A new and improved recovery analysis for iterative hard thresholding algorithms in compressed sensing

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joint with

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The compressed sensing formulation

Let $x \in \mathbb{R}^N$ be a given signal.

Suppose we obtain vector $b \in \mathbb{R}^n$ of noisy linear measurements

$$b = Ax + e,$$

where $A \in \mathbb{R}^{n \times N}$ is the measurement matrix and e is noise.

We assume

- $n < N \implies$ underdetermined system
- x sparse with $k < n$ non-zeros

Algorithms for sparse approximation

- The problem: Find (approximate) k-sparse x from an underdetermined system of linear equations.
- Frame as the nonconvex nonsmooth problem

$$\underset{y \in \mathbf{R}^N}{\text{minimize}} \quad \tfrac{1}{2}\|Ay - b\|_2^2 \quad \text{subject to} \quad \|y\|_0 \leq k$$

  - solve by gradient projection
  - when $Ax = b$, we seek the global solution

Typically, $\|y\|_0 \leq k$ is relaxed to $\|y\|_1 \leq \tau$

$\Longrightarrow$ convex problem

But here, we solve the original l_0-formulation.

Iterative Hard Thresholding (IHT) algorithm

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Iterative Hard Thresholding (IHT): [Blumensath and Davies, 2007]

Inputs: A, b, k and $\alpha \in (0, 1)$.

Initialize: $x^0 = 0$ and $m = 0$.

While some termination criterion is not satisfied, do:

$$x^{m+1} = H_k \left\{ x^m + \alpha A^T (b - Ax^m) \right\}^{(*)}$$

Output: $\hat{x} = x^m$. $\square$

$(*)$ where $H_k(x) : \mathbb{R}^N \to \mathbb{R}^N$ keeps the k largest entries of x.

State-of-the-art analyses

- **Restricted Isometry Property (RIP)**:

$$L_s = 1 - \min_{1 \le \|y\|_0 \le s} \frac{\|Ay\|_2^2}{\|y\|_2^2} \quad \text{and} \quad U_s = \max_{1 \le \|y\|_0 \le s} \frac{\|Ay\|_2^2}{\|y\|_2^2} - 1$$

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- Prove that IHT moves closer to x in each iteration:

$$\|x^{m+1} - x\|_2 \leq \mu(L_{3k}, U_{3k})\|x^m - x\|_2 + \xi(U_{2k})\|e\|_2$$

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- $\Longrightarrow$ If $\mu(L_{3k}, U_{3k}) < 1$,

$$x^m \to x^* \quad \text{such that} \quad \|x^* - x\|_2 \le \frac{\xi(U_{2k})}{1 - \mu(L_{3k}, U_{3k})} \|e\|_2$$

[Blumensath & Davies (2007); Blanchard, CC, Tanner & AT (2010)]

The proportional-growth asymptotic framework

- Claim of compressed sensing: it is possible to sample proportional to the information content (sparsity): guaranteed recovery of x for $n \geq C \cdot k \ln\left(\frac{N}{k}\right)$.

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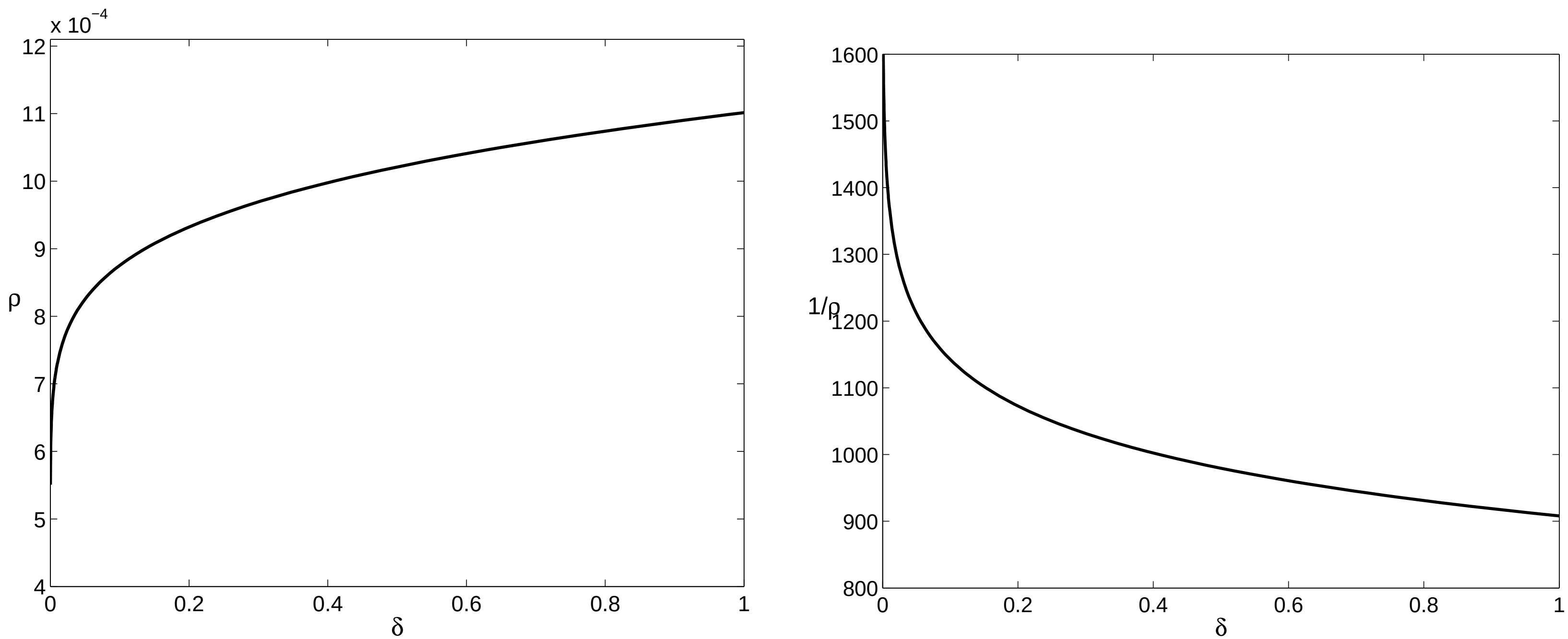
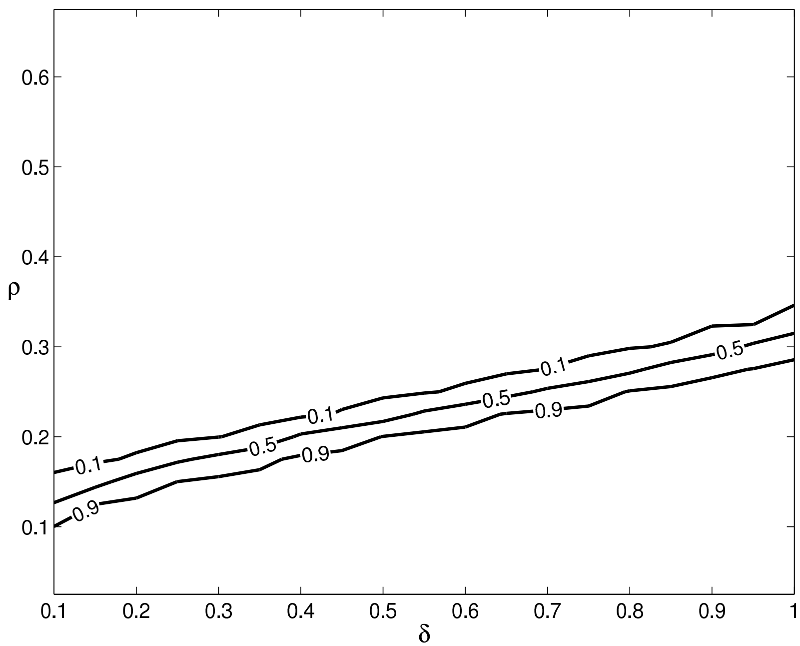
- Defines a phase space for asymptotic analysis.
- For example, RIP bounds for Gaussian matrices $L_k \longrightarrow \mathcal{L}(\delta, \rho)$ and $U_k \longrightarrow \mathcal{L}(\delta, \rho)$ [Bah and Tanner 2010]

RIP phase transition for IHT

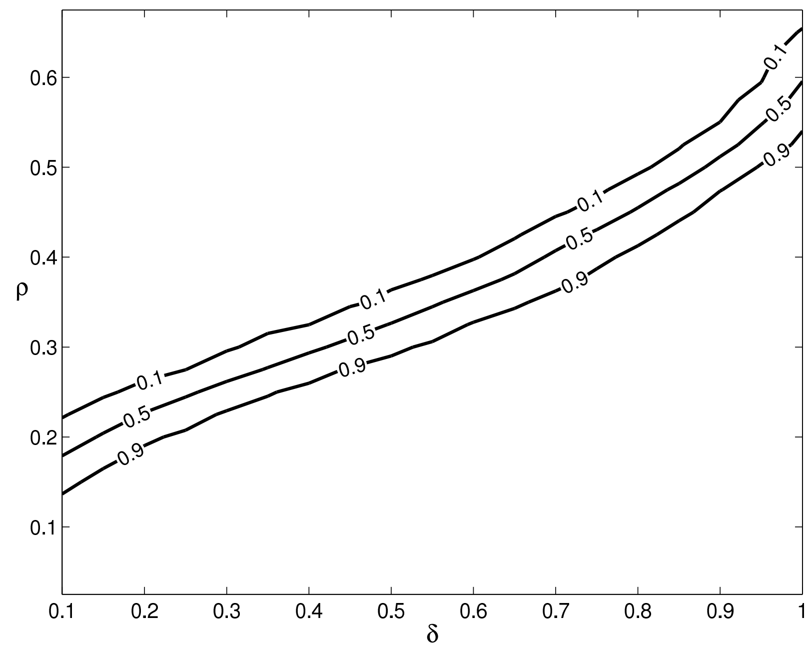


- Recovery guaranteed beneath the phase transition curve
- $n \geq 907k$ measurements needed to guarantee recovery

Empirical phase transitions for IHT



IHT



NIHT

- $\rho \sim 10^{-4}$ for RIP results
- Large gap between theory and average-case behaviour
- NIHT attains the same phase transition as for l_1-relaxation

Recovery guarantees: our approach

Aim: improve sparse-vector recovery guarantees.

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Assumptions:

- **Noiseless case**: $\boldsymbol{e = 0} \implies \boldsymbol{b = Ax}$ and $\boldsymbol{x}$ is $\boldsymbol{k}$-sparse
- Any $\boldsymbol{2k}$ columns of $\boldsymbol{A}$ are linearly independent: $\boldsymbol{L_{2k} < 1}$.

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Then the IHT iterates converge to a fixed point of IHT.

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Approach: derive conditions guaranteeing that

- IHT converges to some fixed point
- x is the only fixed point

$\implies$ IHT converges to x.

Fixed point analysis

A fixed point condition: [Blumensath & Davies]

Let $\bar{x}$ be k-sparse and supported on Γ. Then

$\bar{x}$ is a fixed point of IHT $\iff$ $A_\Gamma^T(b - A\bar{x}) = 0$ and

$$\min_{i \in \Gamma} |\bar{x}_i| \geq \alpha \max_{j \in \Gamma^C} \left| \left\{ A^T(b - A\bar{x}) \right\}_j \right|.$$

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- Any fixed point is a minimum-norm solution on some k-subspace.
- But a minimum-norm solution is not necessarily a fixed point...

Single fixed point condition

Suppose

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$$\implies \|\bar{x}_{\Gamma\setminus\Lambda}\|_2 \geq \alpha \|\left\{A^T(b - A\bar{x})\right\}_{\Lambda\setminus\Gamma}\|_2$$

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$$\implies \|A_\Gamma^\dagger A_{\Lambda\setminus\Gamma} x_{\Lambda\setminus\Gamma}\|_2^2 \geq \alpha^2 \|A_{\Lambda\setminus\Gamma}^T (I - A_\Gamma A_\Gamma^\dagger) A_{\Lambda\setminus\Gamma} x_{\Lambda\setminus\Gamma}\|_2^2.$$

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Theorem: Suppose

$$\|A_\Gamma^\dagger A_{\Lambda\setminus\Gamma} x_{\Lambda\setminus\Gamma}\|_2^2 < \alpha^2 \|A_{\Lambda\setminus\Gamma}^T (I - A_\Gamma A_\Gamma^\dagger) A_{\Lambda\setminus\Gamma} x_{\Lambda\setminus\Gamma}\|_2^2$$

for all $\Gamma \neq \Lambda$. Then x is the only fixed point of IHT.

Analysis for Gaussian matrices

Suppose $A \in \mathbb{R}^{n \times N}$ with entries distributed i.i.d. $N(0, 1/n)$ and suppose x is independent of A. Let Γ be an index set such that $|\Gamma| = k$ and $\Gamma \neq \Lambda$. Then

$$\frac{\|A_\Gamma^\dagger A_{\Lambda\setminus\Gamma} x_{\Lambda\setminus\Gamma}\|_2^2}{\|x_{\Lambda\setminus\Gamma}\|_2^2} = F_\Gamma, \quad \text{where} \quad F_\Gamma \sim \frac{k}{n-k+1} F(k, n-k+1);$$

$$\frac{\|A_{\Lambda\setminus\Gamma}^T (I - A_\Gamma A_\Gamma^\dagger) A_{\Lambda\setminus\Gamma} x_{\Lambda\setminus\Gamma}\|_2^2}{\|x_{\Lambda\setminus\Gamma}\|_2^2} \geq \left(\frac{n-k}{n}\right)^2 R_\Gamma^2,$$

$$\text{where} \quad R_\Gamma \sim \frac{1}{n-k} \chi_{n-k}^2.$$

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Asymptotic large deviations analysis

Recall the proportional-growth asymptotic:

$(k, n, N) \longrightarrow \infty$ such that

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Upper tail bound for F-distribution:

Let $X_n^i \sim \frac{k}{n-k+1} F(k, n-k+1)$ for $i = 1, 2, \ldots, \binom{N}{k}$.

Then there exists a numerically computable function $\mathcal{IF}(\delta, \rho)$ such that for any $\epsilon > 0$,

$$\mathbb{P}\left\{\cap_i \left[X_n^i < \mathcal{IF}(\delta,\rho) + \epsilon\right]\right\} \longrightarrow 1 \ \text{ as } \ n \longrightarrow \infty.$$

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Lower tail bound for normalized χ^2-distribution:

Let $X_n^i \sim \frac{1}{n-k}\chi^2_{n-k}$ for $i = 1, 2, \ldots, \begin{pmatrix} N \\ k \end{pmatrix}$.

Then there exists a numerically computable function $\mathcal{IL}(\delta, \rho)$ such that for any $\epsilon > 0$,

$$\mathbb{P}\left\{\cap_i \left[X_n^i > 1 - \mathcal{IL}(\delta, \rho) - \epsilon\right]\right\} \longrightarrow 1 \text{ as } n \longrightarrow \infty.$$

Comparison with RIP

For $A \in \mathbb{R}^{n\times N}$ Gaussian and $y \in \mathbb{R}^N$ k-sparse independent of A,

$$\frac{\|Ay\|_2^2}{\|y\|_2^2} \sim \frac{1}{n}\chi_n^2.$$

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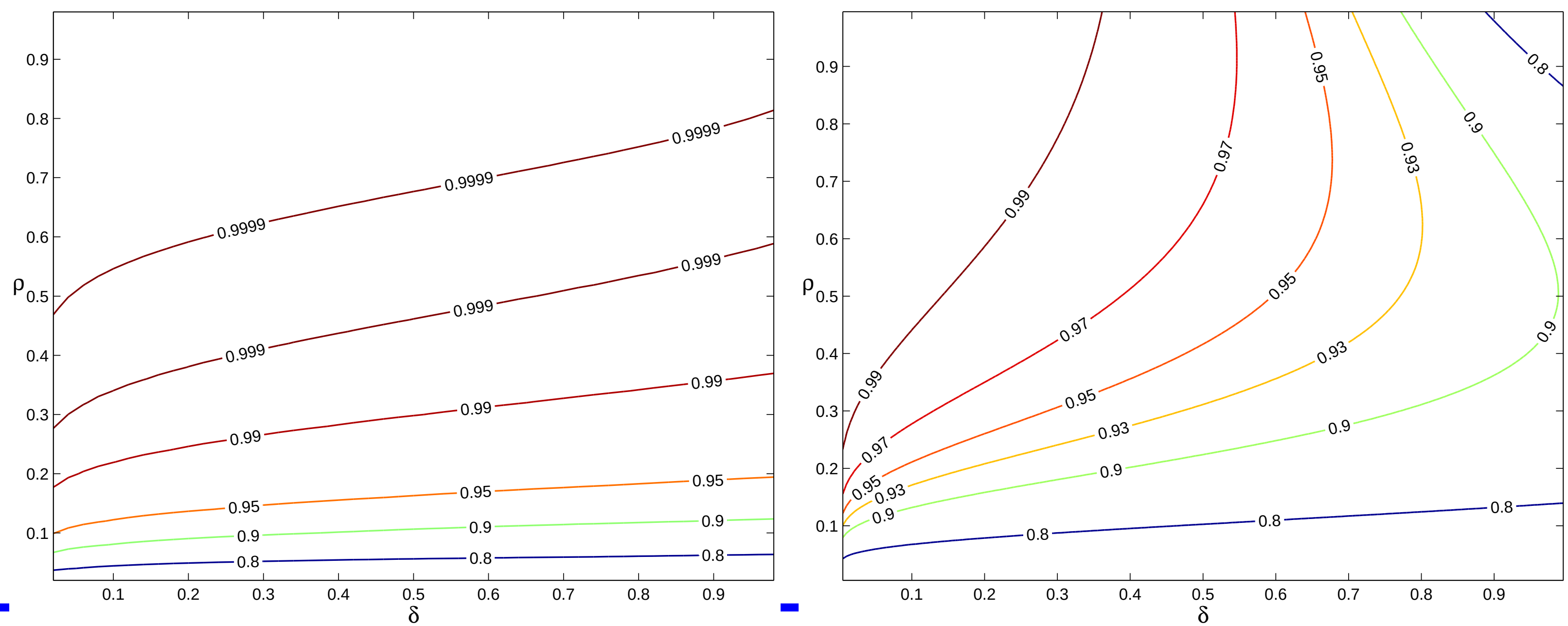

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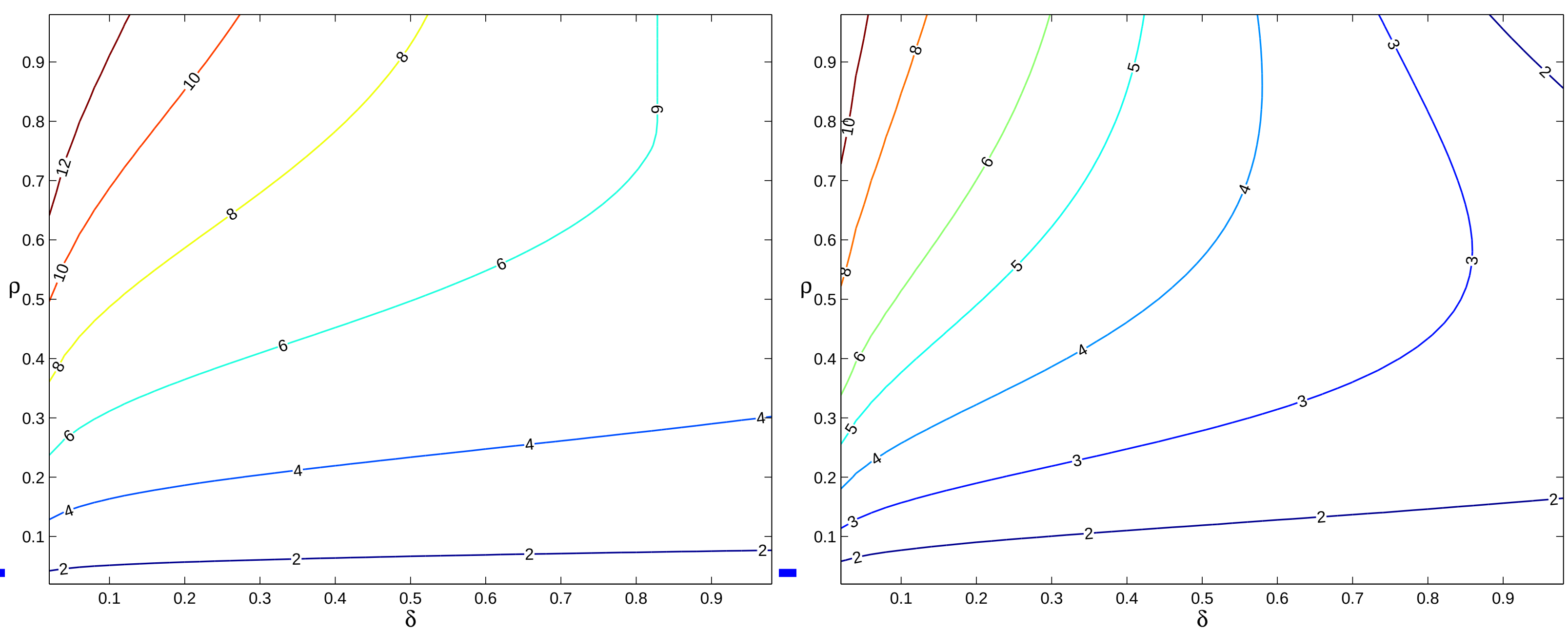

Main recovery result for IHT

Single FP condition: $F_\Gamma < \alpha^2 \left(\frac{n-k}{n}\right)^2 R_\Gamma^2$ for all $\Gamma \neq \Lambda$

$$\overset{(k,n,N)\to\infty}{\Longrightarrow} \sqrt{\mathcal{IF}(\delta,\rho)} < \alpha(1-\rho)[1-\mathcal{IL}(\delta,\rho)].$$

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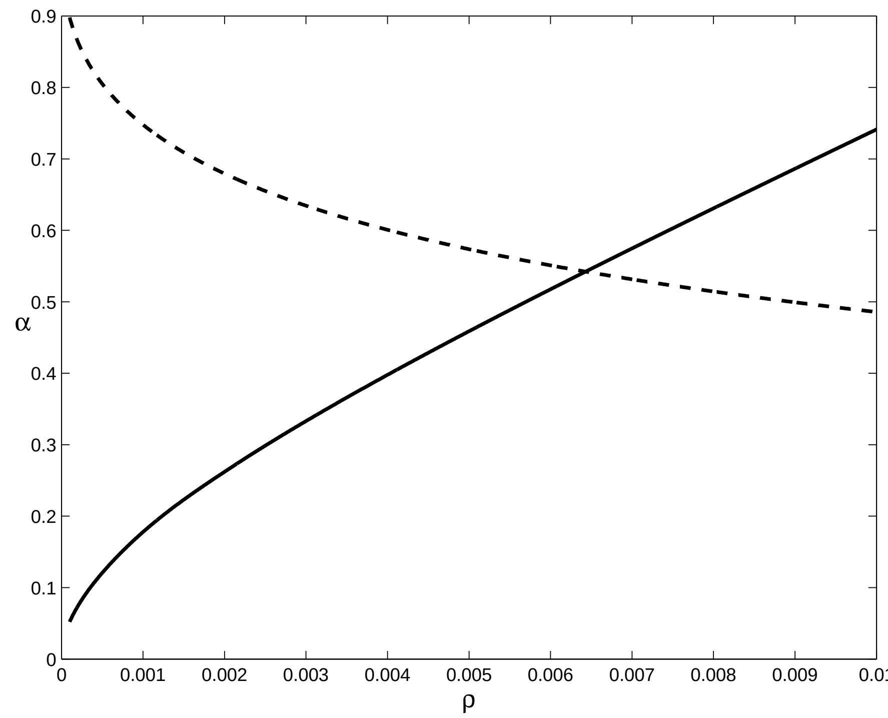

Convergence condition:

$\alpha[1 + U_{2k}] < 1$

$\Downarrow$

$\alpha[1 + \mathcal{U}(\delta, 2\rho)] < 1$.

[Bah and Tanner, 2010]

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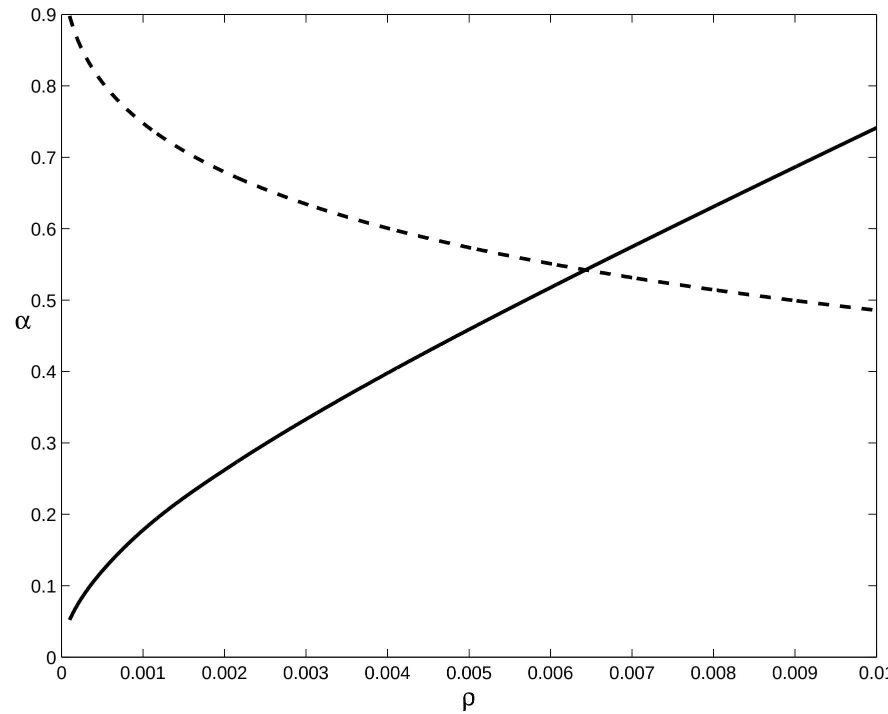


$$\Longrightarrow \quad \frac{\sqrt{\mathcal{IF}(\delta,\rho)}}{(1-\rho)\,[1-\mathcal{IL}(\delta,\rho)]} < \alpha < \frac{1}{1+\mathcal{U}(\delta,2\rho)}$$

Main recovery result for IHT...

Theorem: Let $A \in \mathbb{R}^{n\times N}$ be a Gaussian matrix independent of x and consider the proportional growth asymptotic when $n/N \longrightarrow \delta$ and $k/n \longrightarrow \rho$ as $(k, n, N) \to \infty$. Define

$$\alpha^{min}(\delta,\rho) = \frac{\sqrt{\mathcal{IF}(\delta,\rho)}}{(1-\rho)\left[1-\mathcal{IL}(\delta,\rho)\right]} \quad \text{and} \quad \alpha^{max}(\delta,\rho) = \frac{1}{1+\mathcal{U}(\delta,2\rho)}.$$

If

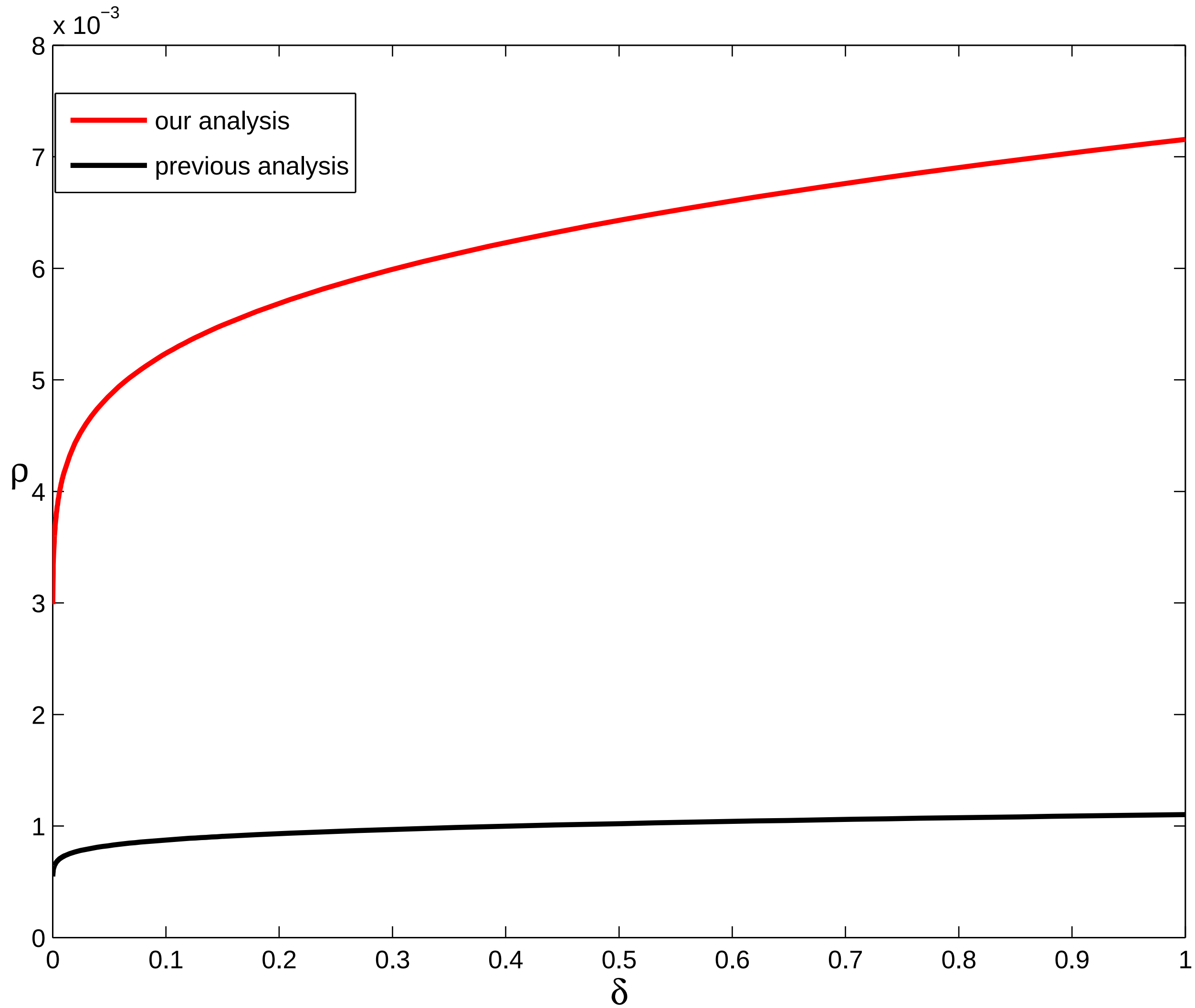
$$\alpha^{min}(\delta,\rho) < \alpha^{max}(\delta,\rho),$$

then IHT converges to x for any α satisfying

$$\alpha \in \left(\alpha^{min}(\delta,\rho), \alpha^{max}(\delta,\rho)\right),$$

with probability tending to 1 exponentially in n.

Phase transition for IHT



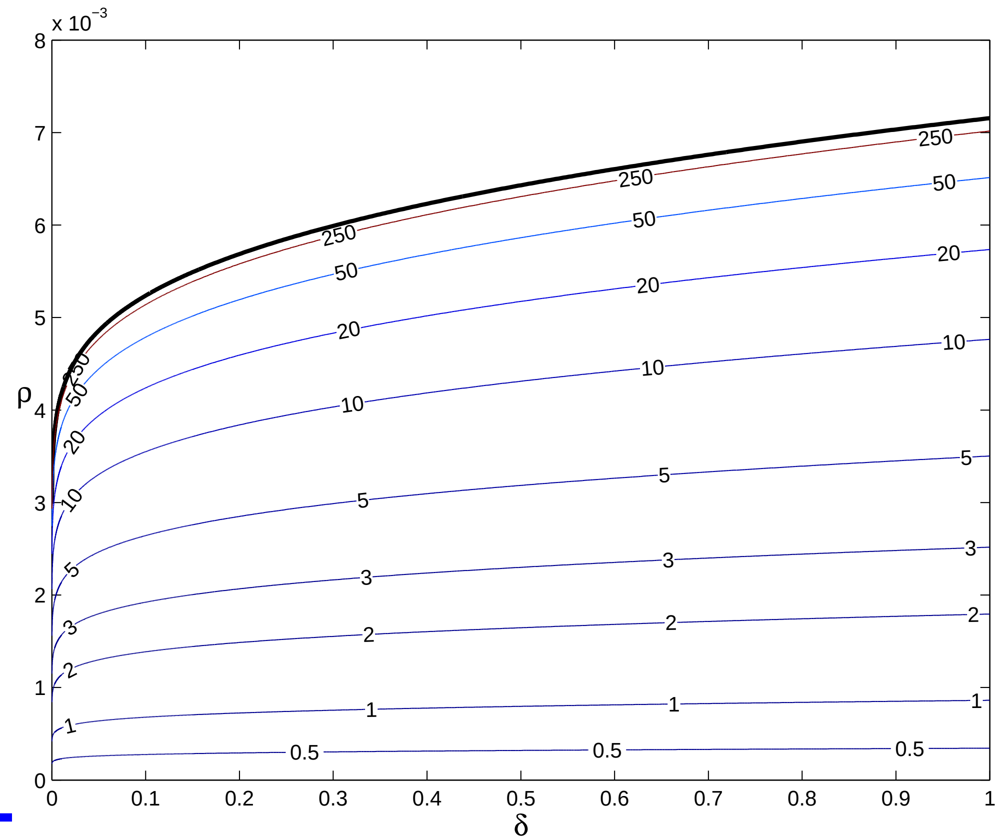
$\longrightarrow$ improvement by a factor of **7** on previous results.

Extension I: the noise case

Gaussian noise model: $b = Ax + e,\ \ e_i \sim N(0, \sigma^2/n)$.

We show that any fixed point $\bar{x}$ satisfies

$$\|\bar{x} - x\|_2 \leq \xi(\delta, \rho) \cdot \sigma.$$



Extension II: IHT variants

Normalised IHT (variable step-size)

- when $\Gamma^{m+1} = \Gamma^m$,

$$\alpha^m = \frac{\|A^T_{\Gamma^m}(b - Ax^m)\|_2^2}{\|A_{\Gamma^m}A^T_{\Gamma^m}(b - Ax^m)\|_2^2}$$

$\longrightarrow$ exact linesearch on the Γ^m face

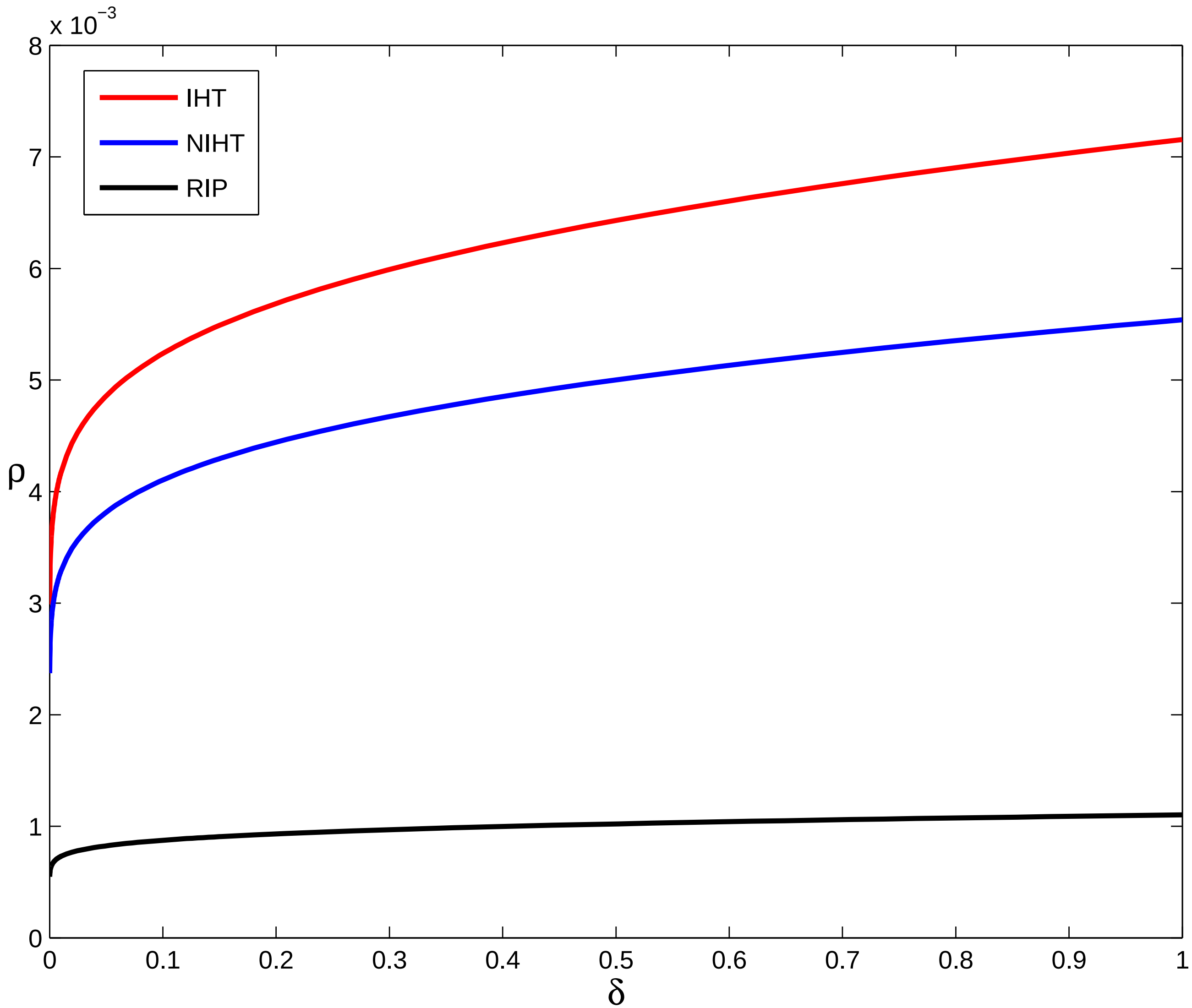
- otherwise employ a ‘sufficient decrease’ strategy.

Fixed points are not well-defined for NIHT

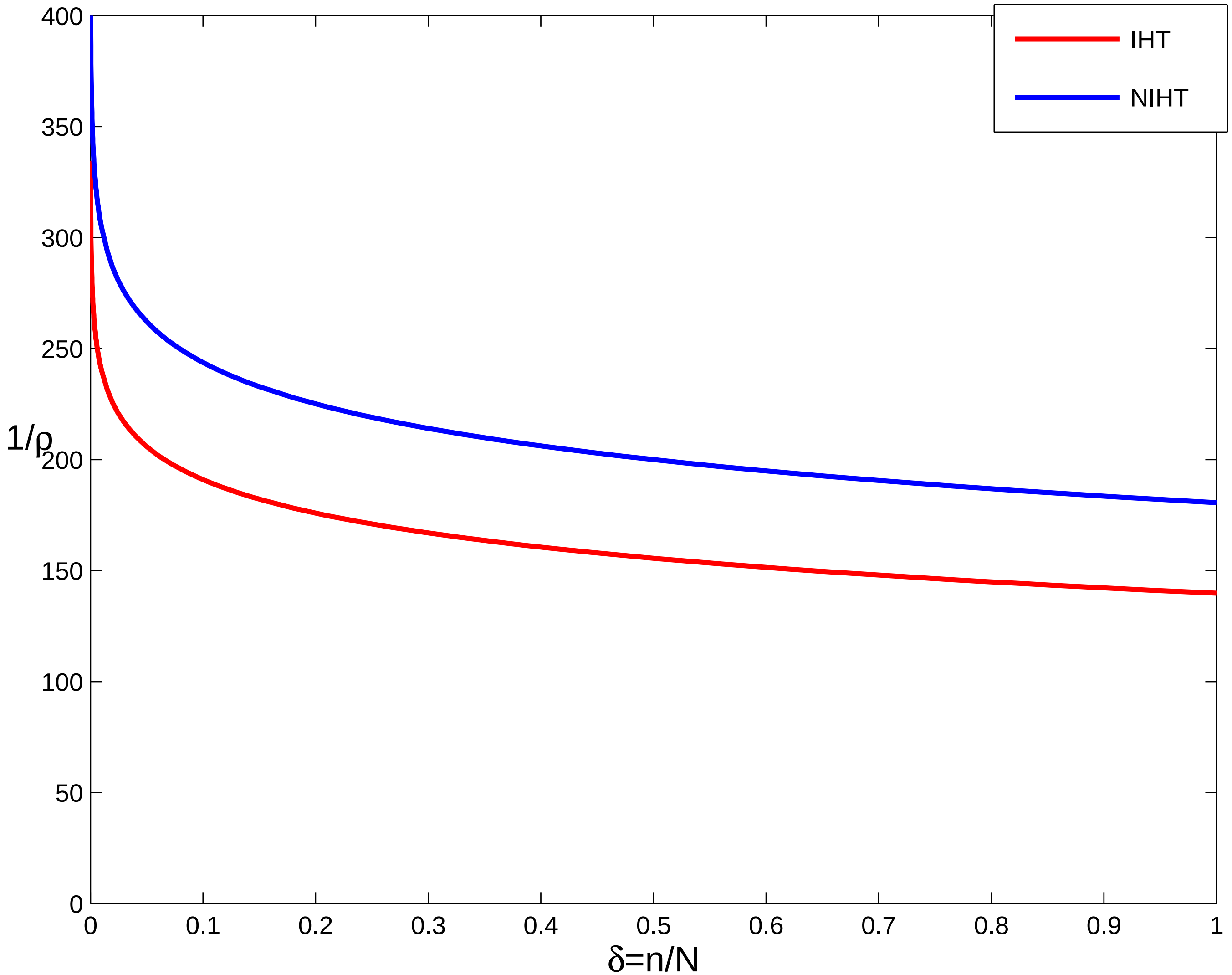
$\longrightarrow$ introduce concept of $\underline{\alpha}$-stable point.

A similar analysis gives an average phase transition for NIHT.

Recovery phase transitions



Inverse of the phase transitions



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Extension III. An even higher phase transition for wavelet trees, recovery if $n > 50k$ (binary).

References

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- *Quantitative recovery conditions for iterative tree projection*; CC, AT (in preparation)
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