Tutorials for Programming Quantum Computers

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Preliminary information

This document serves as an exercises sheet for the *Základy programovania kvantových počítačov (B-ZPKP - ZS 2020/2021 - FEI)* and *Kvantové počítanie (KVATP_I - ZS 2020/2021 - FIIT)* courses. Its main aim is to drill you the basics of mathematical computations used in the course. The mathematics used here is not difficult, but during the course you will encounter using it very often. That is why it is important to learn it to such extent that you will be using it without thinking. To be able to do that, we urge you to go through all the exercises.

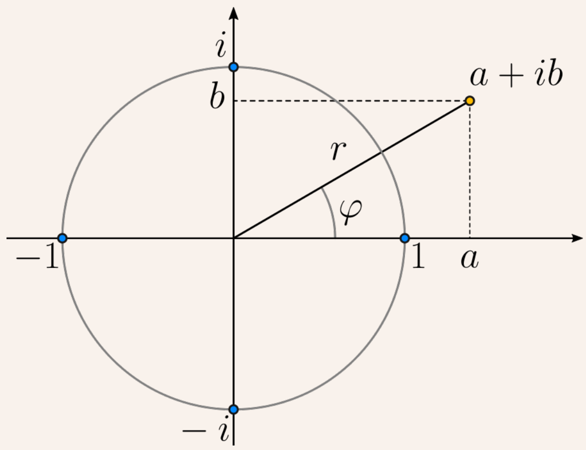
The structure of the document should roughly reflect that of the course progress, but being slightly in advance of the course (especially at the beginning) in order to prepare you for the lectures, which will be easier to comprehend. In between exercises in the yellow boxes you can find some short explanations of used concepts. As said, the exercises are not difficult, but there are some that are more *fun*. These are marked by * symbol. Also for better navigation, or to point you to some particular interesting or important exercises, we include short snippets of text starting by $\triangleright$ symbol.

1 Complex numbers

We can imagine complex number as a pair of real numbers $z = (a, b)$, where a is called its *real part* and b its *imaginary part*. We can perform addition and multiplication of complex numbers:

$$(a,b) + (c,d) = (a+c, b+d), \qquad (a,b)(c,d) = (ac-bd, bc+ad). \tag{1}$$

As a pair of real numbers, complex numbers can be visualized in a plane (called *the complex plane*).



The usual way to write complex numbers is $z = a + ib$, where i determines the imaginary part. This form of writing is just a rewriting of previous pair description. But i here has a deeper meaning. It is not just a symbol for the second dimension, but it stands for $i = \sqrt{-1}$; this is a result of necessity stemming from our need to be able to solve quadratic equations. The definition implies that $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, and so on.

Having this form of quantum numbers, we can confirm Eq. (1):

$$(a+ib) + (c+id) = (a+c) + i(b+d),$$
$$(a+ib)(c+id) = ac + ibc + iad + i^2 bd = (ac-bd) + i(bc+ad).$$

Having a number $z = a + ib$ we call the number $z^* = a - ib$ its *complex conjugate*. This is an important notion as

$$z^* z = (a-ib)(a+ib) = a^2 + b^2 = |z|^2,$$

where $|z|$ is the norm of the complex number. Useful identity for norm is $|w \cdot z| = |w| \cdot |z|$, where w and z are two complex numbers.

There are three widely used forms for complex numbers:

▷ Cartesian form: $z = a + ib$,
▷ trigonometric form: $z = r(\cos\phi + i\sin\phi)$, $r \geq 0$,
▷ polar form: $z = re^{i\phi}$.

Polar and trigonometric forms are essentially the same, because of the identity

$$e^{i\phi} = \cos\phi + i\sin\phi,$$

and both these forms have the same set of parameters:

▷ **norm:** $r = |z|$, which is always a non-negative number, and
▷ **argument:** ϕ which is usually taken to be from interval $\phi \in [0, 2\pi]$.

Considering the complex plane, if we have two numbers $e^{i\phi}$ and $e^{i(\phi+2\pi)}$, we see that they correspond to the sam point and so they are the same number. This implies that the argument is defined up to a period of 2π. Therefore the choice of interval for ϕ is up to us, and sometimes you can see $\phi \in [-\pi, \pi]$ as well.

Translations between the forms:

- ▷ from Cartesian to trigonometric and polar: r is simply the norm of the number z, $r = |z|$, and $\tan\phi = b/a$, with ϕ such that if $a < 0$, it gets additional π as a correction since tan has only period of π, half of the period for the argument. When determining the argument, one need to keep this in mind, or one can make use of standardized 2-argument arctangent function in many calculators and analytic computer programs.
- ▷ from polar or trigonometric to Cartesian: this is a simple conversion, as the trigonometric form already reads $z = r\cos\phi + i(r\sin\phi)$ and so $a = r\cos\phi$, $b = r\sin\phi$.

Exercise 1.1. Evaluate following complex numbers and present their Cartesian form

a) $(7+3i)+(2+i)$

b) $(2-3i)+(4+2i)$

c) $(5+3i)+(2+0i)-(1+3i)$

d) $(3+7i)+(4-2i)-(7+5i)$

Exercise 1.2. Evaluate following complex numbers and present their Cartesian form

a) $(3-5i)(0+2i)$

b) $(3+7i)(4+i)$

c) $(2+i)(2-i)$

d) $(2-3i)(3-2i)$

e) $(3-2i)^2$

f) $(1+2i)^5$

g)* $\displaystyle\sum_{k=1}^{4}(1+i)^k$

▷ Quite often one ends up with a fraction with a complex number in denominator. To get rid of the number, we remember that the sqare of the norm of the complex number is a real number and so we make it in denominator. Let us say the denominator is z. Then we multiply the whole fraction by 1 in form of a fraction z^*/z^*.

Exercise 1.3. Find the norm of these complex numbers

a) $\frac{1}{\sqrt{2}}(1+i)$

b) $3i$

c) $e^{i\pi/4}$

d) $2+i$

e) $2+i\sqrt{5}$

f) $\sqrt{3}e^{i3\pi/8}$

Exercise 1.4. Evaluate following complex numbers and present their Cartesian form

a) $\dfrac{1}{1-i}$

b) $\dfrac{1+i}{3-2i}$

c) $\dfrac{2-3i}{3+2i}$

d) $\dfrac{1-i}{3+2i}-\dfrac{3i}{3-2i}$

e)* $\displaystyle\sum_{k=1}^{4}\frac{1+ki}{1-ki}$

f)* $\displaystyle\prod_{k=1}^{3}\frac{1+ki}{1-ki}$

Exercise 1.5. Convert the numbers between the Cartesian form and polar form

a) $e^{i\pi}$

b) $2e^{-i\pi/2}$

c) $-e^{i\pi/3}$

d) $1\pm i\sqrt{3}$

e) $(1+i)^{12}$

f)* $\left(\dfrac{1-i}{1+i}\right)^7$

Exercise 1.6. Having two complex numbers w and z show following a $z + z^* = 2\Re z$ and $z - z^* = 2i\Im z$, where $\Re z$ and $\Im z$ denote the real and imaginary parts of z respectively,

b $(w \cdot z)^* = w^* \cdot z^*$,

c $|w \cdot z| = |w| \cdot |z|$.

There are cases when we need to compute root(s) of a complex number z, or rather solve equation $z^a = q$ for z, where q and a are given. In this case we express q in the polar form with extra phase $q = |q|e^{i(\theta+2\pi k)}$, where θ is the argument of q. Note that the extra phase does not change the value of number, but plays crucial role in the next step, where we take the a-th root and obtain general solution

$$z = \left[|q|e^{i(\theta+2\pi k)}\right]^{1/a} = |q|^{1/a} e^{i\left(\frac{\theta}{a}+\frac{2\pi k}{a}\right)}.$$

While k is an arbitrary integer, the solutions for k and $k + a$ give the same number and so we can reduce these cases to just $k \in \{0, 1, \dots, a-1\}$.

The case of $a = 2$ simplifies a bit, as in that case the multiple arguments boil down to just $\pm$ sign and the result can be concisely written as $z = \pm\sqrt{|q|}e^{\frac{i\theta}{2}}$.

Exercise 1.7. Find all values of the square root of x, i.e. is such a number z that $z^2 = x$, where

a) $x = \sqrt{4}$ b) $x = \sqrt{1}$ c) $x = \sqrt{-2}$

Exercise 1.8. Find a-th roots of q, i.e. find z if you know $z^a = q$, where

a) $a = 3, q = 1$

b) $a = 2, q = i$

c) $a = 4, q = -i$

d) $a = 2, q = 1 \pm i\sqrt{3}$

Exercise 1.9. Find all b's such that $|a|^2 + |b|^2 = 1$, if

a) $a = \dfrac{1}{\sqrt{2}}$

b) $a = i$

c) $a = \dfrac{1}{\sqrt{N}}$, where N is a parameter

d) $a = \dfrac{1}{3+4i}$

Exercise 1.10. Solve following equations:

a) $z^2 - 2i = 0$

b) $z^3 - 1 = 0$

c)* $z^2 - z^* = 0$

d)* $(1-i)z^2 + 2iz - 1 = 0$

Results.
1.1 a) $9 + 4i$, b) $6 - i$, c) 6, d) 0.
1.2 a) $10 + 6i$, b) $= 5 + 31i$, c) 5, d) $-13i$, e) $= 5 - 12i$, f) $41 - 38i$, g) $-5 + 5i$.
1.3 a) 1, b) 3, c) 1, d) $\sqrt{5}$, e) 3, f) $\sqrt{3}$.
1.4 a) $\frac{1+i}{2}$, b) $\frac{1+5i}{13}$, c) $-i$, d) $\frac{7-14i}{13}$, e) $\frac{-194+244i}{85}$, f) 1.
1.5 a) -1, b) $-2i$, c) $-\frac{1}{2}(1 + i\sqrt{3})$, d) $2e^{\pm i\frac{\pi}{3}}$, e) -64, f) $e^{i\frac{\pi}{2}}$.
1.7 a) $z = \pm 2$, b) $z = \pm 1$, c)$z = \pm i\sqrt{2}$.
1.8 a) 1, $e^{\pm\frac{2\pi i}{3}} = \frac{1}{2}(-1 \pm i\sqrt{3})$, b) $\pm e^{i\pi/4} = \pm\frac{1}{\sqrt{2}}(1+i)$, c) $\pm\cos(\pi/8) \mp i\sin(\pi/8)$, $\pm\sin(\pi/8) \mp i\cos(\pi/8)$, d) $\sqrt{1 \pm i\sqrt{3}} = \pm\sqrt{2}e^{\pm i\frac{\pi}{6}}$.

1.9 All solutions are of the form $b = |b|e^{i\phi}$ for $\phi \in [0, 2\pi]$ and $|b|$ being equal to a) $\frac{1}{\sqrt{2}}$, b) 0, c) $\sqrt{\frac{N-1}{N}}$, d) $\frac{2\sqrt{6}}{5}$.

1.10 a) $z = \pm(1+i)$, b) 1, $e^{\pm\frac{2\pi i}{3}} = \frac{1}{2}(-1 \pm i\sqrt{3})$, c) $z = 0$, $z = 1$, $z = \frac{1}{2}(-1 \pm i\sqrt{3}) = -e^{\pm i\pi/3}$, d) $z = \frac{1}{2}(\pm\sqrt{2} + 1 - i)$.

2 Matrices and their properties

In this course we will rely heavily on matrices. Although they do not allow the most general description of quantum theory and its use, matrices are at the core of the finite-dimensional quantum theory as a way of representing it. As we will be oblivious to the infinities, matrices will be all that we will use here.

Formally *matrices* are rectangular arrays of numbers, e.g.

$$\begin{pmatrix} 3+i & -1 & 0 \\ 2 & 4i & -3+2i \end{pmatrix}$$

is complex matrix of dimension 2×3. Concerning their sizes, there are matrices we will use more often: matrices having dimensions $n \times n$ are called *square matrices*, matrices of dimensions $1 \times n$ (having only one row) and of dimensions $n \times 1$ (having only one column) will represent different kinds of *vectors*. We can also consider matrices of dimension 1×1 which is just a scalar number.

To add two matrices A and B, they have to be of the same size. Their addition is performed element-wise, i.e. $C = A + B$ is such a matrix that for every position (i, j) its element is $c_{ij} = a_{ij} + b_{ij}$. Multiplication is slightly more difficult.

Assume we have two matrices of dimensions $m \times n$ and $n \times k$

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1k} \\ b_{21} & b_{22} & \dots & b_{2k} \\ \vdots & & \ddots & \vdots \\ b_{n1} & b_{n2} & \dots & b_{nk} \end{pmatrix}.$$

Then we can define the matrix product between A and B by $AB = C$, where

$$C = \begin{pmatrix} c_{11} & c_{12} & \dots & c_{1k} \\ c_{21} & c_{22} & \dots & c_{2k} \\ \vdots & & \ddots & \vdots \\ c_{m1} & c_{m2} & \dots & c_{mk} \end{pmatrix},$$

with $c_{ij} = \sum_{x=1}^{n} a_{ix} b_{xj}$; one can remember this as taking i-th row from matric A and j-th column from matrix B and summing products of corresponding elements. Note that in general the product BA is not defined, so the matrix product is not commutative.

The crucial property for our purpose is that matrix product is associative, i.e. having three matrices A, B, and C of proper dimensions, $(AB)C = A(BC)$. This can be generalized for any number of matrices, in example having 5 matrices $ABCDE$ we can put parenthesis in arbitrary places:

$$ABCDE = (AB)((CD)E) = (A((BC)D))E = \dots$$

When a particular operation is associative $(+, \times)$, we usually do not write parentheses at all. However, as we will see later, they will allow us different interpretations of sequences of matrix products, and the right choice of ordering can make the calculations easier.

Exercise 2.1. Compute the following products of matrices:

a) $\begin{pmatrix} 4 & -1 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}$

b) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

c) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & -2 & 4 \\ 3 & 2 & 1 \\ 0 & 5 & -3 \end{pmatrix}$

d) $\begin{pmatrix} 3 & 1 & 0 \\ 2 & -3 & 4 \\ 1 & 2 & -1 \end{pmatrix} \begin{pmatrix} 2 & 1 & -1 \\ 0 & -1 & 2 \\ 4 & 1 & 2 \end{pmatrix}$

Exercise 2.2. Using matrices

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 3 \\ -1 & 2 \end{pmatrix}, \qquad C = \begin{pmatrix} 2 & 4 \\ 1 & -1 \end{pmatrix},$$

show by direct computation that $(AB)C = A(BC)$.

▷ In the next example note the case a) and observe what happens to the middle matrix.

Exercise 2.3. Compute the following products of matrices:

a) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$,

b) $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$,

c) $\frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$.

2.1 Useful terminology

As noted before, we will be particularly interested in products of matrices, where at least one of the dimension is equal to one.

The first case is (scalar or inner) product of matrices of dimensions $1 \times n$ and $n \times 1$:

$$\begin{pmatrix} a_1 & a_2 & \dots & a_n \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = \sum_{x=1}^{n} a_x b_x$$

that results into a matrix of dimension 1×1, a scalar number.

The second case is (outer) product of matrices of dimensions $m \times 1$ and $1 \times k$:

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{pmatrix} \begin{pmatrix} b_1 & b_2 & \dots & b_k \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} & \dots & c_{1k} \\ c_{21} & c_{22} & \dots & c_{2k} \\ \vdots & & \ddots & \vdots \\ c_{m1} & c_{m2} & \dots & c_{mk} \end{pmatrix},$$

where $c_{ij} = a_i b_j$. Such a product of two vectors creates matrix of dimension $m \times k$.

For clarity, we might sometimes put commas between elements in single row matrices, for example

$$\begin{pmatrix} a_1 & a_2 & \dots & a_n \end{pmatrix} = \begin{pmatrix} a_1, & a_2, & \dots, & a_n \end{pmatrix}.$$

▷ Observe carefully the case a), as these types of matrices will be used later to define the *computational base*.

Exercise 2.4. Compute the inner product AB and outer product BA of matrices A and B, where

a) $A = \begin{pmatrix} 1, & 0, & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$,

b) $A = \begin{pmatrix} 1, & 2, & i \end{pmatrix}$, $\quad B = \begin{pmatrix} i \\ 2-i \\ 3 \end{pmatrix}$,

c) $A = \begin{pmatrix} e^{i\omega}, & e^{-i\theta}, & 2 \end{pmatrix}$, $\quad B = \begin{pmatrix} e^{-i\omega} \\ e^{i\theta} \\ -\frac{1}{2} \end{pmatrix}$.

There are many operations which perform changes on matrices. We will often use *transposition* and *conjugate transposition*. Given matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix},$$

its *transpose* is matrix A^T, T indicating the transposition, such that

$$A^T = \begin{pmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{m2} \\ \vdots & & \ddots & \vdots \\ a_{1n} & a_{2n} & \dots & a_{mn} \end{pmatrix}.$$

The transposition simpli flips the element around diagonal. A *conjugate transpose* [or *(Hermitian) adjoint*] of matrix A is matrix $A^\dagger$, $\dagger$ indication the conjugate transposition, which is transposed and all its elements are complex conjugated, i.e.

$$A^\dagger = \begin{pmatrix} a^*_{11} & a^*_{21} & \dots & a^*_{m1} \\ a^*_{12} & a^*_{22} & \dots & a^*_{m2} \\ \vdots & & \ddots & \vdots \\ a^*_{1n} & a^*_{2n} & \dots & a^*_{mn} \end{pmatrix}.$$

Note, that while A had dimensions $m \times n$, both A^T and $A^\dagger$ have dimensions $n \times m$.

Trace of a square matrix A, denoted $Tr(A)$, is the sum of diagonal elements of A, i.e.

$$Tr(A) = Tr \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} = \sum_{i=1}^{n} a_{ii}.$$

Exercise 2.5. Write the transpose and conjugate transpose of following matrices

a) $A = \begin{pmatrix} 1 & 2i \\ -3 & 1+i \end{pmatrix}$

b) $B = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

c) $C = \begin{pmatrix} 1-i, & 2i, & 3 \end{pmatrix}$

d) $D = \begin{pmatrix} -2i \\ -3 \\ 0 \\ 1+i \end{pmatrix}$

▷ In the next example cases c) and d) lead to a specific type of result. Such matrices will be very important later, so keep note of them.

Exercise 2.6. Compute product $A^\dagger A$ where

a) $A = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix}$

b) $A = \begin{pmatrix}1-i\\ 2\\ \sqrt{2}i\end{pmatrix}$

c) $A = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$

d)* $A = \begin{pmatrix}1 & -i & -1+i\\ i & 1 & 1+i\\ 1+i & -1+i & 0\end{pmatrix}$

Exercise 2.7. Compute following traces of matrices

a) $\begin{pmatrix}3 & -3 & -5\\ -2 & 1 & 2\\ 0 & 4 & 5\end{pmatrix}$ b) $\begin{pmatrix}1-x & 1\\ 1 & -1-x\end{pmatrix}$ c) $\begin{pmatrix}-r & t & t\\ t & -r & t\\ t & t & -r\end{pmatrix}$

▷ You should do the next exercise, as the results are very important. Especially observe the form $AA^\dagger - BB^\dagger$ that leads to these results, as it will have a deeper meaning in later parts.

Exercise 2.8. Compute products $A^\dagger A$, $B^\dagger B$ and $AA^\dagger - BB^\dagger$ where

a) $A = \begin{pmatrix}1\\ 0\end{pmatrix}, B = \begin{pmatrix}0\\ 1\end{pmatrix}$

b) $A = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 1\end{pmatrix}, B = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -1\end{pmatrix}$

c) $A = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix}, B = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -i\end{pmatrix}$

Matrices from the last exercise are very important. Please take note of both the matrices A and B and the solutions

$$\sigma_x = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \qquad \sigma_y = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}, \qquad \sigma_z = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.$$

These matrices are called *Pauli matrices* and we will use them plenty. Another important matrix (not only dimension 2) is the *identity matrix* having ones on the diagonal and zeroes everywhere else. In dimension 2 it reads

$$\mathbb{1} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}.$$

2.2 Determinant and inverse of a matrix

Determinant of a square matrix is a scalar number that has a number of important applications. For now we will restrict ourselves to how to calculate the determinant.

For a 2×2 matrix it is simply

$$\det\begin{pmatrix}a & b\\ c & d\end{pmatrix} = ad - bc.$$

Determinants of bigger matrices are calculated using a recursive rule. We choose an arbitrary row (column) and perform decomposition of the matrix along this row (column).

For a general $n \times n$ matrix A the formula reads

$$\det(A) = \sum_{j=1}^{n}(-1)^{i+j}a_{ij}\det(A_{ij}) \text{ (for a fixed i that specifies the row), or}$$

$$\det(A) = \sum_{i=1}^{n}(-1)^{i+j}a_{ij}\det(A_{ij}) \text{ (for a fixed j that specifies the column),}$$

where A_{ij} are $(n-1)\times(n-1)$ submatrices of A, which are obtained from the original one by removing i-th row and j-th column.

For a 3×3 matrix A expanded along the first row this formula gives us

$$\begin{aligned}\det(A) &= \det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33}\end{pmatrix} \\ &= (-1)^{1+1}a_{11}\det\begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33}\end{pmatrix} + (-1)^{1+2}a_{12}\det\begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33}\end{pmatrix} + (-1)^{1+3}a_{13}\det\begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32}\end{pmatrix} \\ &= \quad a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ &\quad - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}.\end{aligned}$$

Exercise 2.9. Compute following determinants

a) $\det\operatorname{diag}(a_1, a_2, \dots, a_d)$, where diag stands for a diagonal matrix with given elements on the diagonal,

b) $\det\begin{pmatrix} 3 & -3 & -5 \\ -2 & 1 & 2 \\ 0 & 4 & 5\end{pmatrix}$,

c) $\det\begin{pmatrix} 1-x & -1 \\ 1 & -1-x\end{pmatrix}$, where x is a parameter,

d) $\det\begin{pmatrix} -r & t & t \\ t & -r & t \\ t & t & -r\end{pmatrix}$; provide a general solution and then evaluate for $t=2, r=-2$ and $t=2, r=4$.

Having a square matrix A, in many cases we can find such matrix A^{-1} that $AA^{-1} = \mathbb{1}$. Matrix A^{-1} is called *inverse of matrix A*. It can be shown that in these cases it does not depend on the order of A and A^{-1} and so $AA^{-1} = A^{-1}A = \mathbb{1}$. Similarly to computing the determinant, there is a general formula for finding A^{-1},

$$A^{-1} = \frac{1}{\det(A)}C^T,$$

where C is the *cofactor matrix* whose elements are $C_{ij} = (-1)^{i+j}\det(A_{ij})$, with A_{ij} being the matrix that is obtained from A by removing i-th row and j-th column. We will not be using this formula extensively, for us the most important point to remember is the existence of an inverse matrix.

Exercise 2.10. Show that Pauli matrices are inverses to themselves, i.e. show that $\sigma_x^{-1} = \sigma_x$, $\sigma_y^{-1} = \sigma_y$ and $\sigma_z^{-1} = \sigma_z$.

▷ In the next example note cases a) and d) in particular, as these, again, foreshadow a very important class of matrices. Try to see how their inverses relate to the original matrices.

Exercise 2.11. Find inverses to the following matrices:

a) $H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1\end{pmatrix}$

b) $A = \begin{pmatrix} 1 & 2 \\ 3 & 4\end{pmatrix}$

c) $Q = \begin{pmatrix} 1 & 2 \\ -i & -2i\end{pmatrix}$

d)* $U = \frac{1}{2\sqrt{2}}\begin{pmatrix} \sqrt{2} & -\sqrt{3}(1+i) \\ \sqrt{3}(1-i) & \sqrt{2}\end{pmatrix}$

Remember that matrices for which $U^{-1} = U^\dagger$ will be very important to us and they are called *unitary matrices*.

Results.

2.1 a) $\begin{pmatrix} 8 & 1 \\ -4 & 7 \end{pmatrix}$, b) $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}$, c) $\begin{pmatrix} 1 & -2 & 4 \\ 6 & 4 & 2 \\ 0 & 15 & -9 \end{pmatrix}$, d) $\begin{pmatrix} 6 & 2 & -1 \\ 20 & 9 & 0 \\ -2 & -2 & 1 \end{pmatrix}$.

2.2 $(AB)C = A(BC) = \begin{pmatrix} 3 & 3 \\ 3 & -15 \end{pmatrix}$.

2.3 a) $\begin{pmatrix} 4 & 3 \\ 2 & 1 \end{pmatrix}$, b) $\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$, c) $\begin{pmatrix} 4 & -3 \\ 1 & 2 \end{pmatrix}$.

2.4 a) $AB = 0$, $BA = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, b) $AB = 4 + 2i$, $BA = \begin{pmatrix} i & 2i & -1 \\ 2-i & 4-2i & 1+2i \\ 3 & 6 & 3i \end{pmatrix}$, c) $AB = 1$,
$BA = \begin{pmatrix} 1 & e^{-i(\omega+\theta)} & 2e^{-i\omega} \\ e^{i(\omega+\theta)} & 1 & 2e^{i\theta} \\ -\frac{1}{2}e^{i\omega} & -\frac{1}{2}e^{-i\theta} & -1 \end{pmatrix}$.

2.5 a) $A^T = \begin{pmatrix} 1 & -3 \\ 2i & 1+i \end{pmatrix}$, $A^\dagger = \begin{pmatrix} 1 & -3 \\ -2i & 1-i \end{pmatrix}$, b) $B^T = -B$, $B^\dagger = B$, c) $C^T = \begin{pmatrix} 1-i \\ 2i \\ 3 \end{pmatrix}$, $C^\dagger = \begin{pmatrix} 1+i \\ -2i \\ 3 \end{pmatrix}$,
d) $D^T = \begin{pmatrix} -2i, & -3, & 0, & 1+i \end{pmatrix}$, $D^\dagger = \begin{pmatrix} 2i, & -3, & 0, & 1-i \end{pmatrix}$.

2.6 a) 1, b) 8, c) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, d) $4\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

2.7 a) 9, b) $-2x$, c) $-3r$.

2.8 a) $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, b) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, c) $\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$.

2.9 a) $\prod_{j=1}^{d} a_j$, b) 1, c) x^2, d) $2t^3 + 3rt^2 - r^3 = 0$.

2.10 $\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = \mathbb{1}$.

2.11 a) $H^{-1} = H$, b) $\frac{1}{2}\begin{pmatrix} -4 & 2 \\ 3 & -1 \end{pmatrix}$, c) no solution, d) $U^{-1} = U^\dagger$.

3 Complex vector space

3.1 Basic definition

Vector space is a set of elements, *vectors*, that we can add together and multiply with a (complex) number, *scalar*. Vectors can be represented as tuples of numbers (or matrices with one dimension being equal to 1); these can be real (Euclidean space, probabilistic space) or complex [(quantum) Hilbert space]. Let us define a notation we shall use for vectors. Namely, in d-dimensional vector space, the vectors will be denoted as[a]

$$|v\rangle = \begin{pmatrix} v_0 \\ v_1 \\ \vdots \\ v_{d-1} \end{pmatrix},$$

where $v_j \in \mathbb{C}$ are called *amplitudes*. Particular system where $d = 2$ is called *qubit*,

$$|v\rangle = \begin{pmatrix} v_0 \\ v_1 \end{pmatrix}.$$

Sometimes, to save space, we will write vectors also as

$$|v\rangle = \begin{pmatrix} v_0, & v_1, & \ldots, & v_{d-1} \end{pmatrix}^T,$$

where T stands for transpose of the vector. First, let us go through some basic properties.

[a] From now on we will deviate from the typical indexing starting from one and, for practical reasons, will be indexing starting from zero.

Exercise 3.1. Compute the combinations of vectors

a) $\begin{pmatrix} 1 \\ 5 \end{pmatrix} + \begin{pmatrix} 3 \\ 8 \end{pmatrix} + \begin{pmatrix} 2 \\ -7 \end{pmatrix}$

b) $\begin{pmatrix} 1+i \\ 1-i \end{pmatrix} - \begin{pmatrix} 1+i \\ 1+i \end{pmatrix}$

c) $\dfrac{2}{1+i} \begin{pmatrix} 1 \\ 1-i \\ 1+i \end{pmatrix} + \dfrac{13}{2-3i} \begin{pmatrix} 2-3i \\ -i \\ 2+3i \end{pmatrix}$

d) $\dfrac{1}{\sqrt{2}}(|u_+\rangle + |u_-\rangle)$, where $|u_\pm\rangle = \dfrac{1}{2} \begin{pmatrix} \pm e^{\pm i\theta} \\ \mp i e^{\pm i\theta} \\ \sqrt{2} e^{\mp i\theta} \end{pmatrix}$

e) $\begin{pmatrix} 1 \\ 2+i \\ 3+2i \\ \vdots \\ d+(d-1)i \end{pmatrix} - \begin{pmatrix} 1-i \\ 2 \\ 3+i \\ \vdots \\ d+(d-2)i \end{pmatrix}$, with d being the dimension,

f) if $|e_j\rangle$ is a vector of zeros except for a single one at j-th component, $j \in \{0, 1, \ldots, d-1\}$ what is $\sum_{j=0}^{d-1} |e_j\rangle$?

▷ You should compute the following exercise, as it can offer better insight into the topic.

Exercise 3.2. Let

$$|u\rangle = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \qquad |v\rangle = \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}$$

a) write $2\,|u\rangle - |v\rangle$,

b) if possible, express vector $|x\rangle = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ as a combination of vectors $|u\rangle$ and $|v\rangle$,

c) if possible, express vector $|y\rangle = \begin{pmatrix} -i \\ 1+2i \\ 2-2i \end{pmatrix}$ as a combination of vectors $|u\rangle$ and $|v\rangle$.

3.2 Inner product, norm, and states

An important notion is that of *inner product*. Inner product is a mapping that assigns a (complex) value to a pair of vectors. Having vectors $|u\rangle$ and $|v\rangle$, the inner product of these vectors will be denoted as $\langle u|v\rangle$. The inner product fulfills conditions:

1. $\langle u+v|w\rangle = \langle u|w\rangle + \langle v|w\rangle$,
2. for $\alpha \in \mathbb{C}$ we have $\alpha\langle u|v\rangle = \langle \alpha^* u|v\rangle = \langle u|\alpha v\rangle$,
3. $\langle u|v\rangle = \langle v|u\rangle^*$,
4. $\langle u|u\rangle \geq 0$ with equality if and only if $|u\rangle = 0$.

Using component definition of a d-dimensional vector, we define

$$\langle u|v\rangle = \sum_{j=0}^{d-1} u_j^* v_j.$$

Two vectors $|u\rangle$ and $|v\rangle$ are called *orthogonal* if $\langle u|v\rangle = 0$. A *norm* ("length") of a vector is defined using inner product as:

$$\||v\rangle\| = \sqrt{\langle v|v\rangle} = \sqrt{\sum_j |v_j|^2}.$$

Norm is always non-negative real number. If the norm of a vector is one, we say the vector is *normalized*. Sometimes, especially when dealing with real-valued vectors, the normalized vectors are called *unit vectors*. These unit vectors represent states.

To make most of our formalism, it is good to define $\langle u|$ as a stand-alone *dual* vector as a conjugate transpose of $|u\rangle$:

$$|u\rangle^\dagger = \langle u| = \begin{pmatrix} u_0^*, & u_1^*, & \ldots, & u_{d-1}^* \end{pmatrix}.$$

Using this conjugate transpose, we can express the inner product simply as a matrix multiplication, which is in line with the definition of the inner product before:

$$\langle u|v\rangle = \begin{pmatrix} u_0^*, & u_1^*, & \dots, & u_{d-1}^* \end{pmatrix} \begin{pmatrix} v_0 \\ v_1 \\ \vdots \\ v_{d-1} \end{pmatrix} = \sum_{j=0}^{d-1} u_j^* v_j.$$

It is usual to call vectors $|v\rangle$ *ket vectors* and $\langle u|$ *bra vectors*, which originates in term for bracket that represents the inner product $\langle u|v\rangle$, which is a bra-ket. This bra-ket notation is also referred to as the *Dirac formalism.*

▷ You should compute all the exercises in this section, as you will use all the skills learnt here in later computations.

Exercise 3.3. Show that bra vector corresponding to the ket vector $a\,|\psi\rangle$ is $a^*\langle\psi|$.

Exercise 3.4. Compute the inner product $\langle u|v\rangle$ of following pairs of vectors

a) $|u\rangle = \begin{pmatrix} 3+2i \\ i \\ 0 \end{pmatrix}, |v\rangle = \begin{pmatrix} 1-i \\ 2+3i \\ 5i \end{pmatrix}$,

b) $|u\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, |v\rangle = \frac{1}{\sqrt{d}} \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$,

c)* $|u\rangle = \frac{1}{\sqrt{d}} \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$, $|v\rangle$ is a d-dimensional vector whose j-th component is $\frac{1}{\sqrt{d}} e^{\frac{2\pi i j}{d}}$,

d)* $|u\rangle = |v\rangle = \begin{pmatrix} 1+i \\ 2-i \\ 3+i \\ \vdots \\ d+(-1)^d i \end{pmatrix}$.

Exercise 3.5. Determine which of the following pairs of vectors are orthogonal.

a) $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$,

b) $|+\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, |-\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$,

c) $\frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$,

d) $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}$.

Exercise 3.6. Normalize following vectors, i.e. for given vector $|v\rangle$ find such number a that $a\,|v\rangle$ is normalized.

a) $\begin{pmatrix} 1 \\ i \\ i^2 \\ \vdots \\ i^{d-1} \end{pmatrix}$,

b) $\begin{pmatrix} 1 \\ \sqrt{k} \\ \sqrt{m-k-1} \end{pmatrix}$, for some integers $k < m$,

c) $\begin{pmatrix} 1 \\ \sqrt{2} \\ \sqrt{3} \\ \vdots \\ \sqrt{m} \end{pmatrix}$, for integer m,

d) $\begin{pmatrix} -24+18i \\ 6+8i \\ 12-9i \end{pmatrix}$,

e)* $\begin{pmatrix} 1 \\ 2 \\ 3 \\ \vdots \\ m \end{pmatrix}$, for integer m.

3.3 Basis

A set of n vectors $\{|v_j\rangle\}_{j=0}^{n-1}$ is *linearly independent* if

$$\sum_{j=0}^{n-1} c_j |v_j\rangle = 0$$

implies $c_j = 0$ for all j. In an d-dimensional vector space we can always find d linearly independent vectors and no more. These d vectors form a basis in given space. If these vectors are orthogonal we say the basis is *orthogonal* and if the basis vectors are on top of that normalized, we say the basis is *orthonormal*.

Note that the definition of linear independence implies that if vectors are linearly dependent, then at least two of the coefficients are non-zero. For example if $c_0 \neq 0$, then we can rearrange the terms to obtain:

$$|v_0\rangle = \sum_{j=1}^{n-1} c'_j |v_j\rangle,$$

where $c'_j = -c_j/c_0$ and at least one of them is not zero.

One can have many different bases in given Hilbert space. Even we restrict only to orthonormal bases, there are infinite possibilities. But some are more useful than others. The *canonical* or *computational* basis $\{|j\rangle\}_{j=0}^{d-1}$, where $|j\rangle$ is a vector of zeros with a single one on j-th component. For example for $d = 3$ the basis vectors are

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad |1\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad |2\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Or in qubit case, where $d = 2$ the basis vectors are

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Note that the numbers inside the ket vectors are just labels and do not represent any numbers.

Although we defined vectors as $n \times 1$ matrices, particular components are just expression in some particular basis, ususally implicitly assuming the computational basis. Each basis can be used to express the vector in. In general vector $|\psi\rangle$ in given basis, $\{|\mu_j\rangle\}_{j=0}^{d-1}$ is given by prescription

$$|\psi\rangle = \sum_{j=0}^{d-1} \langle \mu_j|\psi\rangle \,|\mu_j\rangle \,. \tag{2}$$

Canonical basis is especially useful as the coefficients of any vector are exactly the coefficient in the expansion in the canonical basis:

$$|v\rangle = \begin{pmatrix} v_0 \\ v_1 \\ \vdots \\ v_{d-1} \end{pmatrix} = \sum_{j=0}^{d-1} \langle j|v\rangle \,|j\rangle = \sum_{j=0}^{d-1} v_j \,|j\rangle \,.$$

Exercise 3.7. Express following vectors in the computational basis using Dirac formalism

a) $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$,

b) $\begin{pmatrix} 1 \\ 2i \\ 3-i \end{pmatrix}$,

c) $\begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}$,

d) $\begin{pmatrix} -1 \\ 0 \\ 2+i \\ -i \end{pmatrix}$,

e) $\begin{pmatrix} 1 \\ 2 \\ \vdots \\ d \end{pmatrix}$,

f) $\begin{pmatrix} 1, & 2i, & 3-i \end{pmatrix}$.

▷ The next exercise is a must!

Exercise 3.8. Using both matrix formalism and Dirac formalism, find bra (dual) vectors corresponding to the following ket vectors (assuming $\{|j\rangle\}_j$ is the computational basis)

a) $|a\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i\,|1\rangle)$,

b) $|b\rangle = |0\rangle + (1-i)\,|1\rangle + (2+i)\,|2\rangle + i\,|5\rangle$,

c)* $|c\rangle = \sum_{j=0}^{d-1} e^{i\frac{2\pi j}{d}}\,|j\rangle$.

▷ Burn the cases a) and b) into your mind. We will use them plenty.

Exercise 3.9. Determine whether the following sets of vectors form a basis and if so, whether the basis is orthogonal or orthonormal:

a) $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$,

b) $|+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, |-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$,

c) $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$,

d) $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$,

Exercise 3.10. Let

$$|v_1\rangle = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \qquad |v_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \qquad |v_3\rangle = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}.$$

Check that $\{|v_1\rangle, |v_2\rangle, |v_3\rangle\}$ is an orthonormal basis and using Eq. (2), write the following vectors in this basis.

a) $|a\rangle = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, b) $|b\rangle = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, c) $|c\rangle = \begin{pmatrix} i \\ 0 \\ \sqrt{2} \end{pmatrix}$.

Exercise 3.11. Let us have basis vectors

$$|\mu_1\rangle = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad |\mu_2\rangle = \frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \quad |\mu_3\rangle = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \quad |\mu_4\rangle = \frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}.$$

Express following vectors in this basis using Eq. (2):

a) $|\psi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$, b) $|\psi\rangle = \frac{1}{\sqrt{6}}\begin{pmatrix} 1+i \\ 1-i \\ -1 \\ i \end{pmatrix}$, c) $|\psi\rangle = \frac{1}{2}\begin{pmatrix} e^{i\theta} \\ ie^{i\theta} \\ -ie^{-i\theta} \\ e^{-i\theta} \end{pmatrix}$.

Exercise 3.12*. Find some smallest basis for vector spaces being defined by following parametric families of vectors.

a) $|a\rangle = \begin{pmatrix} x \\ y \\ z \\ x+y \end{pmatrix}$, b) $|b\rangle = \begin{pmatrix} 1+z \\ x+iy \\ x-iy \\ 1-z \end{pmatrix}$, c) $|c\rangle = \begin{pmatrix} z-iy \\ x+iy \\ z+x \end{pmatrix}$.

▷ The following exercise is very good to chase away your boredom for a while.

Exercise 3.13*. Determine whether the following set of vectors forms a basis:

$$|v_1\rangle = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ \dots \\ 0 \\ 0 \end{pmatrix}, |v_2\rangle = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ \dots \\ 0 \\ 0 \end{pmatrix}, |v_3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ \dots \\ 0 \\ 0 \end{pmatrix}, \dots, |v_{d-1}\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \dots \\ 1 \\ 1 \end{pmatrix}, |v_d\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \dots \\ 0 \\ 1 \end{pmatrix}.$$

3.4 Measurements I

Measurements are prime example of where the classical physics and quantum physics differ. In quantum case the measurements do not give us the whole information about measured state and obtaining the particular outcomes is a probabilistic process. The simplest measurement is a *measurement in the same basis as we write the state in* — commonly this is the computational or canonical basis[a].

Let us have a ket vector

$$|v\rangle = \begin{pmatrix} v_0 \\ v_1 \\ \vdots \\ v_{d-1} \end{pmatrix},$$

and suppose the basis in which we expressed it in is the computational basis, i.e. $|v\rangle = \sum_{j=0}^{d-1} v_{j+1} |j\rangle$.

Then the canonical measurement gives us only one outcome from the set $\{0, 1, \dots, d-1\}$. Each outcome j is obtained with probability

$$p_j = |\langle j|v\rangle|^2 = |v_j|^2.$$

The state afterwards is $|j\rangle$. This would be the same for every basis. Current quantum devices usually have a single prescribed measurement basis, which is the same as the computational basis and so this prescription of getting results is the most important for us.

In general, we can, however, imagine measurements in any basis, but previous formulas remain valid. Let us have state $|v\rangle$ as before and a measurement in basis $\{|j\rangle\}_{j=1}^{d}$. Then the measurement yields outcome j with probability $p_j = |\langle j|v\rangle|^2$ and the state after the measurement in $|j\rangle$. The difference is, that the probability now is not equal to $|v_j|^2$, as element v_j corresponds to the computational basis, which is different from the basis in which we measure. Sometimes it can be useful to remember that $p_j = |\langle j|v\rangle|^2 = \langle j|v\rangle^* \langle j|v\rangle = \langle v|j\rangle\langle j|v\rangle$.

Note that the aim of the measurement is to obtain numbers v_j. In order to obtain them, we have to (almost) always perform an infinite number of measurements. With limited time on our hands we cannot do that and have to be satisfied with just approximations of these numbers obtained by performing limited number of measurements. And even then all we find are the approximations to only the norms of the amplitudes $|v_j|$.

[a] So if we have a vector represented in its matrix form, we will always assume that the basis is computational.

▷ Again, the next few exercises are very important for you to understand how measurements work in the quantum world.

Exercise 3.14. Let us consider qubit measurements in the canonical basis. What are particular outcomes probabilities for states (written in the canonical basis)

a) $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$,

b) $|+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$,

c) $|-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$,

d) $|\psi\rangle = \frac{1}{2}\begin{pmatrix} 1 \\ i\sqrt{3} \end{pmatrix}$,

e) $|\phi\rangle = \frac{1}{2}\begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix}$.

Exercise 3.15. What are the particular outcomes $\pm$ probabilities for the state from previous exercise, if we perform measurement in the $\{|\pm\rangle\}$ basis?

Exercise 3.16. Let us have state (in canonical basis)

$$|v\rangle = \frac{1}{5}\begin{pmatrix} 3i \\ 1 \\ 1-i \\ 2+3i \end{pmatrix}.$$

What are the outcome probabilities if we perform measurement in

a) canonical basis $\{|j\rangle\}_{j=0}^{3}$,

b) basis $\{|u_j\rangle\}_{j=0}^{3}$, where $|u_0\rangle = |0\rangle$, $|u_1\rangle = \frac{1}{\sqrt{2}}(|1\rangle + i\,|2\rangle)$, $|u_2\rangle = \frac{1}{\sqrt{2}}(|1\rangle - i\,|2\rangle)$, and $|u_3\rangle = -i\,|3\rangle$,

c) basis $\{|w_j\rangle\}_{j=0}^{3}$, where

$$|w_0\rangle = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad |w_1\rangle = \frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \quad |w_2\rangle = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \quad |w_3\rangle = \frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}.$$

▷ The next exercise is especially important, as it should give you a deeper insight into how measurements work. In particular you should understand what happens if a state undergoes multiple measurements in sequence.

Exercise 3.17. Let us start in state $|0\rangle$ and perform two successive measurements. Let the first measurement be in basis $\{|a_0\rangle, |a_1\rangle\}$ and the following measurement be in the $\{|\pm\rangle\}$ basis. Compute the probability of getting outcomes $(a_0, +)$ and $(a_1, +)$ if

a) $|a_0\rangle = |0\rangle$, $|a_1\rangle = |1\rangle$,

b) $|a_0\rangle = |+\rangle$, $|a_1\rangle = |-\rangle$,

c) $|a_0\rangle = \frac{1}{2}\begin{pmatrix} 1 \\ i\sqrt{3} \end{pmatrix}$, $|a_1\rangle = \frac{1}{2}\begin{pmatrix} i\sqrt{3} \\ 1 \end{pmatrix}$.

▷ If you want to get a better feeling on what to be careful about in measurements, the next exercise is here for you. You will learn about some conditions that need to be fulfilled in order for the measurements to be defined correctly.

Exercise 3.18*. Compute the outcome probabilities if
a) the state is $\frac{1}{3}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and the measurement is in the canonical basis,
b) the state is $|0\rangle$ and the measurement is in basis

$$\left\{\frac{1}{2}\begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\}.$$

Results.

3.1 a) $\begin{pmatrix} 6 \\ 6 \end{pmatrix}$, b) $\begin{pmatrix} 0 \\ -2i \end{pmatrix}$, c) $\begin{pmatrix} 14-i \\ 3-4i \\ -3+12i \end{pmatrix}$, d) $\frac{1}{\sqrt{2}}\begin{pmatrix} i\sin\theta \\ \sin\theta \\ \sqrt{2}\cos\theta \end{pmatrix}$, e) $i\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$, f) $\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$.

3.2 a) $\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$, b) $|x\rangle = |u\rangle - |v\rangle$, c) the task does not have a solutio

3.4 a) $4-7i$, b) $\frac{1}{\sqrt{d}}$, c) 0, d) $\frac{d(d+1)(2d+1)}{6} + d = \frac{d}{6}(2d^2+3d+7)$.

3.5 a) orthogonal, b) orthogonal, c) not orthogonal, d) orthogonal.

3.6 Real normalization factors are a) $\frac{1}{\sqrt{d}}$, b) $1/\sqrt{m}$, c) $\sqrt{\frac{2}{m(m+1)}}$, d) $\frac{1}{35}$, e) $\sqrt{\frac{6}{m(m+1)(2m+1)}}$.

3.7 a) $\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$, b) $|0\rangle + 2i|1\rangle + (3-i)|2\rangle$, c) $|0\rangle - |3\rangle$, d) $-|0\rangle + (2+i)|2\rangle - i|3\rangle$, e) $\sum_{j=0}^{d-1}(j+1)|j\rangle$, f) $\langle 0| + 2i\langle 1| + (3-i)\langle 2|$.

3.8 a) $\frac{1}{\sqrt{2}}(\langle 0| - i\langle 1|)$, b) $\langle 0| + (1+i)\langle 1| + (2-i)\langle 2| - i\langle 4|$, c) $\sum_{j=0}^{d-1} e^{-i\frac{2\pi j}{d}}\langle j|$.

3.9 a) orthonormal basis, b) orthonormal basis c) the vectors are normalized, however, they are not orthogonal; yet, they form a basis, d) the vectors are not normalized and they do not form a basis.

3.10 a) $|a\rangle = \frac{2}{\sqrt{3}}|v_1\rangle + \sqrt{\frac{2}{3}}|v_3\rangle$,
b) $|b\rangle = 2\sqrt{3}|v_1\rangle - \frac{1}{\sqrt{2}}|v_2\rangle - \sqrt{\frac{3}{2}}|v_3\rangle$,
c) $|c\rangle = \frac{1}{3\sqrt{2}}\left[(2+\sqrt{2}i)|v_1\rangle + i\sqrt{3}|v_2\rangle - (2\sqrt{2}-i)|v_3\rangle\right]$.

3.11 a) $|\psi\rangle = \frac{1}{\sqrt{2}}(|\mu_1\rangle + |\mu_4\rangle)$,
b) $|\psi\rangle = \frac{1}{2\sqrt{6}}[(1+i)|\mu_1\rangle + (-1+i)|\mu_2\rangle + (3-i)|\mu_3\rangle + (1+3i)|\mu_4\rangle$,
c) $|\psi\rangle = \frac{1}{\sqrt{2}}[C_-|\mu_1\rangle - iC_-|\mu_2\rangle + iC_+|\mu_3\rangle + C_+|\mu_4\rangle$.

3.12 a) $|a\rangle = \left(\sqrt{2}x + \frac{1}{\sqrt{2}}y\right)|\mu_1\rangle + \frac{\sqrt{6}}{2}y\,|\mu_2\rangle + z\,|\mu_3\rangle$, where

$$|\mu_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\1\end{pmatrix}, |\mu_2\rangle = \frac{1}{\sqrt{6}}\begin{pmatrix}-1\\2\\0\\1\end{pmatrix}, |\mu_3\rangle = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}.$$

b) $|b\rangle = \sqrt{2}\,(|\mu_1\rangle + x\,|\mu_2\rangle + y\,|\mu_3\rangle + z\,|\mu_4\rangle)$, where

$$|\mu_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\1\end{pmatrix}, |\mu_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\1\\0\end{pmatrix}, |\mu_3\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\i\\-i\\0\end{pmatrix}, |\mu_4\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\-1\end{pmatrix}.$$

c) $|c\rangle = \frac{1}{2}[\sqrt{6}(z+x)\,|\mu_1\rangle + \sqrt{2}(x-z+2iy)\,|\mu_2\rangle$, where

$$|\mu_1\rangle = \frac{1}{\sqrt{6}}\begin{pmatrix}1\\1\\2\end{pmatrix}, |\mu_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\1\\0\end{pmatrix}.$$

3.13 The vectors form a basis if and only if d is odd.
3.14 a) $p_0 = 1$, $p_1 = 0$, b) $p_0 = p_1 = 1/2$, c) $p_0 = p_1 = 1/2$, d) $p_0 = 1/4$, $p_1 = 3/4$, e) $p_0 = 1/4$, $p_1 = 3/4$,
3.15 a) $p_+ = p_- = 1/2$, b) $p_+ = 1$, $p_- = 0$, c) $p_+ = 0$, $p_- = 1$, d) $p_+ = p_- = 1/2$, e) $p_+ = \frac{2+\sqrt{3}}{4}$, $p_- = \frac{2-\sqrt{3}}{4}$.
3.16 a) $p_0 = \frac{9}{25}$, $p_1 = \frac{1}{25}$, $p_2 = \frac{2}{25}$, $p_3 = \frac{13}{25}$, b) $p_0 = \frac{9}{25}$, $p_1 = \frac{1}{50}$, $p_2 = \frac{1}{10}$, $p_3 = \frac{13}{25}$, c) $p_0 = 41\%$, $p_1 = 5\%$, $p_2 = 5\%$, $p_3 = 49\%$.
3.17 a) $p(0,+) = 1/2$, $p(1,+) = 0$, b) $p(+,+) = 1/2$, $p(-,+) = 0$, c) $p(a_0,+) = 1/8$, $p(a_1,+) = 3/8$.
3.18 a) state is not normalized, b) the basis is not orthogonal.

4 Qubit transformations

4.1 Transforming between bases

Having two orthonormal bases, we can express a vector in both of them, and having a prescription for the vector in one of the bases, there always exists a transformation rule for expressing it in in the other basis.

Take for example qubit case (two level quantum system), where we have the canonical basis $\{|0\rangle, |1\rangle\}$. In addition we can take another important basis,

$$|+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \qquad |-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix} = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle).$$

One can go between different bases using the substitution rule from above in one direction and for the other direction one can use the inverse rule:

$$|0\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle), \qquad |1\rangle = \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle).$$

Every set of substitution rules that changes one orthonormal basis to another can be identified with a *unitary matrix* U. Unitary matrices have inverses U^{-1} and it turns out that $U^{-1} = U^\dagger$, where $\dagger$ represents the conjugate transpose as previously, i.e. $U^\dagger = \left(U^T\right)^*$, where T stands for transpose ($U^T_{ij} = U_{ji}$). Written component-wise $U^\dagger_{ij} = U^*_{ji}$. Now we have $U^\dagger U = UU^\dagger = \mathbb{1}$, where $\mathbb{1}$ is the identity matrix. All these matrices are of dimension $d \times d$, where d is the dimension of the Hilbert space.

Specific unitaries are the identity matrix $\mathbb{1}$ and in qubit case already seen Pauli matrices:

$$\sigma_x = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}, \qquad \sigma_y = \begin{pmatrix}0 & -i\\i & 0\end{pmatrix}, \qquad \sigma_z = \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}.$$

On the other hand, having a unitary U and a basis $\{|v_j\rangle\}_j$ one can find a new basis by multiplying all original vectors by U and so the basis reads $\{U|v_j\rangle\}_j$.

Exercise 4.1. Write the following vectors in $\{|+\rangle, |-\rangle\}$ basis using the substitution rules:

a) $\frac{1}{\sqrt{2}}(|0\rangle \pm i|1\rangle)$,

b) $\frac{1}{2}\left(|0\rangle - i\sqrt{3}|1\rangle\right)$,

c) $a|0\rangle + b|1\rangle$, with $a, b \in \mathbb{C}$ such that $|a|^2 + |b|^2 = 1$.

Exercise 4.2. Check whether the following matrices are unitary

a) all Pauli matrices,

b) Hadamard matrix $H = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}$,

c) $\begin{pmatrix}1 & 1+i\\0 & \pm 1\end{pmatrix}$,

d) $\frac{1}{\sqrt{3}}\begin{pmatrix}1 & 1 & 1\\1 & \jmath & \jmath^*\\1 & \jmath^* & \jmath\end{pmatrix}$, where $\jmath = e^{\frac{2\pi i}{3}}$.

e) $\frac{1}{2}\begin{pmatrix}1 & -i & -1+i\\i & 1 & 1+i\\1+i & -1+i & 0\end{pmatrix}$,

f) $H_2 = \frac{1}{2}\begin{pmatrix}1 & 1 & 1 & 1\\1 & -1 & 1 & -1\\1 & 1 & -1 & -1\\1 & -1 & -1 & 1\end{pmatrix}$,

g) $\begin{pmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 0 & 1\\0 & 0 & 1 & 0\end{pmatrix}$.

Exercise 4.3. Transform the canonical basis (of suitable dimension) using following unitary matrices and check their orthonormality

a) all Pauli matrices,

b) Hadamard matrix $H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$,

c) $\frac{1}{2}\begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}$,

d) $\frac{1}{2}\begin{pmatrix} 1-i\sqrt{3} & 0 \\ 0 & 1+i\sqrt{3} \end{pmatrix}$,

e) $H_2 = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$,

Exercise 4.4. Use bases from the previous exercise and conjugate transposes of corresponding unitary matrices to show that if U transforms basis $\{|0\rangle, |1\rangle\}$ to basis $\{|w_0\rangle, |w_1\rangle\}$, then $U^\dagger$ transforms basis $\{|w_0\rangle, |w_1\rangle\}$ to basis $\{|0\rangle, |1\rangle\}$.

Exercise 4.5. Let

$$|r_0\rangle = \frac{1}{2}\begin{pmatrix} 1 \\ i\sqrt{3} \end{pmatrix}, \qquad |r_1\rangle = \frac{1}{2}\begin{pmatrix} i\sqrt{3} \\ 1 \end{pmatrix}.$$

Find U such that $U|r_0\rangle = |0\rangle$ and $U|r_1\rangle = |1\rangle$ and compare it with V that has form

$$V = |0\rangle\langle r_0| + |1\rangle\langle r_1|.$$

While we presented unitary matrices as a way of transforming bases, they are also state transformations. So if we have some state $|\psi\rangle$, a unitary matrix defines its change to state $U|\psi\rangle$. For example, we can have state $|0\rangle$ which undergoes transformation under the action of σ_x and then H afterwards

$$\sigma_x|0\rangle = |1\rangle, \qquad H|1\rangle = |-\rangle$$

which we can write together

$$H\sigma_x|0\rangle = H|1\rangle = |-\rangle.$$

We see that the state changes can be composed to more complicated ones. In quantum computation these state changes are called *(quantum) gates*.

In qubit case there are sets of gates which can be composed to do any state transformation, but the details are beyond this course. What we should remember is that Qiskit implements some set of basic gates, including Pauli matrices, Hadamard matrix and the most universal single-qubit case U_3.

▷ While it is important that you go through all the previous exercises, you should certainly have a look at the following exercise and get acquaintained with the form of the matrix U_3. We will use it also in Qiskit.

Exercise 4.6. Let us have two classes of qubit matrices:

$$V = \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix},\ |\alpha|^2 + |\beta|^2 = 1, \qquad U_3(\theta,\phi,\lambda) = \begin{pmatrix} \cos(\theta/2) & -e^{i\lambda}\sin(\theta/2) \\ e^{i\phi}\sin(\theta/2) & e^{i\lambda+i\phi}\cos(\theta/2) \end{pmatrix}$$

Compare the two classes (identify relations between parameters α, β and θ, ϕ, λ) and show that they are unitary. Find also parameters θ', ϕ', and λ' which correspond to $U_3^\dagger(\theta,\phi,\lambda)$, which is an inverse to $U_3(\theta,\phi,\lambda)$.

Exercise 4.7. Check that $(U|\psi\rangle)^\dagger = \langle\psi|U^\dagger$ where $U = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}$, and that $(|\psi\rangle\langle+|)^\dagger = |+\rangle\langle\psi|$, for following vectors $|\psi\rangle$.

a) $|\psi\rangle = |1\rangle$, b) $|\psi\rangle = \frac{1}{2}(|0\rangle + i\sqrt{3}\,|1\rangle)$, c) $|\psi\rangle = \psi_0\,|0\rangle + \psi_1\,|1\rangle$.

▷ The next one is a little bit longer, but if you will complete it, you will have a much better grip at the Pauli matrices.

Exercise 4.8*. Fun with Pauli matrices:
Let δ_{jk} be the Kronecker delta,

$$\delta_{jk} = \begin{cases} 1 & \text{if } j = k, \\ 0 & \text{otherwise,} \end{cases}$$

and ε_{jkl} be the 3-dimensional Levi-Civita symbol

$$\varepsilon_{jkl} = \begin{cases} +1 & \text{if } (j,k,l) \text{ is an even permutation } (1,2,3), (2,3,1), \text{ or } (3,1,2), \\ -1 & \text{if } (j,k,l) \text{ is an odd permutation } (1,3,2), (3,2,1), \text{ or } (2,1,3), \\ 0 & \text{if } i = j,\, j = k, \text{ or } k = i. \end{cases}$$

Check following identities:

a) $\sigma_j^\dagger \sigma_j = \sigma_j^2 = \mathbb{1}$,

b) $\sigma_j \sigma_k = i\varepsilon_{jkl}\sigma_l$, for distinct j, k, and l,

c) $[\sigma_j, \sigma_k] = \sigma_j\sigma_k - \sigma_k\sigma_j = 2i\varepsilon_{jkl}\sigma_l$,

d) $\{\sigma_j, \sigma_k\} = \sigma_j\sigma_k + \sigma_k\sigma_j = 2\delta_{jk}\mathbb{1}$,

e) $\sigma_j\sigma_k\sigma_j = (-1)^{\delta_{jk}}\sigma_k$.

Exercise 4.9*. Suppose $\vec{n} = (n_x, n_y, n_z)^T$ is some real unit vector and $\omega \in [0, \pi]$, show that

$$\mathbb{1}\cos\frac{\omega}{2} + i(n_x\sigma_x + n_y\sigma_y + n_z\sigma_z)\sin\frac{\omega}{2}$$

is a unitary matrix and use it to transform the canonical base.

Exercise 4.10*. Consider a class of matrices given by $U_n = a\mathbb{1}_n + \frac{b}{n}\mathbb{J}_n$, where $\mathbb{1}_n$ is an $n \times n$ identity matrix and $\mathbb{J}_n$ is an $n \times n$ matrix of ones. Find constraints on a and b if U is a unitary matrix.

4.2 Measurements II

Measurements are technically very difficult to realize and, therefore, they are performed only in some particular basis. Usually the canonical basis coincides with the measurement basis, which is by convention chosen to be the z-direction. This is so also in Qiskit. Suppose, first in general dimension d, that we want to measure state $|\psi\rangle$ in some basis $\{|w_j\rangle\}_{j=1}^d$ and we know that $|w_j\rangle = U\,|j\rangle$, where $\{|j\rangle\}_{j=1}^d$ is the basis we can measure in. Then we can use following expression for the outcome probabilities

$$p_j = |\langle w_j|\psi\rangle|^2 = |\,\langle j|\, U^\dagger\, |\psi\rangle\,|^2.$$

In other words,the measurement of the state $U^\dagger\,|\psi\rangle$ in basis $\{|j\rangle\}$ is the same as the measurement of state $|\psi\rangle$ in basis $\{U\,|j\rangle\}$.

In qubit case we are in particular looking for such unitary matrix U that transforms base $\{|0\rangle, |1\rangle\}$ into base $\{|w_0\rangle, |w_1\rangle\}$. Note that it is enough to find U such that $|w_0\rangle = U\,|0\rangle$, as $U\,|1\rangle = e^{i\phi}\,|w_1\rangle$ for some global phase ϕ that does not change the measurements statistics.

Exercise 4.11. Having available only z-measurements, design a process for performing $\{|\pm\rangle\}$ measurement. That is, find the transformation U for the measurement.

▷ The next exercise is a must, as it will be used both in lectures and in our programming exercises.

Exercise 4.12. In CHSH, or Ekert 91, Alice and Bob make use of four measurements in bases

$$\sigma_\omega = \left\{\cos\frac{\omega}{2}|0\rangle + \sin\frac{\omega}{2}|1\rangle, -\sin\frac{\omega}{2}|0\rangle + \cos\frac{\omega}{2}|0\rangle\right\},$$

where $\omega \in \{k\pi/4\}_{k=0}^{3}$. Find such parameters θ, ϕ, and λ for

$$U_3(\theta,\phi,\lambda) = \begin{pmatrix} \cos(\theta/2) & -e^{i\lambda}\sin(\theta/2) \\ e^{i\phi}\sin(\theta/2) & e^{i\lambda+i\phi}\cos(\theta/2) \end{pmatrix}$$

that represent the transformation U for these four measurements that Alice and Bob use.

Results.

4.1 a) $\frac{1}{2}\left[(1\pm i)|+\rangle + (1\mp i)|-\rangle\right]$,
b) $\frac{1}{2\sqrt{2}}\left[(1-i\sqrt{3})|+\rangle + (1+i\sqrt{3})|-\rangle\right]$,
c) $\frac{1}{\sqrt{2}}\left[(a+b)|+\rangle + (a-b)|-\rangle\right]$.

4.2 a) unitary, b) unitary, c) not unitary since d) unitary, e) unitary, f) unitary, g) unitary.

4.3 a) $\{|1\rangle, |0\rangle\}$, $\{i|1\rangle, -i|0\rangle\}$, $\{|0\rangle, -|1\rangle\}$, under the action of σ_x, σ_y, σ_z Pauli matrices respectively,
b) $\{|+\rangle, |-\rangle\}$, c) $\left\{\frac{1}{2}(|0\rangle + \sqrt{3}|1\rangle), \frac{1}{2}(-\sqrt{3}|0\rangle + |1\rangle)\right\}$, d) $\left\{e^{-i\frac{\pi}{3}}|0\rangle, e^{i\frac{\pi}{3}}|1\rangle\right\}$,
e)

$$\frac{1}{2}\begin{pmatrix}1\\1\\1\\1\end{pmatrix}, \quad \frac{1}{2}\begin{pmatrix}1\\-1\\1\\-1\end{pmatrix}, \quad \frac{1}{2}\begin{pmatrix}1\\1\\-1\\-1\end{pmatrix}, \quad \frac{1}{2}\begin{pmatrix}1\\-1\\-1\\1\end{pmatrix}.$$

4.5 $U = V = \frac{1}{2}\begin{pmatrix} 1 & -i\sqrt{3} \\ -i\sqrt{3} & 1 \end{pmatrix}$.

4.7 a) $(U|\psi\rangle)^\dagger = \langle\psi|U^\dagger = \begin{pmatrix} b^*, & a \end{pmatrix}$, $(|\psi\rangle\langle+|)^\dagger = |+\rangle\langle\psi| = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$,
b) $(U|\psi\rangle)^\dagger = \langle\psi|U^\dagger = \frac{1}{2}\begin{pmatrix} a^* - ib^*\sqrt{3}, & -b - ia\sqrt{3} \end{pmatrix}$, $(|\psi\rangle\langle+|)^\dagger = |+\rangle\langle\psi| = \frac{1}{2\sqrt{2}}\begin{pmatrix} 1 & -i\sqrt{3} \\ 1 & -i\sqrt{3} \end{pmatrix}$,
c) $(U|\psi\rangle)^\dagger = \langle\psi|U^\dagger = \frac{1}{2}\begin{pmatrix} a^*\psi_0^* + b^*\psi_1^*, & -b\psi_0^* + a\psi_1^* \end{pmatrix}$, $(|\psi\rangle\langle+|)^\dagger = |+\rangle\langle\psi| = \frac{1}{\sqrt{2}}\begin{pmatrix} \psi_0^* & \psi_1^* \\ \psi_0^* & \psi_1^* \end{pmatrix}$.

4.11 $U = H$.

4.12 σ_0: $\theta = \phi = \lambda = 0$, $\sigma_{\pi/2}$: $\theta = \pi/2$, $\phi = 0$, and $\lambda = \pi$, $\sigma_{\pi/4}$: $\theta = \pi/4$, $\phi = 0$, $\lambda = \pi$, $\sigma_{3\pi/4}$: $\theta = 3\pi/4$, $\phi = 0$, $\lambda = \pi$.

5 Tensor spaces

Tensor products describe composite systems. Having for example two qubits (two two-dimensional spaces), their tensor product lives in a four dimensional space. If the two vectors are

$$|u\rangle = \begin{pmatrix} a \\ b \end{pmatrix} \; |v\rangle = \begin{pmatrix} c \\ d \end{pmatrix}$$

the composition is obtained by prescription:

$$|u\rangle \otimes |v\rangle = \begin{pmatrix} a \begin{pmatrix} c \\ d \end{pmatrix} \\ b \begin{pmatrix} c \\ d \end{pmatrix} \end{pmatrix} = \begin{pmatrix} ac \\ ad \\ bc \\ bd \end{pmatrix}$$

These tensor products define a new linear vector space and so they follow the usual rules defined earlier. The tensor product is similarly defined also for matrices. Let

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \qquad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

then their tensor product is obtained from prescription

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B \\ a_{21}B & a_{22}B \end{pmatrix}.$$

Having the standard qubit basis $\{|0\rangle, |1\rangle\}$, we can easily obtain a multi-qubit basis by tensoring all the combinations. For example in a three-qubit case, we can combine

$$\begin{aligned}
|0\rangle \otimes |0\rangle \otimes |0\rangle &\equiv |000\rangle, & |1\rangle \otimes |0\rangle \otimes |0\rangle &\equiv |100\rangle, \\
|0\rangle \otimes |0\rangle \otimes |1\rangle &\equiv |001\rangle, & |1\rangle \otimes |0\rangle \otimes |1\rangle &\equiv |101\rangle, \\
|0\rangle \otimes |1\rangle \otimes |0\rangle &\equiv |010\rangle, & |1\rangle \otimes |1\rangle \otimes |0\rangle &\equiv |110\rangle, \\
|0\rangle \otimes |1\rangle \otimes |1\rangle &\equiv |011\rangle, & |1\rangle \otimes |1\rangle \otimes |1\rangle &\equiv |111\rangle.
\end{aligned}$$

An easy way of remembering the relation between this tensor product form and matrix (vector) form is that if we take the multi-qubit state as a binary number, then this number in its decimal form determines the position of a one in the composite standard basis with zero indexing the top-most position. For example state $|101\rangle$ if taken as a binary number 101_2 corresponds to the decimal number 5 and so in the matrix notation, the standard basis vector will have one on the sixth position:

$$|101\rangle = \begin{array}{cl} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} & \begin{array}{l} \leftarrow |000\rangle \\ \leftarrow |001\rangle \\ \leftarrow |010\rangle \\ \leftarrow |011\rangle \\ \leftarrow |100\rangle \\ \leftarrow |101\rangle \\ \leftarrow |110\rangle \\ \leftarrow |111\rangle \end{array} \end{array}$$

Naturally, we can have also transformations and measurements on tensor spaces. They can either act on the whole space, or they can act only on part of the system only. For example having a two-qubit space, we can make transformation only on the second qubit:

$$(\mathbb{1} \otimes U) |a\rangle \otimes |b\rangle = |a\rangle \otimes U |b\rangle .$$

Sometimes, when the position of the subspaces might be intelligible, indexing of the spaces is used for clarity. For example having a tri-partite system of parties A, B, and C, the states can be written as $|\psi\rangle_A \otimes |\phi\rangle_B \otimes |\rho\rangle_C$. Or some parts might not be decomposed and we write $|\Psi\rangle_{AB} \otimes |\rho\rangle_C$.

Exercise 5.1. Write the matrix form of the following tensor products:

a) $|+\rangle \otimes |-\rangle$,

b) $|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$

c) $H \otimes H$,

d) $\sigma_x \otimes \sigma_y - \sigma_z \otimes \sigma_x$.

Exercise 5.2. Write the following vectors and matrices in $\mathbb{C}^2 \otimes \mathbb{C}^2$ using Dirac notation and where possible present also the product form:

a) $|\Phi^+\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$, b) $\frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}$, c) $\frac{1}{2} \begin{pmatrix} 1 \\ -i \\ i \\ 1 \end{pmatrix}$

Dirac bra-ket notation is a very useful tool in quantum theory. We have seen its use to compute scalar product, but its application is much wider. At this point let us just note one additional feature of this notation. Having two d-dimensional vectors $|u\rangle$ and $|v\rangle$, we can construct also an outer product of these vectors, $|u\rangle \langle v|$, which produces a matrix with dimension $d \times d$. The usefulness of this construction will become apparent soon.

Exercise 5.3. Compute matrices $|0\rangle \langle 0|$, $|0\rangle \langle 1|$, $|1\rangle \langle 0|$, and $|1\rangle \langle 1|$ and use them to express all Pauli matrices.

Exercise 5.4. Show that in d-dimensional space we can express the identity as

$$\mathbb{1} = \sum_{j=0}^{d-1} |j\rangle \langle j|$$

and show that this form does not depend on the choice of the basis $\{|j\rangle\}_j$.

Exercise 5.5*. Write the following matrices as combinations of tensor products of Pauli matrices and ket-bra symbols from the Dirac notation.

a) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$, b) $\begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}$, c) $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$.

Exercise 5.6. Evaluate following state changes.

a) $(\mathbb{1} \otimes H) |0\rangle \otimes |+\rangle$,

b) $(H \otimes \sigma_x) |01\rangle$,

c) $(\sigma_x \otimes \sigma_z - \mathbb{1} \otimes H) |10\rangle$,

d) $(H \otimes \sigma_x)(\sigma_z \otimes H)(H \otimes \sigma_x)\big(|{+}{-}\rangle - |{-}{+}\rangle\big)$.

▷ We strongly suggest you do the next one.

Exercise 5.7. Consider operation on two qubits that performs NOT on the second qubit if and only if the first qubit is in vector $|1\rangle$. Write it both in the tensor product form and in the matrix form.

5.1 Bell basis

▷ Constructions from the next exercise will be used later; do this exercise and memorize the schemes.

Exercise 5.8. Compare the following three schemes:

Scheme 1: $(\mathbb{1} \otimes \sigma_a)\text{CNOT}(H \otimes \mathbb{1})\,|00\rangle$ with $a \in \{0,1,2,3\}$ and identifying $\sigma_0 = \mathbb{1}$, $\sigma_1 = \sigma_x$, $\sigma_2 = \sigma_y$, and $\sigma_3 = \sigma_z$.

Scheme 2: $\text{CNOT}(H \otimes \mathbb{1})\,|\psi\rangle$, where $|\psi\rangle \in \{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$.

Scheme 3: $(\sigma_x^j \otimes \sigma_z^k)\text{CNOT}(H \otimes \mathbb{1})\,|00\rangle$ with $j,k \in \{0,1\}$ and σ_x^j is σ_x for $j = 1$ and identity otherwise and similarly for σ_z^k.

In all three schemes $\text{CNOT} = |0\rangle\langle 0| \otimes \mathbb{1} + |1\rangle\langle 1| \otimes \sigma_x$. What states are these schemes producing?

In some of the previous exercises we have met some of a very special set of bi-partite qubit states called *Bell states*. These states are

$$\left|\Phi^{\pm}\right\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle), \qquad \left|\Psi^{\pm}\right\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle).$$

These four states form an orthonormal basis and play a crucial role in quantum information applications. Let us compute some of their properties.

▷ The following exercise is important for applications such as quantum teleportation that require measurement in the Bell basis.

Exercise 5.9. Based on previous exercise, design a procedure of measuring in the Bell basis using measurements in computational basis.

▷ The next exercise introduces the notion of *entanglement* and is important in further understanding of the topic.

Exercise 5.10. Show that none of the Bell states can be written in the form of a product of states $|\phi\rangle \otimes |\psi\rangle$ for any qubit states $|\phi\rangle$ and $|\psi\rangle$.

Exercise 5.11. Show that

$$\left|\Psi^{-}\right\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

has the same form in every basis for single qubits, i.e. when the change of the basis is described by a qubit unitary U, we want to show that the state has the same form also after the change of basis $U \otimes U$.

5.2 Selected global operations

We have already used several multi-qubit operations. In this section we present some identities useful in quantum computation and formalize the notation. The standard controlled operation are:

CNOT: Flips the qubit if the control is in the state $|1\rangle$:

$$\text{CNOT}_{A\to B} = |0\rangle\langle 0| \otimes \mathbb{1}_B + |1\rangle\langle 1| \otimes \sigma_x.$$

CZ: Flips the sign, if the control qubit is in state $|1\rangle$:

$$\mathrm{CZ}_{A\to B} = |0\rangle_A \langle 0| \otimes \mathbb{1}_B + |1\rangle_A \langle 1| \otimes \sigma_z.$$

SWAP: Exchanges the positions; it is defined by the action on product states:

$$\mathrm{SWAP}\, |a\rangle_A \otimes |b\rangle_B = |b\rangle_A \otimes |a\rangle_B\,.$$

Toffoli gate: One can see this gate as a reversible AND gate — since a simple AND gate is not reversible, for quantum computation purposes (using reversible unitary gates) one needs to adjust the AND gate. This is achieved by using three qubits. The result of AND operation is stored in the third qubit (to be more precise, the AND of the first two quibit is added modulo 2 to the third qubit) mapping

$$|a\rangle \otimes |b\rangle \otimes |c\rangle \mapsto |a\rangle \otimes |b\rangle \otimes |c \text{ XOR } (a \text{ AND } b)\rangle\,.$$

One can write this also as CCNOT, a NOT operation controlled by two qubits simultaneously.

General controlled operation: In general a controlled U operation has form

$$\text{C-}U_{A\to B} = |0\rangle_A \langle 0| \otimes \mathbb{1}_B + |1\rangle_A \langle 1| \otimes U.$$

In all these cases we might remove $A \to B$ specification if it is clear from the context, or if it is in standard setting, usally from first qubit to the second qubit.

Exercise 5.12. Prove following identities:

a) $\mathrm{CZ}_{A\to B} = \mathrm{CZ}_{B\to A}$ and so we can simply write CZ,

b) $\mathrm{CNOT}_{A\to B} = (\mathbb{1}_A \otimes H) \circ \mathrm{CZ} \circ (\mathbb{1}_A \otimes H)$,

c) $\mathrm{CZ} = (\mathbb{1}_A \otimes H) \circ \mathrm{CNOT}_{A\to B} \circ (\mathbb{1}_A \otimes H)$,

d) $\mathrm{CNOT}_{B\to A} = (H \otimes H) \circ \mathrm{CNOT}_{A\to B} \circ (H \otimes H)$,

e) $\mathrm{SWAP} = \mathrm{CNOT}_{A\to B} \circ \mathrm{CNOT}_{B\to A} \circ \mathrm{CNOT}_{A\to B}$,

f) $\mathrm{CY}_{A\to B} = (\mathbb{1}_A \otimes S^\dagger) \circ \mathrm{CNOT} \circ (\mathbb{1}_A \otimes S)$, where $S = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\frac{\pi}{2}} \end{pmatrix}$,

g) $\mathrm{CH}_{A\to B} = [\mathbb{1}_A \otimes R_y(\pi/4)] \circ \mathrm{CNOT} \circ [\mathbb{1}_A \otimes R_y(-\pi/4)]$, where $R_y(\theta) = \begin{pmatrix} \cos\theta/2 & -\sin\theta/2 \\ \sin\theta/2 & \cos\theta/2 \end{pmatrix}$,

h) $\mathrm{C}\pi(\alpha) = e^{i\alpha/2} R_z(\alpha) \otimes \mathbb{1}$, where $R_z(\alpha) = \begin{pmatrix} e^{-i\alpha/2} & 0 \\ 0 & e^{i\alpha/2} \end{pmatrix}$, and $\pi(\alpha)$ is just multiplication with complex number $e^{i\alpha}$, i.e. it adds a phase α.

Exercise 5.13. Apply the operation $H^{\otimes d} = H \otimes \cdots \otimes H$, where d Hadamard operations are tensored together, on the state $|0\rangle^{\otimes d}$.

Exercise 5.14. Let f be some boolean function $f : \{0, 1, \ldots, N\} \to \{0, 1\}$ and let us have a bipartite unitary operation O_f acting on $\mathbb{C}^N \otimes \mathbb{C}^2$ as

$$O_f\, |x\rangle \otimes |m\rangle = |x\rangle \otimes |m \oplus f(x)\rangle\,,$$

where $\oplus$ is addition modulo 2 and $|m\rangle$ is a state of the canonical basis. Describe this operation in the case of $|m\rangle = |-\rangle$.

Exercise 5.15. Let f be some boolean function $f : \{0, 1, \dots, N\} \to \{0, 1\}$ and let us have a unitary operation R_f acting on $\mathbb{C}^N$ as

$$R_f |x\rangle = (-1)^{f(x)} |x\rangle \, .$$

Supposing we can perform a controlled operation C-R_f, find an operation O_f on a larger system in $\mathbb{C}^N \otimes \mathbb{C}^2$ (where ancillary qubit is the control) such that it determines value of $f(x)$ on the second output.

Results.

5.1 a) $\frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$, b) $\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$, c) $\frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \equiv H_2$, d) $\begin{pmatrix} 0 & -1 & 0 & -i \\ -1 & 0 & i & 0 \\ 0 & -i & 0 & 1 \\ i & 0 & 1 & 0 \end{pmatrix}$.

5.2 a) $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, b) $|1\rangle \otimes |-\rangle$, c) $\frac{1}{\sqrt{2}}(|0\rangle + i\,|1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle - i\,|1\rangle)$.

5.3 $\sigma_x = |0\rangle\langle 1| + |1\rangle\langle 0|$, $\sigma_y = -i\,|0\rangle\langle 1| + i\,|1\rangle\langle 0|$, $\sigma_z = |0\rangle\langle 0| - |1\rangle\langle 1|$.

5.5 a) $|0\rangle\langle 0| \otimes \mathbb{1} + |1\rangle\langle 1| \otimes \sigma_z$, b) $\sigma_x \otimes \sigma_y$, c) $\sigma_z \otimes \sigma_x$.

5.6 a) $|00\rangle$, b) $|+\rangle \otimes |0\rangle$, c) $|0\rangle \otimes |0\rangle - |1\rangle \otimes |+\rangle$, d) $-|+\rangle \otimes |0\rangle + |-\rangle \otimes |1\rangle$.

5.7 $|0\rangle\langle 0| \otimes \mathbb{1} + |1\rangle\langle 1| \otimes \sigma_x$ and $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$.

5.8 All three schemes perform preparation of the Bell states.

5.9 To measure in the Bell basis, we apply $(H \otimes \mathbb{1})$CNOT on the state and then measure in the computational basis.

5.13 $\frac{1}{\sqrt{2^d}} \sum_{j=0}^{2^d-1} |j_2\rangle$, where j_2 is a number j written in base 2.

5.14 $O_f |x\rangle \otimes |-\rangle = (-1)^{f(x)} |x\rangle \otimes |-\rangle$.

6 Partial operations

6.1 Bra-ket notation

Dirac bra-ket notation is a very useful tool in quantum theory. We have seen its use to compute scalar product, but its application is much wider. At this point let us just note one additional feature of this notation. Having two d-dimensional vectors $|u\rangle$ and $|v\rangle$, we can construct also an outer product of these vectors, $|u\rangle\langle v|$, which produces a matrix with dimension $d \times d$. The usefulness of this construction will become apparent soon.

▷ This next set of rules is very important to remember and understand!

Exercise 6.1. Using matrix representation show following:

a) $(c\,|u\rangle)^\dagger = c^*\langle u|$, for $c \in \mathbb{C}$,

b) $(|u\rangle\langle v|)^\dagger = |v\rangle\langle u|$,

c) $(|u\rangle\langle v|)\,|w\rangle = \langle v|w\rangle\,|u\rangle$.

▷ To illustrate previous rules, try following exercise.

Exercise 6.2. Let

$$|u\rangle = \begin{pmatrix} 2+i \\ -3i \\ 1 \end{pmatrix}, \qquad |v\rangle = \begin{pmatrix} 2 \\ -i \\ 1+i \end{pmatrix}, \qquad |w\rangle = \begin{pmatrix} i \\ 0 \\ -3 \end{pmatrix}.$$

Compute following formulas

a) $\langle q|$, where $|q\rangle = (1-i)\,|v\rangle$,

b) $\langle u|v\rangle$,

c) $|w\rangle\langle u|$,

d) $|u\rangle\langle w|$,

e) $(|w\rangle\langle u|)\,|v\rangle$,

f) $(\langle u|v\rangle)\,|w\rangle$.

Exercise 6.3. Let us have $d = 3$ and the computational basis $\{|0\rangle, |1\rangle, |2\rangle\}$. Express all combinations of matrices $E_{jk} = |j\rangle\langle k|$ and using these symbols write following matrices

a) $\begin{pmatrix} 0 & 1 & 0 \\ -i & 2 & 0 \\ 0 & 0 & 1-i \end{pmatrix}$,

b) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbb{1}$,

c) $\begin{pmatrix} 2i & -4 & 0 \\ 1-i & 0 & 3i \\ 1 & 1 & -2i \end{pmatrix}$.

We can now notice that a matrix, let us say C, can be conveniently expressed using the Dirac's notation in the computational basis as

$$C = \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}\,|i\rangle\langle j|\,.$$

This notation especially shines for matrices where a significant number of entries is equal to zero.

But this expression of matrix C holds not only in standard basis but in any orthonormal basis:

$$C = \sum_j\sum_k c'_{jk}\,|w_j\rangle\langle w_k| = \sum_j\sum_k \langle w_j|\,C\,|w_k\rangle\,|w_j\rangle\langle w_k|\,, \tag{3}$$

where $\{|w_j\rangle\}_j$ is the orthonormal basis and elements $c'_{jk} = \langle w_j|\,C\,|w_k\rangle$ represent C in this basis.

Exercise 6.4. Write following matrices in their matrix form:

a) $|\pm\rangle\langle\pm|$,

b) $|+\rangle\langle 0| + |-\rangle\langle 1|$,

c) $|+\rangle\langle +| - |-\rangle\langle -|$,

d) $2\,|s\rangle\langle s| - \mathbb{1}$, where $|s\rangle = \frac{1}{\sqrt{d}}\sum_{j=1}^{d}|j\rangle$ and $\{|j\rangle\}$ is the canonical basis.

▷ The symbol $\mathbb{1}$ for identity (matrix) is very useful in many situations. In the next exercise we provide few of the many rules for the identity that are worth remembering as they will be helpful in many computations.

Exercise 6.5. We have defined $\mathbb{1}$ as an identity matrix. Suppose the dimension of the vector space is d, show that

a) if $\{|j\rangle\}_{j=0}^{d-1}$ is the computational basis, then $\sum_{j=0}^{d-1}|j\rangle\langle j| = \mathbb{1}$,

b) for every vector $|v\rangle$ it holds that $\mathbb{1}\,|v\rangle = |v\rangle$,

c) $\mathbb{1} = \sum_{j=0}^{d-1}|w_j\rangle\langle w_j|$ for any orthonormal basis $\{|w_j\rangle\}_{j=0}^{d-1}$,

d) if $AB = \mathbb{1}$, then also $BA = \mathbb{1}$.

Exercise 6.6. Using matrix representation show following:

a) $(C\,|\psi\rangle)^\dagger = \langle\psi|\,C^\dagger$,

b) $(AB)^\dagger = B^\dagger A^\dagger$,

c) if D is diagonal, then $D^\dagger = D^*$,

where A, B, C, and D are complex square matrices of appropriate dimesnions.

Exercise 6.7*. Express all Pauli matrices and the Hadamard matrix in

a) the $\{|0\rangle, |1\rangle\}$ basis,

b) the $\{|\pm\rangle\}$ basis,

c) the $\left\{\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -i\end{pmatrix}\right\}$ basis,

d) the $\left\{\frac{1}{2}\begin{pmatrix}1\\ i\sqrt{3}\end{pmatrix}, \frac{1}{2}\begin{pmatrix}i\sqrt{3}\\ 1\end{pmatrix}\right\}$ basis.

Using bra-ket form, we can evaluate also matrix sandwiched between vectors:

$$\langle\psi|\,U\,|\phi\rangle = \langle\psi|\left[\sum_j\sum_k u_{jk}\,|j\rangle\langle k|\right]|\phi\rangle = \sum_j\sum_k u_{jk}\langle\psi|j\rangle\langle k|\phi\rangle.$$

In particular the trace of matrix can be expressed as

$$Tr(A) = \sum_{i=0}^{n-1} A_{ii} = \sum_{i=0}^{n-1}\langle i|\,A\,|i\rangle\,.$$

Exercise 6.8. Compute following:

a) $\langle\pm|\,\sigma_x\,|\pm\rangle$, $\langle\pm|\,\sigma_x\,|\mp\rangle$

b) $\langle\pm|\,\sigma_z\,|\pm\rangle$,

c) $\langle s|\,D\,|s\rangle$, where $|s\rangle = \frac{1}{\sqrt{d}}\begin{pmatrix}1, & 1, & \ldots, & 1\end{pmatrix}$ and D is diagonal matrix $D = \sum_{j=1}^{d} d_j\,|j\rangle\langle j|$.

▷ Trace of a matrix does not seem to be very important at this point, but we will see its use later. Again, if you want to be able to efficiently manipulate the traces, have a look at the following exercise.

Exercise 6.9*. Let A and B be $n \times n$ matrices. Show that

a) $\mathrm{Tr}(A+B) = \mathrm{Tr}(A) + \mathrm{Tr}(B)$,

b) $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$,

c) $\mathrm{Tr}(A)$ does not depend on the choice of the base,

d) $p_\phi(|\psi\rangle) = |\langle\phi|\psi\rangle|^2 = \mathrm{Tr}(P_\phi P_\psi)$, where $P_\phi = |\phi\rangle\langle\phi|$ and $P_\psi = |\psi\rangle\langle\psi|$ are states.

Exercise 6.10. Compute traces of the following matrices

a) H,

b) Pauli matrices,

c) $U = \frac{1}{2}\begin{pmatrix} 1 & -i\sqrt{3} \\ -i\sqrt{3} & 1 \end{pmatrix}$,

d) $\mathbb{1}$ in dimension d,

e) $\sum_{j=0}^{d-1} e^{\frac{2\pi i j}{d}} |j\rangle\langle j|$,

f) $\sum_{j=0}^{d-1} e^{\frac{2\pi i j}{d}} |j\rangle\langle j \oplus 1|$, where $\oplus$ is addition modulo d.

6.2 Partial scalar product and measurements

Partial scalar product on $\mathcal{V} = \mathcal{A} \otimes \mathcal{B}, \mathcal{A} = \mathbb{C}^n, \mathcal{B} = \mathbb{C}^m$ is defined as

$$_A\langle i| \left(|a\rangle_A \otimes |b\rangle_B\right) = \langle i|a\rangle |b\rangle_B .$$

The scalar product maps from $\mathcal{V}$ to $\mathbb{C} \otimes \mathcal{B}$, which can be seen as seleceting outcome i and disregarding system $\mathcal{A}$ effectively selecting state $|b\rangle_B$; this happens with probability $|\langle i|a\rangle|^2$.

As in the case of a full system measurement, where scalar products of given state with a set of basis states defined measurements, if the states $|i\rangle_A$ used in the partial scalar product are from an orthonormal basis, they define a *partial measurement.* Let us have state

$$|\psi\rangle_{AB} = \sum_a \sum_b c_{ab} |ab\rangle_{AB} ,$$

then if we have a measurement A in basis $|i\rangle_A$ in which we already expressed the state, the measurement of getting state $|i\rangle_A$ is

$$_A\langle i| \left(|\psi\rangle_{AB}\right) = \sum_a \sum_b c_{ab} \langle i|a\rangle |b\rangle_B = \sum_b c_{ib} |b\rangle_B \equiv |\tilde{\psi}(B|a=i)\rangle .$$

This is an unnormalized vector that should be understood as a probabilistic representation of state on site B given the measurement on site A yield outcome i. The probability of obtaining the outcome i can be recovered from this vector,

$$p(a=i) = \langle\tilde{\psi}(B|a=i)|\tilde{\psi}(B|a=i)\rangle = \sum_b c_{ib}^* c_{ib} = |_A\langle i|\psi\rangle_{AB}|^2.$$

The state we obtain can be renormalized by the square root of the probability, so that it has form

$$|\psi(B|a=i)\rangle = \frac{1}{|_A\langle i|\psi\rangle_{AB}|} |\tilde{\psi}(B|a=i)\rangle .$$

Exercise 6.11. Compute following partial scalar products, determine the final state and determine its probability:

a) $_A\langle 0|\,(|01\rangle_{AB})$,

b) $_A\langle +|\,(|01\rangle_{AB})$,

c) $_A\langle 0|\,(|\Psi^-\rangle_{AB})$, and $_A\langle 1|\,(|\Psi^-\rangle_{AB})$

d) $_A\langle 1|\,\frac{1}{\sqrt{10}}(\sqrt{2}\,|01\rangle_{AB} - i\sqrt{3}\,|10\rangle_{AB} - (\sqrt{3}i + \sqrt{2})\,|11\rangle_{AB})$.

Exercise 6.12. Determine the probabilities with which Bob measures 0 or 1 and in both cases describe the state of Alice:

a) $|+\rangle \otimes |-\rangle$,

b) $\frac{1}{\sqrt{5}}\left[(1-i)\,|01\rangle + \sqrt{3}i\,|11\rangle\right]$,

c) $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$,

d) $\frac{1}{\sqrt{6}}\left(|00\rangle + \sqrt{2}\,|01\rangle - i\sqrt{3}\,|10\rangle\right)$.

▷ The next exercise should be valuable not only because you practice your computation skills, but also because it shows that the order of measurements, despite them being destructive, is not relevant in cases when they are independent (when one does not depend on the outcome of the other).

Exercise 6.13. Let us consider a tripartite state

$$|\psi\rangle = \frac{1}{2\sqrt{2}}\left(\sqrt{2}\,|000\rangle + |010\rangle - |100\rangle + \sqrt{3}\,|011\rangle - |111\rangle\right).$$

If the three parties are Alice-Bob-Charlie, compare the following situations:

1. Alice measures 0 and after her Bob measures 1,
2. Bob measures 1 and after him Alice measures 0.

Give probabilities for both observations in the two situations and express the final state (of Charlie).

6.3 Partial trace and density matrices

Let $\mathcal{V} = \mathcal{A} \otimes \mathcal{B}, \mathcal{A} = \mathbb{C}^n, \mathcal{B} = \mathbb{C}^m$ and let us have state $|\psi\rangle_{AB}$. Having an orthonormal basis on $\mathcal{A}$, $\{|i\rangle_A\}_i$ we can define a *partial trace* over A of this state:

$$\mathrm{Tr}_A[|\psi\rangle_{AB}\langle\psi|] = \sum_i {}_A\langle i|\,(|\psi\rangle_{AB}\langle\psi|)\,|i\rangle_A\,.$$

The result is a matrix that can no longer be expressed as $|\psi\rangle_{AB}\langle\psi|$. Nevertheless, it still can express a state called *mixed*. Such state description can be interpreted as a probabilistic state preparation conditioned on measurement of $|i\rangle$. It reflects possibility that the A part might be inaccessible and this matrix then represents the best knowledge of that state on side of B. The obtained matrix is called the *density matrix* and if it is obtained from a larger state (not necessarily pure in the form $|\psi\rangle_{AB}\langle\psi|$) is called *reduced state*. Similarly as defined for A, we can define partial trace over any other subsystem.

Partial trace provides only one of the explanation of density matrix. Density matrix can be obtained also as a way of mixing states, for example if we prepare state $|0\rangle$ with probability $1/3$ and state $|-\rangle$ with probability $2/3$, the density matrix of this preparation procedure (i.e. the description of a state coming from this procedure) is

$$\rho = \frac{1}{3}\,|0\rangle\langle 0| + \frac{2}{3}\,|-\rangle\langle -| = \frac{1}{3}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} + \frac{1}{3}\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix} = \frac{1}{3}\begin{pmatrix}2 & -1\\ -1 & 1\end{pmatrix}.$$

In the end let us note, that the density matrix does not have to be connected to any reasoning and can be considered as a general way of describing a state. Necessary conditions for matrix ρ to describe state is that $\mathrm{Tr}[\rho] = 1$ and that it is a positive matrix.[a]

[a] This will be explained later, for now think of it as a matrix ρ such that $\langle v|\rho|v\rangle \geq 0$ for all vectors (states) $|v\rangle$.

▷ In the next two exercises notice how the result changes when the original states are separable, i.e. possible to write as a simple tensor product, and when they are not.

Exercise 6.14. Compute partial trace over A for the following states:

a) $|01\rangle$,

b) $|-+\rangle$,

c) $|\psi\rangle_{AB} = \frac{1}{2\sqrt{2}}\left(-i\sqrt{3}\,|00\rangle + \sqrt{3}\,|01\rangle - i\,|10\rangle + |11\rangle\right)$.

Exercise 6.15. Compute partial traces over part A as well as part B for states:

a) $|\psi\rangle = \frac{1}{\sqrt{6}}\left(|00\rangle + \sqrt{2}\,|01\rangle - i\sqrt{3}\,|10\rangle\right)$,

b) $|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$.

▷ Sometimes a matrix may not look like a density matrix of a state and yet it can be, and sometimes it is the other way round. It is important to be able to decide whether a density matrix represent a state or not. This you can practice on the next exercise. While we still do not have good tools to compute positivity of a matrix, we can always use a definition and do it the tedious way. For matrices of dimension 2 this should, however, pose no problem.

Exercise 6.16. Decide whether the following density matrices represent qubit states:

a) Pauli matrices,

b) $\frac{1}{d}\mathbb{1}_d$,

c)* $C = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$,

d)* $D = \frac{1}{2}\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$.

▷ The next exercise shows some basic features of approximate cloning.

Exercise 6.17*. No-cloning theorem states that it is impossible to clone quantum information — unlike in the classical case, one can not make a perfect copy of a quantum state. The best one can do for cloning an arbitrary qubit state is to use Bužek-Hillery approximate copier, which for an arbitrary state $|\psi\rangle_A$ produces

$$|\Phi\rangle_{ABC} = -\sqrt{\frac{2}{3}}\,|\psi\rangle_A\,|\psi\rangle_B\left|\psi^\perp\right\rangle_C + \sqrt{\frac{1}{6}}\,|\psi\rangle_A\left|\psi^\perp\right\rangle_B|\psi\rangle_C + \sqrt{\frac{1}{6}}\left|\psi^\perp\right\rangle_A|\psi\rangle_B\,|\psi\rangle_C\,,$$

where the C part of the system is ancillary (helping) qubit and $\left|\psi^\perp\right\rangle$ is a (unique) qubit state perpendicular to $|\psi\rangle$. What are the reduced states ρ_A, ρ_B, ρ_C?

6.4 Measurements III

Suppose Alice has a state $|v\rangle$ and performs measurement in basis $\{|w_j\rangle\}_j$. We had that she will measure outcome w_j with probability

$$p^A_{w_j}(|v\rangle) = |\langle w_j|v\rangle|^2 = \langle w_j|v\rangle\langle v|w_j\rangle = \mathrm{Tr}\left[P_v P_{w_j}\right] = \mathrm{Tr}\left[P_{w_j} P_v\right],$$

where $P_v = |v\rangle\langle v|$ and $P_{w_j} = |w_j\rangle\langle w_j|$. The above formula works also when Alice has a state described by a density matrix ρ, i.e. $p^A_{w_j}(\rho) = \langle w_j|\,\rho\,|w_j\rangle = \mathrm{Tr}\left[P_{w_j}\rho\right]$.

After Alice measures outcome w_j, the state (if not destroyed) will be $|w_j\rangle$. If Alice hands the state to Bob and provides him with information about the outcome w_j and the basis she measured in, Bob knows the state he has. However, if Alice does not provide this information to Bob, the state he is given will be described by a density matrix

$$\rho_B = \sum_j p^A_{w_j}(\rho) |w_j\rangle \langle w_j| .$$

Note that the description of state ρ_B does not tell any information about the basis or the outcomes even if the form suggests it. But it still can contain some information about either the state Alice had before, or the measurement she performed.

▷ Note in the next two exercises, how Bob can tell (at least statistically), at least to some extent, what state was Alice measuring or what measurement she was performing. This is important in quantum applications in that one can detect whether a state has been manipulated by somebody else before. Do you see how this is an important security feature of the quantum theory?

Exercise 6.18. Let Alice perform measurement in the canonical basis on her state, which she sends to Bob afterwards without telling him the outcomes. Describe the state Bob has using density matrix and determine the probabilities Bob will get for the outcomes of a measurement in the computational basis performed on the handed state, if the original state of Alice was

a) $|a\rangle = |0\rangle$,
b) $|b\rangle = |-\rangle$,
c) $|c\rangle = \frac{1}{2}\left(|0\rangle + i\sqrt{3}|1\rangle\right)$.

Exercise 6.19. Suppose Alice has state

$$|\phi\rangle = \frac{1}{2}\left(|0\rangle + i\sqrt{3}|1\rangle\right)$$

at the beginning and then performs measurement M. The state after the measurement is then sent to Bob without telling him what the measurement was. Compute the density matrix of the state Bob receives and compute probability of him measuring outcome ϕ if he performs measurement in the basis $\{|\phi\rangle, |\phi^\perp\rangle\}$ where $|\phi^\perp\rangle$ is some state orthogonal to state $|\phi\rangle$. For the measurement M use bases

a) $\{|0\rangle, |1\rangle\}$,
b) $\{|+\rangle, |-\rangle\}$,
c) $\{|\phi\rangle, |\phi^\perp\rangle\}$.

Now consider a bipartite system $|\psi\rangle_{AB}$ with one part belonging to Alice and one to Bob. This joint state is measured by Alice (on her subsystem) in some orthonormal basis $\{|w_j\rangle_A\}$. Then the probability that Alice obtains outcome w_j is

$$p^A_{w_j}(|\psi\rangle) = \langle\tilde{\psi}(B|a=j)|\tilde{\psi}(B|a=j)\rangle = {}_{AB}\langle\psi|w_j\rangle_A\langle w_j|\psi\rangle_{AB} = \sum_k {}_{AB}\langle\psi|w_j\rangle_A |v_k\rangle_B \langle v_k|_A \langle w_j|\psi\rangle_{AB},$$

where in the last equality we have expanded identity on Bob's side in some orhtonormal basis $\{|v_k\rangle\}_k$. We can notice that the inclusion of identity creates terms that are squared inner products ${}_B\langle v_k|_A \langle w_j|\psi\rangle_{AB}$ and so we can exchange them to get

$$p^A_{w_j}(|\psi\rangle) = \sum_k {}_B\langle v_k|_A \langle w_j|\psi\rangle_{AB}\langle\psi|w_j\rangle_A |v_k\rangle_B = {}_A\langle w_j| \mathrm{Tr}_B[|\psi\rangle\langle\psi|] |w_j\rangle .$$

This formula shows, that what Alice actually performs is her measurement on effective state $\mathrm{Tr}_B[|\psi\rangle\langle\psi|]$, which is state with knowledge about Bob's part excluded. Furthermore, we can remember, that for any general bipartite state that is expressed via density matrix ρ we have also a general formula

$$p^A_{w_j}(\rho) = {}_A\left\langle w_j\right| \mathrm{Tr}_B[\rho] \left|w_j\right\rangle = \mathrm{Tr}\Big[\rho \left|w_j\right\rangle_A \left\langle w_j\right|\Big].$$

Now that Alice performed a measurement, she can either tell Bob her measurement and outcome she obtained to give him better understanding of what state he has. But if Alice does not tell him her what measurement she performed, Bob's description of state is

$$\begin{aligned}\rho_B &= \sum_j p^A_{w_j}(|\psi\rangle) \left|\tilde{\psi}(B|a=j)\right\rangle \left\langle\tilde{\psi}(B|a=j)\right| = \sum_j \left|\psi(B|a=j)\right\rangle \left\langle\psi(B|a=j)\right| \\ &= \sum_j {}_A\langle w_j|\psi\rangle_{AB}\langle\psi|w_j\rangle_A = \mathrm{Tr}_A[|\psi\rangle\langle\psi|].\end{aligned}$$

Since the partial trace does not depend on choice of the basis, just like the regular trace, this formula shows, that on a bipartite system, Bob can never find out what measurement Alice performed. He can not even find out whether Alice performed any measurement.

▷ The next exercise shows that one can not find by measuring part of the system, whether somebody was measuring on the other part of the system, or what she was measuring. This is a very important feature that might not look very practical, but if it were not so, then there would exist a possibility of a faster-than-light communication. See how quantum theory nicely obeys Einstein's laws of relativity?

Exercise 6.20. Suppose Alice and Bob share a state

$$|\Phi\rangle_{AB} = \frac{1}{2}\Big(|00\rangle + i\sqrt{3}\,|11\rangle\Big)$$

at the beginning and then Alice performs measurement M, but she does not tell this information to Bob. Compute the density matrix of the state Bob has and compute the probability of him measuring outcome ϕ if he performs measurement in the basis $\{|\phi\rangle, |\phi^\perp\rangle\}$ where $|\phi^\perp\rangle = \frac{1}{2}\Big(i\sqrt{3}\,|0\rangle + |1\rangle\Big)$ is some state orthogonal to state $|\phi\rangle = \frac{1}{2}\Big(|0\rangle + i\sqrt{3}\,|1\rangle\Big)$. For the measurement M use bases

a) $\{|0\rangle, |1\rangle\}$, b) $\{|+\rangle, |-\rangle\}$, c) $\{|\phi\rangle, |\phi^\perp\rangle\}$.

Results.

6.2 a) $(1-i)^*\langle v|$, b) $8-i$, c) $\begin{pmatrix} 1+2i & -3 & i \\ 0 & 0 & 0 \\ -6+3i & -9i & -3 \end{pmatrix}$, d) $\begin{pmatrix} 1-2i & -3 & -6-3i \\ -3 & 0 & 9i \\ -i & 0 & -3 \end{pmatrix} = \big(|w\rangle\langle u|\big)^\dagger$, e) $\begin{pmatrix} 1+8i \\ 0 \\ -24+3i \end{pmatrix}$, f) $\begin{pmatrix} 1+8i \\ 0 \\ -24+3i \end{pmatrix}$.

6.3 a) $E_{01} - iE_{10} + 2E_{11} + (1-i)E_{22}$, b) $E_{00} + E_{11} + E_{22}$, c) $2iE_{00} - 4E_{01} + (1-i)E_{10} + 3iE_{12} + E_{20} + E_{21} - 2iE_{22}$.

6.4 a) $\frac{1}{2}\begin{pmatrix} 1 & \pm 1 \\ \pm 1 & 1 \end{pmatrix}$, b) H, c) σ_x, d) $\begin{pmatrix} -r & t & t & \dots & t \\ t & -r & t & \dots & t \\ t & t & -r & \dots & t \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ t & t & t & \dots & -r \end{pmatrix}$, where $t = \dfrac{2}{d}$ and $r = 1-t$.

6.7 a) $\sigma_x = |1\rangle\langle 0| + |0\rangle\langle 1|$, $\sigma_y = i|1\rangle\langle 0| - i|0\rangle\langle 1|$, $\sigma_z = |0\rangle\langle 0| - |1\rangle\langle 0|$, $H = \frac{1}{\sqrt{2}}(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| - |1\rangle\langle 1|)$.
b) $\sigma_x = |+\rangle\langle +| - |-\rangle\langle -|$, $\sigma_y = i|+\rangle\langle -| - i|-\rangle\langle +|$, $\sigma_z = |+\rangle\langle -| + |-\rangle\langle +|$, $H = \frac{1}{\sqrt{2}}(|+\rangle\langle +| +$

$|+\rangle\langle -| + |-\rangle\langle +| - |-\rangle\langle -|$.

c) Denoting $|R\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix}$ and $|L\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -i\end{pmatrix}$ we have

$\sigma_x = i|L\rangle\langle R| - i|R\rangle\langle L|$, $\sigma_y = |R\rangle\langle R| - |L\rangle\langle L|$,

$\sigma_z = |R\rangle\langle L| + |L\rangle\langle R|$, $H = e^{i\frac{\pi}{4}}|L\rangle\langle R| + e^{-i\frac{\pi}{4}}|R\rangle\langle L|$.

d) Denoting $|a\rangle = \frac{1}{2}\begin{pmatrix}1\\ i\sqrt{3}\end{pmatrix}$ and $|b\rangle = \frac{1}{2}\begin{pmatrix}i\sqrt{3}\\ 1\end{pmatrix}$ we have

$\sigma_x = |a\rangle\langle b| + |b\rangle\langle a|$, $\sigma_y = \frac{1}{2}\left(i|a\rangle\langle b| - i|b\rangle\langle a| + \sqrt{3}|a\rangle\langle a| - \sqrt{3}|b\rangle\langle b|\right)$,

$\sigma_z = \frac{1}{2}\left(i\sqrt{3}|a\rangle\langle b| - i\sqrt{3}|b\rangle\langle a| - |a\rangle\langle a| + |b\rangle\langle b|\right)$,

$H = \frac{1}{2\sqrt{2}}\left[(2+i\sqrt{3})|a\rangle\langle b| + (2-i\sqrt{3})|b\rangle\langle a| - |a\rangle\langle a| + |b\rangle\langle b|\right]$.

6.8 a) $\langle\pm|\sigma_x|\pm\rangle = \pm\langle\pm|\pm\rangle = \pm 1$, $\langle\pm|\sigma_x|\mp\rangle = \mp\langle\pm|\mp\rangle = 0$, b) 0, c) $\frac{1}{d}\sum_k d_k$.

6.10 a) 0, b) 0, c) 1, d) d, e) 0, f) 0.

6.11 a) $|1\rangle_B$, $|1\rangle_B$, probability is 1.

b) $\frac{1}{\sqrt{2}}|1\rangle_B$, $|1\rangle_B$, probability is 1/2.

c) With state $|0\rangle_A$: $\frac{1}{\sqrt{2}}|1\rangle_B$, $|1\rangle_B$, probability is 1/2; with state $|1\rangle_A$: $-\frac{1}{\sqrt{2}}|0\rangle_B$, $-|0\rangle_B$, probability is 1/2.

d) $-i\sqrt{\frac{3}{10}}|0\rangle_B - \frac{\sqrt{3}i+\sqrt{2}}{\sqrt{10}}|1\rangle_B$, $-i\sqrt{\frac{3}{8}}|0\rangle_B - \frac{\sqrt{3}i+\sqrt{2}}{\sqrt{8}}|1\rangle_B$, probability is 4/5.

6.12 Denoting Alice's state as $|a_j\rangle$ for outcome j of Bob, we have

a) $p_0 = p_1 = 1/2$, $|a_\pm\rangle = |+\rangle$,

b) $p_0 = 0$, there is no state $|a_0\rangle$, $p_1 = 1$, $|a_1\rangle = \frac{1}{\sqrt{5}}\left[(1-i)|0\rangle + \sqrt{3}i|1\rangle\right]$,

c) $p_0 = p_1 = 1/2$, $|a_0\rangle = |0\rangle$, $|a_1\rangle = |1\rangle$,

d) $p_0 = 2/3$, $|a_0\rangle = \frac{1}{2}\left(|0\rangle - i\sqrt{3}|1\rangle\right)$, $p_1 = 1/3$, $|a_1\rangle = |0\rangle$.

6.13 (1) $p(a=0) = 3/4$, $p(b=1|a=0) = 2/3$, $p(a=0,b=1) = 1/2$, $|\psi\rangle_C = \frac{1}{2}\left(|0\rangle_C + \sqrt{3}|1\rangle_C\right)$,

(2) $p(b=1) = 5/8$, $p(a=0|b=1) = 4/5$, $p(a=0,b=1) = 1/2$, $|\psi\rangle_C = \frac{1}{2}\left(|0\rangle_C + \sqrt{3}|1\rangle_C\right)$.

6.14 a) $|1\rangle_B\langle 1|$, or $|1\rangle_B$, b) $|+\rangle_B\langle +|$, or $|+\rangle_B$, c) $\frac{1}{2}(-i|0\rangle + |1\rangle)(i\langle 0| + \langle 1|)$, or $\frac{1}{\sqrt{2}}(-i|0\rangle + |1\rangle)$.

6.15 a) $\rho_A = \frac{1}{2}|0\rangle_A\langle 0| + i\frac{\sqrt{3}}{6}|0\rangle_A\langle 1| - i\frac{\sqrt{3}}{6}|1\rangle_A\langle 0| + \frac{1}{2}|1\rangle_A\langle 1| = \frac{1}{6}\begin{pmatrix}3 & i\sqrt{3}\\ -i\sqrt{3} & 3\end{pmatrix}$,

$\rho_B = \frac{2}{3}|0\rangle_B\langle 0| + \frac{\sqrt{2}}{6}|0\rangle_B\langle 1| + \frac{\sqrt{2}}{6}|1\rangle_B\langle 0|\frac{1}{3}|1\rangle_B\langle 1| = \frac{1}{6}\begin{pmatrix}4 & \sqrt{2}\\ \sqrt{2} & 2\end{pmatrix}$,

b) $\rho_A = \rho_B = \frac{1}{2}\mathbb{1}_A$.

6.16 a) no (trace is not one), b) yes, c) yes, d) no (not positive).

6.17 $\rho_A = \rho_B = \frac{2}{3}|\psi\rangle_A\langle\psi| + \frac{1}{6}\mathbb{1}_A$, $\rho_C = \frac{1}{3}|\psi^\perp\rangle_C\langle\psi^\perp| + \frac{1}{3}\mathbb{1}_C$.

6.18 a) $\rho^B = |0\rangle\langle 0|$, $p_0^B = 1$, $p_1^B = 0$,

b) $\rho^B = \frac{1}{2}\mathbb{1}$, $p_0^B = p_1^B = 1/2$,

c) $\rho_c^B = \frac{1}{4}|0\rangle\langle 0| + \frac{3}{4}|1\rangle\langle 1|$, $p_0^B = 1/4$, $p_1^B = 3/4$..

6.19 a) $\frac{1}{4}\begin{pmatrix}1 & 0\\ 0 & 3\end{pmatrix}$, 5/8, b) $\mathbb{1}/2$, 1/2, c) $|\phi\rangle\langle\phi|$, 1.

6.20 In all cases the density matrix of Bob's state and, hence, also the outcome probabilities for his measurements are the same, $\rho_B = \frac{1}{4}\begin{pmatrix}1 & 0\\ 0 & 3\end{pmatrix}$ and $p_\phi^B(\rho_B) = 5/8$.