Kaczmarz Iteration with Random Row Permutation

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The Kaczmarz Method

The Kaczmarz iterative method for a linear system

$$A_{m\times n}\vec{x}=\vec{b} \quad (\text{or } AA^*\vec{y}=\vec{b}),$$

performs an affine projection at each step:

$$\vec{x}^{(k+1)}=(I-\mathcal{P}_j)\vec{x}^{(k)}+b_j\vec{a}_j^*.$$

The classical iteration picks j in a circulant fashion:

$$j\in\mathbb{Z}_m.$$

It is equivalent as applying an implicit Gauß-Seidel iteration on AA^* with per-cycle iteration matrix:

$$Q=I-A^*L^{-1}A,$$

where L is the lower triangular part (including the diagonal) of AA^*.

The iteration converges in all cases, to the minimum norm least square solution. (An extended approach must be adopted in the inconsistent cases)

The Shuffling Iteration Scheme

We propose to shuffle all the equations before each cycle of iterations. This is same as randomly picking equations without replacement. In many cases, to have a desirable convergence rate, it, in expectation, suffices to shuffle only once before the entire iteration starts.

The Convergence Estimate

The error reduction rate of the classical Kaczmarz iteration can be bounded by its essential spectral condition number as:

$$\rho^2\leq 1-\frac{C}{\lambda\,\kappa\,\ln r},$$

where $r=\operatorname{rank}(A)$, $\lambda=\lambda_{\max}(AA^*)$, $\kappa=\lambda_{\max}(AA^*)/\lambda_{\min}(AA^*)$ ($\lambda_{\min}$ being the smallest non zero eigenvalue).

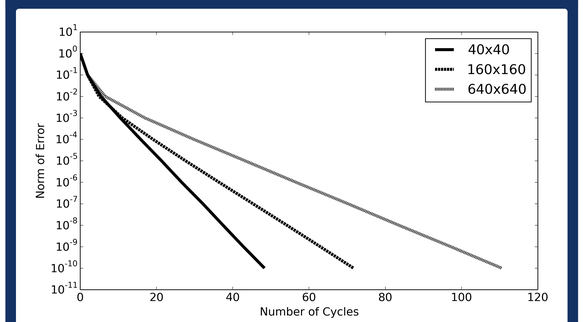
That the estimate is not improvable beyond a constant factor can be illustrated by explicit examples .

The shuffled Kaczmarz iteration has the following bound (with the help of a proper relaxation parameter) for its expected error reduction rate:

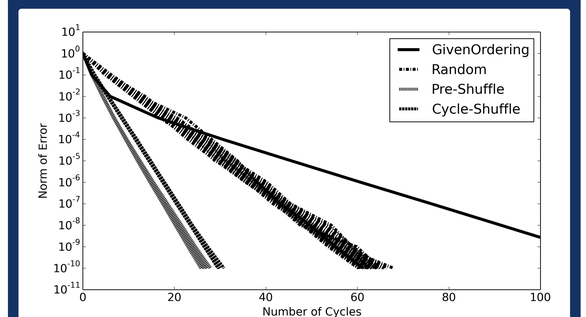
$$\mathbb{E}\left[\rho^2\right]\leq 1-\frac{C}{\kappa}.$$

The actual bound for different row orderings has at least a polynomial concentration up to some finite order. Moreover, if $\|\mathcal{L}(AA^*)\|$ is logarithmically larger than $\|AA^*\|$, then the concentration estimate can not be further improved.

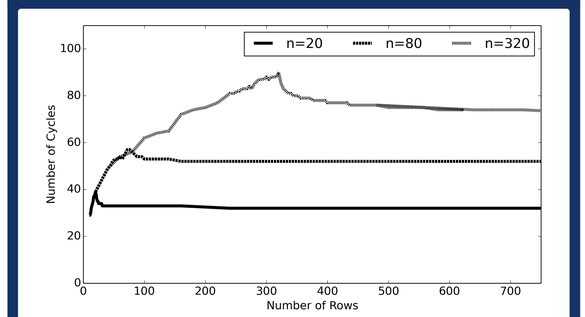
The Logarithm Dependence



Perfomance Comparison



Rank Dependence



The Triangular Truncation

$\mathcal{L}$ yields the strict lower triangular part of a matrix:

$$\mathcal{L}:\begin{pmatrix}\#&\#&\#\\\#&\#&\#\\\#&\#&\#\end{pmatrix}\mapsto\begin{pmatrix}0&0&0\\\#&0&0\\\#&\#&0\end{pmatrix}.$$

As a linear transformator on the matrix space, its norm is logarithmically dependent on the rank of the operand:

$$\|\mathcal{L}(B)\|\leq C\ln r\|B\|.$$

This logarithm norm is attainable, up to unitary diagonal scaling, on a spectral mapping of $\Im(\mathcal{L})$ with the sign function.

It has limited boundedness upon permutation:

$$\|\mathbb{E}_{\sigma\in S_m}\left[(\Im(\mathcal{L})(P_\sigma BP_\sigma^*))^p\right]\|\leq C\|B\|^p.$$

$p<\infty$, and is at least 2.

Example of Arbitrary Slow Convergence in the Classical Iteration

This simple example shows even for a (essentially) well conditioned (with $\kappa=1$) linear system, on which the truncation $\mathcal{L}$ is a contraction, the convergence from the classical iteration scheme can still be arbitrarily slow:

Let the system be homogeneous:

$$A\vec{x}=\vec{0},$$

Each row is defined as:

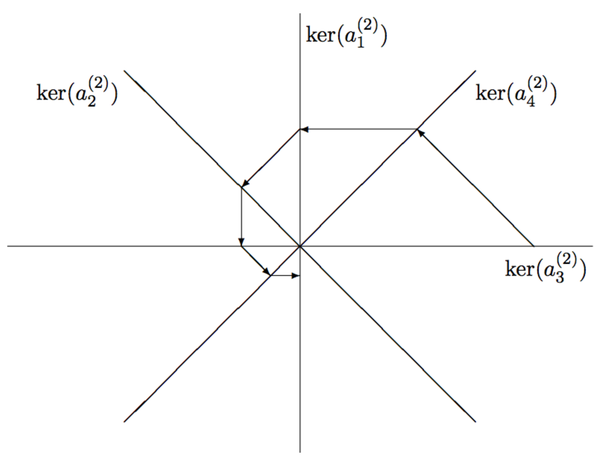
$$\vec{a}_k=\left(\Re(\zeta^k),\,\Im(\zeta^k)\right),$$

where ζ is $\exp(\pi i/m)$.

Then $\kappa=1$ and $r=2$ (neither depend on m) since A is composed of 2×2 orthogonal matrices.

However, as shown in the figure, the geometry simply implies that as $m\to\infty$:

$$\rho=\cos^m(\pi/m)\to 1.$$



References

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