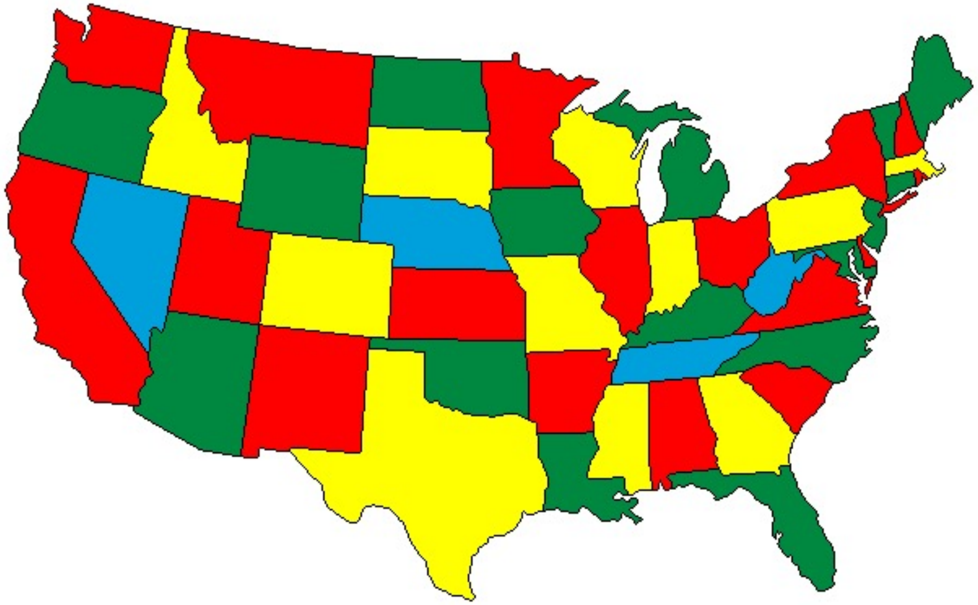
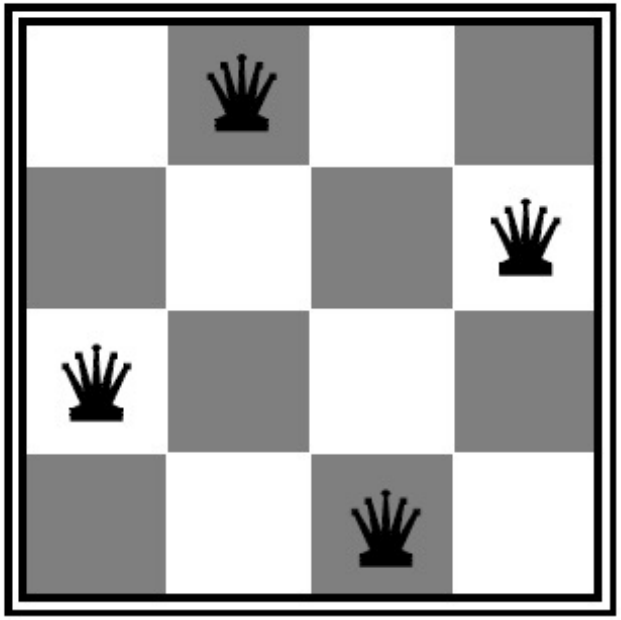
Constraint Satisfaction Problems





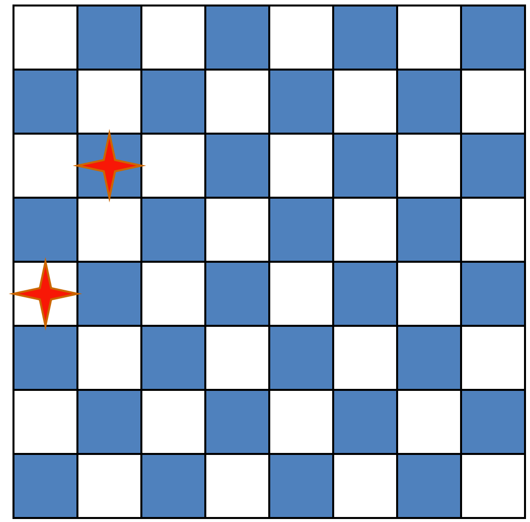
Sanja Lazarova-Molnar

Overview

- Constraint satisfaction is a powerful problem-solving paradigm
  - Problem: set of variables to which we must assign values satisfying problem-specific constraints
  - Constraint programming, constraint satisfaction problems (CSPs), constraint logic programming…
- Algorithms for CSPs
  - Backtracking
  - Constraint propagation
  - Variable and value ordering heuristics

Motivating example: 8 Queens

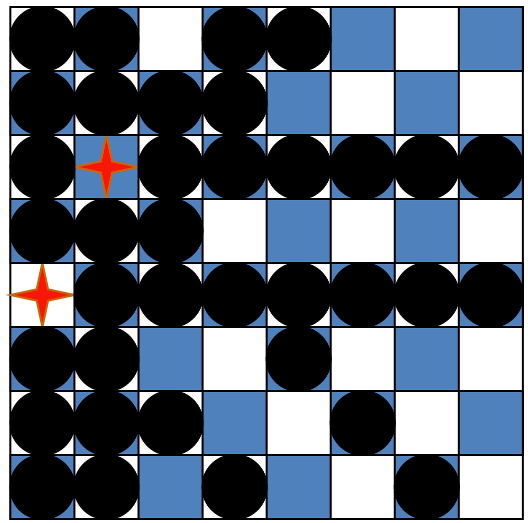
Place 8 queens on a chess board such
That none is attacking another.



Generate-and-test, with no
redundancies → "only" 8^8 combinations

8**8 is 16,777,216

Motivating example: 8-Queens



After placing these two queens, it's trivial to mark the squares we can no longer use

What more do we need for 8 queens?

- Not just a successor function and goal test
- But also
  - a way to propagate constraints imposed by one queen on others
  - an early failure test

CSP Definitions

- **Variables** - $X_1, X_2, \ldots, X_n$ each X_i having a non-empty domain D_i of possible values.
- **Constraints** - $C_1, C_2, \ldots, C_m$ consisting of some subset of variables and specifies allowable combinations of values for that subset.
- **State** - defined by an *assignment* of values to some or all variables ($X_1=v_1$, $X_j=v_j$, ...)
- **Consistent** - assignment that does not violate any constraints.
- **Solution** - complete assignment that satisfies all constraints

CSP Formulation

- **Initial state** - empty assignment: all variables are unassigned.
- **Successor function** - assigns value to an unassigned variable that does not conflict with previously assigned variables
- **Goal test** - complete current assignment
- **Path cost** - constant cost per step

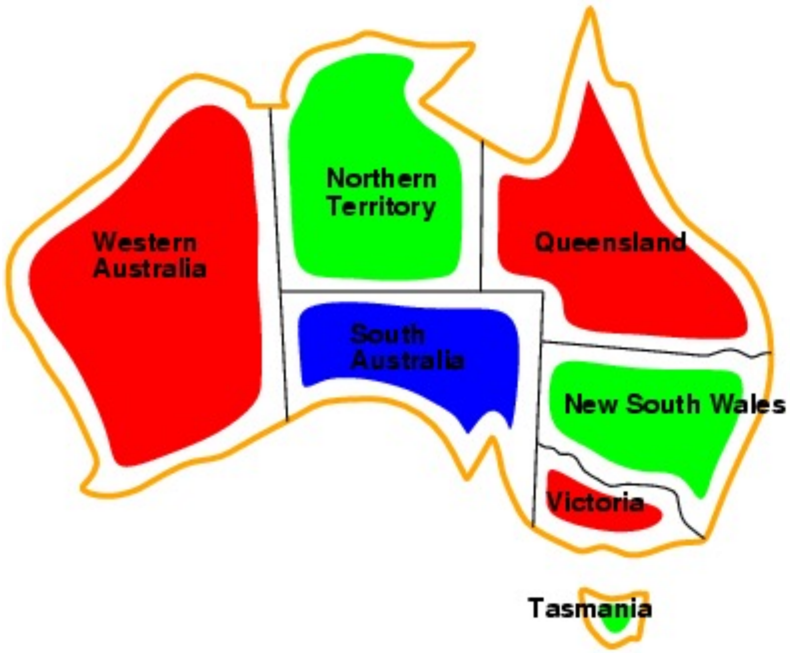
Example: Map Coloring

Variables: WA, NT, Q, NSW, V, SA, T

Domains: {red, green, blue}

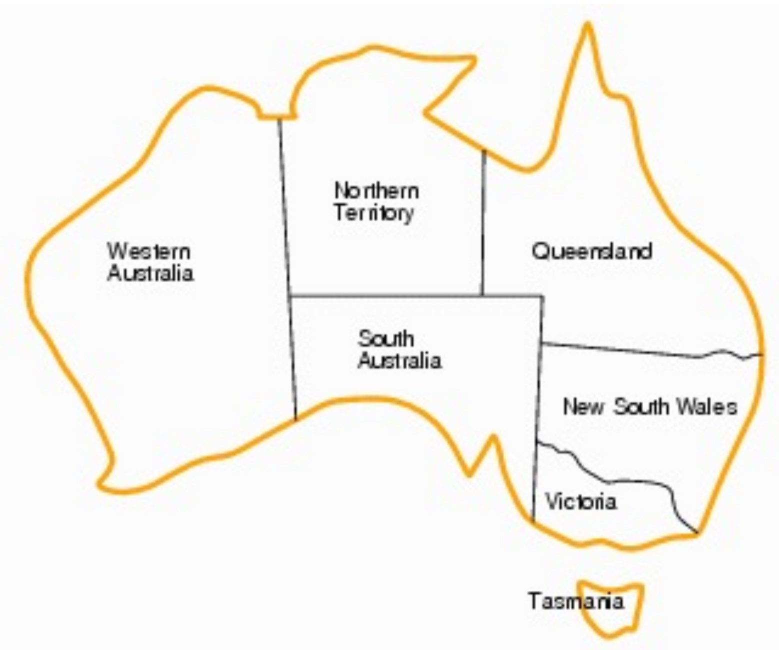
Constraints: adjacent regions must have different colors
e.g., WA ≠ NT, or (WA, NT) in {(red, green), (red, blue), (green, red), (green, blue), (blue, red), (blue, green)}

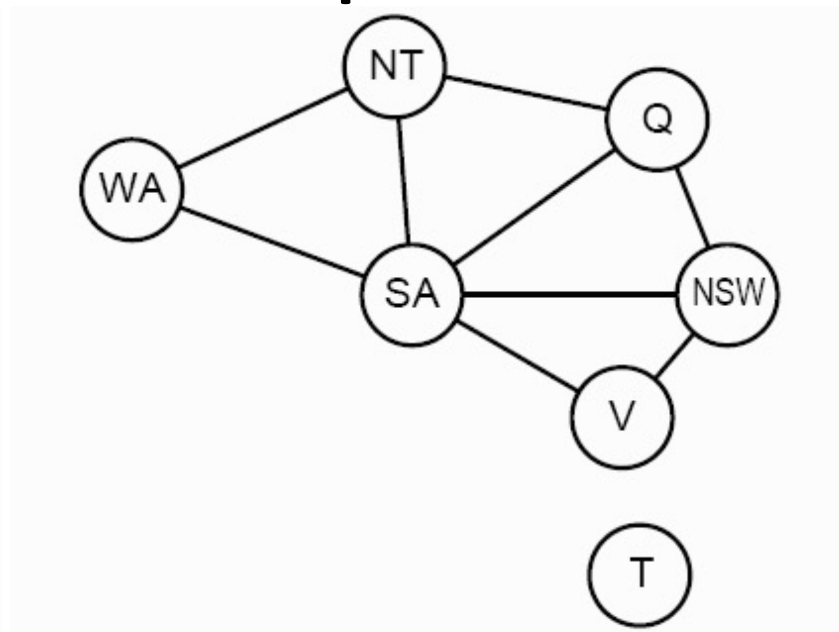
Example: Map Coloring



- State - one of many but not a solution, e.g. WA = red, NT = red, Q = red, NSW = red, V = red, SA = red, T = red
- **Solutions** are *complete* and *consistent* assignments, e.g., WA = red, NT = green, Q = red, NSW = green, V = red, SA = blue, T = green

Constraint Graph



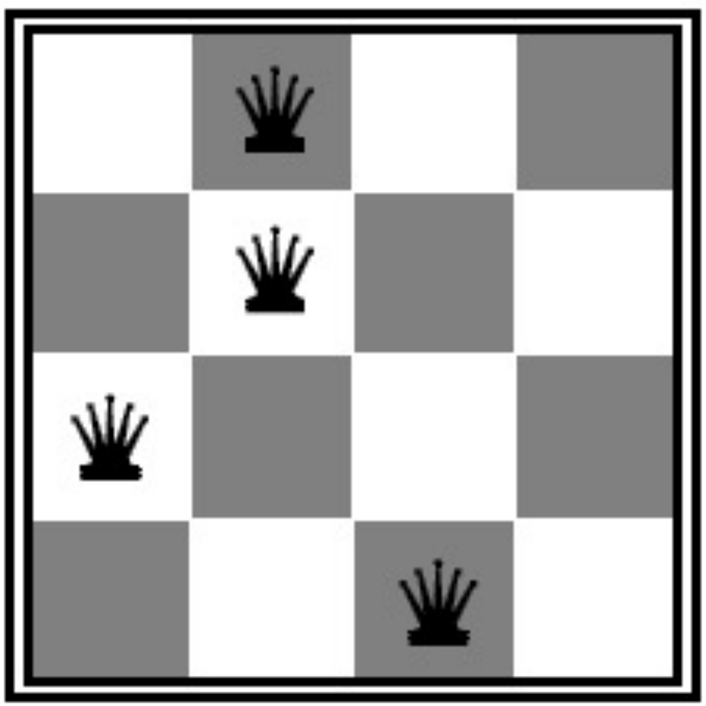


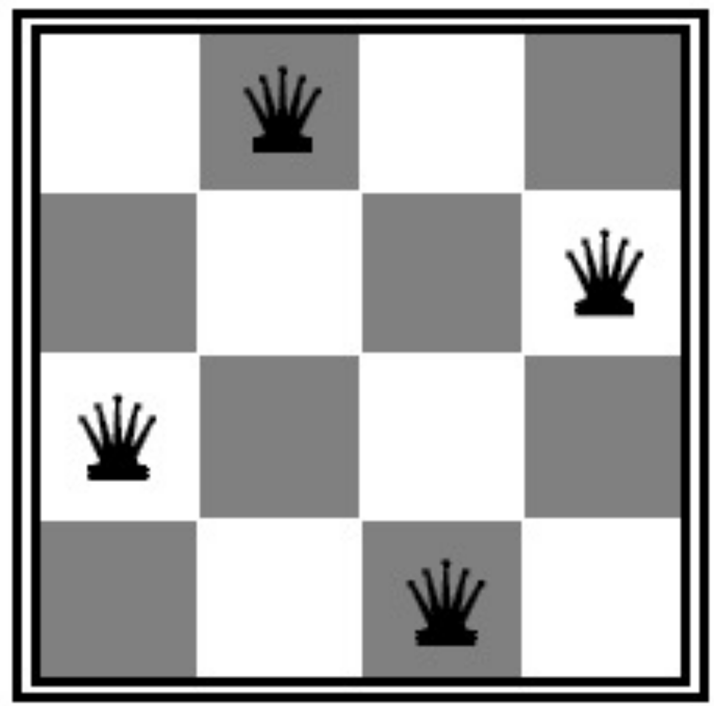
Nodes are variables, arcs show constraints.

- The constraint a != b in map coloring means that no adjacent Australian states are the same color.
- What is meaning of arcs from SA under the a != b constraint?
- What about the fact that there are no arcs to T.
- What is the implication of the arcs between WA, NT and SA?

Example: *n*-queens problem

- Put n queens on an $n \times n$ board with no two queens on the same row, column, or diagonal





Example: N-Queens

- **Variables:** X_{ij}
- **Domains:** {0, 1}
- **Constraints:**

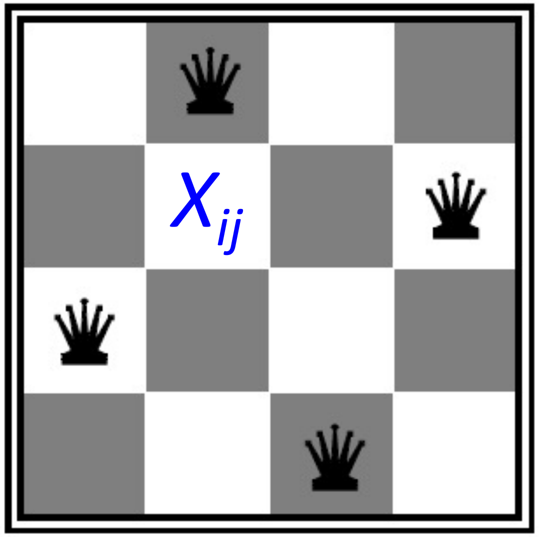
$\sum_{i,j} X_{ij} = N$

$(X_{ij}, X_{ik}) \in \{(0, 0), (0, 1), (1, 0)\}$

$(X_{ij}, X_{kj}) \in \{(0, 0), (0, 1), (1, 0)\}$

$(X_{ij}, X_{i+k,\ j+k}) \in \{(0, 0), (0, 1), (1, 0)\}$

$(X_{ij}, X_{i+k,\ j-k}) \in \{(0, 0), (0, 1), (1, 0)\}$



Example: Cryptarithmetic

- **Variables:** T, W, O, F, U, R

  X_1, X_2

- **Domains**: {0, 1, 2, …, 9}
- **Constraints:**

  $O + O = R + 10 * X_1$

  $W + W + X_1 = U + 10 * X_2$

  $T + T + X_2 = O + 10 * F$

  Alldiff(T, W, O, F, U, R)

  $T \neq 0, F \neq 0$

$$
\begin{array}{cccc}
 & X_2 & X_1 & \\
 & T & W & O \\
+ & T & W & O \\
\hline
F & O & U & R
\end{array}
$$

Example: Sudoku

- **Variables:** X_{ij}
- **Domains:** {1, 2, …, 9}
- **Constraints:**

  Alldiff(X_{ij} in the same *unit*)

					8			4
	8	4		1	6			
			5			1		
1		3	8			9		
6		8		X_{ij}		4		3
		2			9	5		1
		7			2			
			7	8		2	6	
2			3					

Real-world CSPs

- Assignment problems
  - e.g., who teaches what class
- Timetable problems
  - e.g., which class is offered when and where?
- Transportation scheduling
- Factory scheduling

- More examples of CSPs: http://www.csplib.org/

Standard search formulation (incremental)

- **States:**
  - Variables and values assigned so far
- **Initial state:**
  - The empty assignment
- **Action:**
  - Choose any unassigned variable and assign to it a value that does not violate any constraints
    - Fail if no legal assignments
- **Goal test:**
  - The current assignment is complete and satisfies all constraints

Standard search formulation (incremental)

- What is the depth of any solution (assuming n variables)?

  n (this is good)

- Given that there are m possible values for any variable, how many paths are there in the search tree?

  $n! \cdot m^n$ (this is bad)

Backtracking search

- In CSPs, variable assignments are **commutative**
  - For example, *[WA = red then NT = green]* is the same as *[NT = green then WA = red]*
- We only need to consider assignments to a single variable at each level (i.e., we fix the order of assignments)
  - Then there are only $\boldsymbol{m}^{n}$ leaves (n – number of variables and m – number of values)
- Depth-first search for CSPs with single-variable assignments is called **backtracking search**

Example

Example

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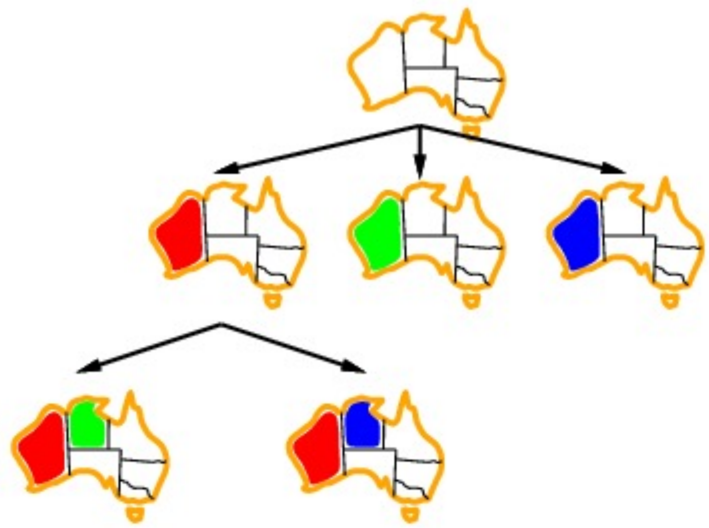

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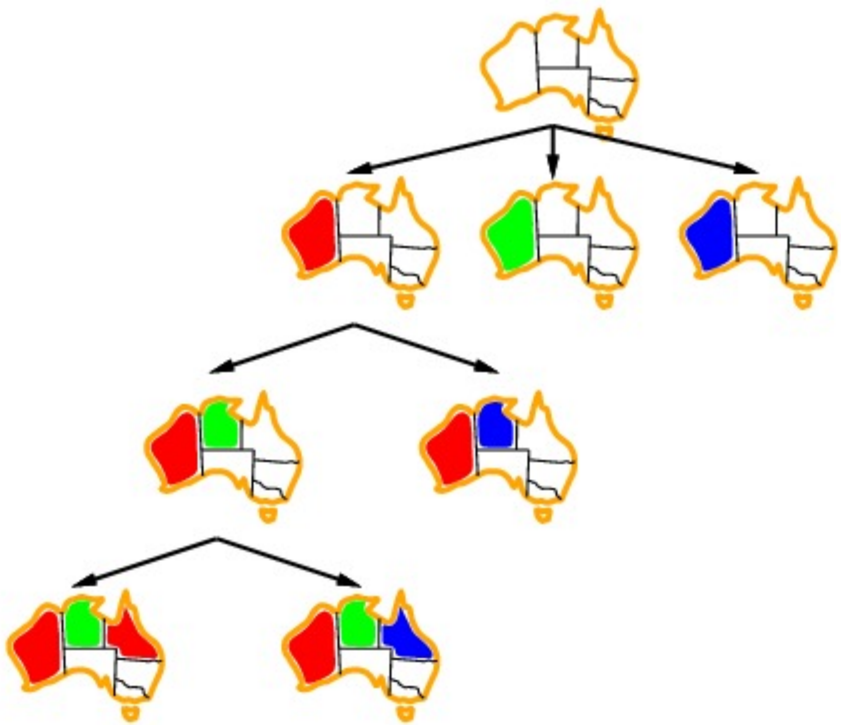
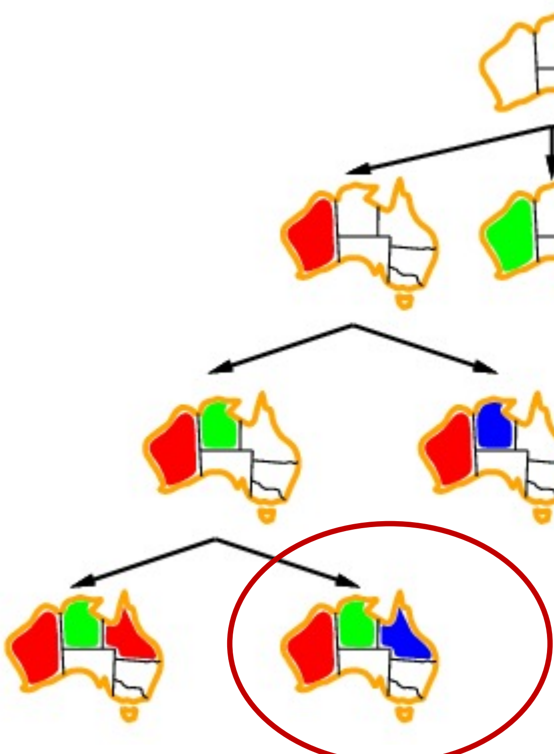


Backtracking

- Constraints on SA will eventually cause failure when WA != Q. When not the same color (bottom right), SA cannot be assigned.
- The algorithm will backtrack to a node with unexplored states.
- For example, such as WA=red, NT=blue.



Basic backtracking algorithm

CSP-BACKTRACKING(PartialAssignment A)

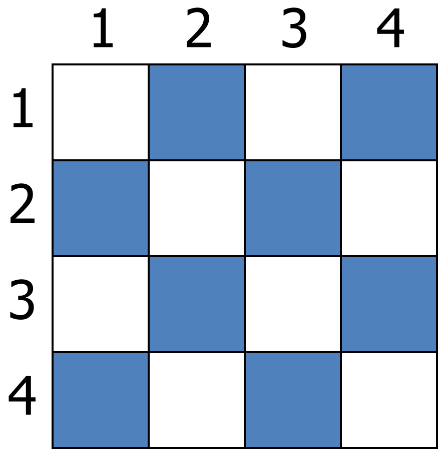
- If A is complete then return A
- X ← select an unassigned variable
- D ← select an ordering for the domain of X
- For each value v in D do

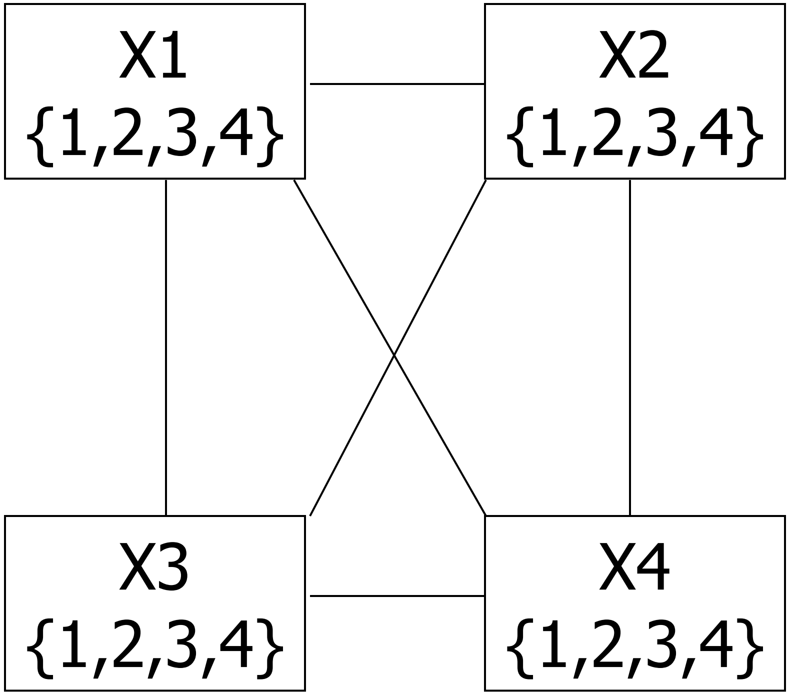
  If v consistent with A then

    - Add (X = v) to A
    - result ← CSP-BACKTRACKING(A)
    - If result ≠ failure then return result
    - Remove (X = v) from A
- Return failure

Start with CSP-BACKTRACKING({})

Example: 4-Queens Problem





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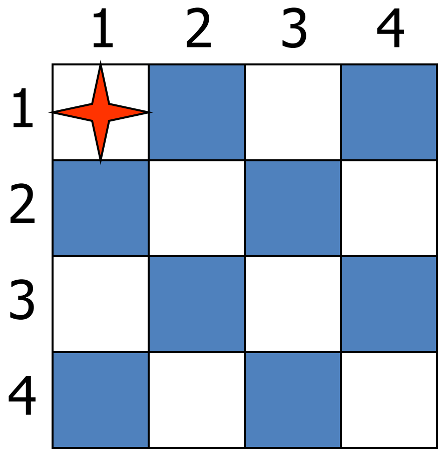

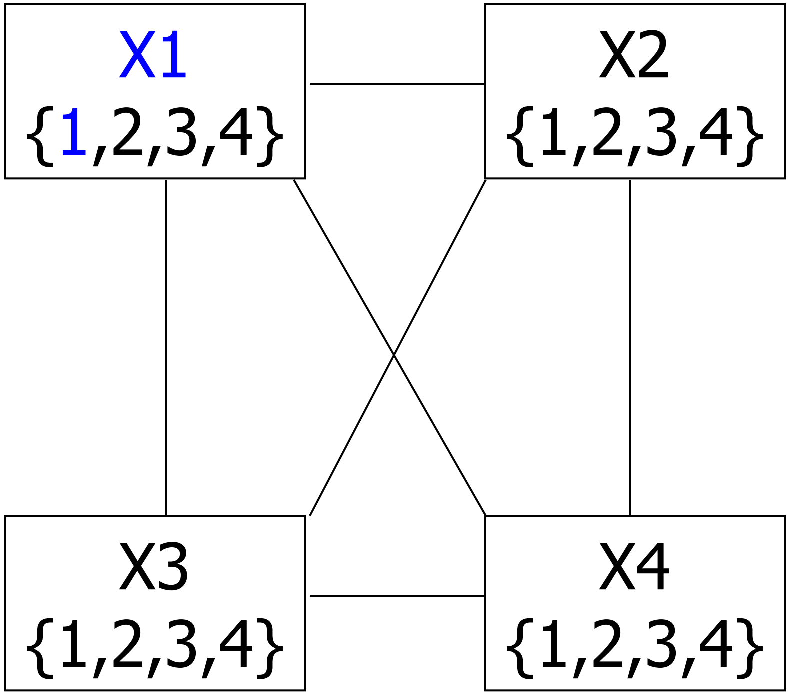

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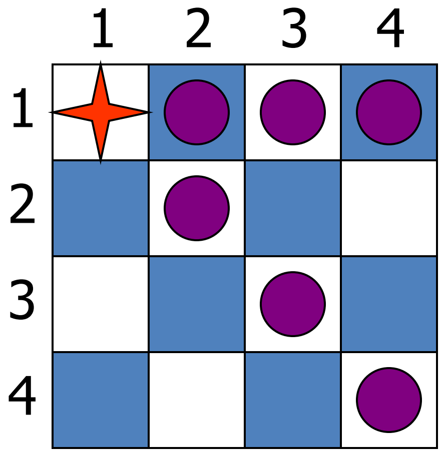

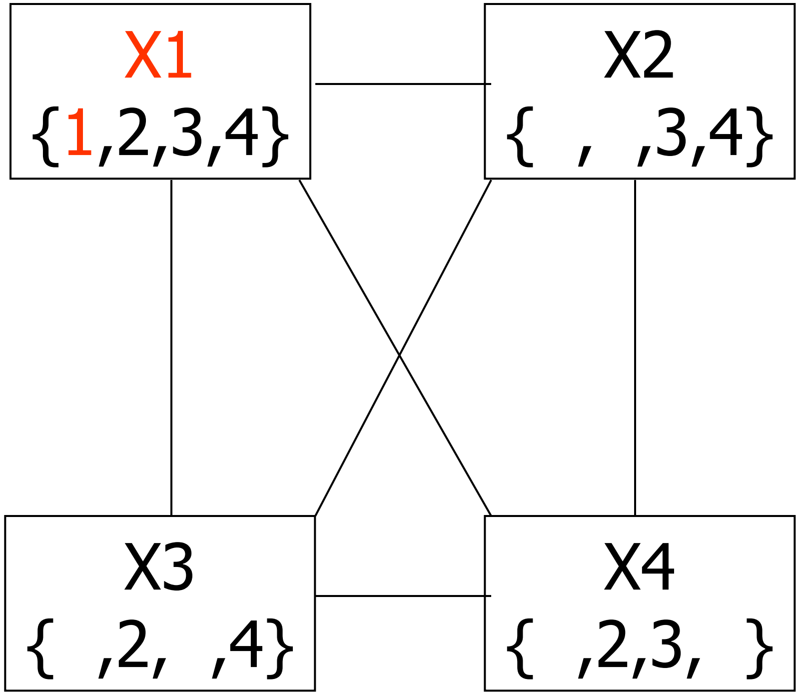

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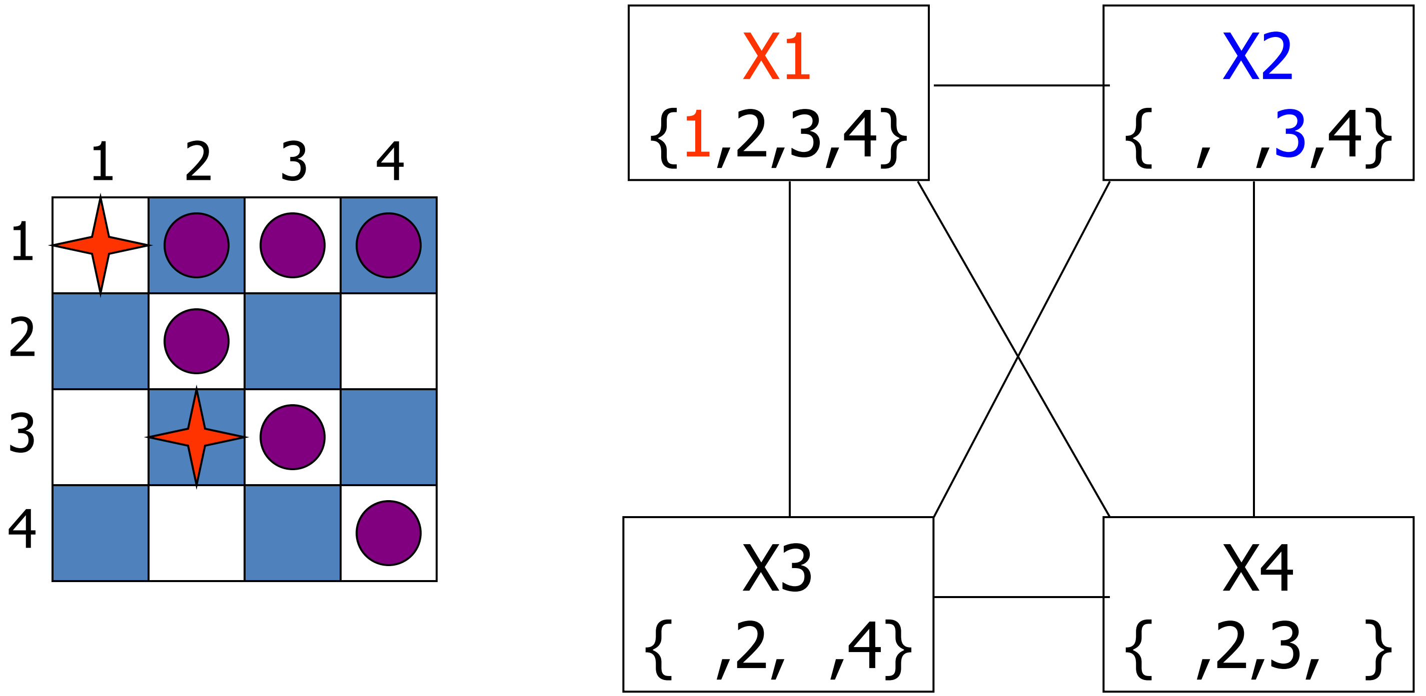

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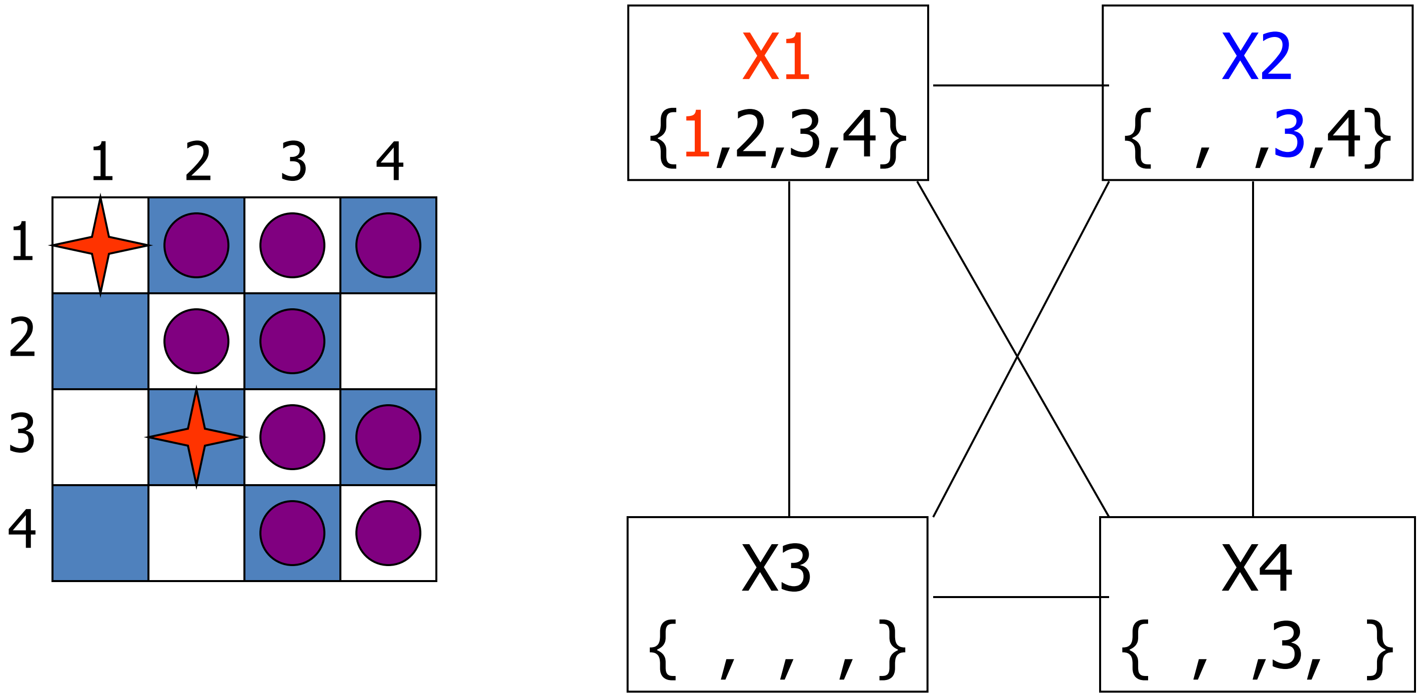

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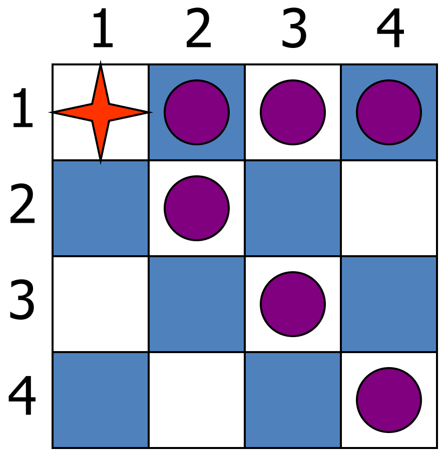

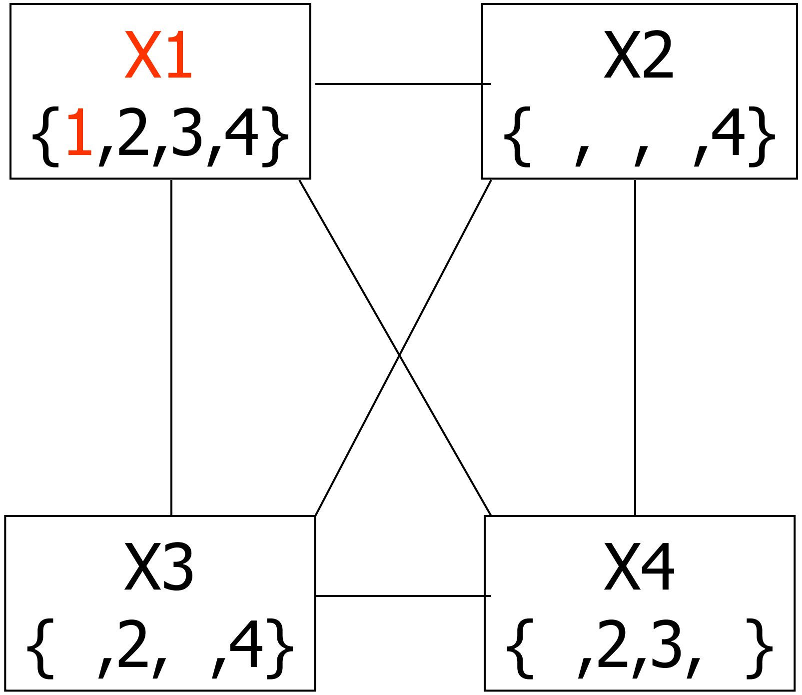

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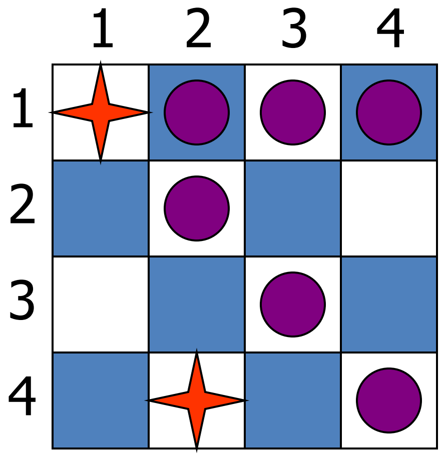

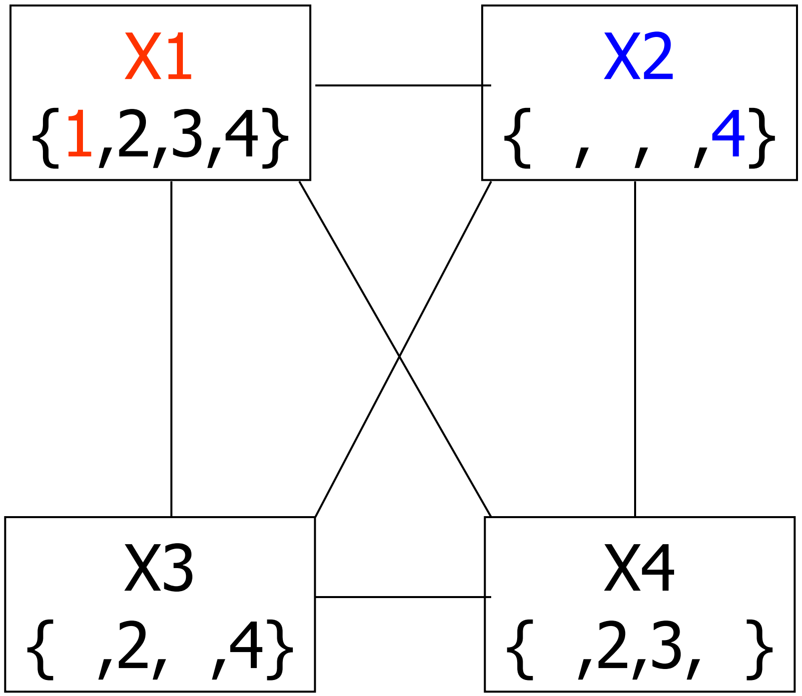

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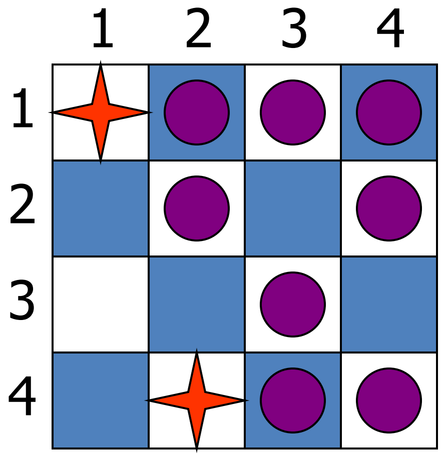

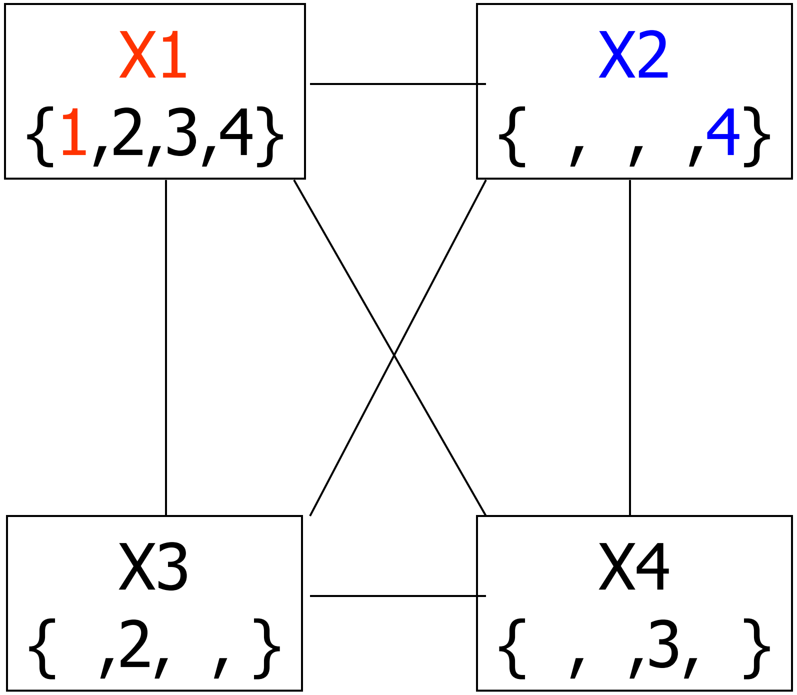

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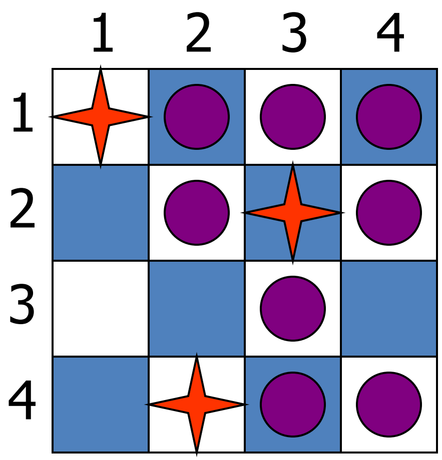

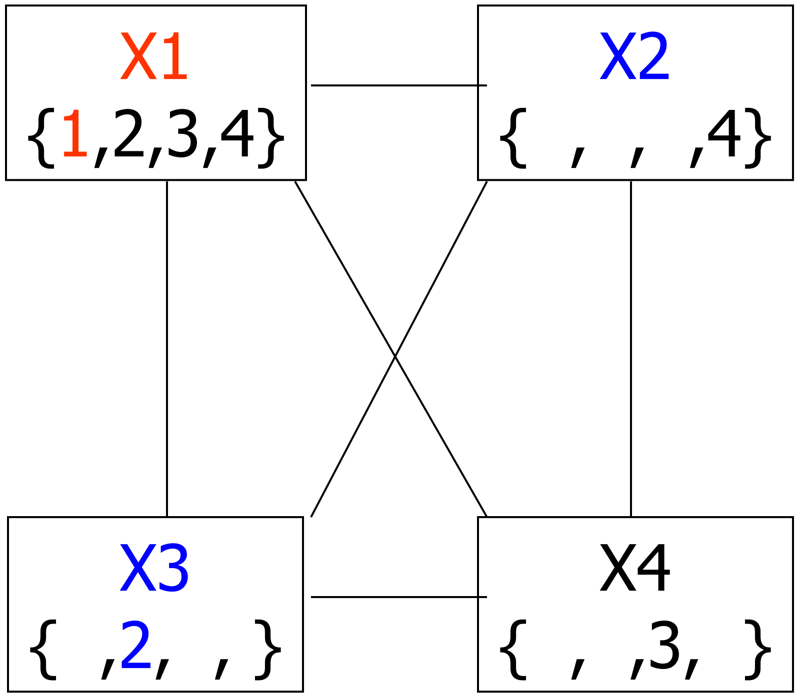

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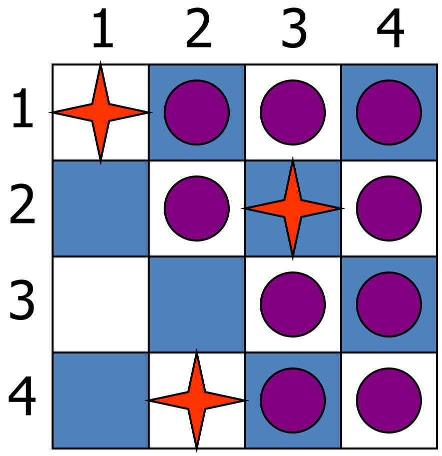

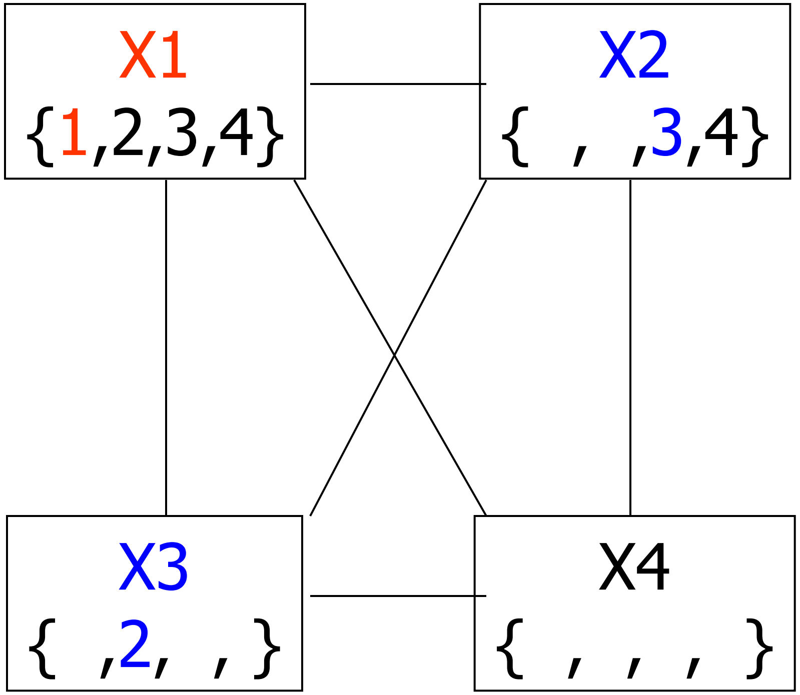

Improving backtracking efficiency

Questions:

- Which variable should be assigned next?
- In what order should its values be tried?
- Can we detect inevitable failure early?

Which variable should be assigned next?

- **Most constrained variable:**
  - Choose the variable with the fewest legal values
  - A.k.a. **minimum remaining values** (MRV) heuristic

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  N.B. Among the variables with the smallest remaining domains, select the one that appears in the largest number of constraints on variables not in the current assignment

Given a variable, in which order should its values be tried?

- Choose the **least constraining value**:
  - The value that rules out the fewest values in the remaining variables

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Which assignment for Q should we choose?

Early detection of failure: Forward checking

- Keep track of remaining legal values for unassigned variables
- Terminate search when any variable has no legal values

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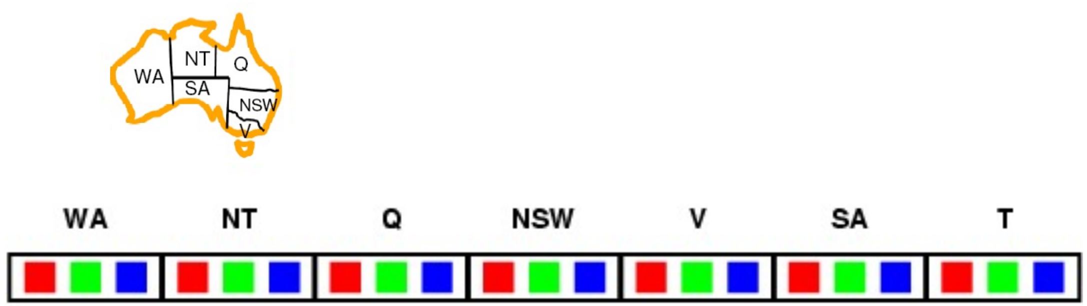

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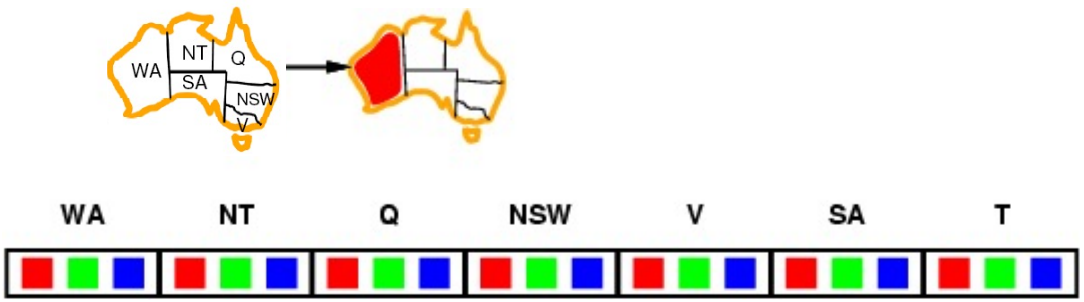

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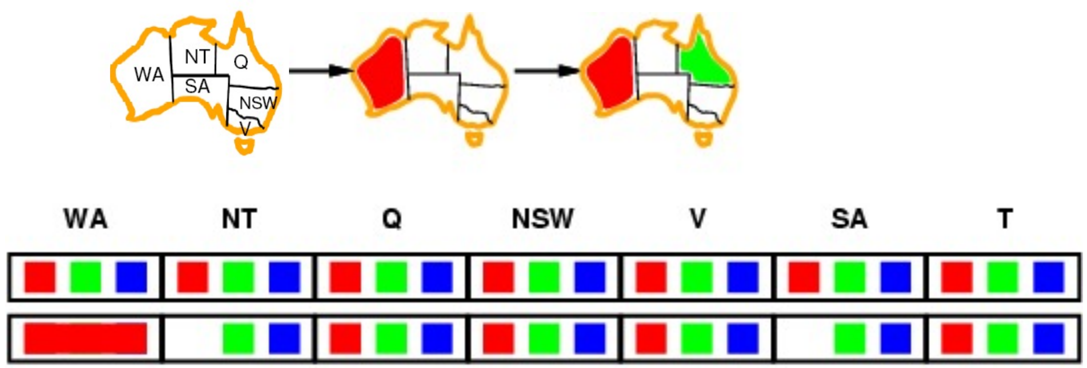

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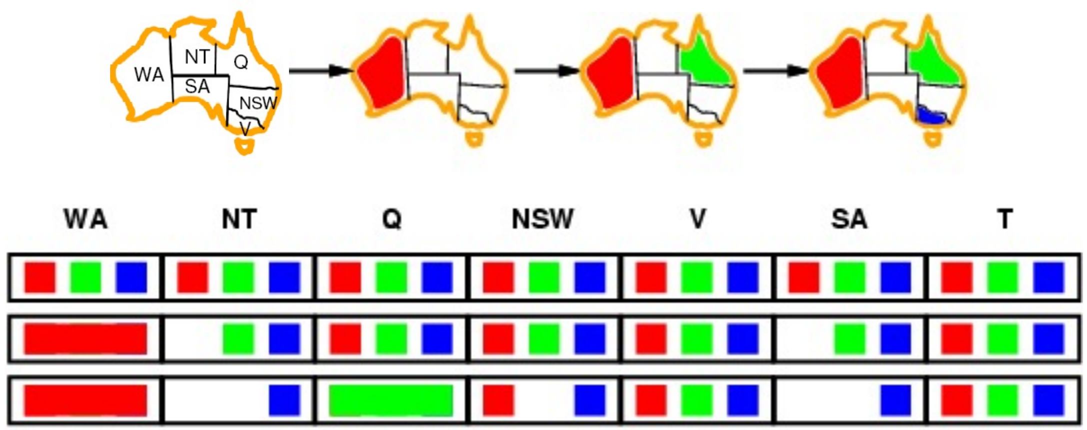
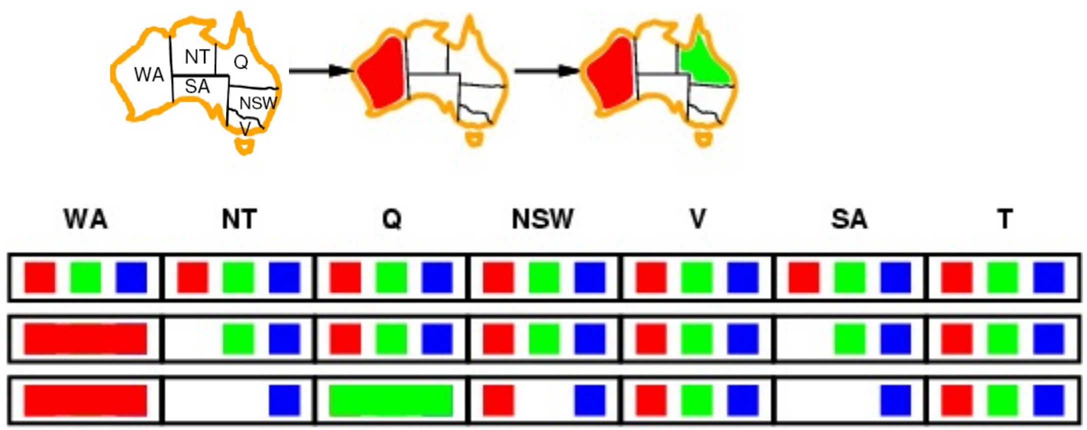


Constraint propagation

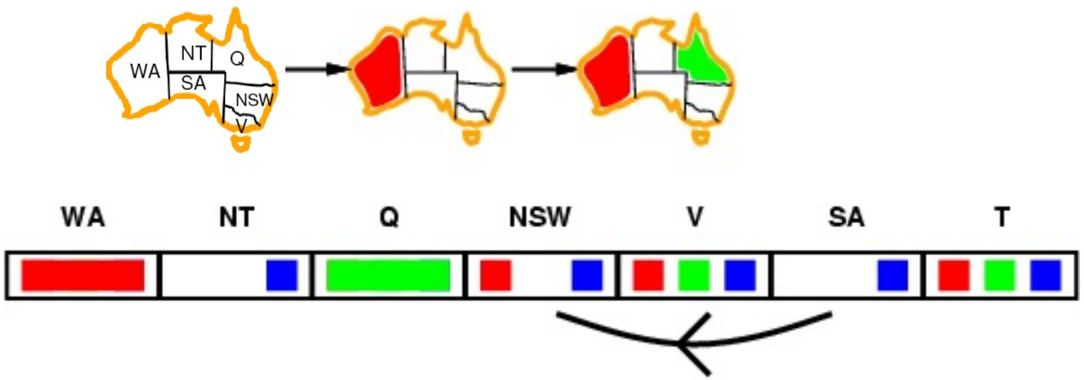
- Forward checking propagates information from assigned to unassigned variables, but doesn't provide early detection for all failures



- NT and SA cannot both be blue!
- **Constraint propagation** repeatedly enforces constraints *locally*

Arc consistency

- Simplest form of propagation makes each pair of variables **consistent:**
  - *X* → *Y* is consistent iff for every value of *X* there is some allowed value of *Y*



Consistent!

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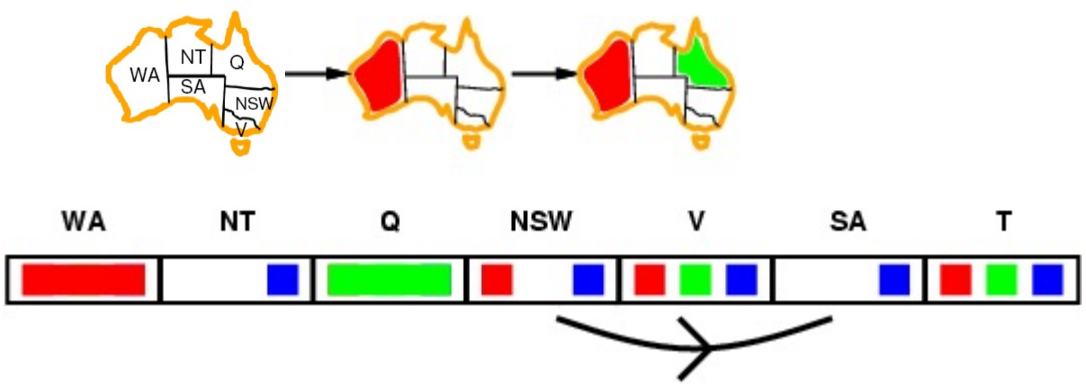

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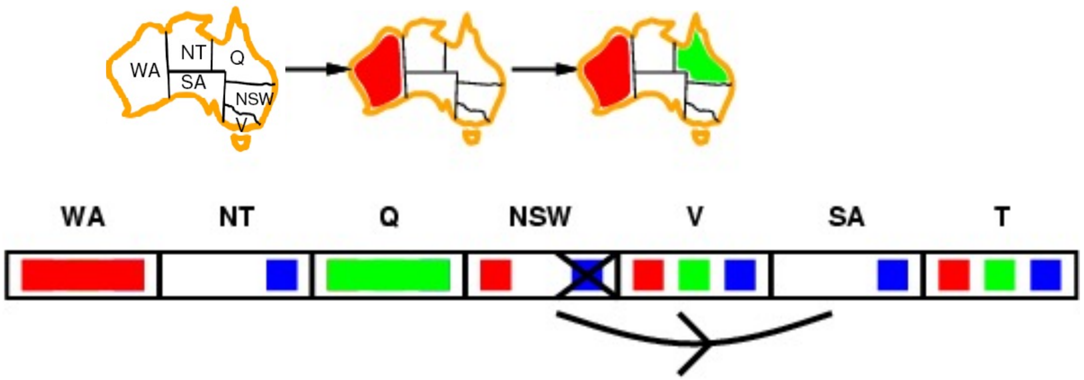

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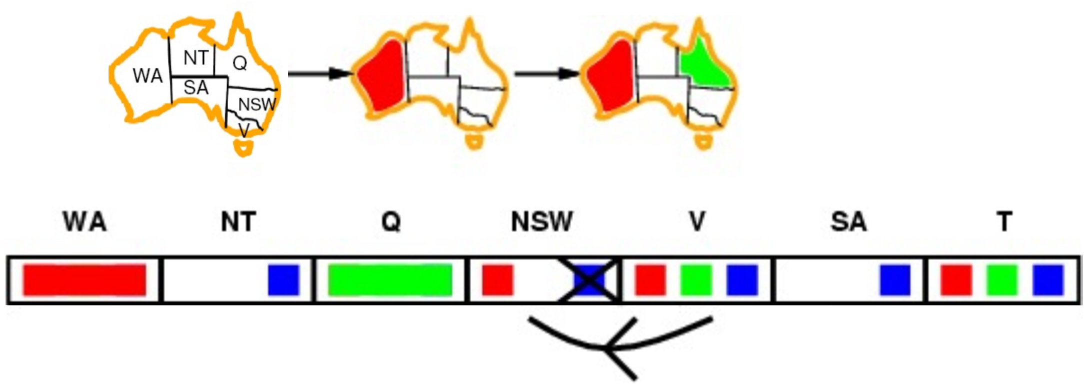

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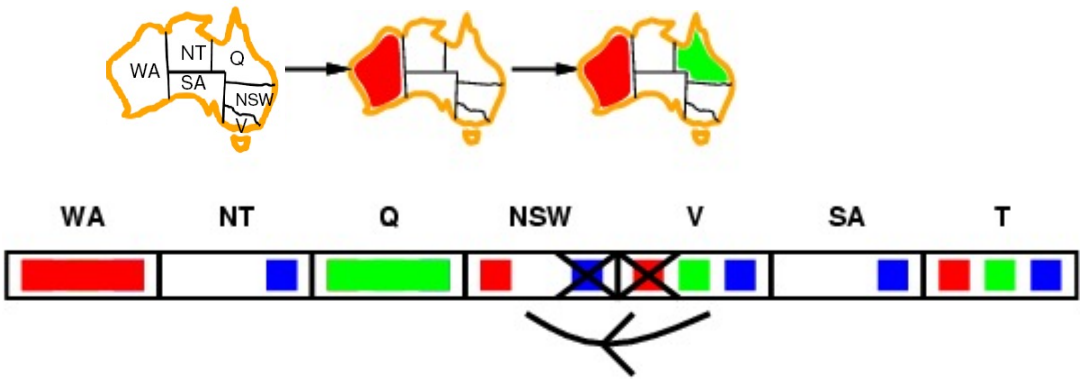

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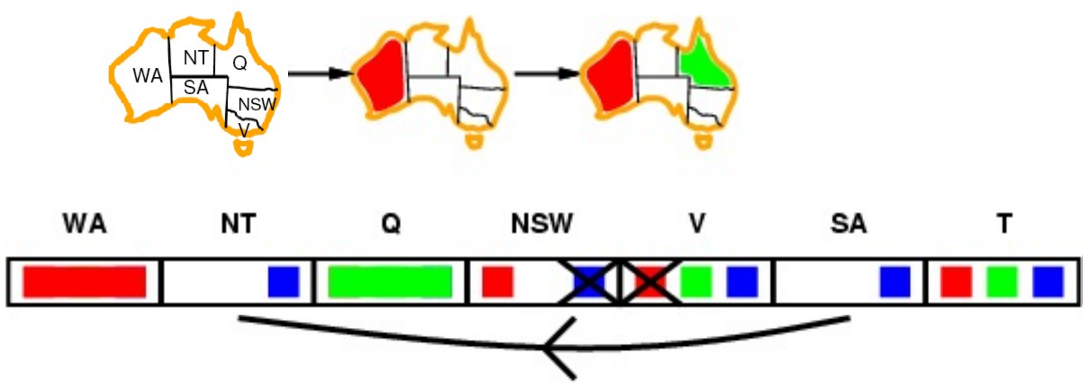

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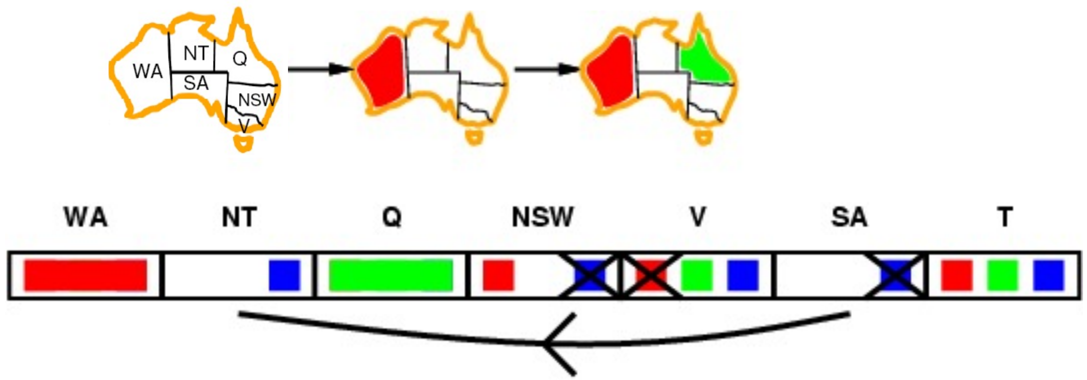
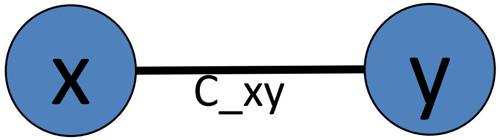


- Arc consistency detects failure earlier than forward checking
- Can be run before or after each assignment

Arc consistency - Example

- Domains
  - D_x = {1, 2, 3}
  - D_y = {1, 2, 3}



- Constraint: X must be <u>less than</u> Y (C_xy)
- C_xy not arc consistent w.r.t. x or y; enforcing arc consistency, we get reduced domains:
  - D'_x = {1, 2}
  - D’_y = {2, 3}

Summary

- CSPs are a special kind of search problem:
  - States defined by values of a fixed set of variables
  - Goal test defined by constraints on variable values
- **Backtracking** = depth-first search where successor states are generated by considering assignments to a single variable
  - **Variable ordering** and **value selection** heuristics can help significantly
  - **Forward checking** prevents assignments that guarantee later failure
  - **Constraint propagation** (e.g., arc consistency) simple yet powerful to constrain values and detect inconsistencies
- Complexity of CSPs
  - NP-complete in general (exponential worst-case running time)