Auto-tuned high-dimensional regression with the TREX: theoretical guarantees and non-convex global optimization

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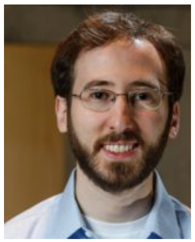

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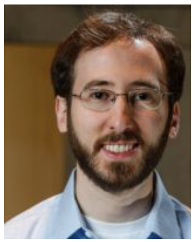


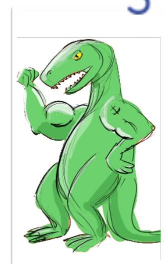

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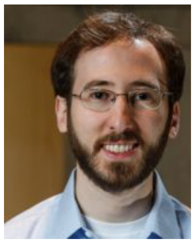


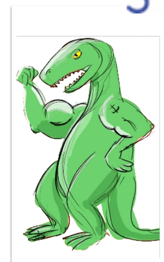

We aim at variable selection in linear regression.
We therefore consider models of the form

$$Y = X\beta^* + \sigma\epsilon, \qquad \text{(Model)}$$

where $Y \in \mathbb{R}^n$ is a response vector, $X \in \mathbb{R}^{n\times p}$ a design matrix, $\sigma > 0$ a constant, and $\varepsilon \in \mathbb{R}^n$ a noise vector.

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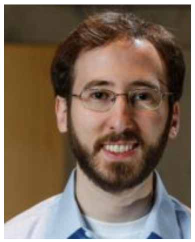


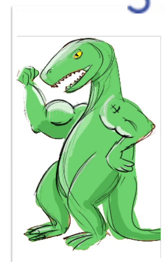

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High-dimensional variable selection in linear regression

$$n\left\{\boldsymbol{Y}\right. = \boldsymbol{X} \times \left.\boldsymbol{\beta}^*\right\}p + \boldsymbol{\sigma}\,\boldsymbol{\epsilon}$$

Standard approach: The LASSO (Tibshirani, 1996)

$$\widehat{\beta}_{\text{Lasso}}(\lambda) \in \underset{\beta \in \mathbb{R}^p}{\operatorname{argmin}} \left\{ \frac{\|Y - X\beta\|_2^2}{n} + \lambda \|\beta\|_1 \right\}. \quad \textbf{(Lasso)}$$

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- Tuning of regularization parameter required

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Novel proposition: The TREX (Lederer and M., AAAI 2015)

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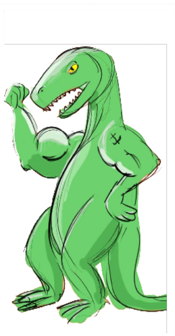

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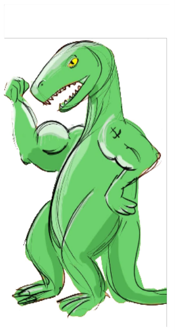

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BUT...

The non-convex TREX objective function can be globally optimally solved by using Second Order Cone Programming.

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1.) The TREX (with e.g. constant a=0.5) can be written as:

$$P^* := \min_{\beta \in \mathbb{R}^p} \left\{ \frac{\|Y - X\beta\|^2}{\max_{j \in \{1,\ldots,p\}} a|x_j^\top (Y - X\beta)|} + \|\beta\|_1 \right\}$$

$$= \min_{\beta \in \mathbb{R}^p} \min_{j \in \{1,\ldots,p\}} \left\{ \frac{\|Y - X\beta\|^2}{a|x_j^\top (Y - X\beta)|} + \|\beta\|_1 \right\}.$$

2.) For each index j this leads to a pair of problem of the form:

$$\min_{\beta \in \mathbb{R}^p} \left\{ \frac{\|Y - X\beta\|^2}{a x_j^\top (Y - X\beta)} + \|\beta\|_1 \quad \text{s.t.} \quad x_j^\top (Y - X\beta) \geq 0 \right\}$$

and

$$\min_{\beta \in \mathbb{R}^p} \left\{ \frac{\|Y - X\beta\|^2}{-a x_j^\top (Y - X\beta)} + \|\beta\|_1 \quad \text{s.t.} \quad -x_j^\top (Y - X\beta) \geq 0 \right\}.$$

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3.) or, in general, 2p problems of the quadratic over linear form:

$$P^*(v) := \min_{\beta \in \mathbb{R}^p} \left\{ \frac{\|Y - X\beta\|^2}{v^\top (Y - X\beta)} + \|\beta\|_1 \quad \text{s.t.} \quad v^\top (Y - X\beta) \geq 0 \right\}.$$

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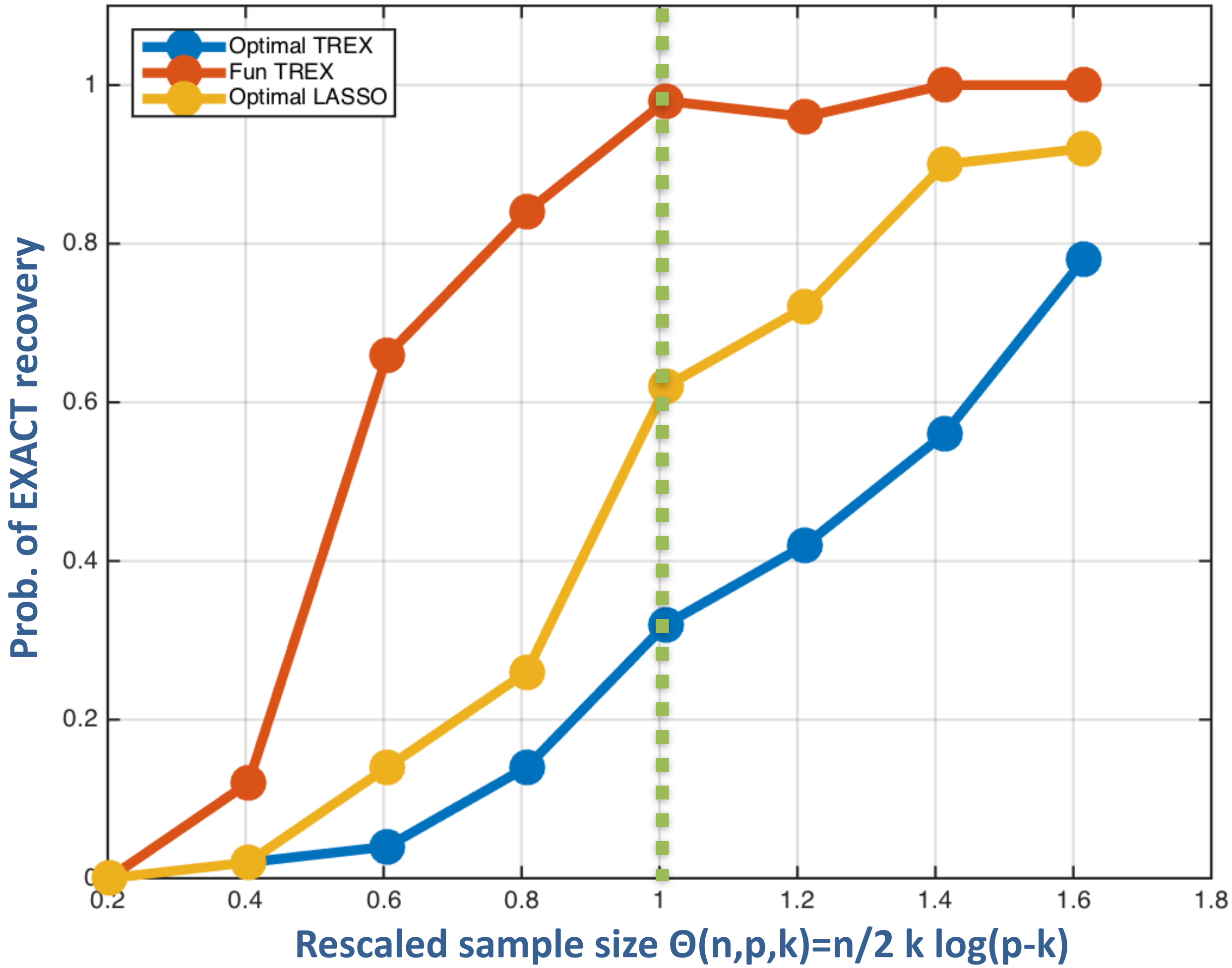
Each problem is a Second-Order Cone Program!

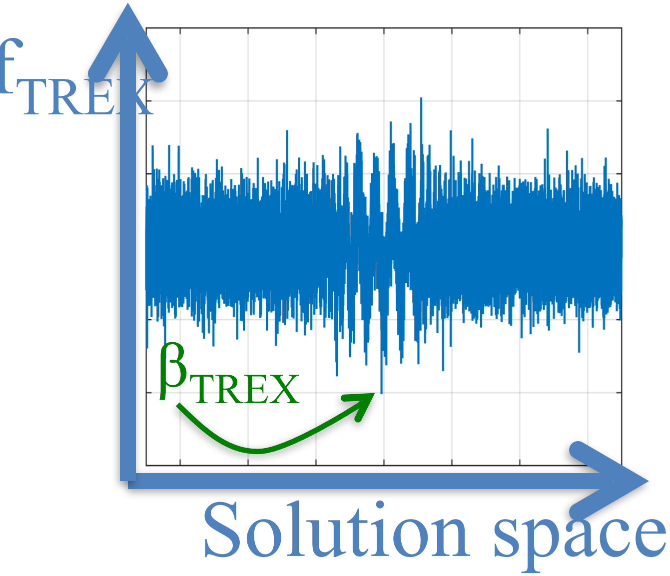
Phase transition of exact recovery with the TREX and the LASSO



p=64

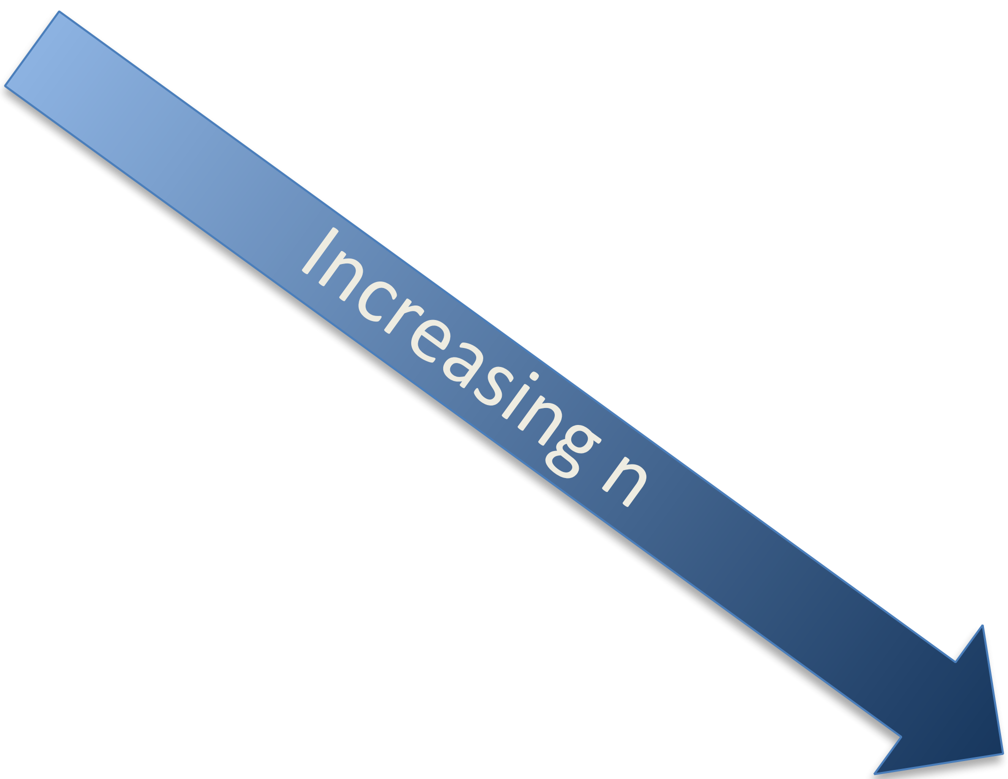
$k=0.2p^{0.75}$

Figure 1: Success probability P[S±(β) = S±(β∗)] of obtaining the correct signed support versus the rescaled sample size θ(n, p, k) = n/[2k log(p − k)] for problem size p=64 with sparsity $k = \lceil 0.20\, p^{0.75} \rceil$. The number of repetitions is 50. The optimal a=0.5 in TREX. The lambda in LASSO is automatically determined by MATLAB. Variable selection using the function gap property (Fun TREX) is shown in red.



Sketching the topology of the TREX

Consider the case where data (p>>n) are generated from a linear model with a sparse β vector with k<<p non-zero entries of equal absolute value.



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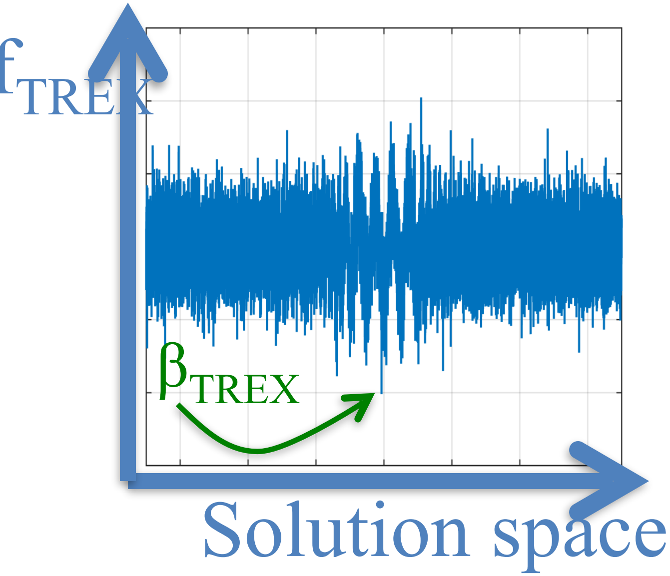

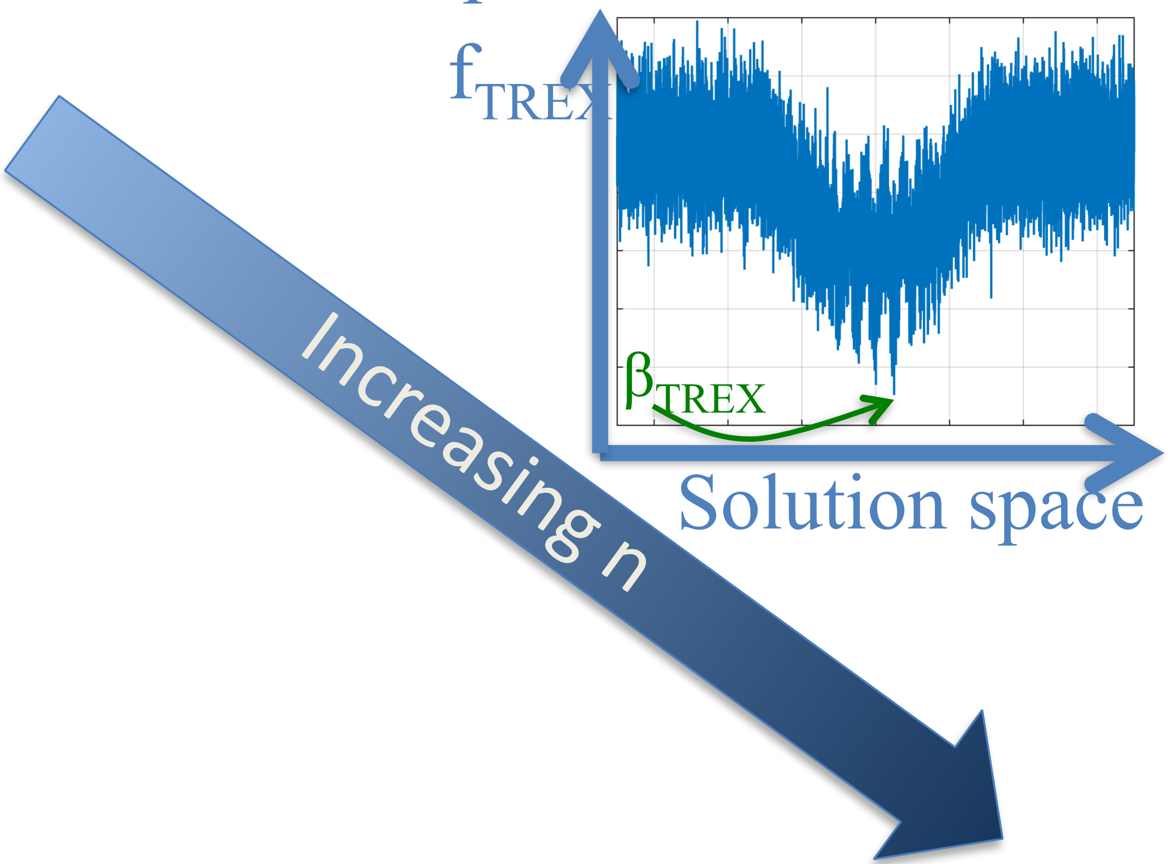

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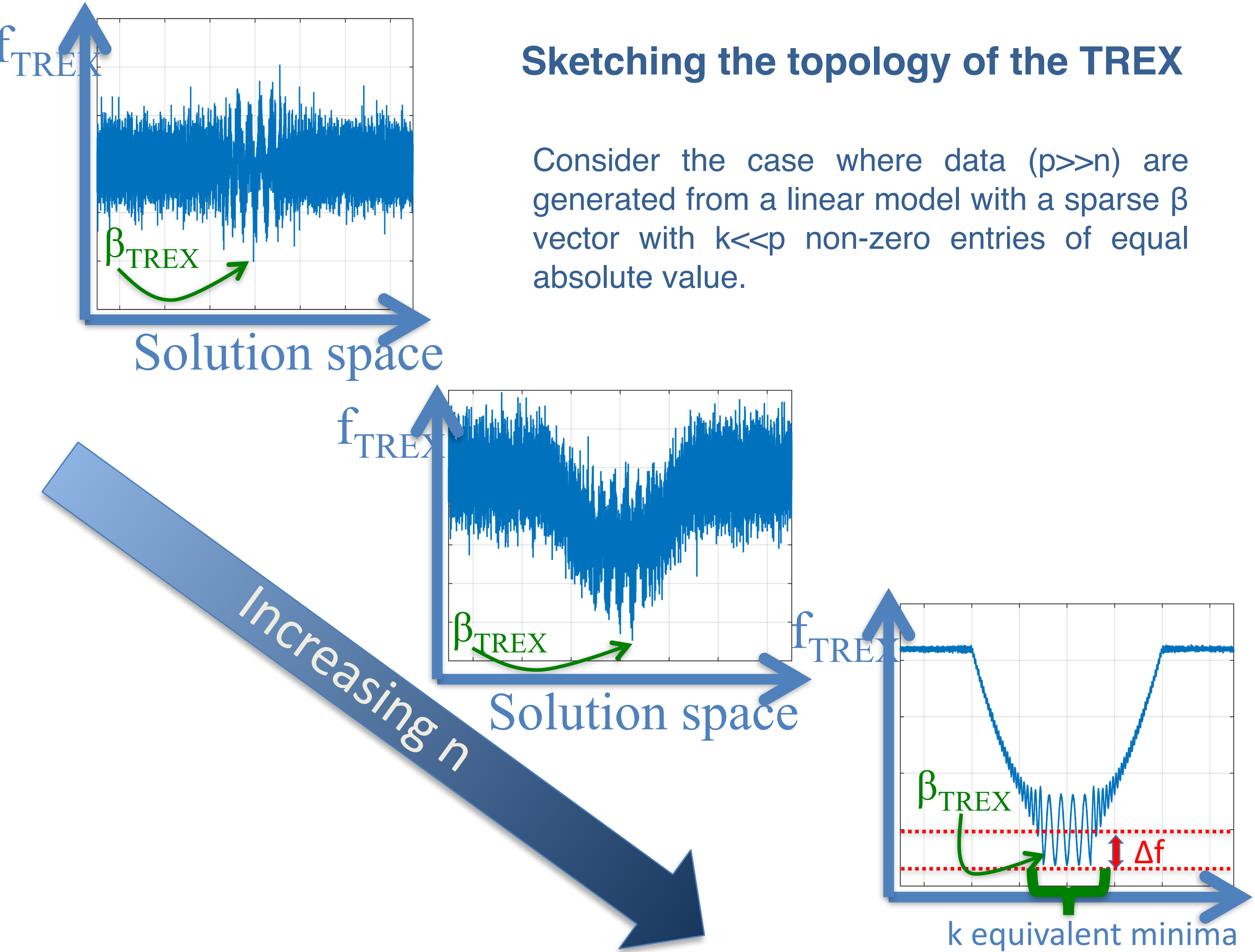

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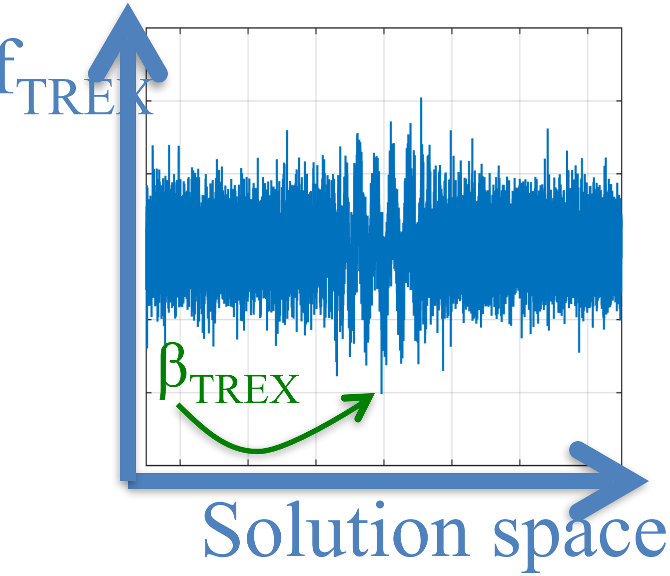

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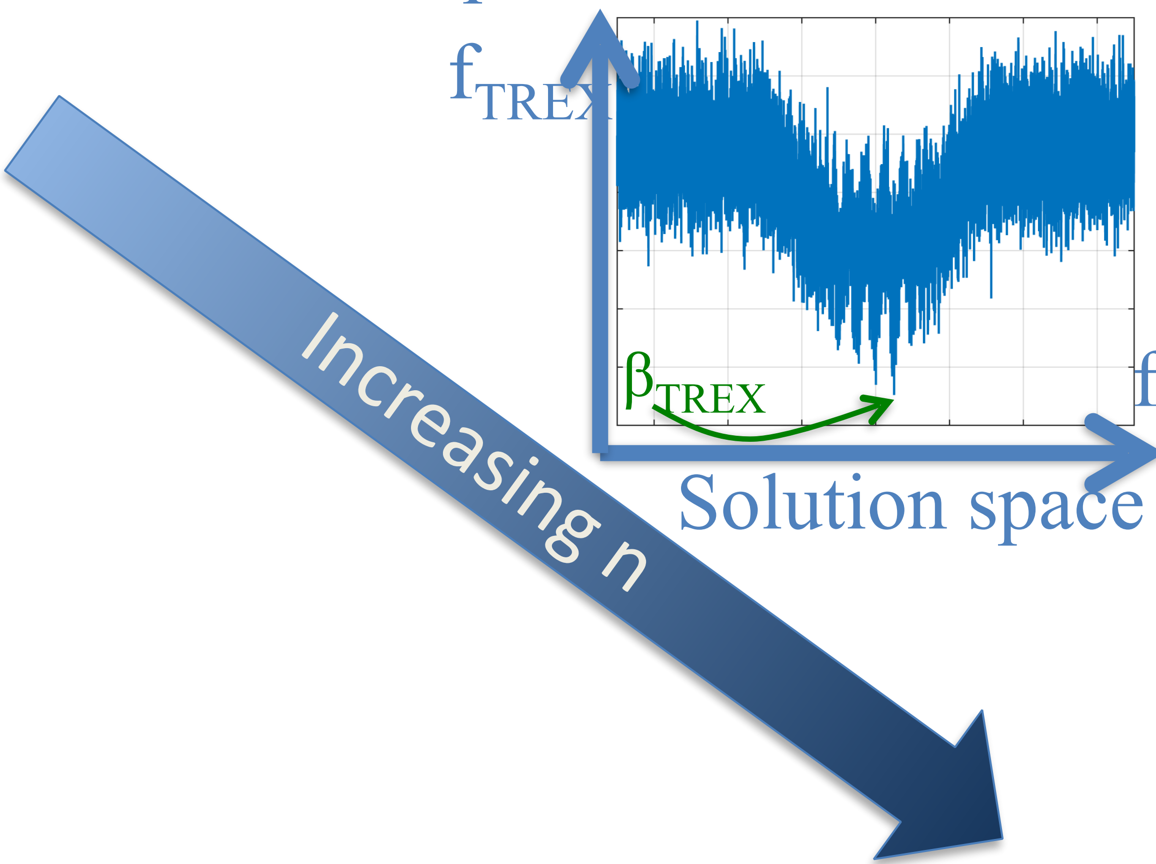
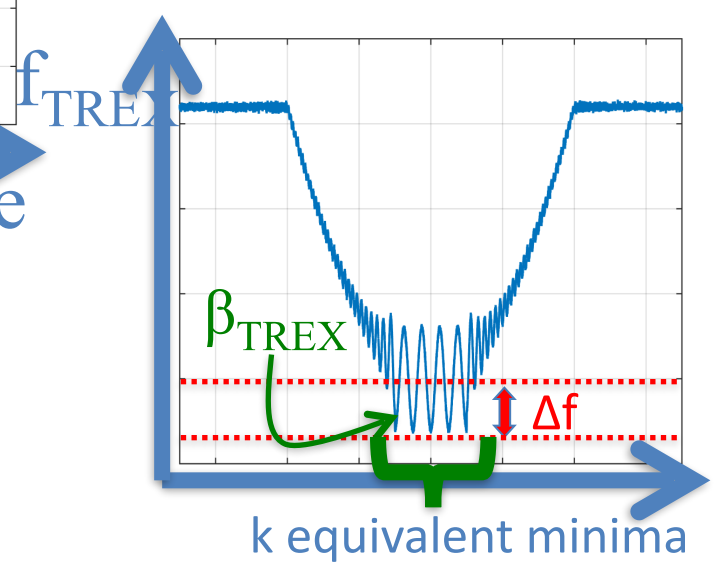


The topology of the objective function can be used as an alternative variable selection method.



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Current solvers for SOCP

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ANY IDEA HOW TO SPEED THINGS UP?