



6-uholník
4.180



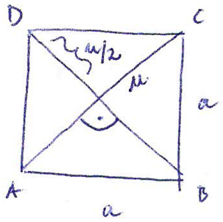
7-uholník
5.180

8-uholník
6.180

Quadrilaterals
- polygons with 4 sides
- the basic quadrilaterals: square, rectangle, parallelogram, rhombus, trapezium, kite (= deltoid) 

- simple (not self-intersecting) 
- complex (self-intersecting, crossed) 

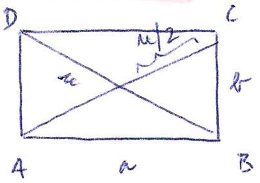
SQUARE



- all sides are equal
- all angles are right (at the vertices)
- u - diagonals; $u = \sqrt{2}a$; they perpendicularly bisect eachother and are of equal length

Perimeter: $\underbrace{O = 4 \cdot a}$ Area: $\underbrace{S = a^2 \;/\; S = \frac{u^2}{2}}$

RECTANGLE

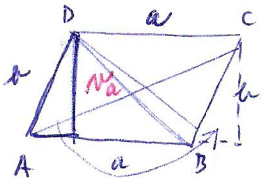


- all angles at the vertices are right
- adjacent sides are unequal in length
- diagonals bisect eachother & are of the same length

$\underbrace{O = 2(a+b)}$

$\underbrace{S = a \cdot b}$

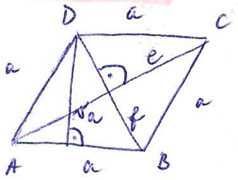
PARALLELOGRAM //RHOMBOID



- a quadrilateral with 2 pairs of parallel sides
- opposite sides are parallel and equal in length
- a parallelogram can be rearranged into a rectangle with the same area

$\underbrace{S = a \cdot v_a}$ $\underbrace{O = 2(a+b)}$

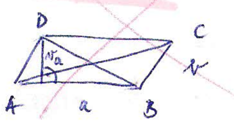
RHOMBUS



- all 4 sides are of the same length
- diagonals e, f bisect eachother & are perpendicular (na polovicu)

$\underbrace{O = 4 \cdot a}$ $\underbrace{S = a \cdot v_a \;/\; S = \frac{e \cdot f}{2}}$

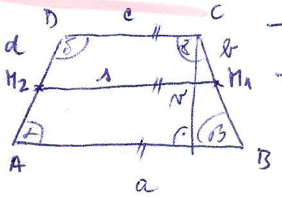
RHOMBOID

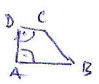


- adjacent sides are of unequal length & angles are <u>not</u> right

$\underbrace{O = 2(a+b)}$ $\underbrace{S = a \cdot v_a}$

TRAPEZIUM

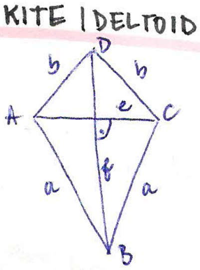


- at least 1 pair of opposite sides is parallel
- a ∥ c → bases of trapezium; b, d → legs of trapezium; v → altitude
- s → midsegment, joins the midpoints of legs s ∥ a or c
- isosceles - legs are of equal length b = d
- right $\alpha = 90°$, therefore $\delta = 90°$ 

$\underbrace{O = a+b+c+d}$ $\underbrace{S = \frac{a+c}{2} \cdot v = s \cdot v}$

↳ midline is the arithmetic mean of the bases

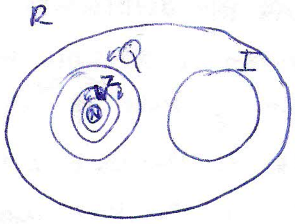
KITE | DELTOID



- 2 pairs of adjacent sides are of equal length
- e, f - diagonals are perpendicular

$\underbrace{O = 2(a+b)}$ $\underbrace{S = \frac{e \cdot f}{2}}$

6. NUMBER THEORY



Number system is made up of several number sets:

- set of natural numbers – denoted by $N = \{1, 2, 3 \dots\}$
- set of whole numbers – $W = \{0, 1, 2, 3, \dots\}$ (set of N + 0 ⇒ N_0 in slovak)
- set of integers – $Z = \{\dots -3, -2, -1, 0, 1, 2, 3, \dots\}$ (W and their opposite values)
- set of rational numbers – $Q = \{ \frac{p}{q} ; p, q \in Z ; q \neq 0 \}$
  ↳ fractional form ↳ q is different from 0
- set of irrational n. – I = numbers which are not rational = numbers with infinite non-reccuring decimal notation e.g. $\sqrt{2}$; π (nekonečný desatinný rozvoj) – nedá sa napísať v tvare zlomku
- set of real numbers – R = union of all number sets

PRIME NUMBER (= prvočíslo) – is a natural number having two different divisors (and they are number 1 and the number itself)

P.N: 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47

COMPOSITE NUMBER (= zložené číslo) – a natural number having <u>at least</u> 3 different divisors

Find all divisors of 12 ⇒ 1, 2, 3, 4, 6, 12 (6 divisors)

9 ⇒ 1, 3, 9 (3 divisors)

→ 1 is not a prime number nor composite number

→ any composite number can be written as a product of primes = PRIME FACTORIZATION

COPRIME NUMBERS = relatively prime numbers (= nesúdeliteľné) – are 2 natural numbers having exactly one common divisor (number 1)

4 and 9; any 2 prime numbers are coprime; 21 and 4; ~~24 and 33~~; 35 and 66; 35 a 44

- **the highest / greatest common divisor**
  - the HCD of 2 or more natural numbers is the greatest divisor common to these numbers.
  - (prime factorization) it is the intersection of prime factorization of these natural numbers
- **the lowest / least common multiple**
  - the LCM of 2 or more natural numbers is the smallest multiple common to these numbers
  - the union of prime factorization of these N numbers.

$$LCM(a, b) = \frac{a \cdot b}{HCD(a, b)}$$

HCD(36, 78) = 2 and also 3 = 6 (tie, kt. sú v oboch)   LCD(36, 78) = 468 $(2^2 \cdot 3^2 \cdot 13)$

P.F.: $36 = 6 \cdot 6 = 2^2 \cdot 3^2$

$78 = 2 \cdot 39 = 2 \cdot 3 \cdot 13$

DIVISIBILITY

Def.: Given $a, b \in N$. Number a is called the divisor of the number b if there is a natural number k such that

$$b = k \cdot a$$

$12 = \overset{k}{4} \cdot 3 \rightarrow 3 | 12$ (3 is the divisor of 12 because there exists a number 4 such that $3 \cdot 4 = 12$)

Criteria for divisibility:

- number is divisible by 2 if last digit ~~ends~~ with 2, 4, 6, 8, 0
- $3/n$ if sum ~~of~~ digits is divisible by 3
- $4/n$ if the last 2-digit number (=dvojčíslie) is divisible by 4
- $5/n$ if the last digit is either 0 or 5
- $6/n$ if it is div. by 2 & 3
- $8/n$ if the last 3-digit number is divisible by 8
- $9/n$ if digit sum is divisible by 9
- $10/n$ last digit is 0
- $12/n$ by 3 & 4

1) Prove: $\forall n \in N: 3/(n-2) \Rightarrow 6/(n^2-5n+6)$ we will use the direct proof and we suppose that $3/(n-2)$

such that

$3/(n-2)\ \exists k \in N; n-2 = 3k$

$n = 3k+2$ substitute for n

these are 2 subsequent numbers ⇒ so $3k-1$ is eg divisible by 2

$n^2-5n+6 = (3k+2)^2 - 5(3k+2)+6 = 9k^2+12k+4-15k-10+6 = 9k^2-3k = 3k(3k-1) =$ definitely (one is always even)

↳ any number multiplied by 3 is divisible by 3 (we see that it's div. by 3)

so we see that it's also divisible by 6 ⇐

$= 6/(n^2-5n+6)$

2) $\forall a,b,c \in N: a/b \wedge a/c \Rightarrow a/(bx+yc)$; $x,y \in Z$ → we suppose that it is true (direct proof again)

$a/b\ \exists k \in N: b = ak$

$a/c\ \exists l \in N: c = l.a$

factor

$bx+yc = ka.x + la.y = a(kx+ly) = a/bx+yc$

3) $\forall x,y \in R^+: (x+y)\left(\frac{1}{x}+\frac{1}{y}\right) \geq 4$ Prove this inequality → using valid axioms

— we expand the brackets

$1+\frac{y}{x}+1+\frac{x}{y} \geq 4$

$\frac{y}{x}+\frac{x}{y} \geq 2$ ← we can multiply inequality eg a variable bcs it's positive (we know that R^+)

$y^2+x^2 \geq 2xy$

$x^2-2xy+y^2 \geq 0$ → formula of the perfect square

$(x-y)^2 \geq 0$

4) $\forall a,b \in R: a^2b^2+1 \geq ab+a+b \quad /\cdot 2$

$2a^2+2b^2+2 \geq 2ab+2a+2b$

$2a^2+2b^2+2-2ab-2a-2b \geq 0$

$a^2+a^2+b^2+b^2+\cancel{1+1}-2ab-2a-2b \geq 0$

$a^2-2ab+b^2+a^2-2a+1+b^2-2b+1 \geq 0$

$(a-b)^2+(a-1)^2+(b-1)^2 \geq 0$

5) $\forall a \in R^+: a^3+1 \geq a+a^2$

$a^3-a^2-a+1 \geq 0$

$a^2(a-1)-(a-1) \geq 0$

$(a-1)(a^2-1) \geq 0$

$(a-1)(a-1)(a+1) \geq 0$

$(a-1)^2(a+1) \geq 0$

$\forall x, y, z \in \mathbb{R}^+: \frac{1}{x} + \frac{1}{y} + \frac{1}{z} \geq \frac{9}{x+y+z}$ $/ \cdot x, y, z$

$yz + xz + xy \geq \frac{9 \cdot x \cdot y \cdot z}{x+y+z}$ $/ \cdot x+y+z$

$yz(x+y+z) + xz(x+y+z) + xy(x+y+z) \geq 9xyz$ $\quad 2xy + 2xy + 2xy$

$xyz + y^2z + yz^2 + x^2z + xyz + xz^2 + x^2y + xy^2 + xyz \geq 9xyz$

$z(y^2 - x^2 - 2xy) + x(y^2 + z^2 - 2yz) + y(x^2 + z^2 - 2xz) \geq \cancel{6xyz}\ 0$

$\cancel{-2xyz - 2xyz - 2xyz \geq 0}$

$z(x-y)^2 + x(y-z)^2 + y(x-z)^2 \geq 0$

HW

1. $\forall n \in \mathbb{N}$: If the number n gives the remainder of 7 when divided by 15, then the number $8n$ gives the remainder of 1 when divided by 5. Prove the implication.

2. For which digit y is the number 68036540 547 80y 750 divisible by 450
   ↳ by 10, 5, 9

3. Cross out 4 digits from the number 7~~9~~0 4~~6~~1 308 754 021 980 65③ ~~0~~2~~1~~ so that the new number is divisible by 12 and it's the smallest number possible.

1) $\forall n \in \mathbb{N}$: $n = 15k + 7 \Rightarrow 8n = 5l + 1$

$n = 15k + 7 \Rightarrow 8n = 120k + 56 \Rightarrow 8n = 120k + 55 + 1 \Rightarrow 8n = 5(24k + 11) + 1 \Rightarrow 8n = 5l + 1$

2) digit sum: 68 $\quad y = 4 \quad 72 : 9 = 8$ ✓

3) 704 613 087 540 219 806 52

7. Quadratic equation

- it's a polynomial equation of the 2^{nd} degree
- it is the equation of the form $ax^2+bx+c=0$; where $a,b,c \in R$; $a \neq 0$ (otherwise → linear)

a,b,c ... coefficients

ax^2 ... quadratic element bx ... linear element c ... absolute term

- acc. to the presence of elements
  - **COMPLETE q.e.** - all 3 elements are present
    $2x^2-x+7=0$ – is complete
  - **INCOMPLETE q.e.** - the one, where either linear element or absolute term or both are missing
    $2x^2+5=0$; $2x^2=-5$; $S=\emptyset$
    $4x^2-3x=0$; $x(4x-3)=0$
    $5x^2=0$; $S=\{0\}$
- **nullified q.e.** - an equation having 0 on one side; any q.e. can be nullified
  $5x^2=2x+3 \Rightarrow 5x^2-2x-3=0$
- **standard q.e.** - an equation whose quadratic coefficient is 1

$$ax^2+bx+c=0 \quad /:a$$
$$x^2+\frac{b}{a}x+\frac{c}{a}=0$$

$$5x^2-2x-3=0 \quad /:5$$
$$x^2-\frac{2}{5}x-\frac{3}{5}=0$$

- an equation is in the standard form if it is nullified and quadratic coefficient is 1
- graph: parabola

Solving methods:

1) Factorization
2) Quadratic formula
3) Method of completing the square
4) Graphical solution - points of intersection of the parabola and the x-axis are the roots of the quadratic equation

Derive the quadratic formula:

$ax^2+bx+c=0$ conditions: $a,b,c \in R$; $a \neq 0$

$$a\left(x^2+\frac{b}{a}x+\frac{c}{a}x\right)=0$$

roznásobíme

$$a\left(x^2+\frac{b}{a}x+\frac{b^2}{4a^2}-\frac{b^2}{4a^2}+\frac{c}{a}x\right)=0$$

$$a\left(x+\frac{b}{2a}\right)^2-\frac{b^2\cancel{a}}{4a^{\cancel{2}}}+c=0$$

$$a\left(x+\frac{b}{2a}\right)^2=\frac{b^2}{4a}-c$$

$$\left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2} \quad /\sqrt{\ }$$

$$x+\frac{b}{2a}=\frac{\pm\sqrt{b^2-4ac}}{2a} \quad /-\frac{b}{2a} \text{ (common denominator)}$$

$$x_{1,2}=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

- The expression under the square root is denoted by capital D. Acc. to the value of the D we can decide on the number of roots:

$D=b^2-4ac$

$D>0$ ... 2 real solutions
$D=0$ ... 1 real solution = double root solution
$D<0$ ... no real solution

$x^2-10x+25=0$

$D=100-4\cdot 25=0$

$\hookrightarrow x=\frac{10}{2}=\underline{\underline{5}}$

→ **Viete's formulae** – give the simple relation between the roots of the equation and the coefficients a, b, c.

$ax^2+bx+c=0 \qquad a,b,c \in \mathbb{R}\ ;\ a \neq 0$

↳ we write the standard form by dividing equation by quadratic coefficient

$x^2+\frac{b}{a}x+\frac{c}{a}=0$

we compare them

if x_1, x_2 are the two roots of the equation, then using prime factorisation it can be written in the form $(x-x_1)(x-x_2)=0 \qquad [\,(x-2)(x+5)=0 \quad x_1=2 \quad x_2=-5\,]$

expand ( ) : $x^2-xx_1-xx_2+x_1x_2=0$

factor x from linear elem. : $x^2-x(x_1+x_2)+x_1 \cdot x_2=0$

$x^2+\frac{b}{a}x+\frac{c}{a}=x^2-x(x_1+x_2)+x_1x_2$

$\frac{b}{a}=-(x_1+x_2)\ /\cdot(-1)$ → we are comparing coefficients of linear elements | → we are comparing absolute terms

$\boxed{-\frac{b}{a}=x_1+x_2}$ $\qquad$ $\boxed{\frac{c}{a}=x_1 \cdot x_2}$

Napr. $x^2-3x-10=0$

$x_1+x_2=3 \qquad x_1 \cdot x_2=-10$

Find the parameter $s \in \mathbb{R}$, so that the equation $sx^2+s+x-2x^2-2sx=0$ has at least 1 solution.

$sx^2-2x^2+x-2sx+s=0$

$x^2(s-2)+x(1-2s)+s=0$

at least 1 = 1 or 2

$D=(1-2s)^2-4 \cdot s(s-2) \geq 0$

$1-4s+4s^2-4s^2+8s \geq 0$

$4s \geq -1$

$s \geq -\frac{1}{4}$

Find the parameter $m \in \mathbb{R}$ so that one root of the q. e. $x^2+mx+5=0$ is greater than the 2nd root by 4. → Viete's formulae

$-\frac{b}{a}=x_1+x_2 \qquad \frac{c}{a}=x_1 \cdot x_2 \qquad x_1=x_2+4$

$-m=x_2+4+x_2$

$5=(x_2+4) \cdot x_2$

$0=x_2^2+4x_2-5$

$0=(x_2-1)(x_2+5)$ → $x_2=1 \quad x_2'=-5$

I $-m=1+4+1$, $m=-6$ $\qquad$ II $-m=-5+4-5$, $m=6$

Find parameter q, so that one root of the eq. $4x^2-15x+q=0$ is the square of the 2nd root

$x_1=x_2^2 \qquad \frac{15}{4}=x_2^2+x_2 \ \&\ \frac{q}{4}=x_2^2 \cdot x_2$

$0=4x_2^2+4x_2-15$

$0=(2x_2+5)(2x_2-3)$ → $-\frac{5}{2}$ & $\frac{3}{2}$

I. $q=4 \cdot \left(-\frac{5}{2}\right)^3$, $q=-\frac{125}{2}$ $\qquad$ II. $q=4 \cdot \left(\frac{3}{2}\right)^3$, $q=\frac{27}{2}$

Find parameter $a \in \mathbb{R}$, so that the difference of the roots of the eq. $(a-1)x^2+(1-5a)x+20=0$ is equal to 3.

$x_1 - x_2 = 3$

$x_1 = 3 + x_2$

$-\frac{1-5a}{a-1} = 3+2x_2$ | $\frac{20}{a-1} = (3+x_2)\cdot x_2 \to \frac{20}{a-1} = 3x_2 + x_2^2$

substitution

$$\frac{-\frac{1-5a}{a-1} - 3}{\frac{2}{1}} = x_2$$

$$-\frac{1-5a-3a+3}{2a-2} = x_2$$

$$-\frac{4-8a}{2a-2} = x_2$$

$$-\frac{\cancel{2}(2-4a)}{\cancel{2}(a-1)} = x_2$$

$$\frac{-2+4a}{a-1} = \frac{4a-2}{a-1} = x_2$$

$$\cancel{\frac{20}{a-1} = 3\cdot\left(-\frac{(2-4a)}{(a-1)}\right) + \left(-\frac{(2-4a)}{(a-1)}\right)}$$

$$\frac{20}{a-1} = 3\cdot\frac{4a-2}{a-1} + \left(\frac{4a-2}{a-1}\right)^2 \quad /\cdot(a-1)^2$$

$$20 = 12a-6 + \frac{16a^2-16a+4}{a-1}$$

~~$0 = 16a^2 - 4a$~~

$$\cancel{20a-20 = 12a^2-12-6a+6+16a^2-16a+4}$$

$$\cancel{0 = 28a^2-42a+18}$$

$$\cancel{0 = 14a^2-21a+9}$$

$$\cancel{D = 441-504}$$

$$20(a-1) = 3\cdot(4a-2)(a-1) + (4a-2)^2$$

$$20a-20 = (12a-6)(a-1) + 16a^2-16a+4$$

$$20a-20 = 12a^2-12-6a+6+16a^2-16a+4$$

$$0 = 28a^2-42a+18$$

$a_1 = 3 \quad a_2 = \frac{3}{2}$

– funkcie – kópia

8. Functions

Function = a correspondence between 2 sets
= a function f is the set of all ordered pairs $[x, y]$ from the cartesian product of real numbers, such that for each real number x there exists <u>at most one</u> real number y such that $y = f(x)$ $\quad ([x, y] \in f)$

$$f = \{[x, y] \in R \times R;\ \forall x \in R;\ \exists \text{ at most one } y \in R;\ y = f(x)\}$$

Domain (= definičný obor) – a set of all real numbers x such that for each x there exists <u>exactly one</u> real number y such that $y = f(x)$
↳ x-os

$$D(f) = \{x \in R;\ \forall x \in R;\ \exists!\ y \in R;\ y = f(x)\}$$

Range (= obor hodnôt) – a set of all real numbers y where for each real number y exist <u>at least one</u> real number x

$$R(f) = \{y \in R;\ \forall y \in R,\ \exists \text{ at least one } x \in R;\ y = f(x)\}$$

The ways of expressing a function

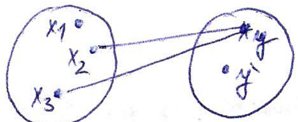
1) **by a graph** – a graphical interpretation of the function
- set of pairs $[x, y]$ which is given by the rule $f(x) = y$

2) **by listing its elements**

3) **by an equation / formula**

4) **by words** – describing the behaviour of the function

- **one-to-one function** (= prostá) = a function f is a one-to-one function if
$\forall x_{1,2} \in D(f);\ x_1 \neq x_2$ also $f(x_1) \neq f(x_2)$ = "for 2 different values from D there are 2 different functions"
  - each x is assigned to <u>different</u> y $\quad (y = x + 1)$
  - if a function is ↗ or ↘ then it is one-to-one
- **many-to-one** = a function f is many-to-one function if to every y are assigned 2 or more x
  - 2 or more x-values are mapped onto y

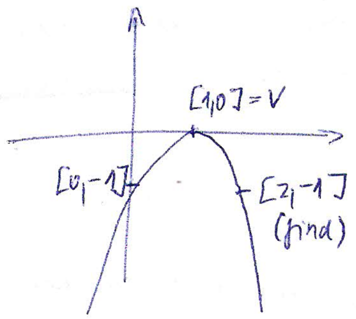


- **inverse function** = let f be a one-to-one function. The inverse function to the function f; denoted f^{-1}; is a set of all ordered pairs $[y, x] \in R \times R$, such that $[x, y] \in f$.
  - in f^{-1} we swap x & y coordinates and also: $D(f) = R(f^{-1})$, $D(f^{-1}) = R(f)$

Notes: – if f is increasing, then f^{-1} is increasing
– if f is decreasing, then f^{-1} is also decreasing

The graphs of f and f^{-1} are symmetric about the line $y = x$

1) The graph of the quadratic function cuts accross the y-axis at the point $[0,-1]$. The maximum of the function is at the point $x=1$ and $R(f)=(-\infty,0\rangle$. Write the eq. of function



$f(x)=y=ax^2+bx+c \rightarrow$ 3 variables $\rightarrow$ we need 3 points to form system

$0=a+b+c \Rightarrow 1=a+b$
$-1=c \qquad\qquad 1-b=a$
$-1=4a+2b+c$

$-1=4(1-b)+2b-1$
$\cancel{-1}=4-4b+2b\cancel{-1}$
$2b=4$
$b=2 \qquad a=-1 \qquad f(x)=-x^2+2x-1$

2) a) Find $a,b\in R$ so that the graph of the function $f: y=a\cdot\sin\left(\frac{x}{2}-\frac{\pi}{2}\right)+b$ passes through: $A[2\pi,8]$; $B[0,2]$. b) $\rightarrow$ how to change the value of b so that the graph of the function passes through the origin?

a) substitute the coordinates for x & y

$A\in f$: $8=a\cdot\sin\left(\frac{2\pi}{2}-\frac{\pi}{2}\right)+b$
$B\in f$: $2=a\cdot\sin\left(\frac{0}{2}-\frac{\pi}{2}\right)+b$

$8=a+b$
$2=-a+b$

$10=2b$
$b=5 \qquad a=3 \qquad f: y=3\sin\left(\frac{x}{2}-\frac{\pi}{2}\right)+5$

b) $f: y=3\sin\left(\frac{x}{2}-\frac{\pi}{2}\right)+b \leftarrow [0,0]$
$0=3\sin\left(-\frac{\pi}{2}\right)+b$
$0=-3+b$
$\underline{b=3}$

$\Rightarrow$ linear function, quadratic function, indirect proportion + fr. function of the 1^{st} degree, exponential + logarithmic function, goniometric f.

3) $f: y=\log_2(x-3)+5 \qquad$ find D, f^{-1}

we state the condition for argument

$x-3>0$
$x>3 \qquad D(f)=(3,\infty)$

$\rightarrow$ the function is one-to-one so we can find its inverse function
$\rightarrow$ we swap y & x

$x=\log_2(y-3)+5$
$x-5=\log_2(y-3)$
$y-3=2^{(x-5)}$
$y=2^{(x-5)}+3 \rightarrow$ the inverse of log function is exponential function (and vice versa)

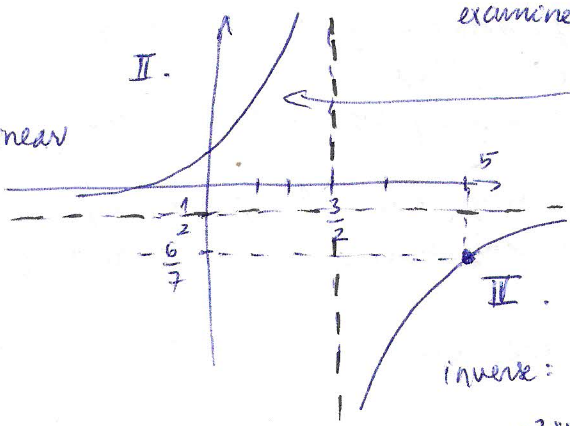
4) $y=\frac{x+1}{-2x+3} \rightarrow$ fr. function of the 1^{st} degree. find asymptotes

vertical: $-2x+3=0$
$x=\frac{3}{2} \parallel y$

horizontal: divide coefficients of linear elements

$y=-\frac{1}{2} \parallel x$

$y=\frac{6}{-7} \quad x=5$



examine properties: $D=R-\{\frac{3}{2}\}$
$R=R-\{-\frac{1}{2}\}$

- hyperbola
- NENO
- increasing over $D(f)$
- not periodic
- no max nor min
- not bounded
- one-to-one

inverse: $x=\frac{y+1}{-2y+3}$
$-2xy-y=-3x+1$
$\cancel{-}y(2x+1)=\cancel{-}(3x\cancel{-}1)$

$y=\frac{3x-1}{2x+1}$

5) Find $a, b \in R$ so that the graph of the function $y = a \cdot \log_2(x+1) + b$ passes through points $A[3,1]$, $B[-\frac{7}{8}, -9]$. Then decide whether $C[-3,-5] \in f$

$A \in f$: $1 = a \cdot \log_2 4 + b$ ← → points are from the logarithmic curve so we can substitute them to the function

$-9 = a \cdot \log_2 \frac{1}{8} + b$

$1 = 2a + b \rightarrow 1 - 2a = b$ $\qquad \log_2 4 = \log_2 2^2 = 2$

$-9 = -3a + b$

$-9 = -3a + 1 - 2a$

$-10 = -5a$

$a = \frac{10}{5} = 2 \qquad b = -3$

$y = 2 \cdot \log_2(x+1) - 3 \qquad C \notin f?$

$-5 = 2 \cdot \log_2(-2) - 3$ → argument must be (+)

6) $V = \log 0{,}01 + 2 \log 10 \cdot \sqrt[4]{10} + \log_5 \frac{1}{125} - \ln e^3$ → 3 ln e

$10^{\frac{1}{4}}$

$U = \log_6 \sqrt{216} + \log_7 \frac{1}{49} - \log_{0{,}5} 8 - \log \frac{1}{0{,}001}$

$\ln x = \log_e x$

$\ln e = \log_e e$

$\frac{1}{\frac{1}{1000}} = 1000 = 10^3$

$10 \cdot 10^{\frac{1}{4}} = 10^{\frac{5}{4}}$

Prove that $U + V = -6$.

1) we evaluate expression V

~~$V = \log \ldots$~~

$V = \log_{10} \frac{1}{100} + 2 \cdot \log_{10} 10^{\frac{5}{4}} + \log_5 5^{-3} - 3 \cdot 1 = -2 + \frac{5}{2} - 3 - 3 = -8 + \frac{5}{2} = \frac{-16+5}{2} = -\frac{11}{2}$

$U = \log_6 6^{\frac{3}{2}} + \log_7 7^{-2} - \log_{\frac{1}{2}} \left(\frac{1}{2}\right)^{-3} - \log_{10} 10^{+3} = \frac{3}{2} - 2 + 3 + 3 = -2 + \frac{3}{2} = \frac{-4+3}{2} = \frac{-1}{2}$

$U + V = -\frac{1}{2} - \frac{11}{2} = -\frac{12}{2} = -6$

7) f: $y = \sqrt{x^2 - 5x + 6} - \frac{\sqrt{x}}{\sqrt{3 + 2x - x^2}}$

$x^2 - 5x + 6 \geq 0 \wedge x \geq 0 \wedge -x^2 + 2x + 3 > 0$

$(x-2)(x-3) \geq 0$

$x^2 - 2x - 3 < 0$

$(x+1)(x-3) < 0$

+ $\quad$ − $\quad$ + (2, 3)

+ $\quad$ − $\quad$ + (−1, 3)

$(-\infty, 2\rangle \cup \langle 3, \infty)$ $\qquad (-1, 3)$ $\qquad D(f) = \langle 0, 2\rangle$

8) $f: y = \frac{x^3 - x^2 + x - 1}{x^2 + 1}$ $\qquad g: x - 1$ $\qquad f \stackrel{?}{=} g$ $\quad$ Decide whether the f, g. are equal

$\frac{x^3 - x^2 + x - 1}{x^2 + 1} = x - 1 \quad / \cdot (x^2 + 1)$

$x^3 - x^2 + x - 1 = x^3 + x - x^2 - 1$

$f(x) = g(x)$

9) Even/odd?

a) $y = \frac{x^2}{x^4-1}$ $D = R - \{\pm 1\}$ ~~MEAN~~ → we can examine $f(x) = f(-x)$

1st condition is fullfilled/satisfied

$f(-x)$ $y = \frac{x^2}{x^4-1}$ ✓ EVEN function

b) $y = x^3 + 2x$ $D = R$ ✓ $f(x) = f(-x)$

$f(-x)$: $y = -x^3 - 2x$ ✗ $f(-x) = -f(x)$

$-f(x) := -(x^3+2x) = -x^3 - 2x$ ✓ ODD

c) $y = \sqrt{x-1}$

$\hookrightarrow x \geq 1$ $D = \langle 1, \infty)$ ✗ NENO

10) $f(x) = \dfrac{tg\,\frac{1}{x}}{\log(x+5) + \log(5-x) + \log(25-x^2)}$ Prove that the function is odd

$\frac{1}{x} \longrightarrow x \neq 0$

1) $x > -5 \wedge x < 5 \wedge (5-x)(5+x) > 0$

$-$ $+$ $-$

-5 5

interval between

$D(f) = (-5, 5) \setminus \{0\}$

$\cos \Rightarrow$ EVEN

$\sin, tg, cotg \Rightarrow$ ODD

2) $f(-x) = -f(x)$

$f(-x) = \dfrac{tg(-\frac{1}{x})}{\log(5-x) + \log(5+x) + \log(25-x^2)}$

odd function

$tg(x) = -tg(x)$

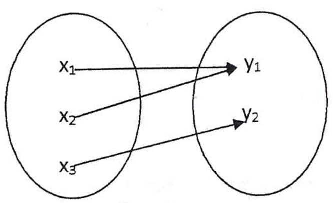
$tg(-\alpha) = -tg\,\alpha$

$-f(x) = \dfrac{-tg(\frac{1}{x})}{\log(x+5) + \log(5-x) + \log(25-x^2)}$ ODD ✓

FUNCTION and FUNCTION PROPERTIES

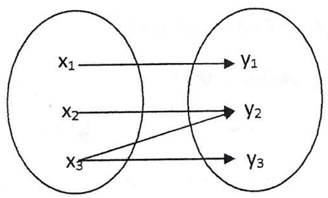
A **function** is a correspondence between two sets which assigns to each element in one set (domain of the function) exactly one value from another set (range of the function).

DOMAIN RANGE



function

DOMAIN RANGE



not function

A **function** is the set of ordered pairs $[x, y] \in$ RxR such that to each real number x there is assigned at most one real number y.

$$f = \{[x, y] \in R \times R;\ \forall x \in R\ \exists at\ most\ one\ y \in R;\ y = f(x)\}$$

R×R ... cartesian product

The **cartesian product** of two two sets A and B is written A×B and is the set of all possible ordered pairs [x; y], where $x \in$ A and $y \in$ B.

e.g. A = {1, 2, 3}

B = {4, 5}

Then: A×B = {[1, 4], [1, 5], [2, 4], [2, 5], [3, 4], [3, 5]}

B×A = {[4, 1], [4, 2], [4, 3], [5, 1], [5, 2], [5, 3]}

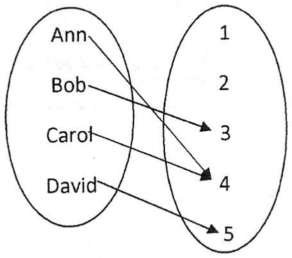
B×B = {[4, 4], [4, 5], [5, 5], [5, 4]}

Domain of a function (D) – the set of permitted x values. It is the set of elements to which the function assigns values

$$D(f) = \{x \in R;\ \forall x \in R\ \exists! y \in R;\ [x, y] \in f\}$$

Range of a function (R) – the set of assigned values of the function

$$H(f) = \{y \in R;\ \forall y \in R\ \exists x \in R;\ [x, y] \in f\}$$

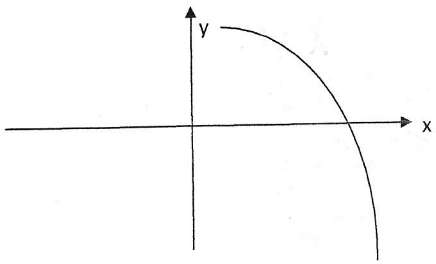


Notice that a function can map more than one element of the domain onto the same element of the range, e.g. Ann→4 and Carol→4. Such Such functions are said to be **many-to-one**.

Functions for which each element of the domain is mapped onto a different element of the range are said to be **one-to-one**: $\forall x_1, x_2 \in D(f);\ x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$

NOTE

A function is one-to-one if and only if each horizontal line intersects the graph of the function just at one point.



WAYS OF EXPRESSING A FUNCTION:

1. naming its elements

$$f = \{[1, 3], [4, 10], [-1, -7]\}$$

2. graph

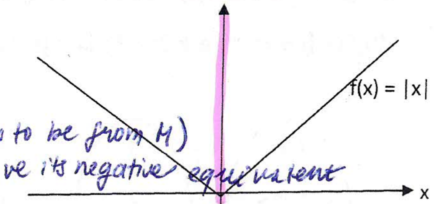
3. y = f(x)

X-axis is the only function which is even & odd
↳ many-to-one function

FUNCTION PROPERTIES **over M ⊂ D(f)**

I. Even and Odd Functions

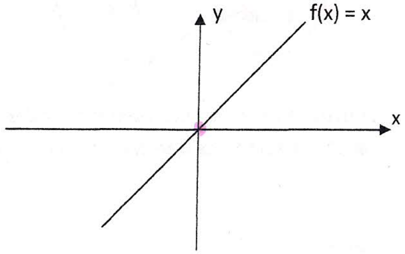
→ **even function** (axial symmetry)



a) $\forall x \in M: (-x) \in M$ (x and −x has to be from M)
↳ has to have it's negative equivalent

b) $f(-x) = f(x)$
↳ graph of an even function is symmetric with respect to the y-axis

→ **odd function** (central symmetry)



a) $\forall x \in M: (-x) \in M$

b) $f(-x) = -f(x)$
↳ symmetric with respect to the ORIGIN

II. Increasing and Decreasing Functions

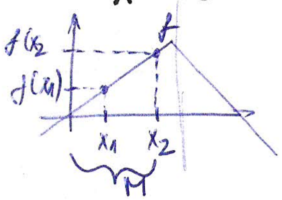
→ **increasing function** over the set M → ("function value at x_1")

$\forall x_1, x_2 \in M: x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$

→ **decreasing function** over the set M

X-OS

$\forall x_1, x_2 \in M: x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$



→ **constant function**

$f(x_1) = f(x_2)$

NOTE

A function is one-to-one if it is increasing or decreasing and not even.

III. Bounded Functions

→ **bounded below function** → "lower boundary" Y-OS

$\exists b \in R; \forall x \in M: f(x) \geq b$

→ **bounded above function** → "upper boundary"

$\exists a \in R; \forall x \in M: f(x) \leq a$
↳ "if any x from the M satisfies the inequality that $f(x) \leq a$"

→ function is **bounded** if it is bounded below and bounded above as well
$\forall x \in M: b \leq f(x) \leq a$

IV. Minimum and Maximum of Functions

X-OS

→ **minimum of the function**: $\exists a \in M, \forall x \in M; x \neq a: f(x) \geq f(a)$

→ **maximum of the function**: $\exists b \in M, \forall x \in M; x \neq b: f(x) \leq f(b)$

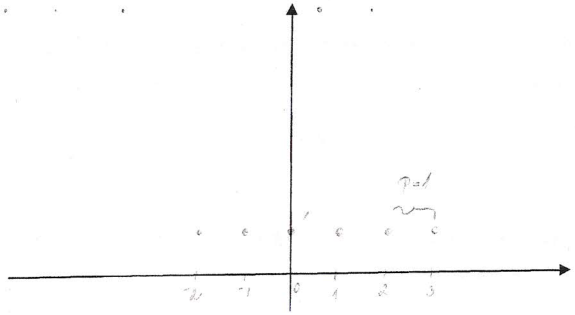
V. Periodic Functions

→ **periodic function** ↙ period of the function

a) $\forall x \in D(f) \Rightarrow (x + kp) \in D(f)$

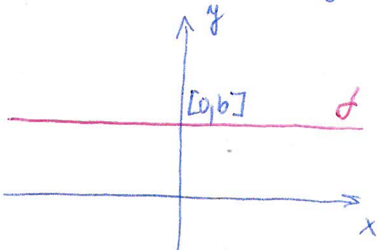
b) $\forall x \in D(f): f(x + kp) = f(x), \; k \in Z$, p ...period of the function p > 0

e.g. f(x) = 1^x, x∈Z ; all goniometric functions → $2\pi = p$ for sin & cos, $\pi = p$ for tan & cotg



Linear function

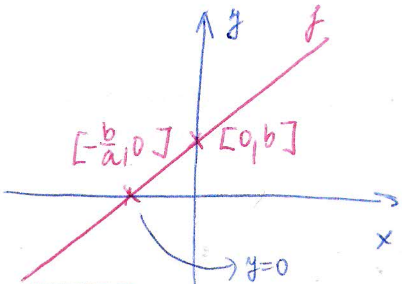
- LF is a function whose equation is in the form $y = ax + b$, where $a, b \in R$.
- graph is a straight line



$\boxed{a=0}$ $y = b$ → constant f.

→ straight line which is parallel to the axis x and it intersects ~~to~~ the y-axis at point [0,b]

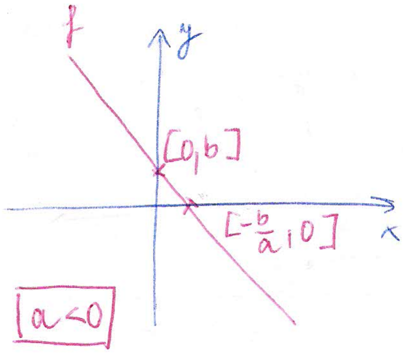
- $D(f) = R$ $R(f) = \{b\}$
- neither I nor D (= not one-to-one)
- bounded
- extrema: max & min at every point of D
- not periodic
- even



$\boxed{a>0}$

→ $y = 0$
$0 = ax + b$
$x = -\frac{b}{a}$

- $D(f) = R$ $R(f) = R$
- I over D(f) (= one-to-one)
- NENO
- not bounded
- no extrema
- not periodic



$\boxed{a<0}$

- $D(f) = R$ $R(f) = R$
- D over D(f) (= one-to-one)
- NENO
- not bounded
- no extrema
- not periodic

NOTE| if $b = 0$

(priama úmernosť)

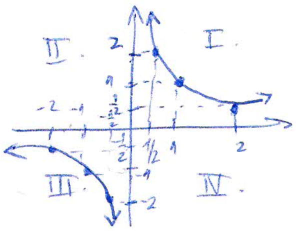
$y = ax$ ⇒ Direct proportion

↳ the graph passes through the origin

Indirect proportion

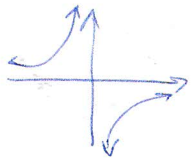
→ A function of the form $y = \frac{k}{x}$, where $k \in R$ and $x \neq 0$ (denominator is not 0) is called an inverse/indirect proportion. The graph of the function is a rectangular hyperbola ⇒ ind. p. always has 2 asymptotes and they are coordinate axes x & y.

$f: y = \frac{1}{x}$ $(k > 0)$



$D(f) = R - \{0\}$
$R(f) = R - \{0\}$
one-to-one
D over D(f)
no boundaries
no extrema

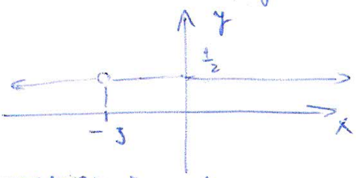
if $k > 0$ → I. & III. q → decreasing
$k < 0$ → II. & IV. q → increasing



Fractional function of the 1st degree

→ A function of the form $y = \frac{ax+b}{cx+d}$, where $a, b, c, d \in R$ and $c \neq 0$; the product $ad \neq bc$.

→ what do we need the conditions for? if $c = 0$ → we get a linear function $y = \frac{2x+3}{5}$



if $ad = bc$ → $y = \frac{x+3}{2x+6} = \frac{x+3}{2(x+3)} = \frac{1}{2}$ → we get a constant function which is not defined at point −3

„discontinuous constant function" (= nespojitá)

→ the graph of a function is a hyperbola → asymptotes won't be coordinate axes x, y but they will be shifted

$y = \frac{x+1}{-2x+3}$ vertical asymptote: $-2x + 3 = 0$ $x = \frac{3}{2}$

horizontal (divide coefficients of the linear element): $y = -\frac{1}{2}$

$y = \dfrac{-5x+7}{-3x+6}$

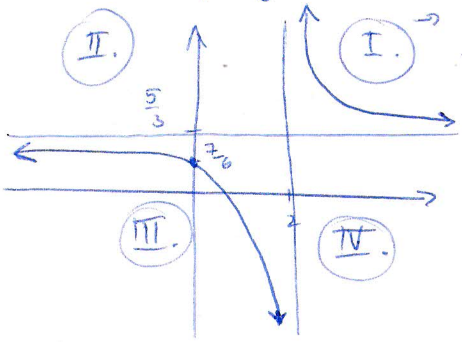
sketch the graph, find asymptotes & examine properties

↳ the f. is the fr. f. of the 1st degree, graph of the f. is a hyper~~para~~bola

↳ at first we need to apply conditions ($ad \neq bc$ ...) → all the conditions are satisfied

↳ asymptotes are ⊥ to eachother

$x = 2$ & $y = \frac{5}{3}$



→ we need to find at least 1 point from the parabola so that we know in which quadrant the graph is

$x = 0 \quad y = \frac{7}{6}$

$D(f) = R - \{2\}$ $\quad R(f) = R - \{\frac{5}{3}\}$

NENO

D over D(f)

no extrema

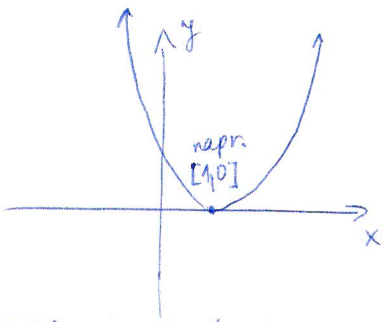
no boundaries

Quadratic function

- QF is a function whose equation is in the form $y = ax^2 + bx + c$, where $a, b, c \in R \wedge a \neq 0$
- graph is a parabola - it is symmetric with respect to the line called the axis of symmetry. Parabola intersects its axis of symm. at a point called the vertex of parabola. We can find out the coordinates of the vertex by method of compeeting the square.

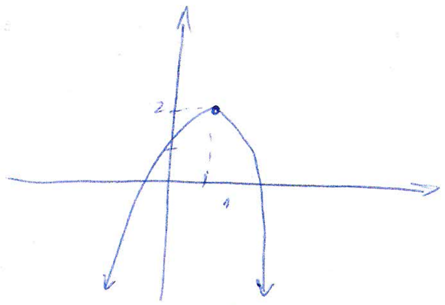
$f(x) = x^2 - 6x + 7 \rightarrow (x^2 - 6x + 9 - 9) + 7 \rightarrow (x-3)^2 - 2 \quad V[3, -2]$

$a > 0 \rightarrow$ CONVEX



- $D(f) = R \quad R(f) = \langle 0, \infty)$
- D over $(-\infty, 1\rangle$
  I over $\langle 1, \infty)$
- many-to-one
- NENO
- BB by 0
- min at 1
- not periodic

$a < 0 \rightarrow$ CONCAVE



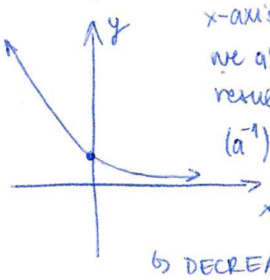
- $D(f) = R \quad R(f) = (-\infty, 2\rangle$
- I over $(-\infty, 1\rangle$ D over $\langle 1, \infty)$
- many-to-one
- NENO
- BA by 2
- max at 1
- not periodic

Exponential & logarithmic function

→ a function of the form $y = a^x$, where $x \in R$, $a > 0$, $a \neq 1$; (base is ⊕ but not 1) is called an <u>exponential function</u>.

→ a graph is exponential curve; we need to know whether $0 < a < 1$ or $a > 1$

$\boxed{0 < a < 1}$

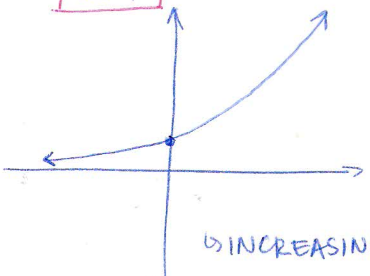


x-axis is its asymptote; we always get a ⊕ result

$(a^{-1}) = \frac{1}{a}$ ⊕

↳ DECREASING

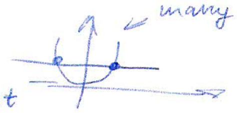
$\boxed{a > 1}$



↳ INCREASING

– the graph of expon. curve passes through point [0,1]

- D(f) = R   R(f) = $(0, \infty)$
- NENO
- monotony funct. (I/D over D(f))
- BB by 0 (x-axis = asymptote)
- no max nor min
- one-to-one (if each x is mapped onto a different y value)
  ↳ if we want to examine it we make a horizontal line test



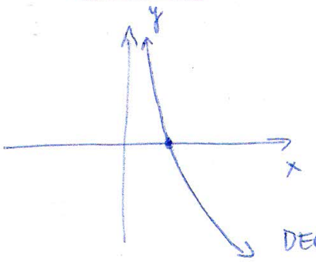
→ exponential f. has inverse function: logarithmic f.

→ A function of the form $y = \log_a x$ (logarithm of x to the base a), where argument x is positive number, base $a > 0$, $a \neq 1$ is called <u>logarithmic function</u>.

→ log f. is the inverse function of exponential function

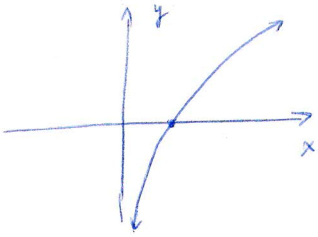
→ graph: logarithmic curve → it passes through point [1,0] ($\log_a 1$ to any base is always 0)

$\boxed{0 < a < 1}$



DECREASING

$\boxed{a > 1}$

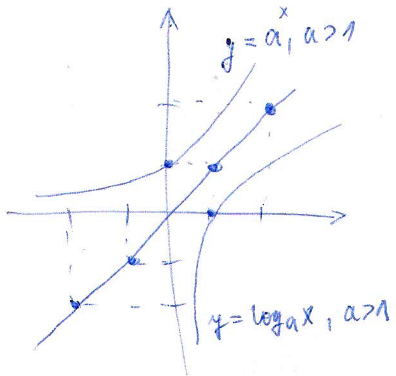


INCREASING

↳ asymptote: y-axis

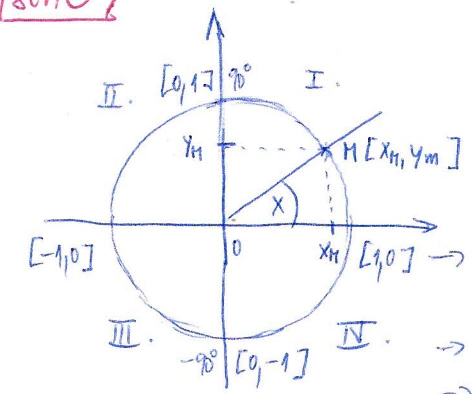
- D(f) = $(0, \infty)$ R(f) = R
- NENO
- not bounded
- no extrema
- monotony → one-to-one)

→ inverse f: exp → log; they swap D & R ⇒ $D(f) = R(f^{-1})$ $D(f^{-1}) = R(f)$
  – graphs are symmetric with respect to IDENTITY =
  = a linear function of the form $y = x$



Trigonometric functions

sine



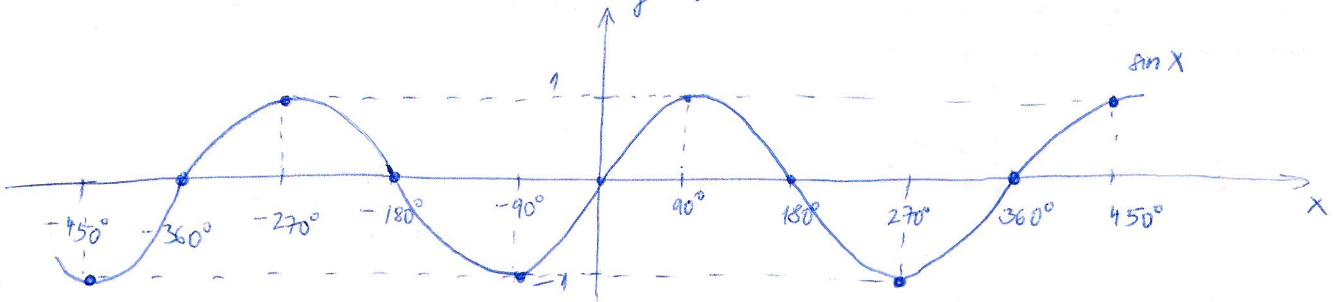
Let's say that there is unit circle with radius of 1 and an angle x. M is the point of intersection of the bracket of the angle x and the unit circle. sin x (of angle) is the y-coordinate of M.

→ if we know that the value of radius of the unit circle is 1 then we can write 4 points of intersection of the circle with the axes.

→ the value of sine / cosine will never be greater than 1 / less than −1

→ graph: sine curve (= sinusoida)

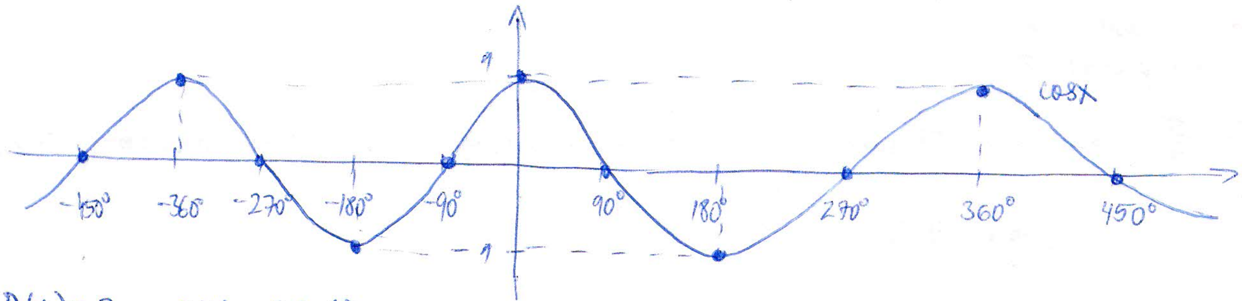
→ axes x and y divide the unit circle into 4 quadrants



- $D(f) = \mathbb{R}$ $R(f) = \langle -1, 1 \rangle$
- ODD – graph is symmetric with respect to the origin $\Rightarrow \sin(-90°) = -\sin 90° = -1$
- periodic – infinite number of intervals over which the function is ↗ / ↘ ; smallest period = 360° (full turn)
- I over $\langle -90° + k360°; 90° + k\cdot 360° \rangle$; $k \in \mathbb{Z}$ D over $\langle 90° + k\cdot 360°; 270° + k\cdot 360° \rangle$ $k \in \mathbb{Z}$
- BOUNDED: BA = 1 BB = −1
- EXTREMA: min at $-90° + k\cdot 360°$; $k \in \mathbb{Z}$ max at $90° + k\cdot 360°$; $k \in \mathbb{Z}$

cosine

→ let's say that there is unit circle with r = 1 and any angle x; then cos x is the x-coordinate of M.



$D(f) = \mathbb{R}$ $R(f) = \langle -1, 1 \rangle$

EVEN – graph is symmetric with respect to the axis y (refl. in a line) $\Rightarrow \cos(-\alpha) = \cos\alpha$

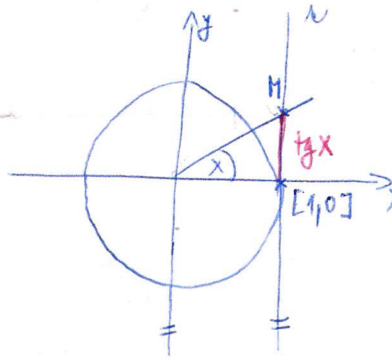
I over $\langle -180° + k\cdot 360°; 0° + k\cdot 360° \rangle$; $k \in \mathbb{Z}$ D over $\langle 0° + k\cdot 360°; 180° + k\cdot 360° \rangle$; $k \in \mathbb{Z}$

BOUNDED: BA = 1 BB = −1

EXTREMA: min at $180° + k\cdot 360°$; $k \in \mathbb{Z}$ max at $0° + k\cdot 360°$; $k \in \mathbb{Z}$

periodic – smallest period: 360°

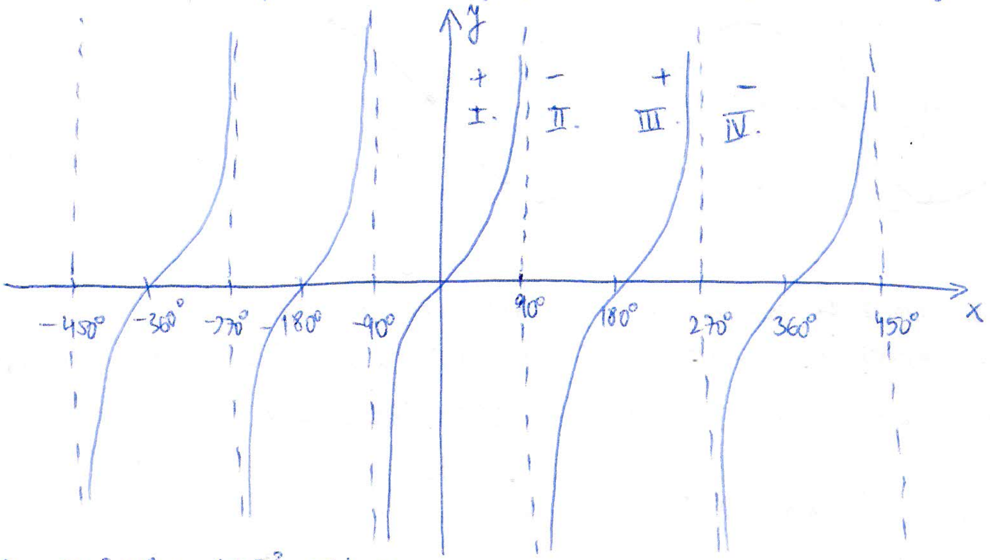
tangent → unit circle; centre: origin of the coordinate system



→ there is $x \in R$ → which is an acute angle
→ now we draw a tangent to the ○ passing through point [1,0] and it is ∥ with the axis y
→ point of contact of the tangent and unit circle is [1,0]
→ M … the intersection of tangent and the bracket of the angle
– the distance of point M from the x-axis is the tangent of the angle x

$tg\,x = \frac{\sin x}{\cos x}$ (= "the ratio of sine to cosine")

↳ we see that there are angles at which tangent is not defined because $\cos x \neq 0$
↳ $x \neq 90° + k \cdot 180°$ → they are the asymptotes to the function tangent



$\tan 30° = \frac{\sin 30°}{\cos 30°} = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{\sqrt{3}}{3}$

- $D(f) = R \setminus \{90° + k \cdot 180°\}$ $H(f) = R$
- ODD – regl. in a point – symm. acc. to the origin
- I over D(f)
- no extrema, no boundaries
- periodic – smallest period: 180°

$\cos x = -\frac{8}{15}$ $x \in (\pi; \frac{3\pi}{2})$ → nesime pre III. kvadrant. Find the values of remaining goniometric functions

$\cos^2 x + \sin^2 x = 1$:

$\left(-\frac{8}{15}\right)^2 + \sin^2 x = 1$

$\frac{64}{225} + \sin^2 x = 1$

$\sin^2 x = \frac{225-64}{225}$

$\sin^2 x = \frac{161}{225}$

$\sin x = - \frac{\sqrt{161}}{225}$

$+ + - -$

$tg\,x = \frac{\sin x}{\cos x} = \frac{-\frac{\sqrt{161}}{\cancel{15}}}{-\frac{8}{\cancel{15}}} = \frac{\sqrt{161}}{8}$

$cotg\,x = \frac{8}{161}$

$\sin 2x = 2 \cdot \left(+\frac{\sqrt{161}}{15}\right) \cdot \left(+\frac{8}{15}\right) = \frac{16 \cdot \sqrt{161}}{225}$

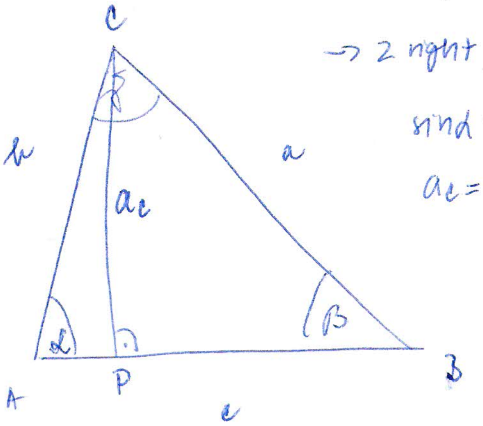
$\cos 2x = \frac{64}{225} - \frac{161}{225} = -\frac{97}{225}$

Sine & Cosine rule

- we use the rules to calculate the lengths of all sides & values of interior ∢ in scalene Δ
- they are used to give a relation between sides & interior ∢ of a scalene Δ

sine rule - the ratios of lengths of the sides to the sines of its angles are equal

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma} = 2r$$ - radius of circumcircle of a Δ



→ 2 right Δs, they are simmilar: ΔAPC and ΔPBC

$\sin\alpha = \frac{a_c}{b}$ $\sin\beta = \frac{a_c}{a}$ → we use trigon. ratio of sine

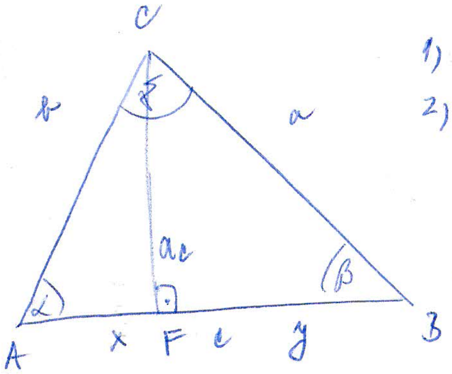
$a_c = b\cdot\sin\alpha$ $a_c = a\cdot\sin\beta$ → we express the heights from both eq.

$b\cdot\sin\alpha = a\cdot\sin\beta$ → compare them, divide them by ∢s

$$\frac{b}{\sin\beta} = \frac{a}{\sin\alpha}$$

cosine rule:

$$a^2 = b^2 + c^2 - 2bc\cdot\cos\alpha$$
$$b^2 = a^2 + c^2 - 2ac\cdot\cos\beta$$
$$c^2 = a^2 + b^2 - 2ab\cdot\cos\gamma$$



1) $x + y = c \quad \ldots \quad y = c - x$

2) pythagorean theorem of ΔAFC & ΔBFC

$b^2 = a_c^2 + x^2 \Rightarrow a_c^2 = b^2 - x^2$

$a^2 = a_c^2 + y^2 \Rightarrow a_c^2 = a^2 - y^2$

3) compare them and substitute y from 1

$b^2 - x^2 = a^2 - y^2$

$b^2 - x^2 = a^2 - (c-x)^2$

$b^2 - \cancel{x^2} = a^2 - c^2 + 2cx - \cancel{x^2}$

$b^2 = a^2 - c^2 + 2cx$

4) ΔAFC: $\cos\alpha = \frac{x}{b} \Rightarrow x = b\cdot\cos\alpha$

5) substitute:

$b^2 = a^2 - c^2 + 2cb\cos\alpha$

$a^2 = b^2 + c^2 + 2bc\cdot\cos\alpha$

$$\left(\frac{1-\cos 2x}{\sin 2x} + \frac{\sin 2x}{1+\cos 2x}\right)\cdot \operatorname{ctg} x = 2$$

nepíšeme, lebo to ideme dokázať

$$\left[\frac{1-(\cos^2 x - \sin^2 x)}{2\sin x\cos x} + \frac{2\sin x\cos x}{1+\cos^2 x - \sin^2 x}\right]\cdot\frac{\cos x}{\sin x} = \cancel{2}$$

$$\left[\frac{\overbrace{1-\cos^2 x+\sin^2 x}^{=2\sin^2 x}}{2\sin x\cdot\cos x} + \frac{2\sin x\cdot\cos x}{2\cos^2 x}\right]\cdot\frac{\cos x}{\sin x} = \cancel{2} \quad \frac{2\cdot\sin^2 x\cdot 2\cdot\cos^2 x + 2\sin x\cdot\cos x\cdot 2\cdot\sin x\cdot\cos x}{2\sin x\cdot\cos x\cdot 2\cdot\cos^2 x} = \frac{\cancel{4\sin^2 x\cdot\cos^2 x}(1+1)}{\cancel{4\sin x\cdot\cos x\cdot\cos^2 x}}\cdot\frac{\cancel{\cos x}}{\cancel{\sin x}} = 2$$

$\sin x \neq 0$ $\quad\cos x \neq 0$

$x \neq 0 + k\pi$ $\quad x \neq \frac{\pi}{2} + k\pi$

Sequence

- a function whose domain is the set of natural numbers

FINITE – has the last term, so we can calculate the total number of terms of the sequence $\{2,4,6,8,10\}$

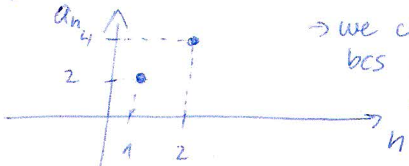
INFINITE – we can't calculate total number of terms, it doesn't have the last term → it continues to infinity $\{2,4,6,8,10\ldots\}$

Ways of expressing the sequence:

1, naming the terms $\{2,4,6,8,10,12\ldots\}$

2, by giving the formula of the n-th term $a_n = 2n$ or $\{2n\}_{n=1}^{\infty}$

3, graph of the sequence



→ we can't join the points of the sequence bcs D is only the set of N

→ graph: set of isolated points

4) recurrence relation – the determination of any term of the sequence using the preceding term → we need to express value of: a_{n+1}, a_1

$\{2n\}_{n=1}^{\infty}$ → 2 methods: difference and quotient method (I. difference, II. quotient)

(I) $a_{n+1} - a_n = 2(n+1) - 2n = 2n + 2 - 2n = 2 \quad / + a_n$

$a_{n+1} = 2 + a_n \;,\; a_1 = 2$

(II) $\dfrac{a_{n+1}}{a_n} = \dfrac{2(n+1)}{2n} \quad / \cdot a_n$

$a_{n+1} = \dfrac{n+1}{n} \cdot a_n \;;\; a_1 = 2$

Sequence properties

Sequence $\{a_n\}_{n=1}^{\infty}$ is a) increasing if any natural number n satisfies the inequality that $a_{n+1} > a_n$ ⇒ (which implies that) $a_{n+1} - a_n > 0$ (the difference of a_{n+1} and a_n term is always a positive number)

$\{5,10,20,25\ldots\}$

b) decreasing if $\forall n \in N: a_{n+1} < a_n \Rightarrow a_{n+1} - a_n < 0$ $\{50,40,30,20,10,\ldots\}$

c) non-increasing & non-decreasing if $\forall n \in N: a_{n+1} \geq a_n \Rightarrow a_{n+1} - a_n \geq 0$ $\{50,40,30,30,20\ldots\}$

d) constant if $\forall n \in N: a_{n+1} = a_n \Rightarrow a_{n+1} - a_n = 0$ $\{5,5,5,\ldots\} = \{5\}_{n=1}^{\infty}$ / $\{5 \cdot n^0\}_{n=1}^{\infty}$

e) alternating / oscillating if $\forall n \in N: a_{n+1} = -a_n$ $\{2,-2,2,-2,\ldots\} = \{(-1)^{n+1} \cdot 2\}_{n=1}^{\infty}$

Sequence $\{a_n\}_{n=1}^{\infty}$ is:

a) bounded below → there is a real number b such that any natural number n satisfies inequality that:

→ if $\exists b \in R; \forall n \in N: a_n \geq b$

b) bounded above → if $\exists a \in R; \forall n \in N: a_n \leq a$

c) bounded → $b \leq a_n \leq a$

derrive the reccurence relation of the sequences :

$\{2.3^{n+1}\}_{n=1}^{\infty}$

$$\frac{a_{n+1}}{a_n} = \frac{\cancel{2}.3^{n+2}}{\cancel{2}.3^{n+1}} = \frac{\cancel{3}^{n}.3^{\cancel{2}}}{\cancel{3}^{n}.\cancel{3}}$$

$$a_{n+1} = 3.a_n \; ; \; a_1 = 18$$

$\left\{\frac{5n}{2n+3}\right\}_{n=1}^{\infty}$

$$a_{n+1} - a_n = \frac{5n+5}{2n+2+3} - \frac{5n}{2n+3} = \frac{(5n+5)(2n+3) - 5n(2n+5)}{(2n+5)(2n+3)} =$$

$$= \frac{10n^2 + 15n + \cancel{10n} + 15 - \cancel{10n} - 25}{4n^2 + 6n + 10n + 15} = \frac{10n^2 + 15n - 10}{4n^2 + 16n + 15} = \frac{5(2n^2 + 3n - 2)}{4n^2 + 16n + 15} \; ??$$

$$\left[\frac{a_{n+1}}{a_n} = \frac{(2n+3)(n+1)}{n(2n+5)}\right]$$

Arithmetic progression

given example: $\{2,4,6,8,10,12,\dots\}$ → we see that 1st term is 2 and each subsequent term is obtained by adding 2 (the constant) to the previous one. We say that the terms progress arithmetically so the sequence is called an arithm. progression.

↳ sequence of terms 2,4,6,8,...

Def A sequence $\{a_n\}_{n=1}^{\infty}$ where $n \in \mathbb{N}$; is called an A.P. if

$\exists d \in \mathbb{R}, \forall n \in \mathbb{N}: a_{n+1} = a_n + d$

↳ each term is equal to the previous one + d

→ d ... "common difference"

① Formula for the n-th term of A.P.

$$a_1$$
$$a_2 = a_1 + d$$
$$a_3 = a_2 + d = a_1 + 2d$$
$$a_4 = a_3 + d = a_1 + 3d$$
$$\vdots$$
$$\boxed{a_n = a_1 + (n-1).d}$$

② Relation between any 2 terms of the progression

$$\left.\begin{aligned} a_r &= a_1 + (r-1)d \\ a_s &= a_1 + (s-1)d \end{aligned}\right\}$$ we solve the system using subtraction of 2 equations simultaneously

$a_r - a_s = (r-1)d - (s-1)d$ → expand brackets

$a_r - a_s = rd - \cancel{d} - sd + \cancel{d}$ → factor d

$$\boxed{a_r - a_s = d(r-s)}$$

③ Any term of an A.P. is the arithmetic mean of preceding and subsequent term

$$a_n = \frac{a_{n-1} + a_{n+1}}{2}$$ → number of terms

④ Sum of n-terms

$$S_n = \frac{(a_1 + a_n).n}{2}$$

Geometric progression

$\{2,4,8,16,32,64,\dots\}$ The 1st term of the sequence is 2 and each subsequent term is obtained by multiplying the previous term by the constant 2. We say that the terms progress geometrically, so the sequence is called a geometric progression.

Def A sequence $\{a_n\}_{n=1}^{\infty}$ is called GP if: $\exists q \in \mathbb{R}, \forall n \in \mathbb{N}: a_{n+1} = a_n \cdot q$ ... q = "common ratio"

① n-th term of GP

$$a_1$$
$$a_2 = a_1 \cdot q$$
$$a_3 = a_2 \cdot q = a_1 \cdot q^2$$
$$a_4 = a_3 \cdot q = a_1 \cdot q^3$$
$$\boxed{a_n = a_1 \cdot q^{n-1}}$$

② Relation between 2 terms (quotient formula)

$$\left.\begin{aligned} a_r &= a_1 \cdot q^{r-1} \\ a_s &= a_1 \cdot q^{s-1} \end{aligned}\right\} \div$$

$$\frac{a_r}{a_s} = \frac{\cancel{a_1} \cdot q^r \cdot \cancel{q^{-1}}}{\cancel{a_1} \cdot q^s \cdot \cancel{q^{-1}}}$$

$$\frac{a_r}{a_s} = q^{r-s}$$

any constant sequence is GP and AP as well

$\{2,2,2,2,\dots\}$

$d=0 \quad q=1$

③ Any term of G.P. is the geom. mean of preceding and subsequent term

$$a_n = \sqrt{a_{n-1} \cdot a_{n+1}}$$

④ Sum of n-terms

$q \neq 1: \quad S_n = a_1 \cdot \frac{q^n - 1}{q - 1}$

if $q = 1: \quad S_n = n \cdot a_1$

The sum of 3 numbers, which are 3 consecutive terms of an A.P., is 30. If the middle term is smaller by 4, the progression is geometric. Find the numbers

AP a_1 $\quad a_1+a_2+a_3=30$

$a_2=a_1+d=10 \quad a_1+a_1+d+a_1+2d=30$

$a_3=a_1+2d \quad 3a_1+3d=30$

$a_1+d=10 \Rightarrow a_2$

GP a_1 $\quad$ ~~$a_1+d-4=a_1\cdot q$~~

$a_2=a_1\cdot q$

$a_3=a_1\cdot q^2$

$a_2=\frac{a_1+a_3}{2}$

AP: $10-d,\ 10,\ 10+d$

GP: $10-d,\ 6,\ 10+d$ → geometric mean: $6=\sqrt{(10-d)(10+d)}\ /^2$

$36=100-d^2$

$d^2=64 \to d=8,\ d=-8$

if $d=8$: $a_1=2$, $a_2=10$, $a_3=18$ $\qquad$ if $d=-8$: $a_1=18$, $a_2=10$, $a_3=2$

Prove that the sequence $\left\{\frac{5n-1}{n+2}\right\}_{n=1}^{\infty}$ is monotonic & BA → monotonic if I or D.

$a_{n+1}-a_n>0$ or $a_{n+1}-a_n<0$ → we are examining the difference of 2 terms

$$\frac{5(n+1)-1}{\cancel{5}(n+1)+2}-\frac{5n-1}{n+2}=\frac{5n+4}{\cancel{5n}+3}-\frac{5n-1}{n+2}=\frac{(5n+4)(n+2)-(5n-1)(n+3)}{(n+3)(n+2)}=\frac{5n^2+10n+4n+8-(5n^2+15n-n-3)}{n^2+5n+6}=$$

$$=\frac{\cancel{5n^2}+\cancel{14n}+8-\cancel{5n^2}-\cancel{14n}+3}{(n+3)(n+2)}=\frac{11}{(n+3)(n+2)}>0 \to \text{INCREASING F.}$$

↳ D is a set of N ⇒ we can only substitute numbers 1, 2, 3, … denominator will be allways ⊕

→ list 5 terms of sequence $\left(=\left\{\frac{5n-1}{n+2}\right\}\right)$

$a_1=\frac{4}{3} \quad a_2=\frac{9}{4} \quad a_3=\frac{15}{4} \quad a_4=\frac{19}{6} \quad a_{98}=\frac{489}{100}=4{,}89$ → BA by 5 :

$\frac{5n-1}{n+2}\le 5 \quad /\cdot(n+2)\ \forall n\in N!$

$\cancel{5n}-1\le \cancel{5n}+10$

$-1\le 10$ ✓

Given the A.P. such that $a_4+a_7+a_{10}=15$; $a_5+a_8+a_{11}=9$. Decide and prove whether there is k, such that $a_k=0$ and m such that $a_m=1$.

$a_1+5d+a_1+6d+a_1+9d=15$

$a_1+4d+a_1+7d+a_1+10d=9$

$3a_1+18d=15$

$3a_1+21d=9\ /\cdot(-1)$

$-3d=6$

$d=-2$

$3a_1-36=15$

$3a_1=51$

$a_1=17$

$a_k=a_1+(k-1)d$

$0=17-(k-1)2$

$0=17-2k+2$

$2k=19$

$k=\frac{19}{2}$ ✗ a_n

↳ $n\notin N$!

$1=17-(m-1)2$

$0=16-2m+2$

$2m=18$

$m=9$ ✓

Prove that $\sin 2x, \cos x, \frac{1}{2tgx}$ $x \in (0, \pi)$ form 3 consecutive terms of a GP.

→ we use geometric mean:

$\cos x = \sqrt{\frac{\cancel{2\sin x \cos x}}{\frac{1}{2tgx}}}$

$\cos x = \sqrt{\sin 2x \cdot \frac{1}{2tgx}}$

$\cos x = \sqrt{2\sin x \cdot \cos x \cdot \frac{\frac{1}{1}}{2 \cdot \frac{\sin x}{\cos x}}}$

$\cos x = \sqrt{\cancel{2}\cancel{\sin x} \cdot \cos x \cdot \frac{\cos x}{\cancel{2} \cdot \cancel{\sin x}}} = \sqrt{\cos^2 x}$

$\underline{\cos x = \cos x}$

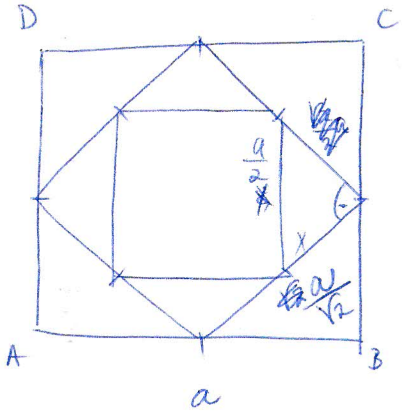
$\log 16, \log 8, \log 4 \to AP?$

$\cancel{\log_{10} 16 = x}$ Stvorce

$\log 8 = \frac{\log 16 + \log 4}{2}$

$2 \cdot \log 8 = \log 16 \cdot 4$

$64 = 64$ ✓



$\cancel{2 \cdot \left(\frac{\sqrt{2}}{2}a\right)^2 = x^2}$ $\quad 2 \cdot \left(\frac{a}{2}\right)^2 = x^2$

$\cancel{2 \cdot \frac{2}{4} a^2 = x^2}$ $\quad \frac{2 \cdot a^2}{\cancel{4}\, 2} = x^2$

$\frac{\frac{a}{\sqrt{2}}}{\frac{2}{1}} = \left(\frac{a}{2\sqrt{2}}\right)^2 \cdot 2 = x^2$

$\frac{a^2}{\cancel{8}\, 4} \cdot \cancel{2} = x^2$

$x = \frac{a}{2}$

a^2

$\frac{a^2}{2}$ GP: $q = \frac{1}{2}$

$\frac{a^2}{4}$

STATEMENTS

One characteristic of the mathematical language is its **logic, exactness** and **explicitness.**

In mathematics, we express ideas by mathematical symbols, arranged in meaningful patterns, called **mathematical sentences.**

In general, the language of mathematics uses 4 categories of symbols:

1. symbols for **ideas** (munbers and elements) – digits

2. symbols for **relations** (which indicate how ideas are connected or related to one another) - =, >, ≤, ...

3. symbols for **operations** (which indicate what is done with the ideas) - +, :, ...

4. symbols for **punctuation** (which indicate the order in which the mathematics is to be completed) – decimal point, comma, brackets, ...

Mathematical sentences that cannot be judged true or false, such as $x + 7 = 13$, are called **open sentences.** Mathematical sentences such as $6 + 7 = 13$, $14 + 7 = 13$, which contain enough information to be judged true or false, are called **closed sentences = mathematical statements.**

A **mathematical statement** – is a mathematical sentence that contains enough information to be judged true or false.

A statement can be: **true** – 1

false – 0

No statement may be both true and false at the same time in the same context.

e.g.

S: Bratislava is the capital of Slovakia. (true statement)

T: $2 + 5 = 6$ (false statement)

A **hypothesis** – is an unproven statement which is usually thought to be true and for which, usually, a lot of supporting evidence can be found.

An **axiom** – is a statement which is assumed to be true, and is used as a basis for developing a system. Any system of logic starts by saying clearly what axiom it uses.

In everyday speaking and writing we make constant use of **compound statements.**

A **compound statement** – is a statement formed from two or more simple statements by joining these statements with **connectives.** A **connective** is a word (or words) used to form compound statements: *and; or; if ..., then ...; if and only if*

1. CONJUNCTION (logical product)	and	$A \wedge B$
2. DISJUNCTION (logical sum)	or	$A \vee B$
3. IMPLICATION (conditional)	if ..., then ...	$A \Rightarrow B$
4. EQUIVALENCE (biconditional)	if and only if	$A \Leftrightarrow B$

A	B	A′	$A \wedge B$	$A \vee B$	$A \Rightarrow B$	$A \Leftrightarrow B$
1	1					
1	0					
0	1					
0	0					

Negation of mathematical statements

To each mathematical statement V it is possible to create a statement V′which negates that meaning of the statement. Such a statement V′is called the **negation of the statement** V.

Statement	*Its negation*
Number 7 is divisible by 2.	**It is not true that** number 7 is divisible by 2. or Number 7 **is not** divisible by 2.
Today is Wednesday.	**It is not true that** today is Wednesday. or Today **is not** Wednesday.
Peter is wearing a brown sweater.	Peter **is not** wearing a brown sweater. Peter is wearing a **blue** sweater. – IT IS NOT CORRECT !!!

NOTE

Negation V′has to contain all the possibilities which are not included in the statement V.

Certain phrases may cause problems, therefore we need to learn them.

Statement	*Negation*
every ... is ...	at least one ... is not ...
at least one ... is ...	no one ... is ...
at least n ... is ...	at most (n – 1) ... is ...
at most n ... is ...	at least (n + 1) ... is ...

e.g.

V: Every triangle is obtuse. V´: At least one triangle is not obtuse.

V: The equation $2x - 1 = 5$ has at most 1 root. V´: The equation $2x - 1 = 5$ has at least 2 roots.

Compound statement	*Its negation*
$A \wedge B$	$A' \vee B'$
(Ann **and** Sue are coming)	(Ann isn´t coming **or** Sue isn´t coming)
$A \vee B$	$A' \wedge B'$
(Ann **or** Sue are coming)	(Ann isn´t coming **and** Sue isn´t coming)
$A \Rightarrow B$	$A \wedge B'$
(**If** Ann is coming, **then** Sue is coming as well.)	(Ann is coming **and** Sue isn´t coming)
$A \Leftrightarrow B$	$(A \wedge B') \vee (A' \wedge B)$
(Ann is coming **if and only if** Sue is coming)	(Ann is coming and Sue isn´t coming **or** Ann isn´t coming and Sue is coming)

OTHER PROPERTIES:

- $(A')' \Leftrightarrow A$
- $(A \Rightarrow B) \Leftrightarrow (A' \Rightarrow B')$
- $(A \Leftrightarrow B)$ is the same as $(A \Rightarrow B) \wedge (B \Rightarrow A)$
- $[A \wedge (B \wedge C)] \Leftrightarrow [(A \wedge B) \wedge C]$
- $[A \vee (B \vee C)] \Leftrightarrow [(A \vee B) \vee C]$

A **statement formula** – is an expression which contains statements connected with conjunction, disjunction, implication and equivalence.

A **tautology** – is a compound statement, which is always true, regardless the truth value of the statements it is composed of.

QUANTIFIED STATEMENTS

Quantified statements – are statements, which in connection with other expression or statements form a new statement, but with a different meaning. These statements contain so called **quantifiers**, which are of two types:

a) generality quantifier - $\forall$ ... for all

(It expresses the amount of elements of a given set, for which certain property is common)

b) existential quantifier - $\exists$ ... there exists

$\exists!$ ... there exists exactly one

(It expresses a guess of the amount of elements of the set, for which certain property is common)

e.g.

V: $\forall x \in R \;\; \exists y \in R: \; x + y = 2$

V´: $\exists x \in R \;\; \forall y \in R: \; x + y \neq 2$

$(A \wedge B)' = (A' \vee B')$
$(A \vee B)' = (A' \wedge B')$
$(A \Rightarrow B)' = (A \wedge B')$
$(A \Leftrightarrow B)' = (A \wedge B') \vee (A' \wedge B)$

Statement - mathematical sentence which can be judged T/F
Open sentence - can't be judged whether it is T/F
Hypothesis - unproven statement
Axiom - fixed statement
Compound statement - made up of 2 simple statements, joined by connectives (implication, ...)
Negation of a statement - statement which has opposite value
Quantified statement - it contains either existential / universal quantifier ($\exists, \exists!$, at least, ...)
- negation: swap them

Given the true statements A, B and the false statement C. Find out if the following compound statements are true:

a) $(A \Rightarrow B) \Leftrightarrow (A \wedge B') - 0$

$1 \Rightarrow 1 \Leftrightarrow 1 \wedge 0$

$1 \Leftrightarrow 0 \rightarrow$ False statement

b) $[A \wedge (B \vee C)] \Leftrightarrow [(A \wedge B) \vee (A \wedge C)] - 1$

$1 \wedge 1 \Leftrightarrow 1 \vee 0$

$1 \Leftrightarrow 1 \rightarrow$ True.

State the truth values of the given statements and their negations:

a) $\forall a, b \in R: a = b \Leftrightarrow a^2 = b^2$ F neg: $\exists a, b \in R: [(a = b) \wedge (a^2 \neq b^2)] \vee [(a \neq b) \wedge (a^2 = b^2)]$

$-1 \neq 1 \leftarrow 1 = 1$
$-4 \neq 4 \quad (-4)^2 = 4^2 \quad V$

b) $\forall a, b \in R^+: a < b \Rightarrow a^2 < b^2$ T neg: $\exists a, b \in R: a < b \wedge a^2 \geq b^2$ (impl. is false if $T \Rightarrow F$).
(0 nie je ani ⊕ ani ⊖)

Form the negation of these statements:

a) all the multiples of number 3 are odd numbers F (3, 6, 9 ...)
N: There is at least one multiple of 3 which is even

b) the diff. of 2 N is a N (nat. numb.) F (2 - 5 ...)
there is at least 1 pair of N, ~~and their~~ whose difference is not N

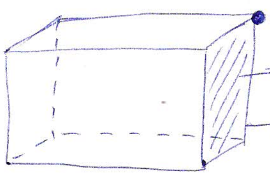
c) the sums of lengths of $\forall 2$ sides in a triangle is > than the length of the 3rd side T
—— " —— at least one Δ is $\leq$ —— " ——

$\forall \Delta: a + b > c \quad \exists \Delta: a + b \leq c$

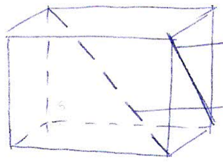
d) No prime number is an even number. F (2!)
There exists at least 1 prime n. which is an even number.

Solid Geometry

- a special branch of geometry, the geometry of 3-dimensional Euclidean space dealing with the measurements of volumes and surface areas of 3-dimensional figures called solids (3 dimensions: lenght, width, height)



→ vertex = the point at which 3 edges meet
→ face = it is a plain surface enclosed by the edges
→ edge (hrana) = it is the line segment defining the outline of the solid



→ face diagonal - a diagonal which lies entirely in one face
→ space diagonal - it doesn't lie in one face

Division of solids

- Round s. (cone, cylinder, sphere)
- Angular s. (prism, cuboid, cube, pyramid) - base: polygon

- Rotational s. (round s.) - made by the rotation of the planimetric figure
- Non-rotational s. (angular solids)

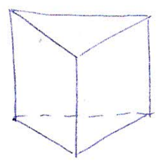
- Polyhedra - solids with flat faces = faces are planimetric figures (triangle, square...)
- Non-polyhedra - solids with circular base and their lateral surface is a curved surface

The surface area of the solid
↳ the total area of the exterior surface of the figure / the sum of areas of all faces of the solid

Volume of the solid
↳ the total space of the solid occupied by its surface

• ANGULAR SOLIDS

→ Prism (hranol) - it is a figure which has 2 parallel congruent bases and the bases can be any polygon (triangle, square, rectangle...)



→ if the base is Δ ⇒ triangular prism
→ if the base is equilateral Δ ⇒ regular prism
→ if the lateral faces are $\perp$ to the base ⇒ right prism
→ if the — " — are rectangle ⇒ rectangular prism (CUBOID)

Δ base: $V = B \cdot h$

$S = 2 \cdot B + \underbrace{L}_{\text{area of lateral surface}}$

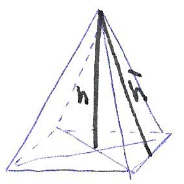
rectangular prism: $V = B \cdot h = a \cdot b \cdot c$

$S = 2(ab + bc + ac)$

square-based prism → cube: $V = a^3$

$S = 6 \cdot a^2$

→ Pyramid - base is square, triangle... there is only one base and the lateral faces are Δs



h ... height of the solid
h' ... slant height

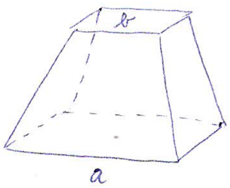
$V = \frac{1}{3} B \cdot h$

$S = B + L$

base is square: $a^2 + \left(\frac{a \cdot h'}{2}\right) \cdot 4$

base is rectangle: $a \cdot b + \left(\frac{a \cdot h_a}{2}\right) \cdot 2 + \left(\frac{b \cdot h_b}{2}\right) \cdot 2$

↳ if we have square-based pyramid, triangles will be isosceles
↳ if we perform the section of the pyramid by the plane parallel with the base, we get **truncated pyramid** (zrezaný ihlan)



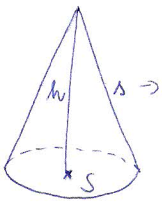
lateral face is trapezium

$$S = B_1 + B_2 + L = B_1 + B_2 + \frac{1}{2}(a+b).h \cdot 4$$

$$V = \frac{1}{3}h\left(B_1 + \sqrt{B_1 . B_2} + B_2\right)$$

- **ROUND SOLIDS**

→ **Cone** – it's rotational solid made by the rotation of a triangle, its base is a circle



$$V = \frac{1}{3}B.h$$

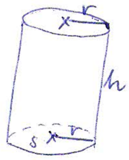
$$S = B + L = \pi r^2 + \pi r s = \pi r(r+s)$$

– **truncated cone** (zrezaný kužel) – made by the section of the cone by the plane parallel with the base

$$S = \pi r_1^2 + \pi r_2^2 + \pi s(r_1 + r_2)$$

$$V = \frac{1}{3}\pi h(r_1^2 + r_1 \cdot r_2 + r_2^2)$$

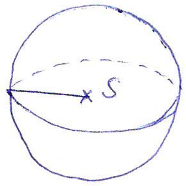
→ **Cylinder** (valec) – round solid with 2 circular bases which are parallel and they are connected by a curved surface



$$V = \pi r^2 h$$

$$S = 2\pi r^2 + 2\pi r h$$

→ **Sphere** – is a solid which is bounded by enclosed surface and every point from the surface is equidistant from a fixed point called the center



$$V = \frac{4}{3}\pi r^3$$

$$S = 4\pi r^2$$

Proof

– a sequence of statements made up of axioms*, assumptions, definitions, formulas... leading to the establishment of the truth value of one final statement

we distinguish
- proof of a statement
- proof of mathematical implication ($A \Rightarrow B$)

I. Proof of a statement

1) $\boxed{\text{Direct proof}}$ – it is a proof in which all assumptions are true and all the arguments are valid.

$A \Rightarrow A_1 \Rightarrow A_2 \Rightarrow \dots \Rightarrow V$ We suppose an axiom A and prove that statement V follows using * valid

example: Prove that $\forall x \in R$ satisfy the inequality $\frac{6x+8}{x^2+1} \le 9$ V $/ \cdot (x^2+1)$ $\Rightarrow 6x+8 \le 9(x^2+1)$ A_3 $\Rightarrow$

→ original statement; x^2+1 always ⊕

$6x+8 \le 9x^2+9$ A_2 $\Rightarrow 0 \le 9x^2-6x+1$ A_1

$0 \le (3x-1)(3x-1)$ → formula of the perfect square $= (3x-1)^2$ A

↳ it's an axiom

❗ Prove that: $\forall x, y \in R^+: \frac{x^4+y^4}{2} \ge \frac{x+y}{2} \cdot \frac{x^3+y^3}{2}$ $/ \cdot 4$

$2(x^4+y^4) \ge (x+y)(x^3+y^3)$ → expand

$2x^4+2y^4 \ge x^4+xy^3+yx^3+y^4 \quad / -x^4, -y^4$

$x^4+y^4-xy^3-yx^3 \ge 0$ (factor y^3 from 2 midterms) and x^3 from 1st and last term

$x^3(x-y)+y^3(y-x) \ge 0$ → factor -1

$x^3(x-y)-y^3(x-y) \ge 0$ → factor $x-y$

$(x-y)(x^3-y^3) \ge 0$

$(x-y)(x-y)(x^2+xy+y^2) \ge 0$

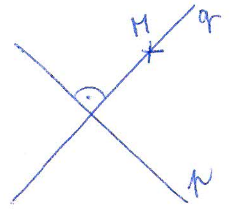
$(x-y)^2(x^2+xy+y^2) \ge 0$ ⊕ ⊕ ⊕ ⊕

↳ $x, y \in R^+$

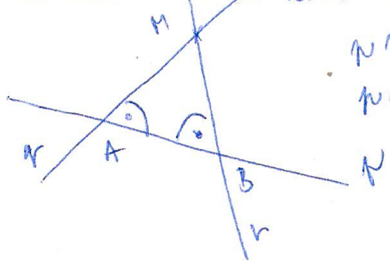
2) $\boxed{\text{Indirect proof}}$ = proof by contradiction – instead of proving statement V we make a false assumption (we negate it... V') and using * we get ~~to~~ the contradiction (spor) with V' (V negated). That's why our assumption was false and the original statement V is true.

$V' \Rightarrow C_1 \Rightarrow C_2 \Rightarrow C_3 \Rightarrow \dots C_n$

Prove that we can draw at most 1 perpendicular to the line p which passes through $M \notin p$



V': we can draw at least 2 perpendiculars to the line p...



$p \cap q = \{A\}$

$p \cap r = \{B\}$

→ we get $\triangle ABM$ → sum of interior ∡ must be exactly 180°, but here ∡MAB + ∡ABM + ∡AMB > 180° AXIOM!

→ our assumption was false so it means that we can only draw one perpendicular

II. Proof of implication

1) $\boxed{\text{Direct proof}}$ – we start with statement A and valid axioms will lead us to the statement B

$$A \Rightarrow A_1 \Rightarrow A_2 \Rightarrow \dots \Rightarrow B$$

$\forall n \in N$: n is even $\Rightarrow n^2$ is even $\qquad (n = 3k+2)$

If n is even, $n = 2k$; $k \in N$

Then $n^2 = 4k^2 = 2 \cdot 2k^2$ → we see that it is divisible by 2 → so it is even

2) $\boxed{\text{Indirect proof}}$ – we prove the contrapositive (obmenená veta) of the implication $A \Rightarrow B$ (A implies B) ... $B' \Rightarrow A'$.

– using logical steps we end with A'

– they are logically equivalent ($A \Rightarrow B$; $B' \Rightarrow A'$)

→ it is proof by contrapositive of an implication

$$B' \Rightarrow B_1 \Rightarrow B_2 \Rightarrow \dots \Rightarrow A'$$

$\forall n \in N$: $3 \mid (n^2+2) \Rightarrow 3 \nmid n$

contrapositive: $\underline{3 \mid n} \Rightarrow 3 \nmid (n^2+2)$

if $3 \mid n$, $\exists k \in N : n = 3k$ ... then $n^2+2 = 9k^2+2$ ⇒ we see that it isn't possible to factor 3 ...

... $3 \nmid (9k^2+2) \Rightarrow 3 \nmid (n^2+2)$

3) $\boxed{\text{Proof by contradiction}}$

$(A \Rightarrow B)' = A \wedge B'$

Prove that:

$\forall n \in N$: $3 \mid n^2 \Rightarrow 3 \mid n$ ✓

$3 \mid n^2 \wedge 3 \nmid n$ ✗ $\quad$ if $3 \nmid n$, $n = 3k+1$ or $n = 3k+2$

if $\underline{3 \mid n^2}$, $n^2 = 3k$

1) $n^2 = (3k+1)^2 = 9k^2 + 6k + 1 = 3(3k^2+2k) + 1$ ... $\underline{3 \nmid n^2}$ ... we see that it is $\underline{\text{contradiction}}$ SPOR

2) $n^2 = (3k+2)^2 = 9k^2 + 12k + 4 \overset{3+1}{=} 3(3k^2+4k+1) + 1$

Absolute value of a real number

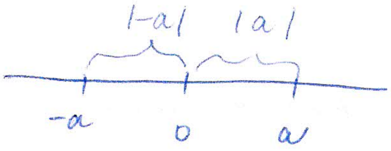
The absolute value | modulus of a real number a is a's numerical value without regard to its sign.

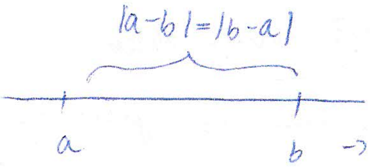
Def The absolute value of a real number a is denoted by two vertical bars $|a|$ and is defined:

$$|a| = \begin{cases} a, & a \geq 0 \\ -a, & a < 0 \end{cases}$$

→ Geometric interpretation

From a geometric point of view, the absolute value of a real number is the number's distance from 0 along the number line and more generally, the absolute value of difference of 2 real numbers is the distance between them.





→ 2 different real numbers

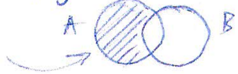
→ Properties

1) $|a| \geq 0$
2) $|a| = 0 \Leftrightarrow a = 0$
3) $|a| = |-a|$
4) $|a| \geq a$
5) $|a+b| = |b+a|$
6) $|a-b| = |b-a|$
7) $|a \cdot b| = |b \cdot a|$ (comutative)
8) $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}, b \neq 0$
9) $|a+b| \leq |a| + |b|$ — $|-4+5| = 1$, $|-4| + |5| = 9$
10) $|a-b| \geq |a| - |b|$ — $|2-7| = 5$, $|2| - |7| = -5$

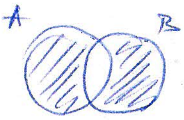
Interval - is the subset of the numberline

Interval / set operations:

1) INTERSECTION $A \cap B$ - set of elements such that ...
   $A \cap B = \{x \in R;\ x \in A \wedge x \in B\}$ $\wedge$ - at the same time
   $A \cap B = \emptyset$ ... A, B = "disjoint"
   e.g. $A = \{1,2,3\}$ $B = \{5,6,7\}$ $A \cap B = \emptyset$
   A = weekdays B = weekend
2) UNION $A \cup B$ - set of numbers such that...
   $A \cup B = \{x \in R;\ x \in A \vee x \in B\}$
3) DIFFERENCE $A - B$



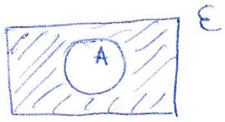
   $A - B = \{x \in R,\ x \in A \wedge x \notin B\}$ - but not B; the sets are not comutative
   $B - A = \{x \in R,\ x \in B \wedge x \notin A\}$
4) SYMMETRIC DIFFERENCE
   $A \triangledown B = \{x \in R;\ x \in A \vee x \in B \wedge x \notin A \cap B\}$



prvky, kt. nepatria do prieniku

5) COMPLEMENT - A is a subset of universal set

$A \subset \mathcal{E}$



$A' = \{x \in R,\ x \in \mathcal{E} \wedge x \notin A\}$ A ... working days, ℰ ... week days } A' = weekend

Venn diagrams - graphical interpretation of intervals, interval operations

+ INTERVALY

Given the universal set Z and its subsets A, B, C.

$A = \{n \in N : 3 | n \wedge \overset{n<10}{n - 10 < 0}\} = \{3, 6, 9\}$

$B = \{n \in N : |n-4| < 3\} = \{2, 3, 4, 5, 6\}$ List the elements of the set $[(A \cup B) \cap C]'_Z$ ↗ "complement over the set Z (of integers)"

$C = \{n \in N : n^2 - 2n - 3 \leq 0\} = \{\cancel{-1}, \cancel{0}, 2, 3\}$ $\boxed{n \in N}$

$Z = \{n \in N : n \leq 10\} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

lebo $\boxed{n \in N}$!!

~~B: $n - 4 < 3$ or $n + 4 > 3$~~

~~$n < 7$ $n < -1$~~

$B: -3 < n - 4 < 3 \quad | + 4$

$1 < n < 7$

$C: (n+1)(n-3) \leq 0$

$+ \quad \ominus \quad +$

$-1 \quad 3$

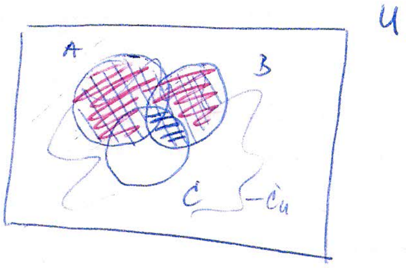
$A \cup B = \{2, 3, 4, 5, 6, 9\}$

$(A \cup B) \cap C = \{2, 3\}$

$[(A \cup B) \cap C]'_Z = \{1, 4, 5, 6, 7, 8, 9, 10\}$ → všetko okrem 2,3

Using the venn diagrams find the following set: → shade the region

$(A \cup B) - (B - C'_u)$



1) $C'_u = ?$ 2) $B - C'_u$ (patria B ale nie C'_u

3) $A \cup B$ ////

4) $A \cup B$ V ale nie //// $(B - C'_u)$

Factorial – for any $n \in \mathbb{N}$ the notation $n!$ is used for product of all naturals less/equal to n

$$n! = n \cdot (n-1)(n-2) \dots 3 \cdot 2 \cdot 1 \quad \text{and} \quad 0! = 1$$

Combination number

$$\binom{n}{k} = \frac{n!}{(n-k)!\,k!} \; ; n,k \in \mathbb{N} ; n \geq k$$

↳ ("n above k")

→ **properties of comb. number**:

1) $\binom{n}{n} = 1 \rightarrow \frac{n!}{0!\,n!} = \frac{1}{1} = 1$

2) $\binom{n}{0} = 1 \rightarrow \frac{n!}{n!\,0!} = \frac{1}{1} = 1$

3) $\binom{n}{1} = n \rightarrow \frac{n!}{(n-1)!\,1!} = \frac{n(n-1)!}{(n-1)!\,1!} = n$

the definition of combination number

4) $\binom{n}{k} = \binom{n}{n-k} \rightarrow \binom{n}{n-k} = \frac{n!}{n-(n-k))!\,(n-k)!} = \frac{n!}{k!\,(n-k)!} = \binom{n}{k}$

5) $\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1} \rightarrow \binom{n}{k} + \binom{n}{k+1} = \frac{n!}{(n-k)!\,k!} + \frac{n!}{(n-k-1)!\,(k+1)!} =$

$$= \frac{n!}{(n-k)(n-k-1)!\,k!} + \frac{n!}{(n-k-1)!\,(k+1)k!} = \frac{n!(k+1) + n!(n-k)}{\underbrace{(n-k)(n-k-1)!}_{(n-k)!}\,\underbrace{(k+1)k!}_{(k+1)!}} =$$

→ factor $n!$

$$= \frac{n!\,(k+1+n-k)}{(n-k)!\,(k+1)!} = \frac{n!\,(n+1)}{(n-k)!\,(k+1)!} = \frac{(n+1)!}{(n-k)!\,(k+1)!} = \binom{n+1}{k+1} \rightarrow \frac{(n+1)!}{\underbrace{(n-k+1-1)!}_{(n-k)!}\,(k+1)!}$$

6) $\binom{n}{k+1} = \frac{n-k}{k+1} \cdot \binom{n}{k} \rightarrow \binom{n}{k+1} = \frac{n!}{(n-k-1)!\,(k+1)!} \cdot \frac{n-k}{n-k} = \frac{n!\,(n-k)}{(n-k)!\,(k+1)!} =$

$$= \frac{n!\,(n-k)}{(n-k)!\,(k+1)\,k!} = \frac{n-k}{k+1} \cdot \frac{n!}{(n-k)!\,k!} = \frac{n-k}{k+1} \cdot \binom{n}{k}$$

Combinatorics

→ a branch of mathematics which studies finite countable numbers / structures

- **Variations without repetition** – take set A of n-different elements and we choose k-elements from set A (such that $k \leq n$) such that the elements **can't repeat** and the order of the elements **is important**. Each such choice is called the k-th class variation of n-elements.

$$V(k,n) = \frac{n!}{(n-k)!}$$

(= "the total number of the k-th class variation of n-elements without repetition is equal to ...")

→ if $k = n$ $\quad V(n,n) = \frac{n!}{(n-n)!} = n!$ .... PERMUTATIONS WITHOUT REPET. = a special type of variations where $k = n$.

- **Variations with repetition** – .... elements **can repeat** and the order of the elements is important. Each such choice is called the k-th class variation of n-elements with repetition.

$$V'(k,n) = n^k$$

- **Permutations with repetition** – (order is important) – given elements of n-different types. The number of permutations that we form from n_1-elements of the 1^{st} type, n_2-elements of the 2^{nd} type ..., n_k-elements of the k^{th} type is calculated by the formula:

$$P_{n_1, n_2, n_3 \ldots n_k}(n) = \frac{n!}{n_1!\, n_2!\, n_3! \ldots n_k!}$$

↳ $n_1 + n_2 + n_3 \ldots n_k = n$

- **Combinations without repetition** – .... the elements **can't repeat** and **the order** of the elements **isn't important**. Each such choice is called the k-th class combination.

$$C(k,n) = \binom{n}{k} = \frac{n!}{(n-k)!\,k!}$$

- **Combinations with repetition** – .... elements **can repeat** and **the order isn't important**. Each such choice is called the k-th class combination with repetition.

$$C'(k,n) = C(k, n+k-1) = \binom{n+k-1}{k}$$

Examples! V – 10 digits – you have to create code (4-digit) – they can/can't repeat
– total number of license plates in a city
– password in a mobile phone

P – 10 romantic books, 5 sci-fi books & 3 historical → we want to arrange them on the book shelves

C – I want to buy 2 portions of ice cream and there are 10 different flavours to choose from
– team of players from 30 students

STATISTICS

Statistics deals with information from data

It is the science of making effective use of numerical data relating to groups of individuals or experiments.

A **data set** (or **dataset**) is a collection of data; it is a finite set

A dataset has several characteristics which define its structure and properties. These include the number and types of the attributes and the various **statistical measures**:

1. The **arithmetic mean** of a list of elements is the sum of all of the terms divided by the number of elements in the list.

If n-numbers are given, each number denoted by x_i, where $i = 1, 2, \ldots, n$, the arithmetic mean is: equal to
1 over n (where n is number of elements of the set), times the

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$ sum of values x_i where the index i is a natural number

arithm. mean of the data set is = to ...

2. The **mode** is the value that occurs the most frequently in a dataset.

The mode of a data sample {1, 3, 6, 6, 6, 6, 7, 7, 12, 12, 17} is 6.

Given the list of data {1, 1, 2, 4, 4}, the mode is not unique, the data set is said to be bimodal.

stredná hodnota

3. The **median** of a finite list of numbers is found by arranging all the numbers from the lowest value to the highest value and picking the middle. If there is an even number of items, then there is no single middle value, the median is defined to be the arithmetic mean of the two middle values.

smerodajná odchýlka

4. The **standard deviation** is a measure of how spread out a distribution is.
shows how much the variation is from the average

$$s = \sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(x_i - \bar{x}\right)^2}$$ sum of values x_i − arithm. mean whole squared

It shows how much variation there is from the average.

rozptyl

5. The **statistical dispersion** is the square of the standard deviation:

$$s^2 = \frac{1}{n}\sum_{i=1}^{n}\left(x_i - \bar{x}\right)^2$$

štatistický súbor

Ways of picturing the statistical data set:

a) histogram – a graphical display of tabular frequencies, shown as adjacent rectangles

The total area of the histogram is equal to the number of data.

b) frequency polygons – they serve the same purpose as histograms, but are especially helpful in comparing sets of data

c) circle diagram

PROBABILITY

Probability is the measure of how likely an event is. It describes the chance that an event will occur. When solving probability tasks, two conditions have to be considered:

- the number of events is finite
- all the outcomes of an experiment are equally likely

Outcome is the result of an experiment or activity.

Sample space is the set of all possible outcomes of an experiment. - kocka: $S = \{1, 2, 3, 4, 5, 6\}$.

Event is the subset of the sample space relating to an experiment. 1 na kocke ...

For example, suppose that the sample space for an experiment in which a coin is tossed three times is: {HHH, HHT, HTH, THH, HTT, THT, TTH, TTT} and let E = {HHH, HHT, HTH, THH}. Then E is the event in which at least two „heads" are obtained.

LAPLACE SCHEME:

- number of favourable outcomes

$P(E) = \frac{m}{n}$, where **m** is the number of ways that an event can occur and **n** is the total number of outcomes.

- total n. of outcomes

Probabilities range from 0 to 1. If an event is absolutely certain to occur, the probability is 1 and such an event is called a **certain event**.

e.g. *If it is Thursday, the probability that tomorrow is Friday is certain.*

If an event is impossible and will never occur, the probability is 0 and such an event is called an **impossible event**.

e.g. *The probability of getting a number greater than 6 when a die is thrown once.*

Opposite event

If A is an event within the sample space S of an experiment, the complement of A (denoted A´) consists of all outcomes in S that are not in A and it is called an **opposite event**. set of all outcomes that are not included in an A.

P(A´) = 1 – P(A)

e.g. *A: 6 will be tossed at least twice.*

A´: 6 will be tossed at most once.

Mutually exclusive events - "vzájomne sa vylučujúce"

Two events are said to be **mutually exclusive** if the occurrence of one automatically excludes the possibility of the other occurring.

e.g. In tossing a coin, if the outcome is a head this excludes the possibility of obtaining a tail on that trial.

$P(A \cup B) = P(A) + P(B)$

Not – mutually exclusive events - if the occurence of both is probable

Suppose we want the probability of one card selected from a pack being either a heart or a queen. These events **are not** mutually exclusive as the card could be both a heart and a queen (= the queen of hearts)

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Independent events

Two events are said to be independent if the result of the second event is not affected by the result of the first event. If A and B are independent events, the probability of both events occuring is the product of the probabilities of the individual events.

$P(A \cap B) = P(A) . P(B)$

e.g. Rolling 4 on a single 6-sided die and then rolling 1 on a second roll of the die.

Analytic expression of the line in the plane

I. Parametric equations

- given line p and the directional ~~vector~~ $\vec{AB}$ (which is $\parallel$ with the line p & there is another point $X \in p$
→ they are collinear
→ 2 vectors are parallel ⇒ they are lineary dependent
↳ $\vec{AX} \parallel \vec{AB}$

$\vec{AX} = \lambda . \vec{u}$; $\lambda \in R$
↳ "$\vec{AX}$ is a scalar multiple of vector $\vec{u}$."

↳ coordinates of vector AX are calculated by the difference:

$X - A = \lambda . \vec{u}$ $\quad /+A$ ⇒ parametric expression: if we want to write PE we need $A, \vec{u}$ (point from line)

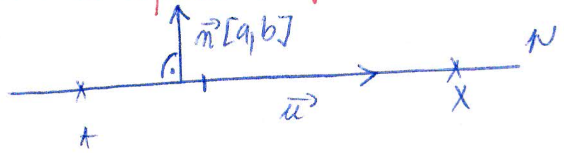
$X = A + \lambda . \vec{u}$, $\lambda \in R$ ... we get PE of the line

$X[x, y]$ $\quad A[a_1, a_2]$, $\vec{u} = [u_1, u_2]$ $\quad$ then PE:

$p: x = a_1 + \lambda u_1$
$\quad y = a_2 + \lambda u_2$; $\lambda \in R$

} these are parametric equations ~~expression~~ ~~equations~~

II. General equation of the line

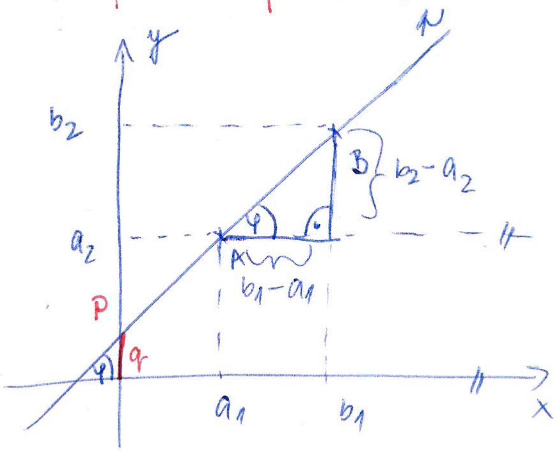


→ line p given by point A and direction vector $\vec{u}$
→ we need normal vector = vector which is $\perp$ to the $\vec{u}$ or the line itself

line segment .... $\lambda \in \langle 0, 1 \rangle$
half line ... $\lambda \geq 0$

$p: ax + by + c = 0$ $\quad$ a, b are coordinates of $\vec{n}$
$\quad$ x, y — " — $X \in p$
$\quad$ $c \in R$ ... absolute term

III. Slope-intercept form



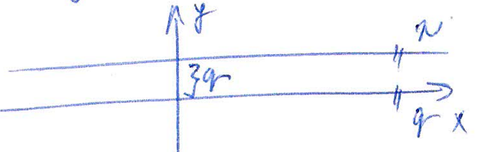
φ... angle between the line and axis x ... $\varphi = \measuredangle(p, x)$
↳ we can find the same angle if we draw a parallel line with axis x
↳ triangle is $\triangle$, we use trig. ratio for tangent

$$tg\,\varphi = \frac{b_2 - a_2}{b_1 - a_1} = k \text{ ... "slope of the line"}$$

$p: y = kx + q$ $\quad$ q... distance between origin and point of intersection of line & y-axis ... P $\quad q = |0, P|$
φ... directional angle

tangent of the angle between p & x-axis

Note: if $p \parallel$ x-axis

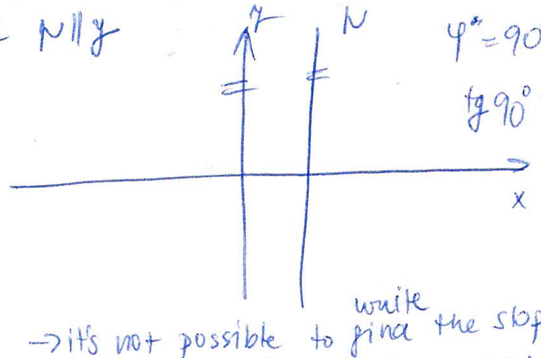


$\varphi = 0°$
$tg\, 0° = 0$

constant linear function = f. in which slope intercept is 0

$p: 0 . x + q = y$
$\quad y = q$

if $p \parallel y$



$\varphi = 90°$
$tg\, 90° = \nexists$

→ it's not possible to ~~find~~ write the slope intercept form bcs tangent is not defined at 90°

Relations among the coefficients of General equation

$p: a_1 x + b_1 y + c_1 = 0$

$q: a_2 x + b_2 y + c_2 = 0$

If $p \equiv q$ (coincident) $\Rightarrow \underbrace{[a_1, b_1]}_{\text{normal vector}} = \underbrace{k \cdot [a_2, b_2]}_{\text{multiplied by the same scalar}} \wedge c_1 = k \cdot c_2 \qquad k \in R \setminus \{0\}$

e.g. $p: 2x + 5y - 3 = 0 \quad \vec{n}_p = [2, 5] \quad c_1 = 3$

$q: -6x - 15y + 9 = 0 \quad \vec{n}_q = [-6, -5] \quad c_2 = 9$

$k = -3$ } they are coincident

If $p \parallel q \Rightarrow [a_1, b_1] = k \cdot [a_2, b_2] \wedge c_1 \neq k \cdot c_2$

e.g. $p: 2x + 5y - 3 = 0$

$q: -6x - 15y + 1 = 0$

} only parallel

If $p \perp q$ (perpendicular) $\rightarrow$ the scalar product of their normal vectors is 0

$[a_1, b_1] \cdot [a_2, b_2] = 0$

$\vec{n}_p \cdot \vec{n}_q = 0$

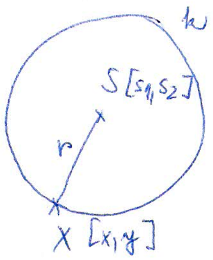
$\rightarrow$ in space: only parametric equation

Circle

- set of points which are equidistant from the fixed point = centre of the circle
$k = \{X \in E_2 : |SX| = r\}$ r... radius

- set of points in the plane which are the same or smaller distance from the centre of the O
$K = \{X \in E_2 : |SX| \leq r\}$

Analytic expression



$|SX| = r$ $\vec{SX} = [x - s_1, y - s_2]$

$\hookrightarrow r = \sqrt{(x-s_1)^2 + (y-s_2)^2} \; /^2$

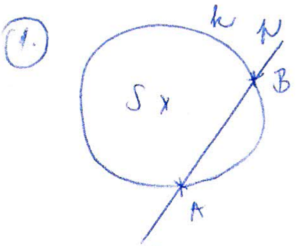
$\boxed{r^2 = (x-s_1)^2 + (y-s_2)^2}$ ⇒ central equation of the circle

$r^2 = x^2 - 2xs_1 + s_1^2 + y^2 - 2ys_2 + s_2^2$

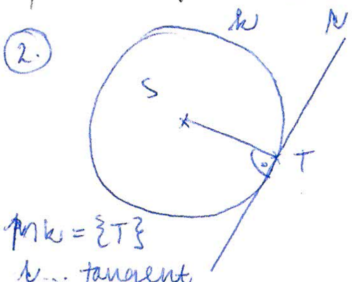
$0 = x^2 + s_1^2 \underbrace{-2xs_1}_{a} \underbrace{-2ys_2}_{b} + \underbrace{y^2 + s_2^2 - r^2}_{c}$ $a, b, c \in R$

$\boxed{0 = x^2 + y^2 + ax + by + c}$ ⇒ general equation of the circle

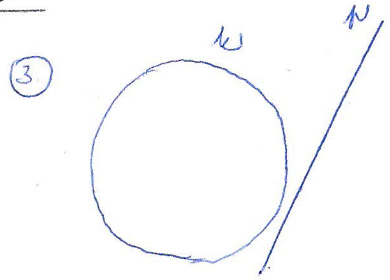
Mutual position of O and a line



$p \cap k = \{A, B\}$
p... secant
(the line intersects/ cuts accross)



$p \cap k = \{T\}$
t... tangent
T... point of contact
→ radius at the T and the tangent form the right ∢



$p \cap k = \emptyset$
p... neither secant nor tangent

$p: a_1x + b_1y + c_1 = 0$
$k: x^2 + y^2 + a_2x + b_2y + c_2 = 0$

} system of 2 equations: it is non-linear substitution, we express 1 of the variables from the eq. of the line and substitute it into the equation of a O ⇒ we get a quadratic equation.
if $D < 0$ ... 3.
if $D = 0$ ... line is tangent
if $D > 0$ ... secant

Conic section: a section which is made by cutting the cone by the plane which is ∥ with the base of the cone

Find the value of the $q \in R$ so that the line $r: x+q$ is tangent to the circle $k: x^2+y^2-10x-12y+53=0$.

$$x^2+(x+q)^2-10x-12(x+q)+53=0$$

$$x^2+x^2+2xq+q^2-10x-12x-12q+53=0$$

$$2x^2+x(2q-22)+q^2-12q+53=0$$

$$D=(2q-22)^2-4\cdot 2(q^2-12q+53)\cancel{=0}$$

$$D=4q^2-88q+484-8q^2+96q-424$$

$$D=-4q^2+8q+60=0 \quad /:(-4)$$

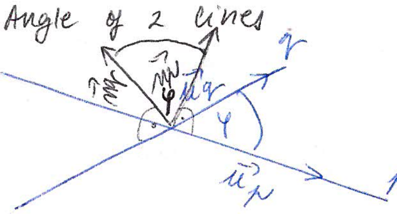
$$q^2-2q-15=0$$

$$(q+3)(q-5)=0$$

$$q_1=-3 \qquad q_2=5$$

Metric relations

1) Angle of 2 lines



→ given lines p, q → intersecting lines
→ φ... acute angle
→ we have $\vec{u}_p$ & $\vec{u}_q$, then the ∡ between them is equal to:

$$\boxed{\cos\varphi = \frac{|\vec{u}_p \cdot \vec{u}_q|}{|\vec{u}_p| \cdot |\vec{u}_q|}}$$

= "absolute value of scalar product of dir. vectors"
= "magnitudes of the directional vectors"

- there is absolute value bcs the angle of 2 lines is always of acute angle → it means that it will be from the 1^{st} quadrant

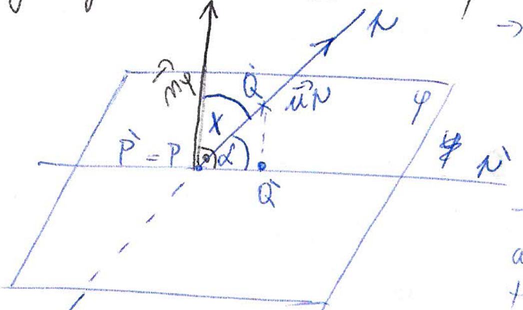
$$\cos\varphi = \frac{|\vec{m}_p \cdot \vec{m}_q|}{|\vec{m}_p| \cdot |\vec{m}_q|}$$

$$\varphi = ∡(p, q) = ∡(\vec{u}_p, \vec{u}_q) = ∡(\vec{m}_p, \vec{m}_q)$$

↳ it is ~~angle~~ between ... and also ...

2) Angle of the line and the plane

→ the line cuts accross the plane @ point $P \in \varphi, p$
→ to find the angle ~~wee~~ need to find orthogonal projection of line to the plane



→ p'... orthogonal / perpendicular projection of line p onto the plane $\varphi(\rho\varphi)$
→ α ... angle between line p & plane φ is in fact the angle between line p and its orthogonal projection to the plane

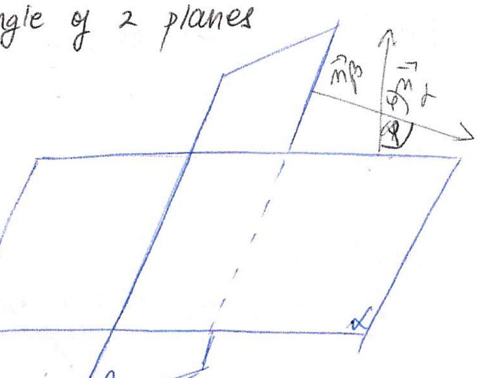
→ normal vector is ⊥ to φ and p as well
→ X... complementary angle $(\frac{\pi}{2} - \alpha)$ → angle between $\vec{m}_\varphi$ and $\vec{u}_p$

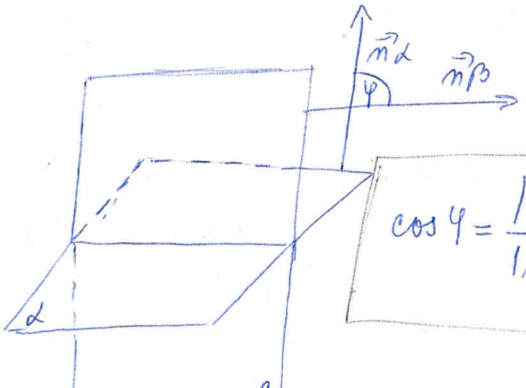
$$\boxed{\underbrace{\cos\left(\frac{\pi}{2} - \alpha\right)}_{\sin\alpha} = \frac{|\vec{u}_p \cdot \vec{m}_\varphi|}{|\vec{u}_p| \cdot |\vec{m}_\varphi|}}$$

$$\varphi = ∡(p, \varphi) = ∡(p, p')$$

$\sin\alpha$ (ak posunieme $\cos\alpha$ o 90° doľava dostaneme $\sin\alpha$)

3) Angle of 2 planes





$$\boxed{\cos\varphi = \frac{|\vec{m}_\alpha \cdot \vec{m}_\beta|}{|\vec{m}_\alpha| \cdot |\vec{m}_\beta|}}$$

PERPENDICULARITY OF THE LINES & PLANES (7 axioms)

A_1: If the line p is perpendicular ($\perp$) to two intersecting lines of the plane α then it is ⊥ to the plane ($p \perp \alpha$)

A_2: There is exactly one line which is ⊥ to the plane and passes through the point which is not from the plane

A_3: All lines which are perpendicular to the same plane are parallel

A_4: There is exactly one plane which is ⊥ to the line and passes through the point which is not from the line

A_5: All planes that are perpendicular to the same line are parallel

A_6: If the line p is not ⊥ to the plane α, then there is exacty one plane β which passes through the line p and is ⊥ to α $(p \subset \beta)$
↳ subset (lies in β)

A_7: Given planes α, β and line p from α ($p \subset \alpha$). If the line p is $\perp$ to the plane β then α is also $\perp$ to β

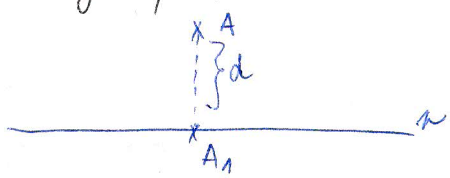
$\boxed{\text{Distance of 2 points}}$ – d. o. the points A and B is the length of the line segment $\overline{AB}$

→ 2 points in the plane (ak v space → $A[a_1, a_2, a_3]$; $B[a_1, a_2, a_3]$ a porom $\sqrt{\ldots + (b_3 - a_3)}$)

$|A, B| = |\overrightarrow{AB}| = \sqrt{(b_1-a_1)^2 + (b_2-a_2)^2}$

↳ magnitute of the vector AB

Distance of a point and a line



→ is in fact length of the line segment AA_1 where A_1 is the orthogonal projection of the point A onto the line

→ distance of A from its o.p. onto the line

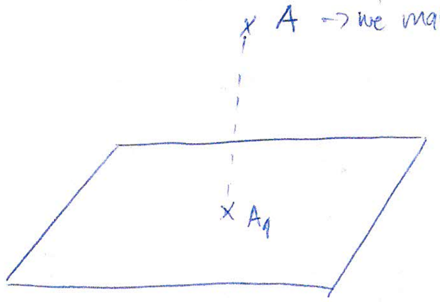
$d(A, p) = |A, A_1|$ ... A_1 ... perp. projection

- If coordinates of A are $[a_1, a_2]$ & there is GE of a line p: $0 = ax + by + c$

$d(A, p) = \dfrac{|a_1 \cdot a + b \cdot a_2 + c|}{\sqrt{a^2+b^2}}$ → absolute value (bcs the distance can't be negative)

→ magnitude of the normal vector of a line

Distance of a point and a plane



A → we make $\perp$ projection of a point to the plane

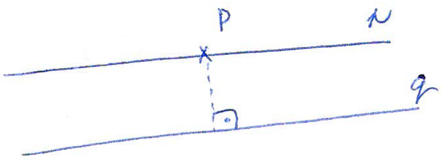
→ ... onto the plane

$d(A, \alpha) = |AA_1|$

- $A[a_1, a_2, a_3]$ $\alpha: ax + by + cz + d = 0$

$d(A, \alpha) = \dfrac{|a \cdot a_1 + b \cdot a_2 + c \cdot a_3 + d|}{\sqrt{a^2+b^2+c^2}}$ → mag. of $\vec{n}_\alpha$

Distance of parallel lines ⇒ is the distance of point from a line



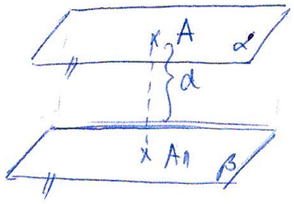
$d(p, q) = d(P, q)$; $P \in p$

→ if p, q are given by general equations:

p: $ax + by + c_1 = 0$ q: $ax + by + c_2 = 0$ → only absolute terms are different

$d(p, q) = \dfrac{|c_2 - c_1|}{\sqrt{a^2+b^2}}$

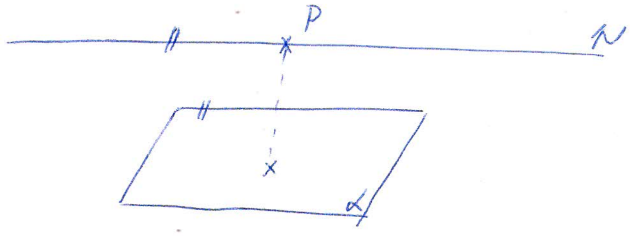
Distance of 2 parallel planes → is the distance of point $\in \alpha$ and its perpendicular projection onto the plane β $d(\alpha, \beta) = |AA_1|$



$\alpha: ax + by + cz + d_1 = 0$ $\beta: ax + by + cz + d_2 = 0$

$d(\alpha, \beta) = \dfrac{|d_2 - d_1|}{\sqrt{a^2+b^2+c^2}}$

Distance of a line from a plane which is ∥ with the line



→ distance of a point from the line and its orthogonal projection onto the plane

SYSTEMS (=sústavy)

"by"
2x2 System = the system of 2 linear equations in 2 variables
- solution set: the ordered pair [ ]
- methods for solving: 1) elimination →
$$2x+y-3=0$$
$$5x-y+5=0$$
$$7x=-2$$
$$x=-\frac{2}{7}$$
we ~~eliminate~~ (have to remove) 1 of the variables from both equations & then solve linear equation in 1 variable (we sum them up)

→ then we substitute x to $1^{st}/2^{nd}$ equation ⇒ y

$S=\{[-\frac{2}{7}; ...]\}$

2) substitution
3) graphical

2) we express 1 variable from 1 equation and substitute it into the other equation

$2x+y-3=0$ ... $y=3-2x$

$5x-y+5=0$ ← (we substitute the binomial

$5x-3+2x+5=0$

⋮

3) ~~It's in fact~~ The 2 equations represent 2 lines in the plane (E_2) so there are 3 positions:

parallel

$2x+3y=1$
$2x+3y=5$
$0=-4$

$S=\emptyset$

the system is contradiction
↳ solution set is empty set

lines coincide

$2x+3y=1 \Rightarrow x=\frac{1-3y}{2}$
$2x+3y=1$
$0=0$

↳ system has infinite number of solutions: we express one variable

$S=\{[\frac{1-3y}{2}; y]\}$

they depend on eachother

lines intersect

$2x+y=3$
$x-y=5$
$3x=8$
$x=\frac{8}{3}$

$y=-\frac{4}{3}$

$S=\{[\frac{8}{3}; -\frac{7}{3}]\}$ ←

↳ when we sketch graphs of the lines they will intersect at point

3x3 System - the system of 3 linear equations in 3 variables
- solution set: ordered triplet
- methods of solving: we express 1 variable from 1 equation and substitute the variable into the 2 remaining equations → we get a 2 by 2 system
- triangular form, determinance

→ the system of a linear and quadratic equation is called a non-linear system and it is solved by substitution method

$x-2y+5=0$
$2x^2-3x+5y-7=0$

⋮