Inexact Proximal-Gradient Methods and Linearly-Convergent Stochastic-Gradient Methods

Mark Schmidt

Joint work with Nicolas Le Roux, Francis Bach, Michael Friedlander

INRIA - SIERRA Project - Team
Laboratoire d'Informatique de l'École Normale Supérieure
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Outline

Composite Convex Optimization Problems

- We consider composite optimization problems:

$$\min_{x \in \mathbb{R}^d} f(x) := g(x) + h(x),$$

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- Often, g is a data-fitting term, and h is a regularizer,

$$\min_{x \in \mathbb{R}^d} \sum_{i=1}^{N} l_i(x) + \lambda r(x).$$

- A well-studied example is ℓ_1-regularized least squares,

$$\min_{x \in \mathbb{R}^d} \|Ax - b\|^2 + \lambda \|x\|_1.$$

Convergence Rates of Proximal-Gradient Methods

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- Proximal-gradient methods have the same convergence rates as [accelerated] gradient methods for smooth optimization.
  [Nesterov, 2007, Beck & Teboulle, 2009]

Non-Simple Regularizers and Big-N Problems

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For example,

1. Overlapping-group ℓ_1-regularization,

$$h(x) := \lambda \sum_{g \in \mathcal{G}} \|x_g\|,$$

2. Data-fitting with a large number of samples N,

$$g(x) := \sum_{i=1}^{N} f_i(x).$$

Non-Simple Regularizers and Big-N Problems

We can often efficiently approximate these quantities:

1. For overlapping-group ℓ_1-regularization, we can use an inexact proximity operator,
$$y \approx \mathrm{prox}[x].$$

2. For data-fitting with a large number of samples N, we can use a subsample of the f_i,
$$\frac{1}{|\mathcal{B}|}\sum_{i \in \mathcal{B}} f_i'(x) \approx \frac{1}{N}\sum_{i=1}^{N} f_i'(x) = f'(x).$$

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But, we may lose the convergence rates with these approximations.

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2. **Linearly-Convergent Stochastic-Gradient Methods**:
   - We show that using an increasing sample of the f_i functions achieves a linear convergence rate.
   - We propose a method that achieves a linear convergence rate but only evaluates a single f_i on each iteration.

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- We can equivalently write this as the quadratic optimization

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- The solution is the proximal-gradient algorithm:

$$x_{k+1} = \mathrm{prox}_{\alpha_k}[x_k - \alpha_k g'(x_k)].$$

Special case of Projected-Gradient Methods

- Projected-gradient methods are a special case:

$$h(x) = \begin{cases} 0 & \text{if } x \in \mathcal{C} \\ \infty & \text{if } x \notin \mathcal{C}. \end{cases}$$

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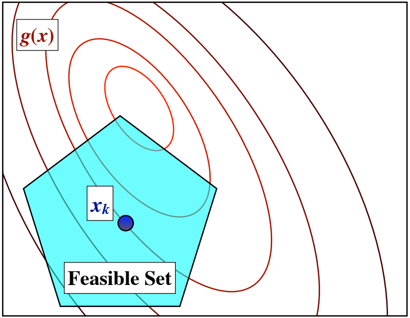

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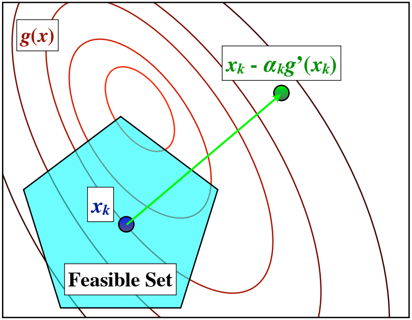

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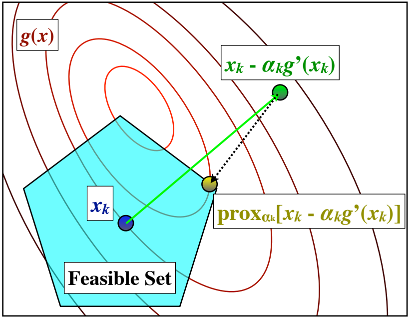

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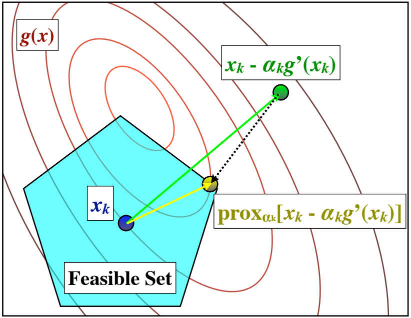

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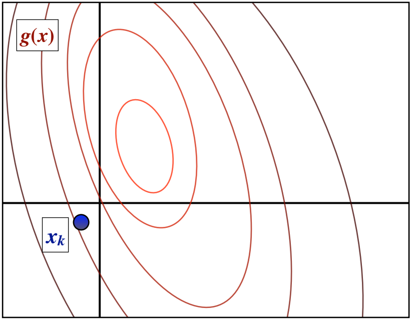

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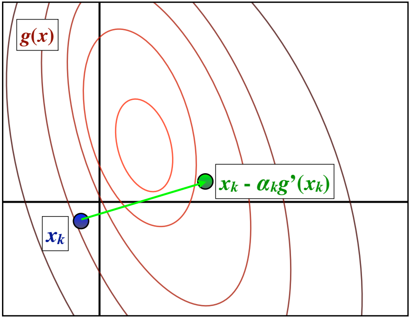

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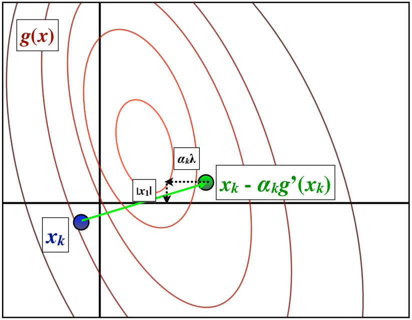

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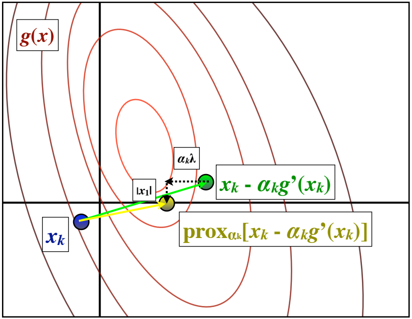

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- But for smooth problems accelerated gradient methods have faster rates [Nesterov, 1983]:

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- For composite problems accelerated proximal-gradient methods have these same rates:

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    2. *Group ℓ_1-Regularization*.
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    5. *Simplex constraints*.
    6. *Euclidean cone constraints*.

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Inexact Proximal-Gradient Methods

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Inexact Proximal-Gradient Methods

- We can efficiently approximate the proximity operator for:
  1. *Total-variation regularization and generalizations like the graph-guided fused-LASSO*.
  2. *Nuclear-norm regularization and other regularizers on the singular values of matrices*.
  3. *Overlapping group ℓ_1-regularization with general groups*.
  4. *Positive semi-definite cone*.
  5. *Combinations of simple functions*.

Summary of Contribution

Many recent works use inexact proximal-gradient methods:

- Cai et al. [2010], Liu & Ye [2010], Schmidt & Murphy [2010], Barbero & Sra [2011], Fadili & Peyré [2011], Ma et al. [2011].

Summary of Contribution

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Our question:

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Our contribution:

- *Inexact proximal-gradient methods can achieve the fast convergence rates, if the errors are appropriately controlled*.

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- Proximal-gradient methods with decreasing error magnitude:
  [Patriksson, 1995, Combettes, 2004]
  - Do not consider convergence rates.

Problem Setting and Algorithm

- We consider the problem

$$\min_{x \in \mathbb{R}^d} \; g(x) + h(x).$$

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- The accelerated proximal-gradient method uses

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where

$$y_k = x_k + \beta_k(x_k - x_{k-1}),$$

and the sequence $\{\beta_k\}$ is chosen to give a faster rate.

Central Assumptions and Notation

- In all our results we assume:
  - g is convex and g' is L-Lipschitz continuous,

$$||g'(x) - g'(y)|| \leq L||x - y||, \forall x, y.$$

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  - $g + h$ attains its minimum at a certain x_*.
  - The step size α_k is set to $1/L$.
  - The gradient g' is computed with an error e_k.
  - x_k is an ε_k-approximate solution of the proximity operator,

$$\frac{L}{2}||x_k - y||^2 + h(x_k) \leq \varepsilon_k + \min_{x \in \mathbb{R}^d} \left\{ \frac{L}{2}||x - y||^2 + h(x) \right\}.$$

    (we can use a duality gap to check this condition)

Fast Convergence Rates of Proximal-Gradient Methods

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Proximal-Gradient	$O(1/k)$	$O((1-\mu/L)^k)$
Accelerated Proximal-Gradient	$O(1/k^2)$	$O((1-\sqrt{\mu/L})^k)$

- We give conditions on the sequences of gradient errors $\{e_k\}$ and proximity errors $\{\varepsilon_k\}$ that preserve these rates.

Convexity - Basic Proximal-Gradient Method

Proposition 1. *If the sequences $\{||e_k||\}$ and $\{\sqrt{\varepsilon_k}\}$ are summable then the basic proximal-gradient method achieves*

$$f\left(\frac{1}{k}\sum_{i=1}^{k} x_i\right) - f(x_*) = O(1/k).$$

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- E.g., $\|e_k\|$ and $\sqrt{\varepsilon_k}$ could decrease as $O(1/k^{1+\delta})$ for $\delta > 0$.
- If they decrease as $O(1/k)$, then we get $O((\log k)^2/k)$. (see the paper for the constant factors)

Convexity - Accelerated Proximal-Gradient Method

Proposition 2. *If the sequences* $\{k||e_k||\}$ *and* $\{k\sqrt{\varepsilon_k}\}$ *are summable then the accelerated proximal-gradient method achieves*

$$f(x_k) - f(x_*) = O(1/k^2),$$

with $\beta_k = (k-1)/(k+2)$.

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- E.g., $\|e_k\|$ and $\sqrt{\varepsilon_k}$ could decrease as $O(1/k^{2+\delta})$ for $\delta > 0$.
- As in previous work, our analysis indicates the accelerated method is more sensitive to errors.

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- For *twice-differentiable* functions, equivalent to $g''(x) \succeq \mu I, \forall x$.
- Here, we can obtain linear convergence rates.

Strong Convexity - Basic Proximal-Gradient Method

Proposition 3. *If the sequences* $\{||e_k||\}$ *and* $\{\sqrt{\varepsilon_k}\}$ *are in* $O(\rho^k)$ *for* $\rho < (1-\mu/L)$ *then the basic proximal-gradient method achieves*

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- If they converge with $\rho > (1 - \mu/L)$, the rate is $O(\rho^k)$.
- If they converge with $\rho = (1 - \mu/L)$, the rate is $O(k(1 - \mu/L)^k)$.

Strong Convexity - Accelerated Method

Proposition 4. *If the sequences $\{||e_k||^2\}$ and $\{\varepsilon_k\}$ are in $O(\rho^k)$ for $\rho < (1 - \sqrt{\mu/L})$ then the accelerated proximal-gradient method achieves*

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with $\beta_k = (1 - \sqrt{\mu/L})/(1 + \sqrt{\mu/L})$.

- We also obtain a bound on the iterates because

$$\frac{\mu}{2}||x_k - x_*||^2 \leq f(x_k) - f(x_*).$$

Outline

CUR-like factorization with the ℓ_2-norm

We consider the factorization of Mairal et al. [2011] to approximate a matrix W using a subset of rows and columns:

$$\boxed{\min_X \frac{1}{2}||W - WXW||_F^2 + \lambda_{\text{row}} \sum_{i=1}^{n_r} ||X^i||_p + \lambda_{\text{col}} \sum_{j=1}^{n_c} ||X_j||_p.}$$

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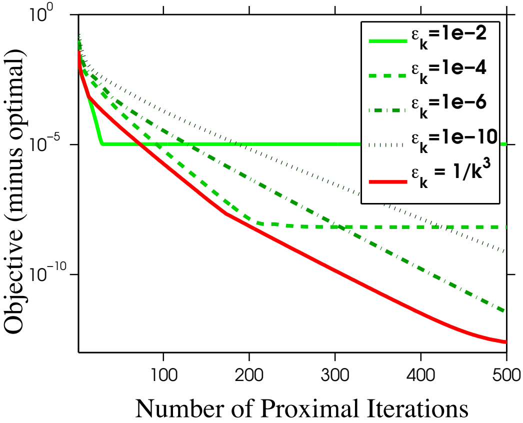
- For appropriate p, yields sparse rows and sparse columns.
- Previous work used $p = \infty$, since there is no known exact algorithm for $p = 2$.
- We use the proximal-Dykstra algorithm to compute an approximate proximity operator with $p = 2$.
- Duality gap ensures ε_k-optimality of approximate proximity.

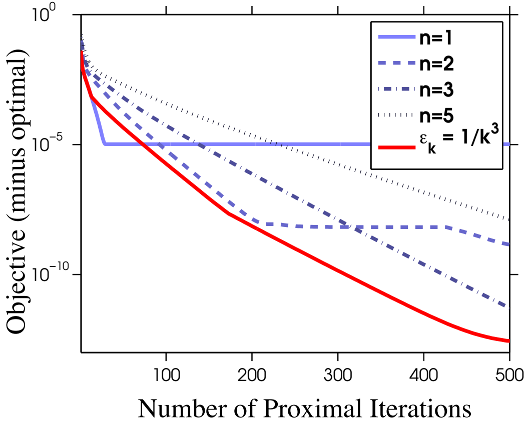
Comparison against a fixed prox solution accuracy

Using an optimal ε_k sequence compared to a fixed precision for the approximate proximity:



Comparison against a fixed number of prox iterations

Using an optimal ε_k sequence compared to running a fixed number of proximal iterations:



Discussion

- Inexact proximal-gradient methods may be useful in other applications: *total-variation or nuclear-norm regularization*.
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- We would like to handle an unknown L and μ.
- We would like to adaptively update $||e_k||$ and ε_k.
- We would like to analyze proximal-Newton methods.
- Villa et al. [2011] and Jiang et al. [2011] have independently analyzed accelerated proximal-gradient methods (convex g).

Summary

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- But, they require the calculation of the proximity operator.
- Many authors have recently applied these methods under an inexact proximity operator.
- We show that the convergence rates are preserved if the inexactness is appropriately controlled

Outline

Strongly Convex and Smooth Big-N Problems

- We now focus to problems of the form

$$\min_{x \in \mathbb{R}^d} g(x) := \frac{1}{N} \sum_{i=1}^{N} f_i(x),$$

where each f_i' is L-Lipschitz continuous and g is μ-strongly convex.

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where each f_i' is L-Lipschitz continuous and g is μ-strongly convex.

- Includes ℓ_2-regularization of any convex loss functions,

$$f_i(x) := \frac{\lambda}{2} \|x\|^2 + l_i(x).$$

- We are interested in the case where N is large.

Stochastic Gradient Methods for Big-N Problems

- Stochastic gradient (SG) methods use iterations of the form

$$x^{k+1} = x^k - \alpha_k f'_{i_k}(x^k),$$

  where i_k is selected uniformly among the set $\{1, \dots, n\}$.
- Appealing because the iteration cost is independent of N.

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- Appealing because the iteration cost is independent of N.
- But SG iterations have a sublinear convergence rate

$$\mathbb{E}[g(x^k)] - g(x^*) = O(1/k).$$

- This is optimal if only accessing the function through unbiased function/gradient measurements.

Full Gradient Methods for Big-N Problems

- But, for finite data sets better rates are possible.
- For example, we could use the full gradient (FG) method,

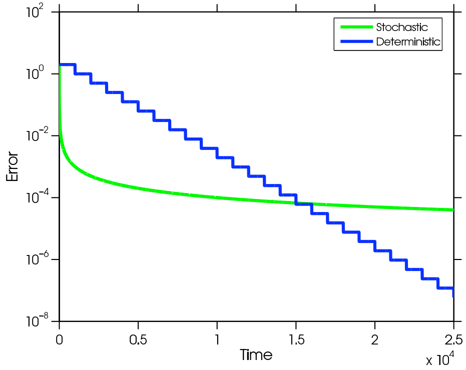
$$x^{k+1} = x^k - \alpha_k g'(x^k) = x^k - \frac{\alpha_k}{N}\sum_{i=1}^{N} f_i'(x^k).$$

- This method achieves a linear convergence rate,

$$g(x^k) - g(x^*) = O(\rho^k).$$

- But, FG iterations are N times more expensive than SG iterations.

Stochastic vs. Full Gradient Methods



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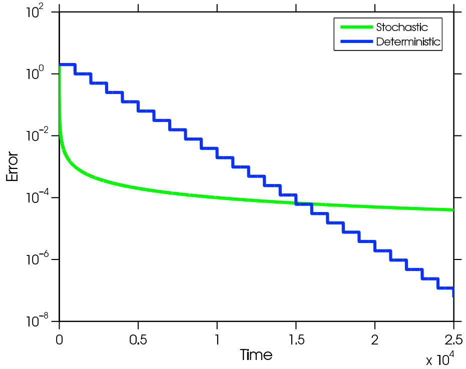

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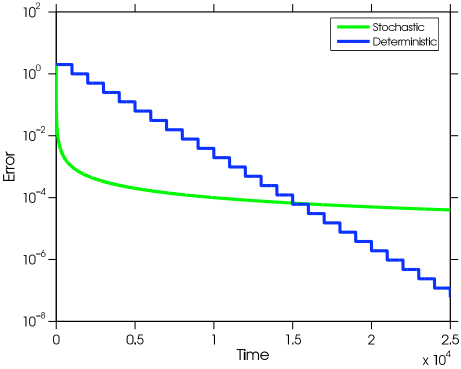
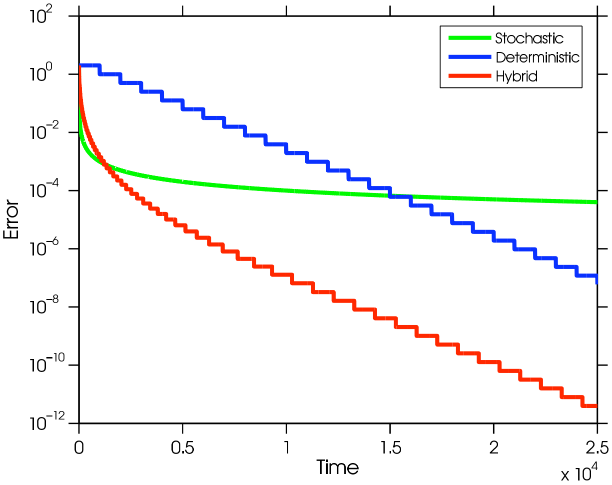


- Stochastic makes great progress initially, but slows down.
- Determinstic makes steady progress, but is expensive.
- Can we design hybrid methods with the best of both worlds?

Motivation for Hybrid Methods



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- *Momentum, gradient averaging, iterate averaging, stochastic version of FG methods*:

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  - Linear convergence, but only up to a fixed tolerance.
- *Hybrid Methods, Incremental Average Gradient*:

  [Bertsekas, 1997, Blatt et al., 2008]

  - Linear rate, but iterations make full passes through the data.

Two ideas for achieving a linear rate

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- Idea #1: **Control the sample size** to interpolate between the stochastic and deterministic method.
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- Idea #2: **Build a sequence of estimates** that converge to $g'(x_k)$ as $\|x_{k-1} - x_k\| \to 0$.
  (only looks at a single f_i on each iteration)

Outline

Hybrid Deterministic-Stochastic Methods

- A common variant of SG methods uses a batch $\mathcal{B}_k$ instead of a single example,

$$x^{k+1} = x^k - \frac{\alpha_k}{|\mathcal{B}_k|}\sum_{i\in\mathcal{B}_k} f_i'(x_k).$$

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- We can choose the batch sizes to achieve linear convergence.
- Early iterations are cheap like SG iterations.

Incremental Gradient Method Error Bounds

Under standard assumptions on the f_i', by choosing $|\mathcal{B}_k|$ to satisfy

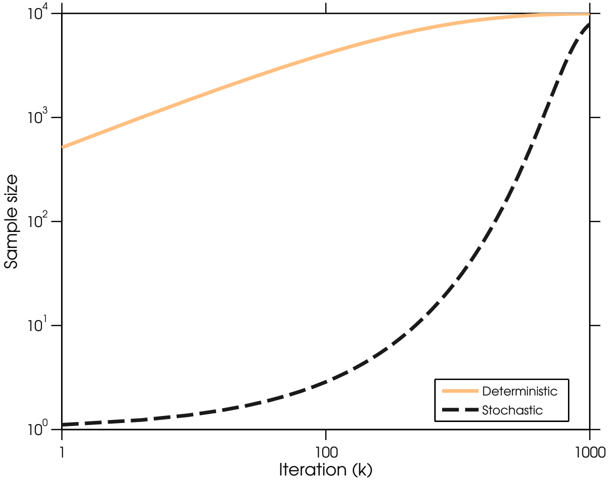
$$\frac{N - |\mathcal{B}_k|}{N} \frac{1}{|\mathcal{B}_k|} = O(\gamma^k),$$

for any $\epsilon > 0$ we have

$$\mathbb{E}[f(x_k) - f(x_*)] = [f(x_0) - f(x_*)]O([1 - \mu/L + \epsilon]^k) + O(\sigma^k),$$

where $\sigma = \max\{\gamma, 1 - \mu/L\}$.

Batch Schedule needed for Linear Rate



Improved Rates with Newton-like Scaling

- The algorithm may converge slowly if μ/L is small.
- We can also analyze a Newton-like algorithm

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where d_k is the solution of

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- We can then show rates using a modified μ and L based on the Hessian approximation H_k.

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- We made an implementation, where we use the *L-BFGS* quasi-Newton Hessian approximation.

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$$\bar{f}(x_k + \alpha_k d_k) < \bar{f}(x_k) + \eta \alpha g(x_k)^T d_k,$$

on the sampled objective

$$\bar{f}(x_k) = \frac{1}{|\mathcal{B}_k|} \sum_{i \in \mathcal{B}_k} f_i(x_k).$$

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- By increasing the batch size this eventually reduces to a conventional line-search quasi-Newton method, inheriting the global and local convergence guarantees of this method.

Batch-Size Selection in Stochastic Gradient Methods

We performed experiments comparing three algorithms:

- Deterministic: Conventional L-BFGS quasi-Newton method.
- Stochastic: Constant step-size stochastic gradient descent.
- Hybrid: An L-BFGS quasi-Newton method with batch size

$$|\mathcal{B}_{k+1}| = \lceil \min\{1.1 \cdot |\mathcal{B}_k| + 1, M\} \rceil.$$

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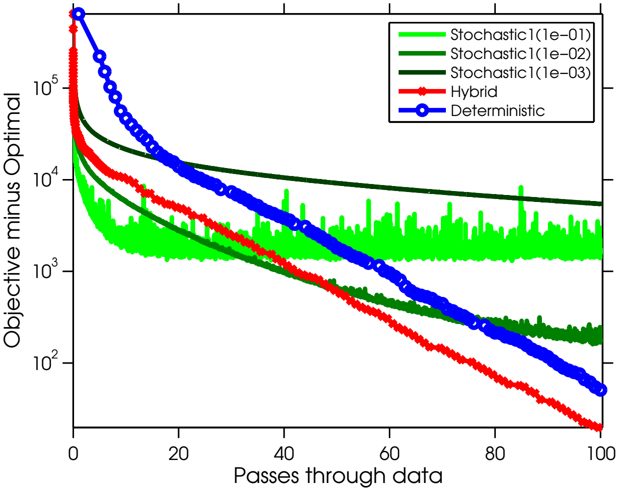
We trained a conditional random fields (CRF) on the CoNLL-2000 noun-phrase chunking shared task (chain-structure).

Evaluation on Chain-Structured CRFs

Results on chain-structured conditional random field:



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Stochastic Average Gradient

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where

$$y_i^k = \begin{cases} f_i'(x_k) & \text{if } i = i_k, \\ y_i^{k-1} & \text{otherwise.} \end{cases}$$

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  - Randomized version of the incremental aggregated gradient (IAG) algorithm of Blatt et al. [2008].

Convergence Rate of SAG: Small Steps

With a step size of $\alpha_k = \frac{1}{2NL}$, the SAG iterations satisfy

$$\mathbb{E}[\|x^k - x^*\|^2] \leq C\left(1 - \frac{\mu}{8LN}\right)^k$$

- A linear rate with iterations that are independent of N!

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- A linear rate with iterations that are independent of N!
- But, with this step size the performance is similar to the FG and IAG methods.

Convergence Rate of SAG: Big Steps

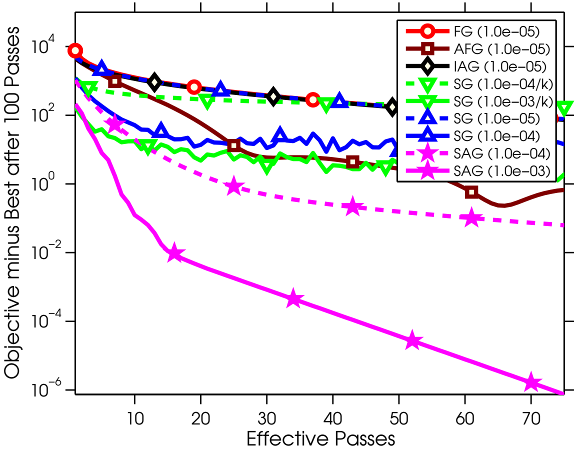
If we have enough data, the SAG iterations have a faster convergence rate with a larger step size:

If $\frac{\mu}{L} \geq \frac{8}{N}$, with a step size of $\alpha_k = \frac{1}{2N\mu}$, the SAG iterations satisfy

$$\mathbb{E}[g(x^k) - g(x^*)] \leq C\left(1 - \frac{1}{8N}\right)^k$$

Comparison of SAG to FG and SG Methods

Comparing SAG to a variety of FG and SG methods:



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Part 1:

- You can have the fast convergence rates of proximal-gradient methods, even if you can't compute the proximity operator.
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Part 2:

- If you have a large finite data set, there are some options in between stochastic and exact gradient methods.
  - M. Friedlander, M. Schmidt. **Hybrid Deterministic-Stochastic Methods for Data Fitting**. *Accepted to SISC*, 2012.
  - N. Le Roux, M. Schmidt, F. Bach. **A Stochastic Gradient Method with an Exponential Convergence Rate for Strongly-Convex Optimization with Finite Training Sets**. *Submitted*, 2012.