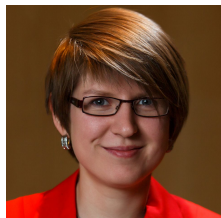


Auto-tuned high-dimensional regression with the TREX: theoretical guarantees and non-convex global optimization

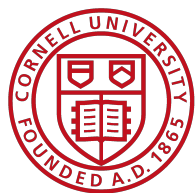
Jacob Bien¹, Irina Gaynanova¹, Johannes Lederer¹, Christian L. Müller^{2,3}

¹Cornell University, Ithaca ²New York University, ³Simons Center for Data Analysis, New York



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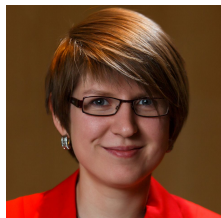
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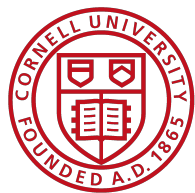
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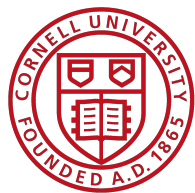
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We aim at **variable selection** in linear regression.
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$$Y = X\beta^* + \sigma\epsilon, \quad (\text{Model})$$

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High-dimensional variable selection in linear regression

The diagram illustrates the linear regression equation $Y = X\beta^* + \sigma\epsilon$ with dimension annotations. The response variable Y is represented by a purple vertical rectangle, with a curly brace to its left labeled n . The design matrix X is a large purple rectangle. The coefficient vector β^* is a light purple vertical rectangle, with a curly brace to its right labeled p . The error term ϵ is a purple vertical rectangle, preceded by the scalar σ . The equation is shown with an equals sign and a multiplication sign. The vector β^* is highlighted with a light purple background and two dark purple squares at its top and bottom, indicating the high-dimensional nature of the variable selection problem.

$$\begin{matrix} n \\ \left\{ \right. \end{matrix} \mathbf{Y} = \mathbf{X} \times \underbrace{\begin{matrix} \beta^* \\ \left. \right\} p \end{matrix}} + \sigma \epsilon$$

Standard approach: The LASSO (Tibshirani, 1996)

$$\hat{\beta}_{\text{Lasso}}(\lambda) \in \operatorname{argmin}_{\beta \in \mathbb{R}^p} \left\{ \frac{\|Y - X\beta\|_2^2}{n} + \lambda \|\beta\|_1 \right\}. \quad (\text{Lasso})$$

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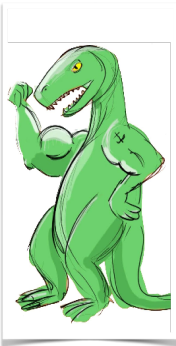
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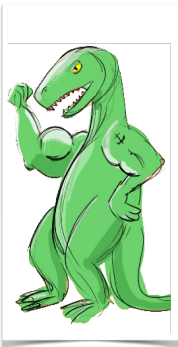
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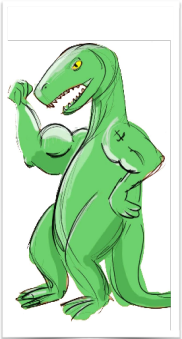


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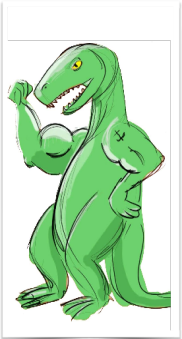
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BUT...

The **non-convex** TREX objective function can be **globally optimally** solved by using Second Order Cone Programming.

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1.) The TREX (with e.g. constant $a=0.5$) can be written as:

$$\begin{aligned} P^* &:= \min_{\beta \in \mathbb{R}^p} \left\{ \frac{\|Y - X\beta\|^2}{\max_{j \in \{1, \dots, p\}} a |x_j^\top (Y - X\beta)|} + \|\beta\|_1 \right\} \\ &= \min_{\beta \in \mathbb{R}^p} \min_{j \in \{1, \dots, p\}} \left\{ \frac{\|Y - X\beta\|^2}{a |x_j^\top (Y - X\beta)|} + \|\beta\|_1 \right\}. \end{aligned}$$

2.) For each index j this leads to a pair of problem of the form:

$$\min_{\beta \in \mathbb{R}^p} \left\{ \frac{\|Y - X\beta\|^2}{\textcolor{red}{a}x_j^\top (Y - X\beta)} + \|\beta\|_1 \quad \text{s.t.} \quad x_j^\top (Y - X\beta) \geq 0 \right\}$$

and

$$\min_{\beta \in \mathbb{R}^p} \left\{ \frac{\|Y - X\beta\|^2}{-\textcolor{red}{a}x_j^\top (Y - X\beta)} + \|\beta\|_1 \quad \text{s.t.} \quad -x_j^\top (Y - X\beta) \geq 0 \right\}.$$

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3.) or, in general, $2p$ problems of the quadratic over linear form:

$$P^*(v) := \min_{\beta \in \mathbb{R}^p} \left\{ \frac{\|Y - X\beta\|^2}{v^\top (Y - X\beta)} + \|\beta\|_1 \quad \text{s.t.} \quad v^\top (Y - X\beta) \geq 0 \right\}.$$

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Each problem is a Second-Order Cone Program!

Phase transition of exact recovery with the TREX and the LASSO

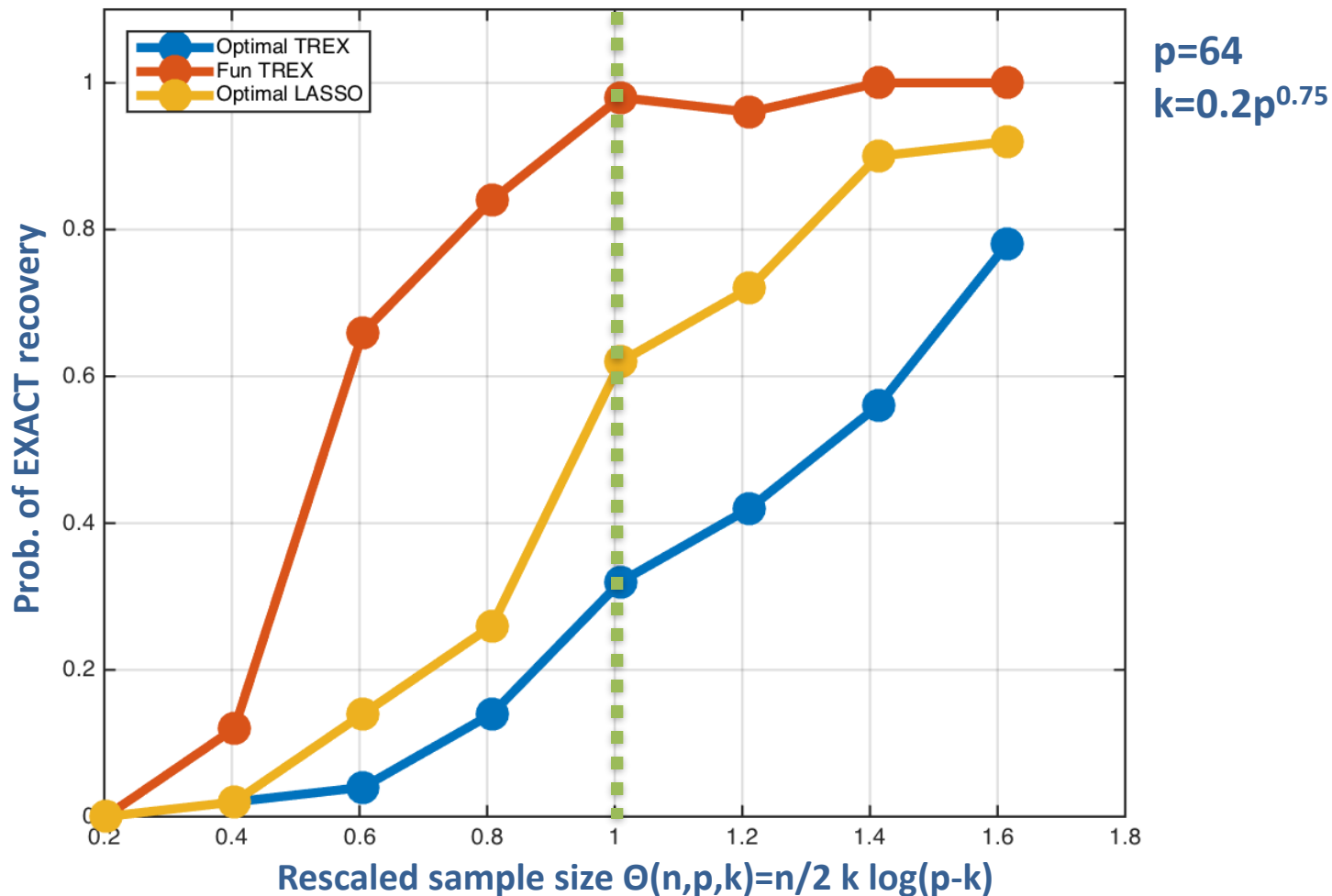
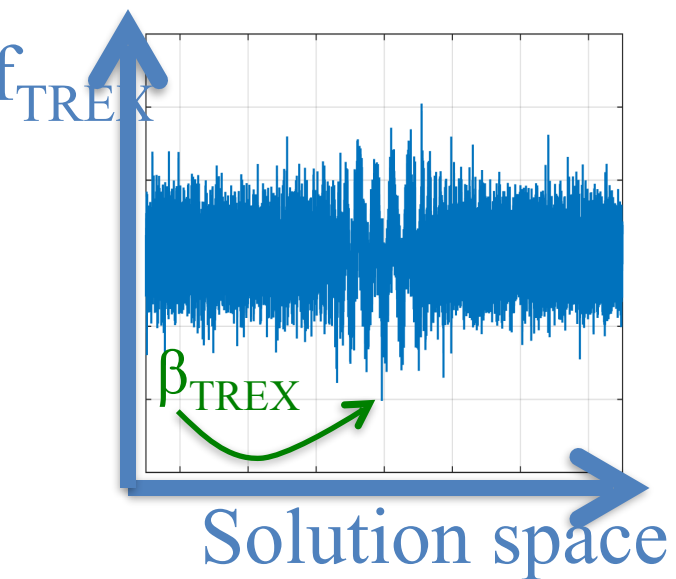
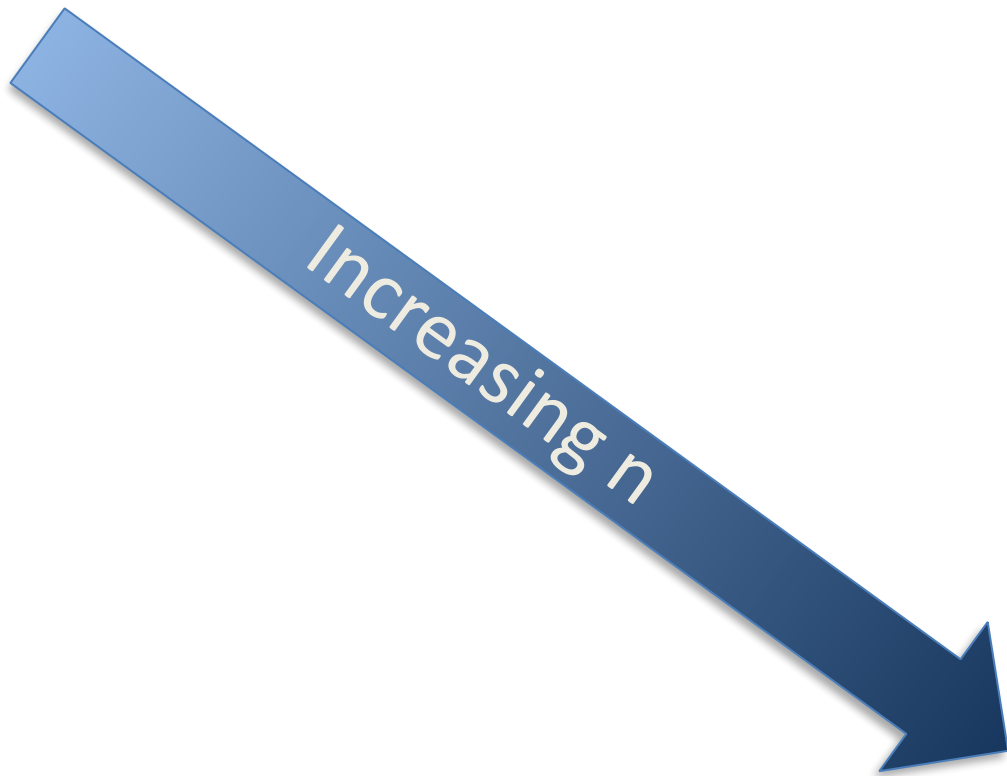


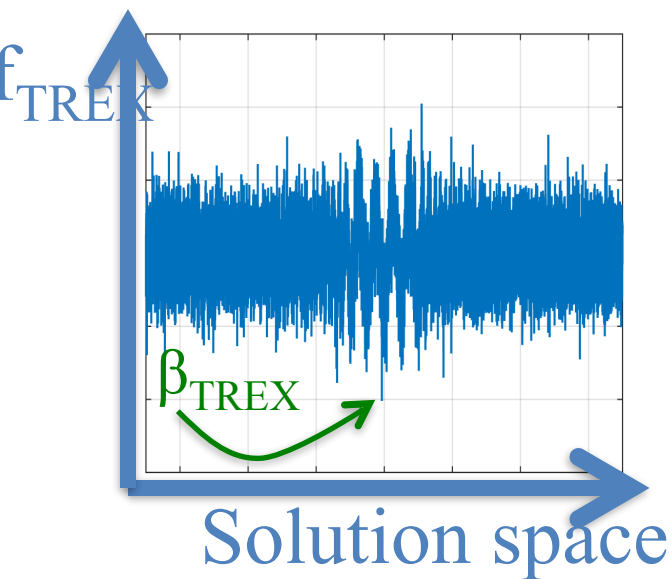
Figure 1: Success probability $P[S_{\pm}(\beta) = S_{\pm}(\beta^*)]$ of obtaining the correct signed support versus the rescaled sample size $\theta(n, p, k) = n/[2k \log(p - k)]$ for problem size $p=64$ with sparsity $k = \lceil 0.20 p^{0.75} \rceil$. The number of repetitions is 50. The optimal $a=0.5$ in TREX. The lambda in LASSO is automatically determined by MATLAB. Variable selection using the function gap property (Fun TREX) is shown in red



Sketching the topology of the TREX

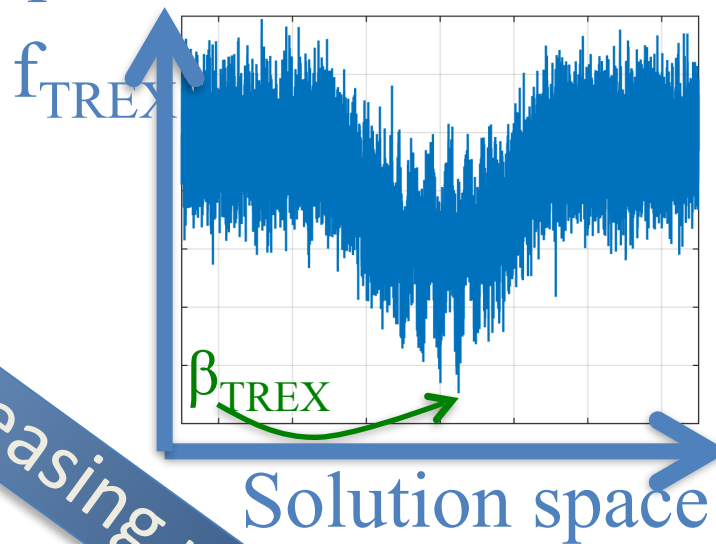
Consider the case where data ($p \gg n$) are generated from a linear model with a sparse β vector with $k \ll p$ non-zero entries of equal absolute value.



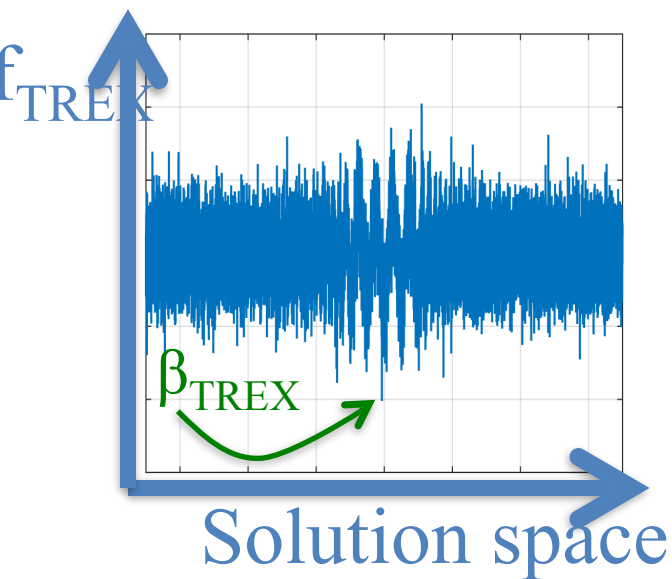


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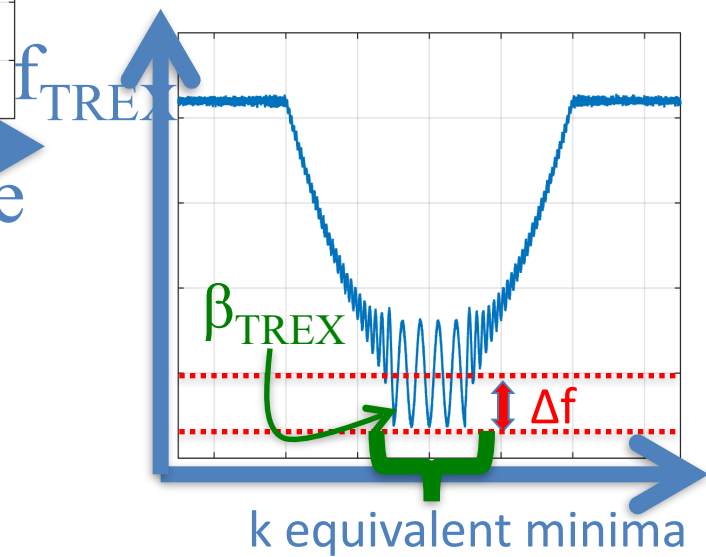
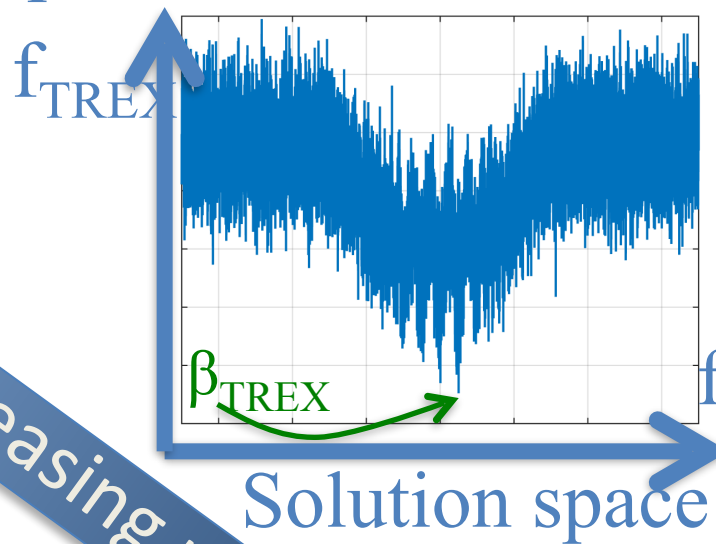


Increasing n

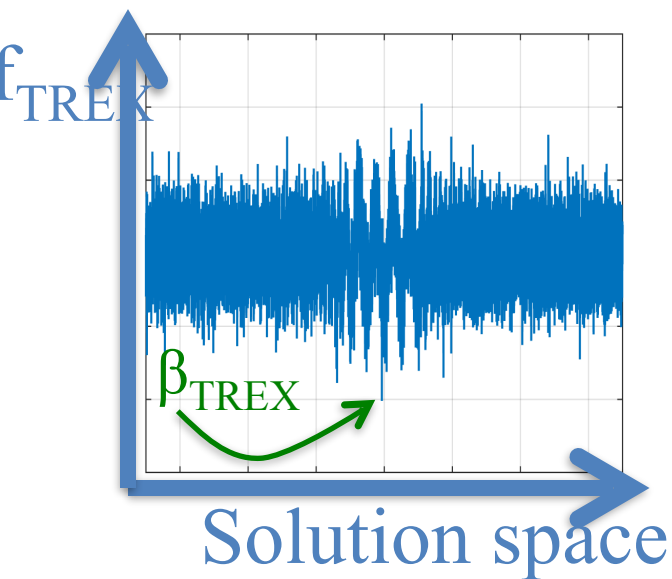


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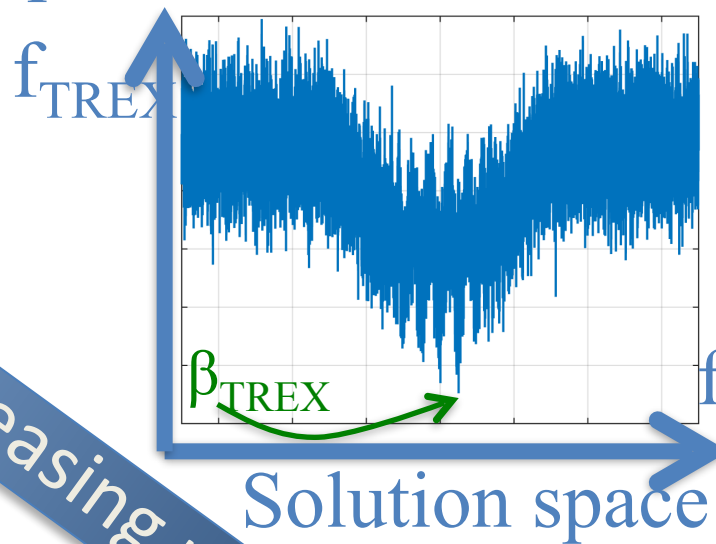


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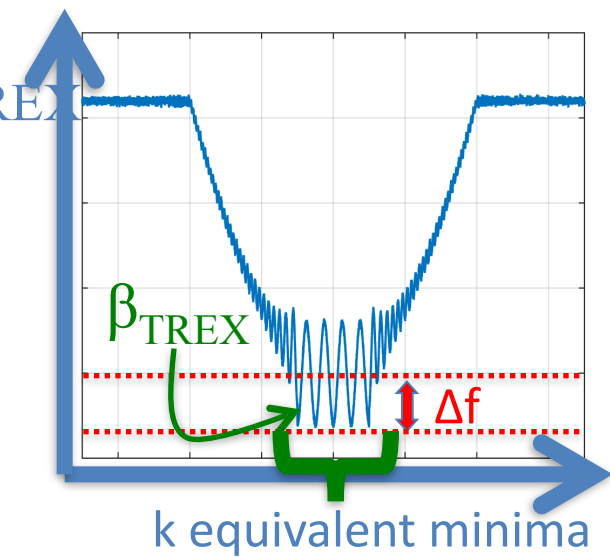


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The topology of the objective function can be used as an alternative variable selection method.



Increasing n

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ANY IDEA HOW TO SPEED THINGS UP?