Game Theory

Lecture notes for MATH11090 & MATH09002

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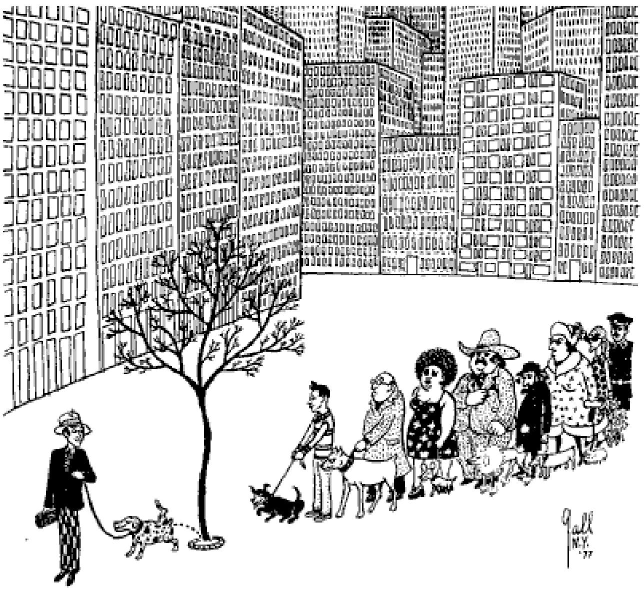
Tragedy of the Commons: Depletion of Shared Resources

Tragedy of the commons *is a* **dilemma** *arising from the situation in which*

- *multiple individuals,*
- *acting independently,*
- *and solely and rationally consulting their own self-interest,*

will ultimately **deplete a shared limited resource** *even when it is clear that it is not in anyone's long-term interest for this to happen.*



(Wikipedia)

Game: Tragedy of the Commons

N students want to **share internet connection** of total bandwidth 1

- Student P_i decides to use $s_i \in S_i = [0, 1]$ portion of the bandwidth
- Quality of the connection deteriorates with increasing total bandwidth usage
- It makes sense to model the payoffs as follows:

$$\pi_i(s_1, \ldots, s_N) = \begin{cases} 0 & \sum_j s_j \geq 1 \\ s_i(1 - \sum_j s_j) & \text{otherwise.} \end{cases}$$

Problem: Find all pure Nash equilibria of this game.

Approach: Via the Pure Best Response theorem (a generalized version thereof for N players): a **profile** of pure strategies $s = (s_1, \ldots, s_N)$ is NE if each student's strategy is the best pure response to the strategies of the others.

Tragedy of the Commons: Finding Pure NE (1)

Viewpoint of student P_i (me):

- The others use up $t_i \stackrel{\text{def}}{=} \sum_{j \neq i} s_j$ portion of the bandwidth.
- If $t_i \geq 1$, then I get 0 payoff whatever I do: *any* pure strategy is my best pure response!
- If $t_i < 1$, my best pure response is to choose s_i maximizing $s_i(1 - t_i - s_i)$.
  - This is a concave quadratic function of the variable s_i
  - Maximum is obtained by taking derivative and setting it to 0: $s_i = \frac{1}{2}(1 - t_i)$

Each potential pure NE has to fall into one of these categories:

- CASE 1: $t_i < 1$ for all i, i.e., no group of $N - 1$ students completely saturate the bandwidth
- CASE 2: $t_i \geq 1$ for at least one i, i.e., at last one group of $N - 1$ students completely exhaust the bandwidth

Tragedy of the Commons: Finding Pure NE (2)

CASE 1: Let us look for a NE in which $t_i < 1$ for all i. Then

- The optimal response of each player P_i is $s_i = (1 - t_i)/2$; which implies that $s_i = 1 - t_i - s_i = 1 - \sum_j s_j$ (*)
- If we let $c = \sum_j s_j$, then $s_i = 1 - c$ for all i. Plugging this into (*):

$$1 - c = 1 - N(1 - c) \quad \Rightarrow \quad 1 - c = \tfrac{1}{N+1} \quad \Rightarrow \quad s_i = \tfrac{1}{N+1} \quad \text{for all } i.$$

CASE 2: If a NE $s = (s_1, \ldots, s_N)$ exists in which $t_i \geq 1$ for some i then, by the definition of payoffs, *all* players must have a zero payoff in it. If it was the case that $t_j < 1$ for some j, then player j could get a nonzero payoff by choosing $s_j < 1 - t_j$. Therefore,

$$t_i \geq 1 \quad \text{for all} \quad i. \tag{1}$$

On the other hand, any combination of strategies s satisfying (1) must be a NE.

Tragedy of the Commons: Finding Pure NE (3)

Summary: Set of **all pure NE:**

$$\underbrace{\{(\tfrac{1}{N+1}, \ldots, \tfrac{1}{N+1})\}}_{\text{tragic equilibrium}} \cup \underbrace{\{s = (s_1, \ldots, s_N) \ : \ s_i \in [0,1], \sum_{j \neq i} s_j \geq 1 \text{ for all } i\}}_{\text{disastrous equilibrium!}}.$$

Properties of the "tragic" NE:

$$\textbf{each player's payoff} = s_i(1 - \sum_j s_j) = \tfrac{1}{(N+1)^2}$$

$$\textbf{total payoff to all players} = \tfrac{N}{(N+1)^2} \approx \tfrac{1}{N}$$

The **coordinated** choice $s_i = 1/(2N)$ for all (why this?), would give

$$\textbf{total payoff} = \sum_{i=1}^{N} [s_i(1 - \sum_j s_j)] = N \times \tfrac{1}{2N} \times \tfrac{1}{2} 1/(4N) = \tfrac{1}{4} \gg \tfrac{1}{N}.$$

Game: War of Attrition ("waiting game")

[Longman Dictionary] **War of attrition:** *a struggle in which you harm your opponent in a lot of small ways, so that they become gradually weaker.*

Two players **compete for a resource** which has value v to both.

- Both players chose a time $t_i \geq 0$ until which they are willing to persist in the contest: $S_1 = S_2 = [0, \infty)$
- Payoffs decrease linearly with time at rate $\alpha > 0$, equally to both
- The resource is won by the one who quits last (a tie $\Rightarrow$ both lose!)

$$\pi_1(t_1, t_2) = \begin{cases} v - \alpha t_2 & \text{if } t_1 > t_2 \quad (P_1 \text{ wins}) \\ -\alpha t_1 & \text{if } t_1 \leq t_2 \quad (P_1 \text{ loses}) \end{cases}$$

$$\pi_2(t_1, t_2) = \begin{cases} v - \alpha t_1 & \text{if } t_2 > t_1 \quad (P_2 \text{ wins}) \\ -\alpha t_2 & \text{if } t_2 \leq t_1 \quad (P_2 \text{ loses}) \end{cases}$$

Problem: Find all pure Nash equilibria.

War of Attrition: Finding Pure NE (1)

$$\text{Recall:} \quad \pi_2(t_1, t_2) = \begin{cases} v - \alpha t_1 & t_2 > t_1 \quad (P_2 \text{ wins}) \\ -\alpha t_2 & t_2 \leq t_1 \quad (P_2 \text{ loses}) \end{cases}$$

First observation:

- Each player can guarantee a zero payoff by choosing $t_i = 0$
- Therefore, **no player can have a negative payoff in a NE**

Best pure response analysis:

- **CASE 1:** $t_1 < v/\alpha$ (that is: $v - \alpha t_1 > 0$)
  - If $t_2 = 0$, P_2 gets a **zero payoff**
  - If $0 < t_2 \leq t_1$, P_2 **loses** and gets the **negative payoff** $-\alpha t_2$
  - If $t_1 < t_2$, P_2 **wins** and gets the **positive payoff** $v - \alpha t_1$
- **If (t_1, t_2) is a NE, then $t_1 < v/\alpha \Rightarrow t_2 > t_1$ (*1)**
- **CASE 2:** $t_1 \geq v/\alpha$ (that is: $v - \alpha t_1 \leq 0$)
  - If $t_2 = 0$, P_2 gets a **zero payoff**
  - If $0 < t_2 \leq t_1$, P_2 **loses** and gets the **negative payoff** $-\alpha t_2$
  - If $t_1 < t_2$, P_2 **wins** and gets the **negative payoff** $v - \alpha t_1$
- **If (t_1, t_2) is a NE, then $t_1 \geq v/\alpha \Rightarrow t_2 = 0$ (*2)**

War of Attrition: Finding Pure NE (2)

The entire previous slide is valid if we swap the indices 1 and 2 since the game is symmetric.

Our findings so far: If (t_1, t_2) is a NE, then the following statements must hold (**(*3) and (*4) follow by symmetry** from (*1) and (*2))

$$t_1 < v/\alpha \quad \Rightarrow \quad t_2 > t_1 \qquad (*1)$$

$$t_1 \geq v/\alpha \quad \Rightarrow \quad t_2 = 0 \qquad (*2)$$

$$t_2 < v/\alpha \quad \Rightarrow \quad t_1 > t_2 \qquad (*3)$$

$$t_2 \geq v/\alpha \quad \Rightarrow \quad t_1 = 0 \qquad (*4)$$

Further consequences: If (t_1, t_2) is a NE, then

$$t_1 < v/\alpha \overset{(*1)}{\Rightarrow} t_2 > t_1 \overset{(*3)}{\Rightarrow} t_2 \geq v/\alpha \overset{(*4)}{\Rightarrow} t_1 = 0 \qquad (*5)$$

$$t_2 < v/\alpha \overset{(*3)}{\Rightarrow} t_1 > t_2 \overset{(*1)}{\Rightarrow} t_1 \geq v/\alpha \overset{(*2)}{\Rightarrow} t_2 = 0 \qquad (*6)$$

$$t_1 = 0 \quad \text{or} \quad t_2 = 0 \quad \text{(follows from (*2) and (*5))} \qquad (*7)$$

War of Attrition: Finding Pure NE (3)

Summary of results so far:

- (*7) says that in a NE, we must have either $t_1 = 0$ or $t_2 = 0$
- If $t_1 = 0$, then by (*5), $t_2 \geq v/\alpha$
- If $t_2 = 0$, then by (*6), $t_1 \geq v/\alpha$

So we have managed to narrow down a relatively small and well-described set T which **must contain all pure NE:**

$$T \overset{\text{def}}{=} \underbrace{\{(0, t_2) \ : \ t_2 \geq v/\alpha\}}_{T_1} \cup \underbrace{\{(t_1, 0) \ : \ t_1 \geq v/\alpha\}}_{T_2}.$$

Comments:

- As far as we know at this point of the analysis, it is possible that the actual set of NE is even smaller than this.
- In fact, it might be that a pure NE pair does not even exist (as in Matching Pennies)!
- However, we are lucky: it is easy to verify that **every pair of strategies in T is a NE.**

War of Attrition: Finding Pure NE (4)

Checking that every $(t_1, t_2) \in T_1 \cup T_2$ is a NE:

It is enough to do this for T_1 only, the analysis is identical for T_2 by symmetry.

- As we have seen on the "Finding pure NE (1)" slide (CASE 1): **any $t_2 > t_1$ is the best response of P_2 to $t_1 = 0$** (in particular, $t_2 \geq v/\alpha$ is).
- As we have seen on the "Finding pure NE (1)" slide (CASE 2 with swapped indices): **$t_1 = 0$ is the best pure response of P_1 to $t_2 \geq v/\alpha$.**

The book

James N. Webb, Game Theory: Decisions, Interactions and Evolution, Springer, 2007

claims that $(0, v/\alpha)$ and $(v/\alpha, 0)$ are the only pure NE.

Moral: Do not always believe a book! Or your instructor ;-)

War of Attrition: The Process of Finding Pure NE

- We have first obtained two **necessary conditions** (*1) and (*2): conditions that any NE pair (t_1, t_2) must satisfy
- We have then used symmetry to derive analogous conditions for the other player, obtaining (*3) and (*4)
- If (*1)+(*2)+(*3)+(*4) was a system of linear or nonlinear conditions, we would know what to do:
  - solve the system, and
  - for each solution check whether it is a NE (via Pure Best Response thm)
- However, (*1)+(*2)+(*3)+(*4) is NOT a standard system of equations: How to find all solutions (t_1, t_2) of this **system of conditions?**
  - We have looked more deeply at what these conditions say, by examining the relationships between them
  - In particular, we have obtained new derived condition (*5), which together with (*2) implies condition (*7), which turned out to be **very illuminating** and enabled us to come up with a nice "normal" description of a set (T) containing all pure NE.
  - It turned out that T contained NE only

Zero-Sum Games

In a zero-sum game $\pi_2 = -\pi_1$.

- Letting $f = \pi_2$, we can write $G = (\{P_1, P_2\}, S_1 \times S_2, f)$.

Player

- P_1 is interested in maximizing his payoff $-f$ ($=$ minimizing his loss f)
- P_2 wants to maximize his payoff f

If P_1 chooses $s_1 \Rightarrow$ cannot lose more than $\sup_{s_2 \in S_2} f(s_1, s_2)$

- This would be the actual loss if P_2 happened to know that P_1 was playing s_1
- This is because choosing s_2 in this way guarantees to P_2 the maximum payoff

Conservative Strategies

"He sees David Cameron as his role model"

Minimax Conservative Strategy: Player 1

If player P_1 is **risk-averse (conservative)**, she would want to pick a strategy **minimizing the worst-case loss function**

$$u_1(s_1) \stackrel{\text{def}}{=} \sup_{s_2 \in S_2} f(s_1, s_2).$$

Definition

Conservative strategy $\hat{s}_1$ minimizing the worst-case loss of P_1 is called the **minimax strategy** and the resulting loss

$$\hat{u}_1 \stackrel{\text{def}}{=} u_1(\hat{s}_1) = \inf_{s_1 \in S_1} \sup_{s_2 \in S_2} f(s_1, s_2)$$

is called the **conservative value of P_1.**

By playing her conservative strategy, P_1 can ensure with complete confidence that

- her loss will be at most $\hat{u}_1$, or equivalently,
- her payoff will be at least $-\hat{u}_1$.

Maximin Conservative Strategy: Player 2

In complete analogy, the conservative strategy of P_2 would be to **maximize her worst case payoff function**

$$u_2(s_2) \stackrel{\text{def}}{=} \inf_{s_1 \in S_1} f(s_1, s_2).$$

Definition

Conservative strategy $\hat{s}_2$ maximizing the worst-case payoff of P_2 is called the **maximin strategy** and the resulting payoff

$$\hat{u}_2 \stackrel{\text{def}}{=} u_2(\hat{s}_2) = \sup_{s_2 \in S_2} \inf_{s_1 \in S_1} f(s_1, s_2)$$

is called the **conservative value of P_2.**

By playing her conservative strategy, P_2 can ensure with complete confidence that

- her payoff would be at least $\hat{u}_2$, or equivalently,
- her loss will be at most $-\hat{u}_2$.

A Lemma Needed to Prove Minimax Inequality

Lemma

For any f, S_1, S_2 and $(s_1', s_2') \in S_1 \times S_2$,

$$\inf_{s_1 \in S_1} f(s_1, s_2') \leq f(s_1', s_2') \leq \sup_{s_2 \in S_2} f(s_1', s_2) \tag{2}$$

Proof.

Too trivial to even say it's trivial! □

Minimax Inequality

Minimax Inequality

Theorem (Minimax Inequality)

$$\hat{u}_2 \stackrel{def}{=} \sup_{s_2 \in S_2} \inf_{s_1 \in S_1} f(s_1, s_2) \leq \inf_{s_1 \in S_1} \sup_{s_2 \in S_2} f(s_1, s_2) \stackrel{def}{=} \hat{u}_1 \tag{3}$$

Proof.

By applying supremum over $s_2' \in S_2$ to the chain of inequalities (2), we obtain

$$\sup_{s_2' \in S_2} \inf_{s_1 \in S_1} f(s_1, s_2') \leq \sup_{s_2 \in S_2} f(s_1', s_2).$$

Taking infimum over $s_1' \in S_1$ in both sides of the last inequality gives

$$\sup_{s_2' \in S_2} \inf_{s_1 \in S_1} f(s_1, s_2') \leq \inf_{s_1' \in S_1} \sup_{s_2 \in S_2} f(s_1', s_2)$$

$\square$

Optimization Duality and Conservative Strategies

Weak duality results in optimization can be viewed from the Game Theoretic perspective as

- **conservative** game-playing
- between **two** players
- in a **zero-sum** game

Consider the optimization problem

$$\begin{aligned}
&\text{minimize} \quad && f_0(x) \\
&\text{subject to} && \\
& && f_1(x) \leq 0 \\
& && f_2(x) \leq 0 \\
& && \cdots \\
& && f_m(x) \leq 0
\end{aligned} \tag{4}$$

where $f_0, f_1, \ldots, f_m : X \rightarrow R$ are arbitrary real-valued functions.

Primal and Dual Problems

Let us define

- $S_1 = X, \; S_2 = R_+^m = \{y \in R^m \; : \; y_i \geq 0, \; i = 1, \ldots, m\}$
- $f(x,y) = f_0(x) + \sum_{i=1}^m y_i f_i(x)$ **Lagrangian = payoff function**
-

$$g(x) \stackrel{\text{def}}{=} \sup_{y \in Y} f(x,y) = \begin{cases} f_0(x) & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ +\infty & \text{otherwise.} \end{cases} \tag{5}$$

- $h(y) \stackrel{\text{def}}{=} \inf_{x \in X} f(x,y)$

Consider the following pair of **primal** and **dual** problems

$$p^* \stackrel{\text{def}}{=} \inf_{x \in X} g(x) \stackrel{\text{def}}{=} \hat{u}_1 \quad (P) \qquad \text{and} \qquad d^* \stackrel{\text{def}}{=} \sup_{y \in Y} h(y) \stackrel{\text{def}}{=} \hat{u}_2 \quad (D).$$

Note: (P) is equivalent to the original problem (4). Why?

Weak Duality for General Pair of Optimization Problems

Theorem (Weak duality)

The optimal values p^* and d^* of the primal and dual optimization problems (P) and (D) satisfy $p^* \geq d^*$.

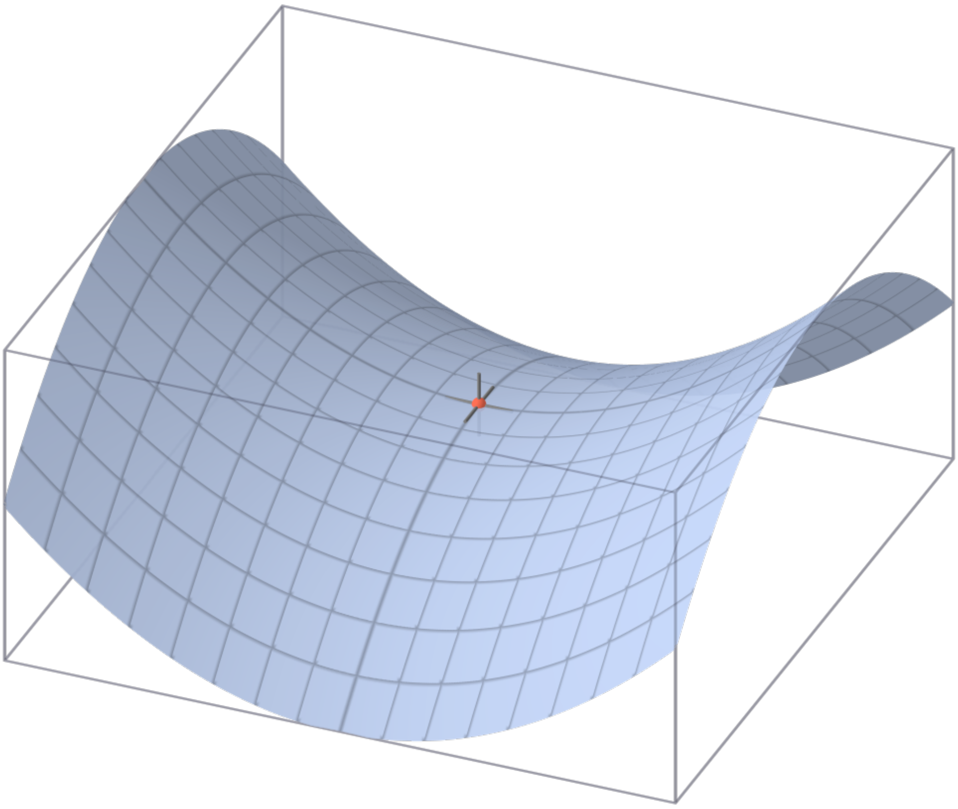
Proof.

Follows from Minimax Inequality since $\hat{u}_1 = p^*$ and $\hat{u}_2 = d^*$ $\square$

This result

- gives the basic relationship that holds between (P) and (D) under **no assumptions**
- is one of the reasons for using the term **duality**

Saddle Points

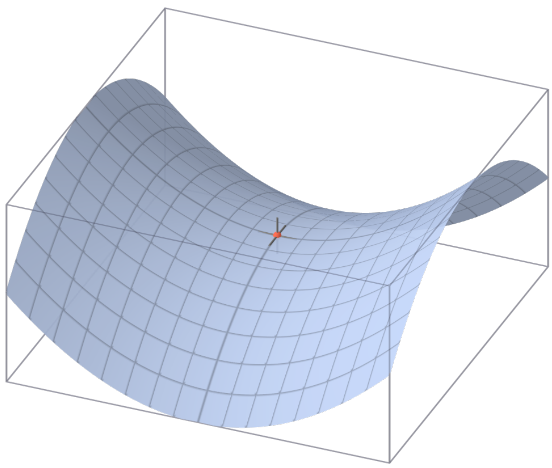


Saddle Points

Definition

Pair of strategies $(s_1^*, s_2^*) \in S_1 \times S_2$ is a **saddle point** of f if

$$\inf_{s_1 \in S_1} f(s_1, s_2^*) = f(s_1^*, s_2^*) = \sup_{s_2 \in S_2} f(s_1^*, s_2). \tag{6}$$



Minimax Equality

Theorem (Minimax Equality)

If a saddle point (s_1^*, s_2^*) exists then

$$d^* \equiv \hat{u}_2 \overset{def}{=} \sup_{s_2 \in S_2} \inf_{s_1 \in S_1} f(s_1, s_2) = f(s_1^*, s_2^*) = \inf_{s_1 \in S_1} \sup_{s_2 \in S_2} f(s_1, s_2) \overset{def}{=} \hat{u}_1 \equiv p^*.$$

The common value $\hat{u} \overset{def}{=} \hat{u}_2 = \hat{u}_1$ is called the **value of the game.**

Proof.

The leftmost and rightmost expressions in the below chain of inequalities are both equal to $f(s_1^*, s_2^*)$ by the definition of a saddle point:

$$\inf_{s_1 \in S_1} f(s_1, s_2^*) \le \underbrace{\sup_{s_2 \in S_2} \inf_{s_1 \in S_1} f(s_1, s_2) \overset{(3)}{\le} \inf_{s_1 \in S_1} \sup_{s_2 \in S_2} f(s_1, s_2)}_{\text{Minimax Inequality}} \le \sup_{s_2 \in S_2} f(s_1^*, s_2).$$

$\square$

Convex-Concave Games

Definition

A two-person zero-sum game where

- S_1 and S_2 are convex sets, and
- $f(s_1, s_2)$ is convex in s_1 and concave in s_2

is called a **convex-concave game.**

Convex-Concave Games Have Saddle Points

Theorem (Existence of Saddle Points)

Let $G = (\{P_1, P_2\}, S_1 \times S_2, f)$ be a convex-concave game and assume that

(i) *S_1 and S_2 are closed and bounded sets,*

(ii) *f is defined on $S_1 \times S_2$,*

(iii) *f is continuous.*

Then **a saddle point exists.**

Proof.

Nontrivial, have to omit it. □

Game Value and Strong Duality

Corollary (Value of Convex-Concave Games)

Any convex-concave game G satisfying the assumptions of the previous theorem has a value. That is $\hat{u}_1 = \hat{u}_2$.

Corollary (Strong Duality in Convex Optimization)

Strong duality holds between the between the pair of convex optimization problems (P) and (D). That is $p^* = d^*$.

Proof.

Theorem "Existence of Saddle Points" ensures that a saddle point exists, "Minimax Equality" then implies that $p^* = \hat{u}_1 = \hat{u}_2 = d^*$. □