On Harman's "Unreachable Points" Puzzle

Peter Richtárik

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Abstract

In his blog post "Unreachable points"[1], Radoslav Harman posed the following puzzle:

> "In the 2D plane there is a circular disk D and a point a inside it. For each point b on the boundary of D let T_b be the intersection of D and the line passing through the midpoint of the line segment $[a, b]$ and perpendicular to it. What is the set of points of D which do not lie on any of the segments T_b?"

In this short note we give a simple proof of a general version of this puzzle and add a few more insights. Our solution approach highlights the interplay between geometry, convex analysis, matrix theory and optimization, and is perhaps suitable as an exercise for graduate students.

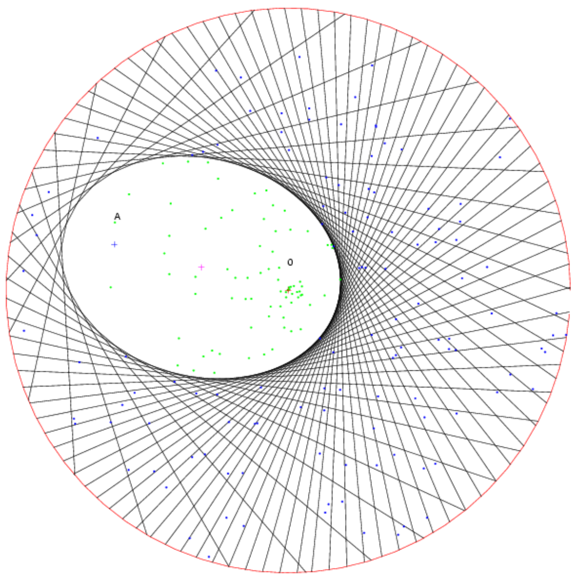

Figure 1: The white shape inside, the set of unreachable points, is an ellipsoid whose center is the midpoint between a and the center of the circle. It lies inside the disk if a does. It is analytically described by Theorems 1 and 6 in this note.

We will assume without loss of generality that D is the unit Euclidean ball in $\mathbf{R}^n$. In the first result we show that the set of unreachable points is a level set of a certain simple convex function: norm perturbed by a linear term.

[1] http://radoslav-harman.blogspot.com/2010/02/nedosiahnutelne-body.html

Theorem 1 (Level Set)**.** *Let $a \in \mathbf{R}^n$ with $\|a\| \leq 1$. Furthermore, for $b \in \mathbf{R}^n$ let*

$$T_a(b) = \{x \in \mathbf{R}^n \ : \ \langle x - \tfrac{1}{2}(a+b), a-b \rangle = 0\}.$$

Then the set of points unreachable by any of the hyperplanes $T_a(b)$ for $\|b\| = 1$ is given by

$$S_a \stackrel{def}{=} \bigcap_{\|b\|=1} [T_a(b)]^c = \left\{ x \in \mathbf{R}^n \ : \ \|x\| - \langle a, x \rangle < \frac{1 - \|a\|^2}{2} \right\},$$

where $[T_a(b)]^c = R^n \backslash T_a(b)$, i.e., the complement of $T_a(b)$ in $\mathbf{R}^n$.

Proof. It is easy to see that for $\|b\| = 1$ we have

$$T_a(b) = \left\{ x \in \mathbf{R}^n \ : \ \langle b, x \rangle = \langle a, x \rangle + \frac{1 - \|a\|^2}{2} \right\}. \tag{1}$$

Let us now fix x and ask whether there exists b of unit norm such that $x \in T_a(b)$. It can be shown using a continuity argument together with the Cauchy-Schwarz inequality that the function $b \mapsto \langle b, x \rangle$ maps the unit sphere onto the interval $[-\|x\|, \|x\|]$. This, together with (1) implies such b exists if and only if

$$-\|x\| \leq \langle a, x \rangle + \frac{1 - \|a\|^2}{2} \leq \|x\|. \tag{2}$$

Since $\langle a, x \rangle + \|x\| \geq -\|a\|\|x\| + \|x\| = \|x\|(1 - \|a\|) \geq 0 \geq (\|a\|^2 - 1)/2$, the left-hand side inequality in (2) is always satisfied. Therefore, x does not lie in $T_a(b)$ for any b of unit norm precisely when the right-hand side inequality in (2) is violated, which proves the theorem. $\square$

Corollary 2. *If $\|a\| < 1$, then S_a is a convex set containing 0 and a. If $\|a\| = 1$, then $S_a = \emptyset$.*

Proof. It follows from Theorem 1 that S_a is a level set of the convex function $f(x) = \|x\| - \langle a, x \rangle$ and hence it is convex. That $0 \in S_a$ is trivial. To show that $a \in S_a$ it is enough to note that $\|a\| < (1 + \|a\|)/2$ and multiply both sides by $1 - \|a\|$. If $\|a\| = 1$, then all $x \in S_a$ must satisfy $\|x\| - \langle a, x \rangle < 0$. Since $\|x\| - \langle a, x \rangle \geq \|x\| - \|a\|\|x\| = 0$, S_a must be empty. $\square$

Since S_a is empty if a has norm one, from now on we will assume that $\|a\| < 1$.

Theorem 3 (Enclosing Ball 1)**.** *All points of S_a have norm strictly less than $\frac{1}{2}(1 + \|a\|)$. The bound is tight.*

Proof. Choose $x \in S_a$ and let $x = tx'$, where $t \geq 0$ and $\|x'\| = 1$. Then from Theorem 1 we know that $t\|x'\| - t\langle a, x' \rangle < \frac{1}{2}(1 - \|a\|^2)$, and consequently

$$\|x\| = t < \frac{1 - \|a\|^2}{2(1 - \langle a, x' \rangle)} \leq \frac{1 - \|a\|^2}{2(1 - \|a\|)} = \frac{1 + \|a\|}{2}.$$

For any positive ϵ small enough so that $t = \frac{1}{2}(1 + \|a\|) - \epsilon > 0$, let $x_\epsilon = ta/\|a\|$. Note that

$$\|x_\epsilon\| - \langle a, x_\epsilon \rangle = t - t\|a\| = \tfrac{1}{2}(1 - \|a\|^2) - \epsilon(1 - \|a\|) < \tfrac{1}{2}(1 - \|a\|^2),$$

and hence by Theorem 1, $x_\epsilon \in S_a$. If we let $\epsilon \to 0$, then $\|x_\epsilon\| \to \frac{1}{2}(1 + \|a\|)$. $\square$

The following is a technical result which we will use in proving that S_a is an ellipsoid.

Lemma 4. *Let a and $x \in \mathbf{R}^n$ satisfy $\langle a, x\rangle + (1 - \|a\|^2)/2 < 0$. Then $\|x\| > -\langle a, x\rangle + (\|a\|^2 - 1)/2$.*

Proof. Let $\alpha = \langle a, x\rangle$. Then $-\alpha = -\langle a, x\rangle \leq \|a\|\|x\|$ and hence $\|x\| \geq -\alpha/\|a\|$. It therefore suffices to show that $-\alpha/\|a\| > -\alpha + (\|a\|^2 - 1)/2$, which can be simplified to $\alpha < \frac{1}{2}\|a\|(1 + \|a\|)$. However, we know by assumption that $\alpha < (\|a\|^2 - 1)/2$ and therefore it is enough to prove that $\|a\|^2 - 1 \leq \|a\|(1 + \|a\|)$, which is straightforward. □

Lemma 5. *The following hold:*

(i) $I_n - aa^T \succ 0$, *and*

(ii) $\frac{I_n - aa^T}{1 - \|a\|^2} \succeq I_n$.

Proof. Since $1 - \|a\|^2 > 0$ and $I_n \succ 0$, the first statement follows from the second. Inequality (ii) is equivalent to $I_n - aa^T \succeq (1 - \|a\|^2)I_n$, which in is turn equivalent to $\|a\|^2 I_n \succeq aa^T$. We thus only need to show that for all vectors $x \in \mathbf{R}^n$, $\|a\|^2 x^T I_n x \geq x^T aa^T x$, which follows from the Cauchy-Schwarz inequality. □

Theorem 6 (Ellipsoid)**.** *The set of unreachable points S_a is a full-dimensional ellipsoid given by*

$$S_a = \{x \in \mathbf{R}^n \ : \ (x - v)^T B (x - v) < r^2\},$$

where $B = I_n - aa^T \succ 0$ governs its shape, its center is $v = a/2$ and the radius is $r = \frac{1}{2}\sqrt{1 - \|a\|^2}$.

Proof. We know from Theorem 1 that $S_a = \{x \in \mathbf{R}^n \ : \ \|x\| < \langle a, x\rangle + (1 - \|a\|^2)/2\}$. Lemma 4 says that we can square both sides of this inequality without having to worry that we have introduced new solutions. Letting $t = \|a\|^2$, we have

$$\begin{aligned}
S_a &= \{x \in \mathbf{R}^n \ : \ x^T x < \langle a, x\rangle^2 + (1 - t)^2/4 + \langle a, x\rangle(1 - t)\} \\
&= \{x \in \mathbf{R}^n \ : \ x^T(I_n - aa^T)x < 2\langle(1 - t)a/2, x\rangle + (1 - t)^2/4\} \\
&= \{x \in \mathbf{R}^n \ : \ x^T B x < 2x^T B v - v^T B v + v^T B v + (1 - t)^2/4\} \\
&= \{x \in \mathbf{R}^n \ : \ (x - v)^T B (x - v) < v^T B v + (1 - t)^2/4\} \\
&= \{x \in \mathbf{R}^n \ : \ (x - v)^T B (x - v) < r^2\}.
\end{aligned}$$

□

We will now show that S_a is tightly circumscribed by the ball with center $a/2$ and radius $\frac{1}{2}$.

Theorem 7 (Enclosing Ball 2)**.** *The distance of all points of S_a from $a/2$ is strictly less than $\frac{1}{2}$. This bound is tight.*

Proof. Let H be the open ball with center $v = a/2$ and radius $\frac{1}{2}$. Then

$$C = \{x \in R^n \ : \ (x - v)^T [4I_n](x - v) < 1\}.$$

From Theorem 6 we know that

$$S_a = \left\{x \in R^n \ : \ (x - v)^T \left[\frac{4(I_n - aa^T)}{1 - \|a\|^2}\right](x - v) < 1\right\}.$$

Inclusion $S_a \subseteq C$ now follows from part (ii) of Lemma 5.

To establish tightness, let $x_\epsilon \in S_a$ be as in the proof of Theorem 3. If we let $\epsilon \to 0$, then

$$\left\| x_\epsilon - \frac{1}{2}a \right\| = \left(\frac{1}{2}(1 + \|a\|) - \epsilon - \frac{1}{2}\|a\| \right) \to \frac{1}{2}.$$

□

In the next result we will show that *all* the hyperplanes $T_a(b)$, for b of unit norm, are *supporting* to S_a. We will do so by studying the behavior of the linear function $x \mapsto \langle b - a, x\rangle$ on $T_b(a)$ and on S_a. Note that this function is constant on $T_a(b)$, achieving on it the value $\langle b - a, \frac{a+b}{2}\rangle = \frac{1}{2}(1 - \|a\|^2)$. In particular, we will prove that the supremum of this function over S_a is also equal to $\frac{1}{2}(1 - \|a\|^2)$, and we give a formula for the intersection of $T_a(b)$ and the closure of S_a.

Theorem 8 (Supporting Hyperplanes). *If b is of unit norm, then the hyperplane $T_a(b)$ is a supporting hyperplane to S_a and, moreover,*

$$\operatorname{cl} S_a \cap T_a(b) = \left\{ \frac{1 - \|a\|^2}{2 - 2\langle a, b\rangle} b \right\}.$$

Proof. Using the characterization of S_a given in Theorem 6, it can be shown using the Karush-Kuhn-Tucker optimality conditions that the unique optimal point x^* and optimal value Opt of the problem

$$Opt \overset{\text{def}}{=} \max_{x \in \operatorname{cl} S_a} \langle c, x\rangle$$

are given by

$$x^* = v + r\frac{B^{-1}c}{\|c\|_B^*} \tag{3}$$

and

$$Opt = \langle c, v\rangle + r\|c\|_B^*, \tag{4}$$

where $v = a/2$, $r = \frac{1}{2}\sqrt{1 - \|a\|^2}$ and $B = I_n - aa^T$ are as in Theorem 6, and $\|c\|_B^* = (c^T B^{-1} c)^{1/2}$. Let us first compute $y = B^{-1}(b - a)$. Since $B = I_n - aa^T$, we will guess that y is of the form $b - \alpha a$, and then compute α. Since $By = y - aa^T y = b - \alpha a - \langle a, b\rangle a + \alpha\|a\|^2 a$, it is enough to solve for α from $\alpha + \langle a, b\rangle - \alpha\|a\|^2 = 1$. We get $\alpha = (1 - \langle a, b\rangle)/(1 - \|a\|^2)$, whence

$$B^{-1}(b - a) = b - \frac{1 - \langle a, b\rangle}{1 - \|a\|^2} a = \frac{b - \|a\|^2 b - a + \langle a, b\rangle a}{1 - \|a\|^2}, \tag{5}$$

and

$$\|b - a\|_B^* = \sqrt{(b - a)^T B^{-1}(b - a)} = \frac{1 - \langle a, b\rangle}{\sqrt{1 - \|a\|^2}}. \tag{6}$$

We now have all the ingredients needed to evaluate x^* and Opt:

$$x^* = \frac{a}{2} + \frac{1}{2}\sqrt{1 - \|a\|^2}\frac{\frac{b - \|a\|^2 b - a + \langle a, b\rangle a}{1 - \|a\|^2}}{\frac{1 - \langle a, b\rangle}{\sqrt{1 - \|a\|^2}}} = \frac{a}{2} + \frac{1}{2}\frac{b - \|a\|^2 b - a + \langle a, b\rangle a}{1 - \langle a, b\rangle} = \frac{1 - \|a\|^2}{2 - 2\langle a, b\rangle} b,$$

$$Opt = \langle b - a, \tfrac{1}{2}a\rangle + \frac{1}{2}\sqrt{1 - \|a\|^2}\frac{1 - \langle a, b\rangle}{\sqrt{1 - \|a\|^2}} = \tfrac{1}{2}(1 - \|a\|^2).$$

The rest follows from the discussion preceding this theorem. □

We are now ready to prove that the points 0 and a are the foci of the ellipsoid S_a.

Corollary 9 (Foci)**.** *The points* 0 *and* a *are the foci of the ellipsoid* S_a*. In particular,*

$$S_a = \{x \in \mathbf{R}^n \ : \ \|x\| + \|x - a\| < 1\}.$$

Proof. It is enough to show that for all points x on the boundary of S_a we have $\|x\|+\|x-a\| = 1$. Let x be any point on the boundary of S_a. It can be shown easily that 0 lies in the interior of S_a and hence $x \neq 0$. Let $b = x/\|x\|$ and consider the point z obtained as the intersection of $T_a(b)$ and $\operatorname{cl} S_a$. Theorem 8 tells us that $x = tb$ for some $t > 0$. Notice that because also $x \in T_b(a)$, the right triangles $[x, b, \frac{1}{2}(a + b)]$ and $[x, a, \frac{1}{2}(a + b)]$ are identical, whence $\|a - x\| = \|b - x\|$. Finally,

$$\|x\| + \|x - a\| = \|x\| + \|x - b\| = \|b\| = 1.$$

□