Kaczmarz Iteration with Random Row Permutation

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The Classical Kaczmarz Method in a Nutshell

- Linear System:

$$A_{m\times n}\vec{x} = \vec{b} \quad (\text{ or } AA^*\vec{y} = \vec{b})$$

- Affine Projection:

$$\vec{x}^{(k+1)} = (I - \mathcal{P}_j)\vec{x}^{(k)} + b_j\vec{a}_j, \quad j \in \mathbb{Z}_m$$

- Iteration Matrix:

$$Q = I - A^*L^{-1}A$$

- Iterated Solution:

$$\vec{x} = \mathcal{P}_{\ker(A)}\vec{x}^{(0)} + A^{\dagger_{1,2,4}}\vec{b}$$

Existing Estimate

- Classical Iteration (Meany, 1969)

$$\rho^2 \leq 1 - \det(AA^*)$$

- Random Iteration (Strohmer/Vershynin, 2009)

$$\rho^2 \leq 1 - \frac{\lambda_{\min}}{\mathrm{tr}(AA^*)}$$

- As POCS (e.g. Deutsch, 1997)

$$\rho^2 \leq 1 - \prod_k \left(1 - \theta_{(\vec{a}_k^\perp, \cap_{j>k} \vec{a}_j^\perp))}\right)$$

- Variations: Block, Extended, etc.

Our Contributions

- The Convergence Estimate

$$\rho^2 \le 1 - \frac{C}{\lambda\kappa \ln r}$$

Each factor can be illustrated by example.

- The Shuffled Iteration

$$\rho^2 \le 1 - \frac{C}{\kappa}$$

(with cheap Relaxation plan)

The Triangular Truncation

- The Cartesian Decompostion

$$\mathcal{L} = \Re(\mathcal{L}) + i\Im(\mathcal{L})$$

- The Logarithm Norm

$$\|\Im(\mathcal{L})(\mathrm{sgn}(\Im(\mathcal{L})(\vec{1}\vec{1}^*)))\|_\infty \to \frac{2}{\pi} \ln r$$

(Unbounded on L^1 and L^∞; e.g. Toeplitz matrices; can be classified)

- Limited Boundedness Upon Permuation

$$\|\mathbb{E}_\sigma \left[(\Im(\mathcal{L})(P_\sigma B P_\sigma^*))^p\right]\|_\infty \leq C\|B\|_\infty^p, \quad p \leq \tilde{p} < \infty$$

Key Message

ONE Shuffle Suffices!