Is Greedy Coordinate Descent a Terrible Algorithm?

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Optimization and Big Data, 2015

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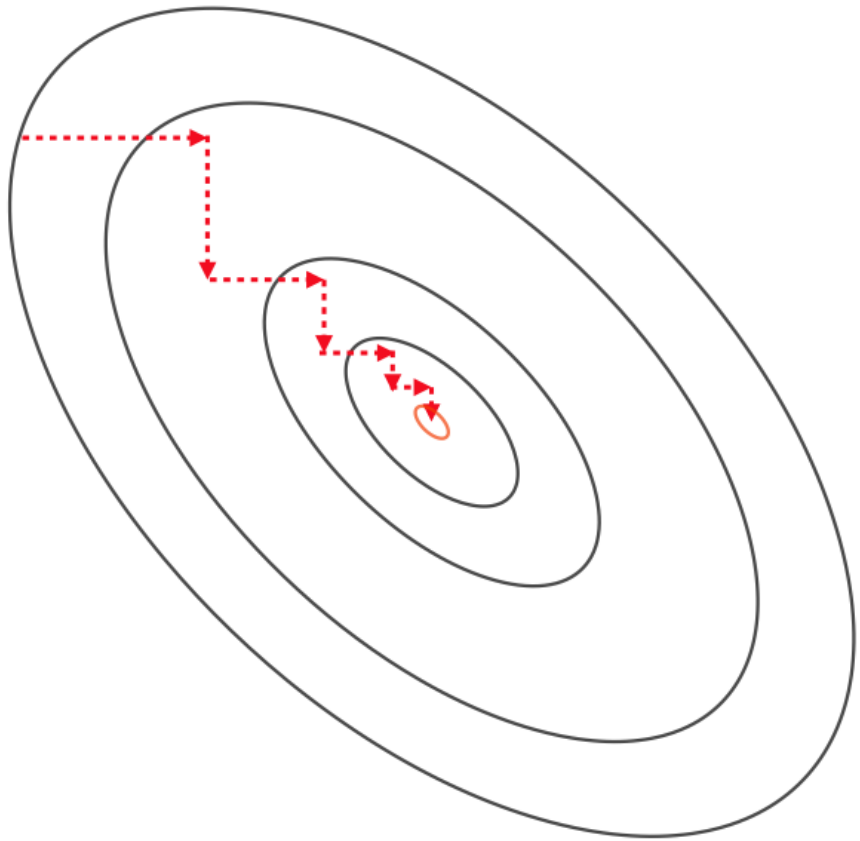

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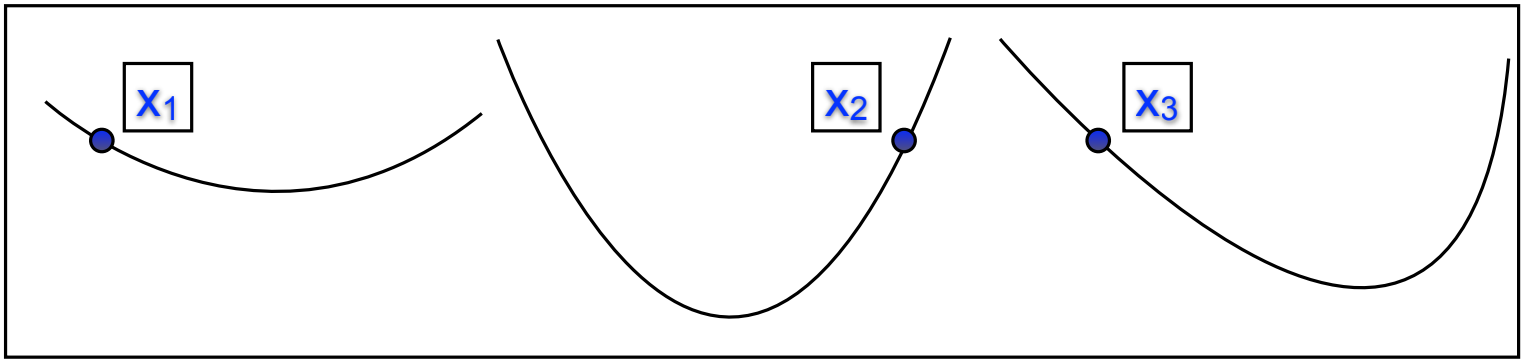

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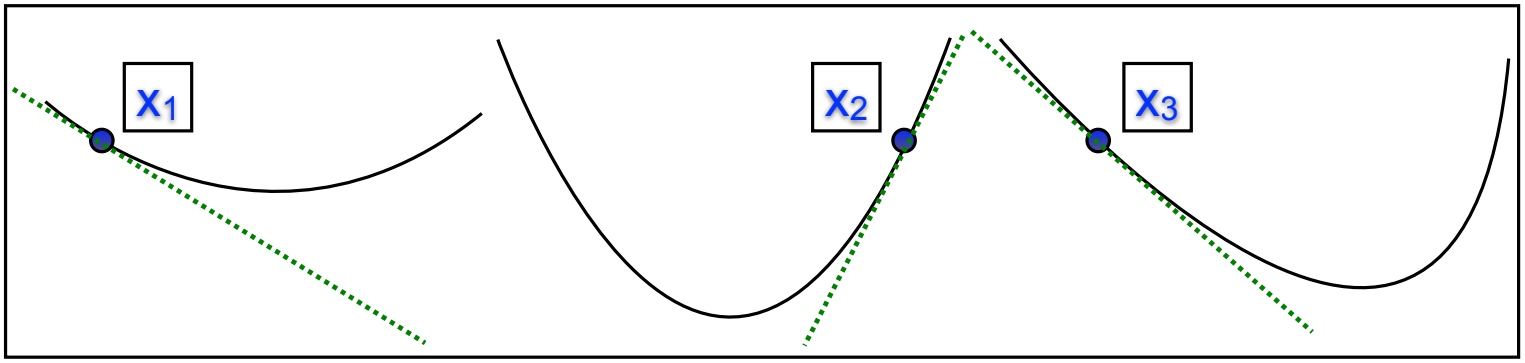

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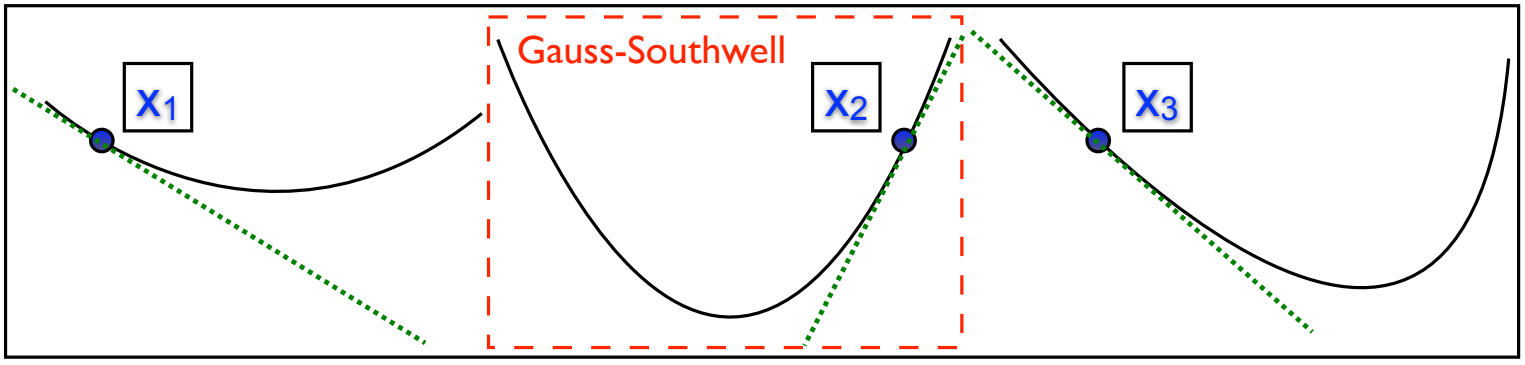

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- But this theory disagrees with practice...

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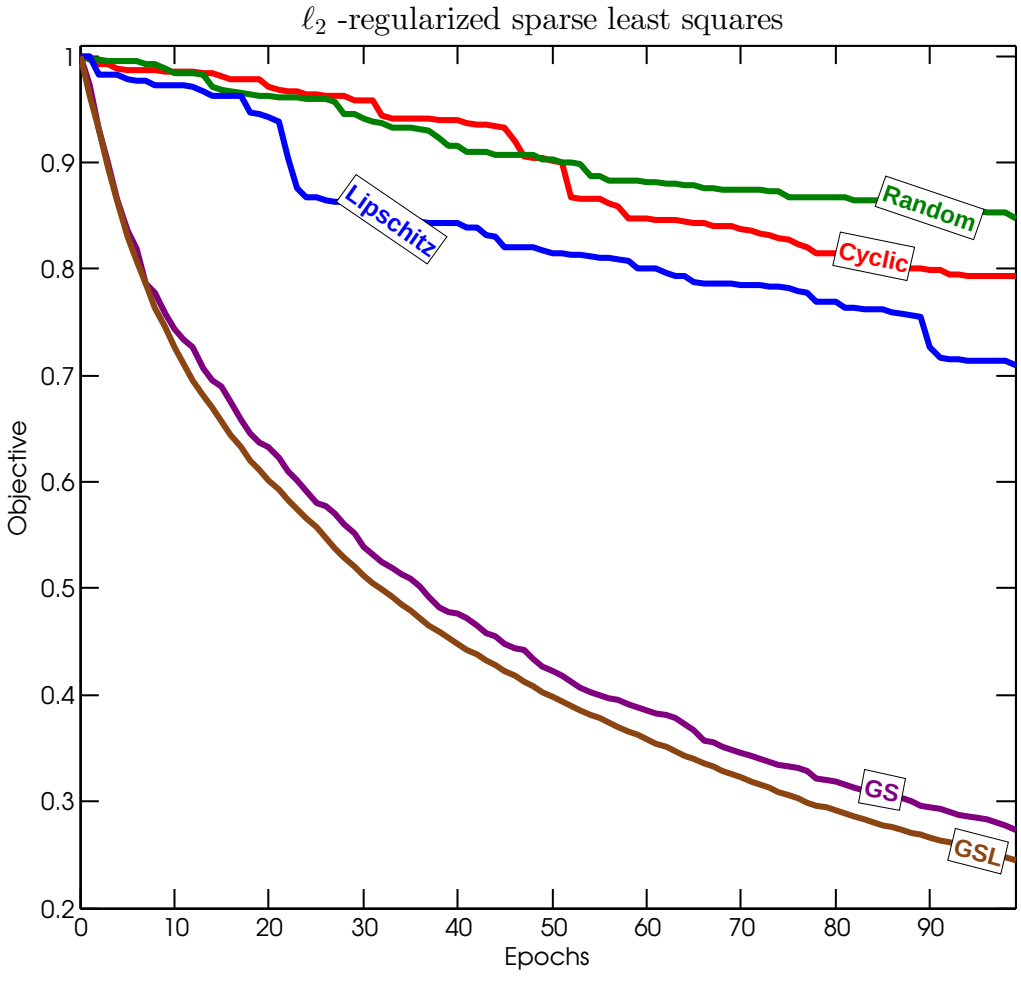

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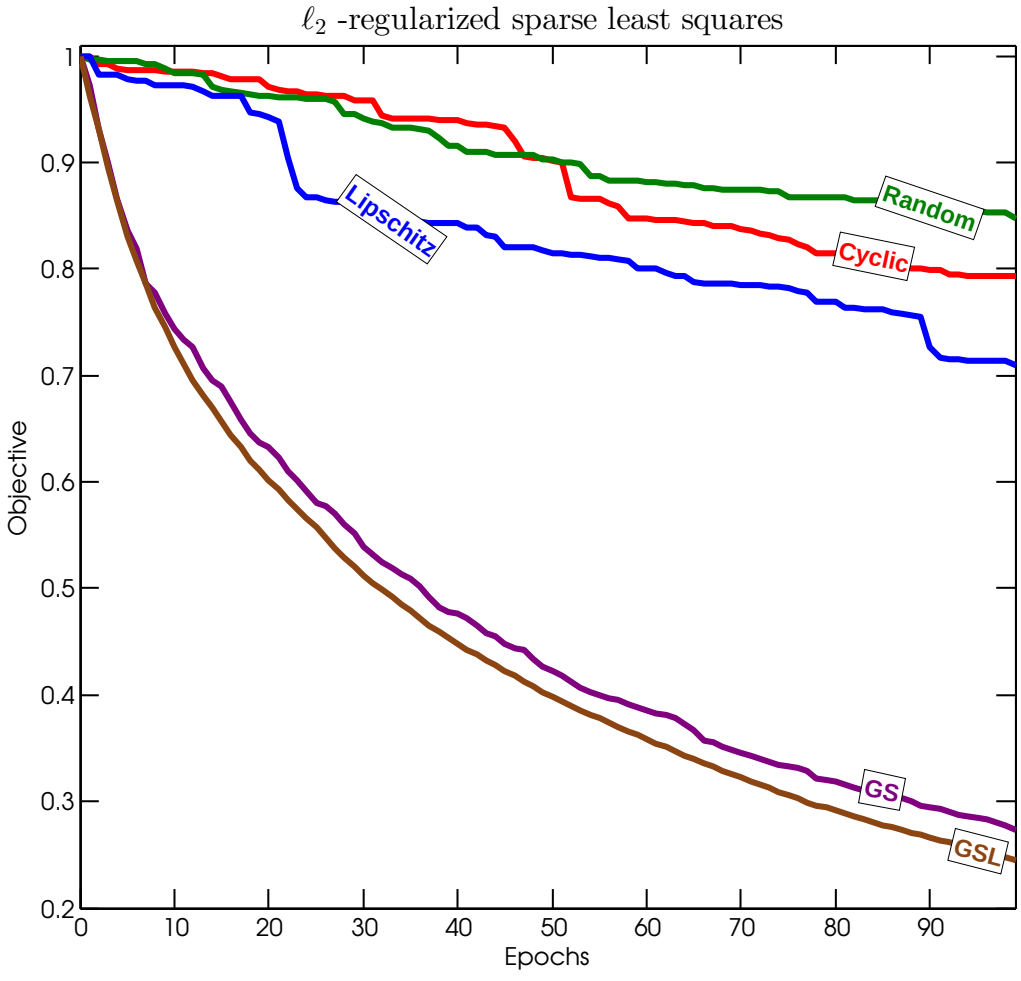

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- This work: refined analysis of GS.

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- There are basically two problems where this is true:

$$h_1(x) = f(Ax) + \sum_{i=1}^{n} g_i(x_i), \quad \text{or} \quad h_2(x) = \sum_{(i,j)\in E} f_{ij}(x_i, x_j) + \sum_{i=1}^{n} g_i(x_i),$$

where f and f_{ij} are smooth, A is a matrix, E are edges in a graph. (g_i can be general non-degenerate convex functions)

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- h_1 includes least squares, logistic regression, Lasso, and SVMs.

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- h_2 includes quadratics, graph-based label propagation, and probabilistic graphical models.

$$\text{E.g., } \min_{x\in\mathbb{R}^n} \frac{1}{2}x^T Ax + b^T x = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_i x_j + \sum_{i=1}^{n} b_i x_i.$$

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    [Meshi et al., 2012]
  - Dense quadratic: max degree $= (n-1)$, average degree $= (n-1)$.
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  - GS can be approximated as nearest neighbour problem.

    [Dhillon et al., 2011, Shrivastava & Li, 2014].

Notation and Assumptions

- We focus on the convex optimization problem

$$\min_{x \in \mathbb{R}^n} \; f(x),$$

where ∇f is coordinate-wise L-Lipschitz continuous,

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  is a convex function for some $\mu > 0$.
- If twice-differentiable, equivalent to

$$\nabla^2_{ii} f(x) \leq L, \quad \nabla^2 f(x) \succeq \mu I.$$

Convergence of Randomized Coordinate Descent

- Coordinate descent with constant step size $\frac{1}{L}$ uses

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- With i_k chosen uniformly from $\{1, 2, \ldots, n\}$ [Nesterov, 2010],

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- Compare this to the rate of gradient descent,

$$f(x^{k+1}) - f(x^*) \leq \left(1 - \frac{\mu}{L_f}\right)[f(x^k) - f(x^*)].$$

- Since $Ln \geq L_f \geq L$, coordinate descent is slower *per iteration*, but n coordinate iterations are faster than one gradient iteration.

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same rate as randomized [Boyd & Vandenberghe, 2004, §9.4.3].

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- Otherwise, GS can be faster by as large as n.

Comparison for Separable Quadratic

- In f is a quadratic with diagonal Hessian, we can show

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  - In this case, GS can actually be faster than gradient descent.
- μ_1 is harmonic mean of λ_i divided by n, $H(\lambda)/n$:
  - $H(\lambda)$ is dominated by minimum of its arguments.
  - If each worker takes λ_i time to finish a task on their own, $H(\lambda)/n$ is time needed when 'working together' [Ferger, 1931].

Fast Convergence with Bias Term

- Consider the linear-prediction framework in statistics,

$$\underset{x,\beta}{\operatorname{argmin}} \sum_{i=1}^{n} f(a_i^T x + \beta) + \frac{\lambda}{2}\|x\|^2 + \frac{\sigma}{2}\beta^2,$$

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- Typically $\sigma << \lambda$ to avoid biasing against a global shift.
- This is an instance of h_1 where GS has the most benefit.

Rates with Different Lipschitz Constants

- Consider the case where we have an L_i for each coordinate,

$$|\nabla_i f(x + \alpha e_i) - \nabla_i f(x)| \leq L_i |\alpha|,$$

and we use a coordinate-dependent stepsize,

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- In this setting, we get a rate of

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- Since $L = \max_i L_i$, this is faster if $L_{i_k} < L$ for any i_k.
- But rate is the same in the worst case, even if L_i are distinct.
- Let's consider the effect of exact coordinate optimization on L_{i_k}.

Gauss-Southwell with Exact Optimization

- Exact coordinate optimization chooses the stepsize minimizing f.
- We can get the same rates for randomized/GS because

$$\begin{aligned} f(x^{k+1}) &= \min_{\alpha}\{f(x^k - \alpha\nabla_{i_k} f(x^k)e_{i_k})\} \\ &\leq f\left(x^k - \frac{1}{L_{i_k}}\nabla_{i_i} f(x^k)e_{i_k}\right) \\ &\leq f(x^k) - \frac{1}{2L_{i_k}}[\nabla_{i_k} f(x^k)]^2, \end{aligned}$$

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- But theory again disagrees with practice:
  - Empirically, exact optimization is much faster.

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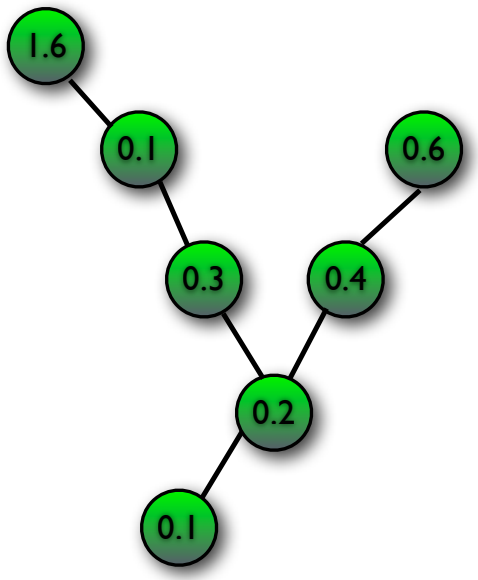

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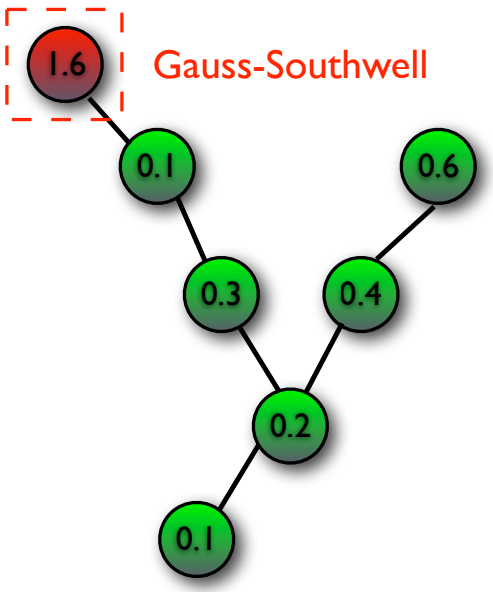

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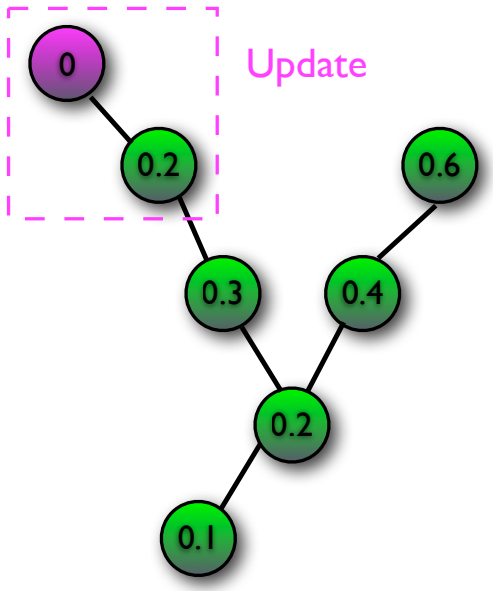

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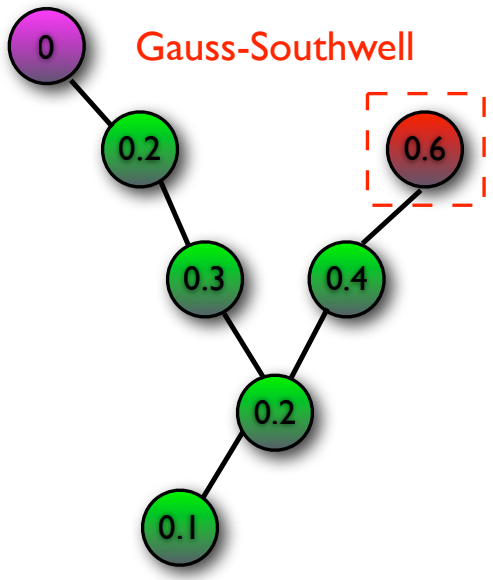

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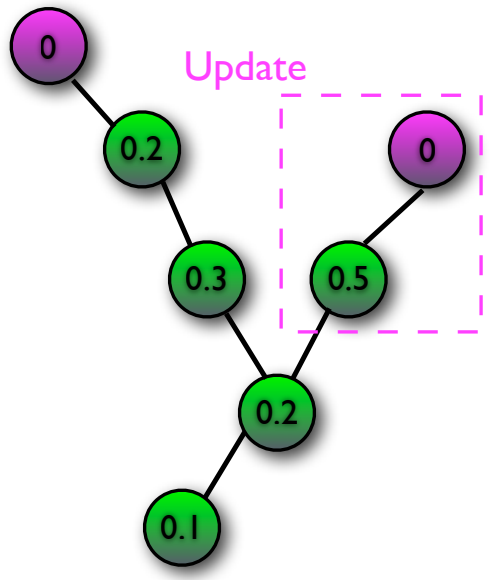

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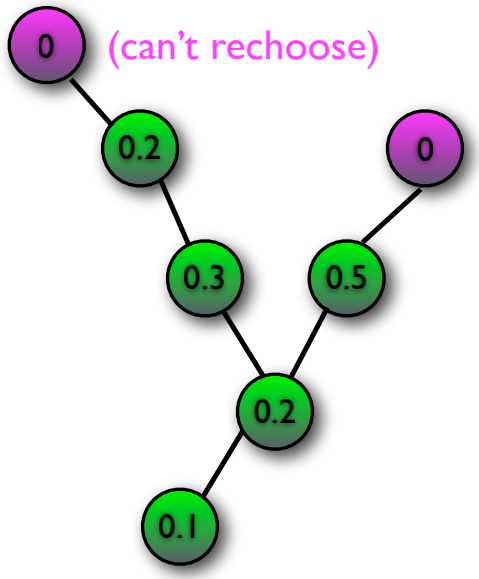

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$$f(x^k) - f(x^*) = O\left(\left(1 - \frac{\mu_1}{\max\{L_2^G, L_3^G\}}\right)^k\right)[f(x^0) - f(x^*)].$$

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- Similar for h_1: edges between variables non-zero in same row.

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- Faster than uniform sampling when the L_i are disinct.
- This could be faster or slower than the GS rule.
- In the separalbe quadratic case:
  - With one large λ_i, Lipschitz sampling is faster.
  - With one small λ_i, GS is faster.
- So which should we use?

Gauss-Southwell-Lipschitz Rule

- We obtain a faster rate by using L_i in the GS rule,

$$i_k = \underset{i}{\text{argmax}} \frac{|\nabla_i f(x^k)|}{\sqrt{L_i}},$$

  which we call the Gauss-Southwell-Lipschitz (GSL) rule.

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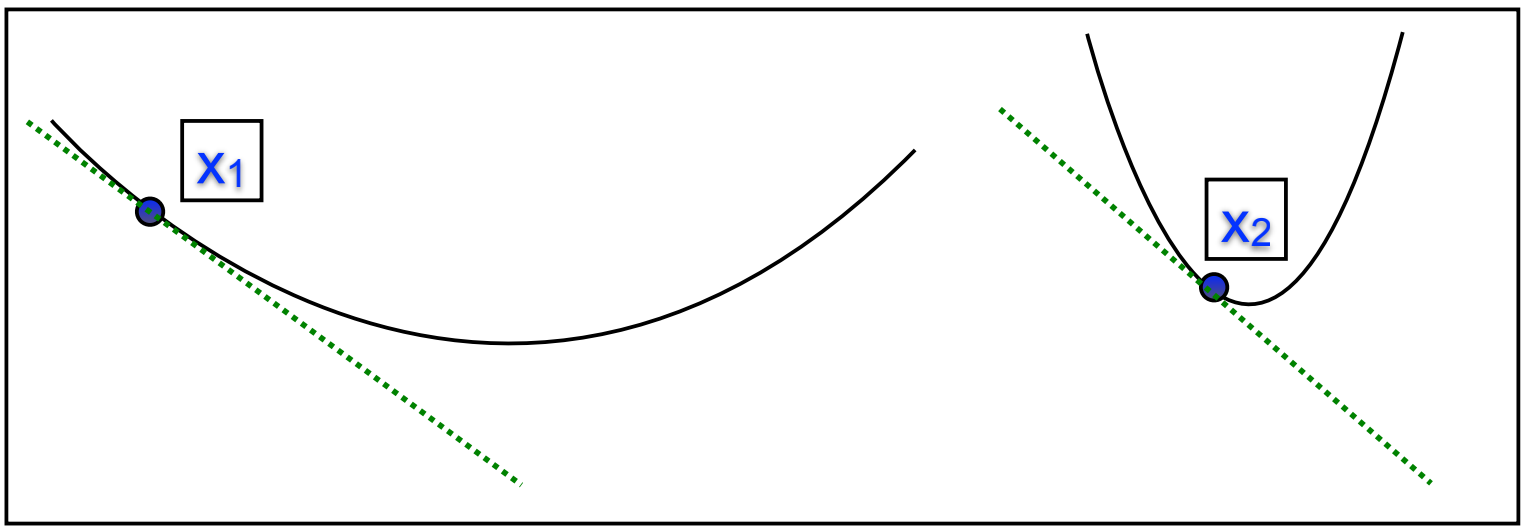

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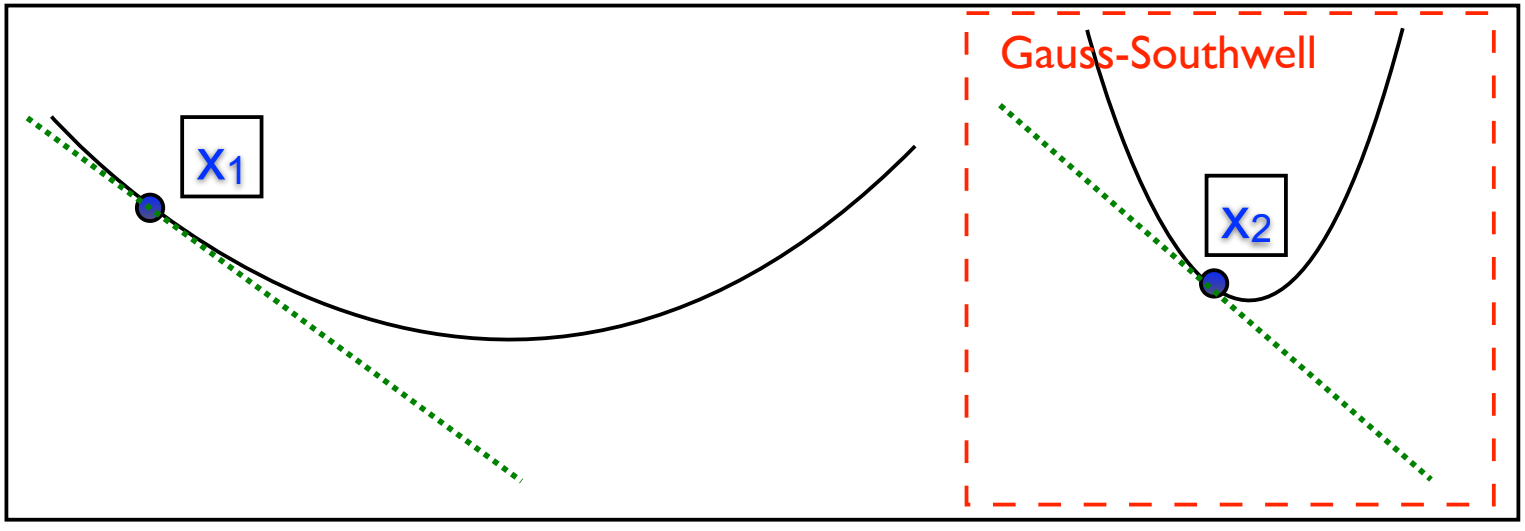

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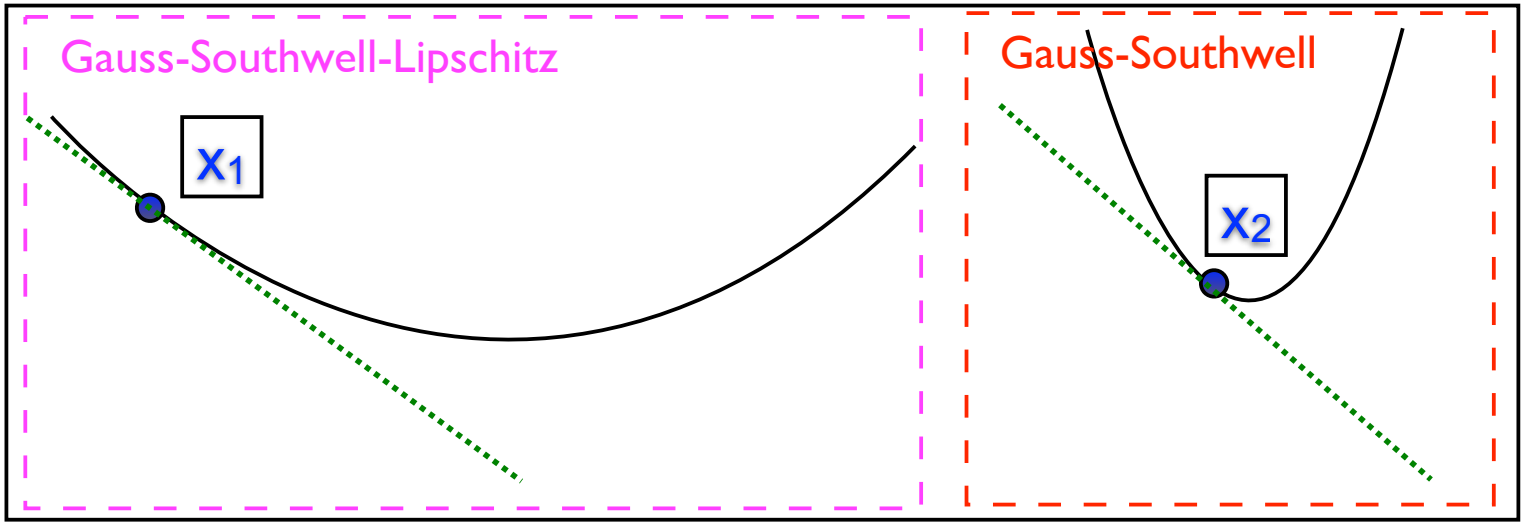

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- Analysis gives tighter bound on maximum improvement rule, used in certain applications.

[Della Pietra et al., 1997, Lee et al., 2006]

Gauss-Southwell-Lipschitz as Nearest Neighbour

- Consider a special case of h_1,

$$\min_x \; h_1(x) = \sum_{i=1}^{n} f(a_i^T x),$$

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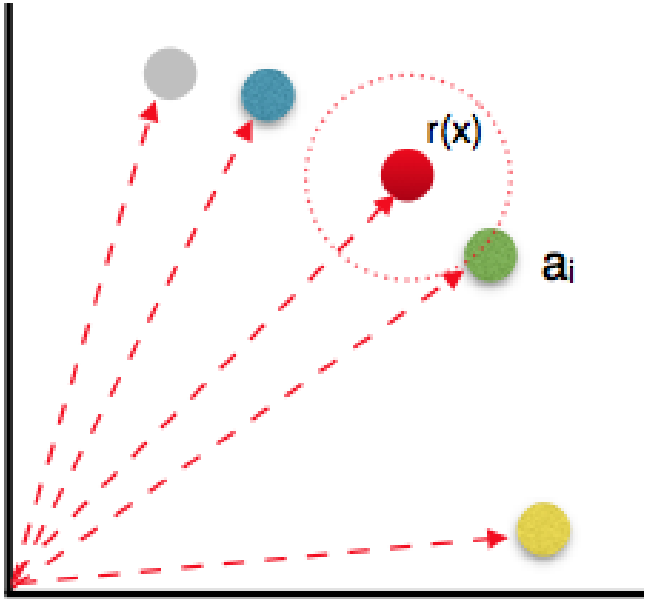

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- Using $L_i = \gamma\|a_i\|^2$, exact GSL as a nearest neighbour problem,

$$\operatorname*{argmin}_i \frac{1}{2}\left\|r(x^k) - \frac{a_i}{\sqrt{\gamma}\|a_i\|}\right\|^2 = \frac{1}{2}\|r(x^k)\|^2 - \frac{a_i^T r(x^k)}{\sqrt{\gamma}\|a_i\|} + \frac{1}{2\gamma}.$$

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- With additive error,

$$|\nabla_{i_k} f(x^k)| \geq \|\nabla f(x^k)\|_\infty - \epsilon_k,$$

  we have a fast rate if $\epsilon_k \to 0$ fast enough.
- With constant additive error, only get a certain solution accuracy.

Proximal Coordinate Descent

- Important application of coordinate descent is for problems

$$\min_{x \in \mathbb{R}^n} F(x) \equiv f(x) + \sum_i g_i(x_i),$$

  where f is smooth and g_i might be non-smooth.
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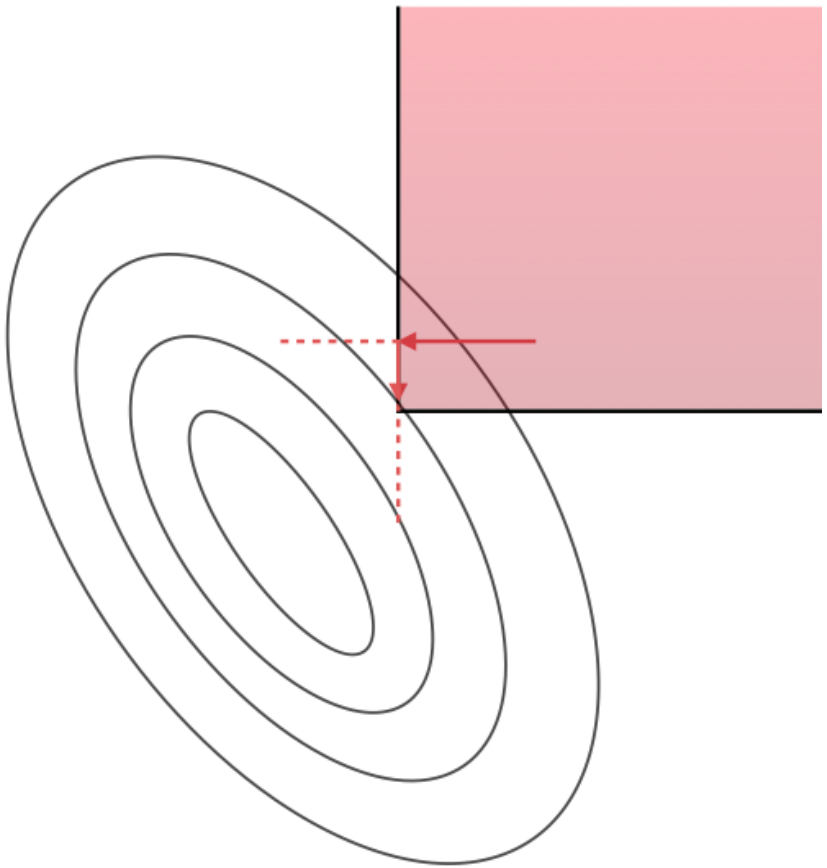

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- GS-q: Maximize progress under quadratic approximation of f.

$$i_k = \underset{i}{\operatorname{argmin}} \left\{ \min_d f(x^k) + \nabla_i f(x^k) d + \frac{L d^2}{2} + g_i(x_i^k + d) - g_i(x_i^k) \right\}.$$

  - Least intuitive, but has the best theoretical properties.
  - Generalizes GSL if you use L_i instead of L.

Proximal Gauss-Southwell Convergence Rate

- For the GS-q rule, we show that

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  where $\epsilon^k \to 0$ measures non-linearity of g_i that are not updated.

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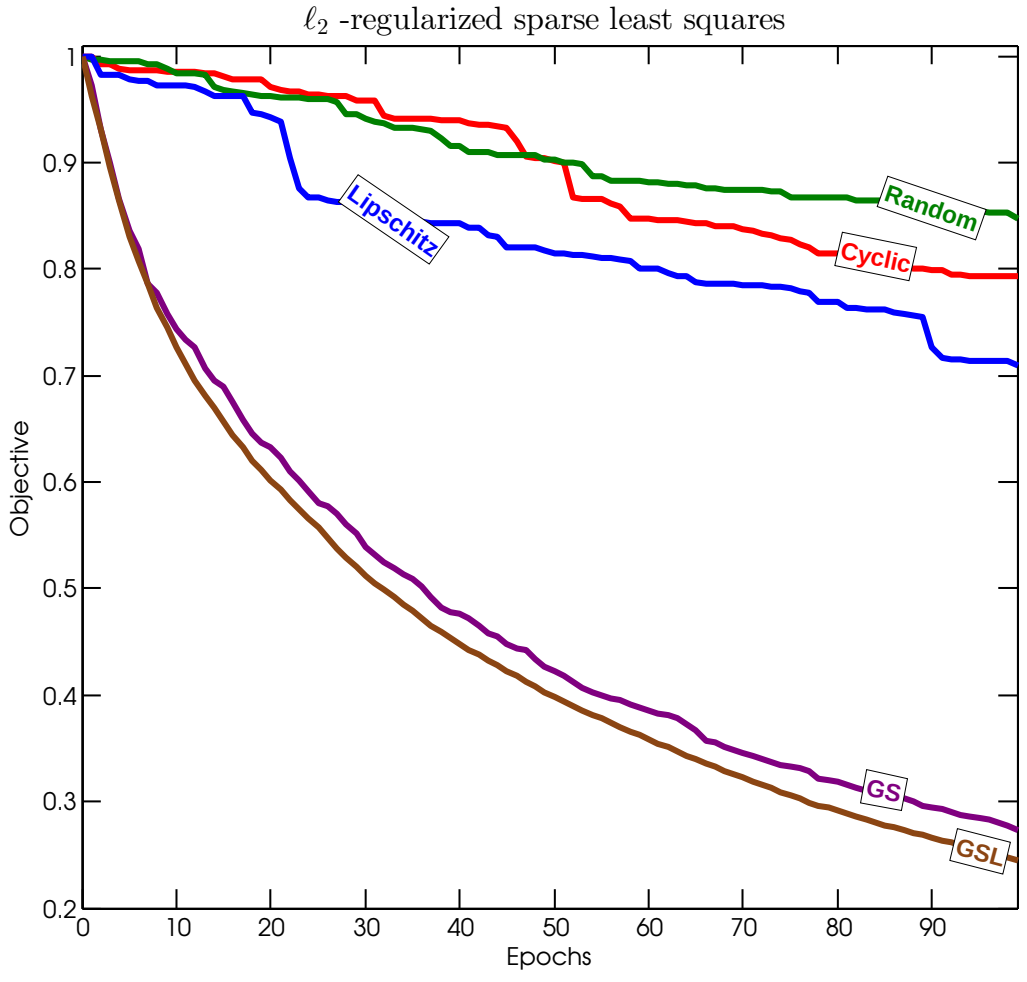
- But one final time theory disagrees with practice:
  - All three rules seem to work pretty well.
  - Though GS-r works badly if you use the L_i.

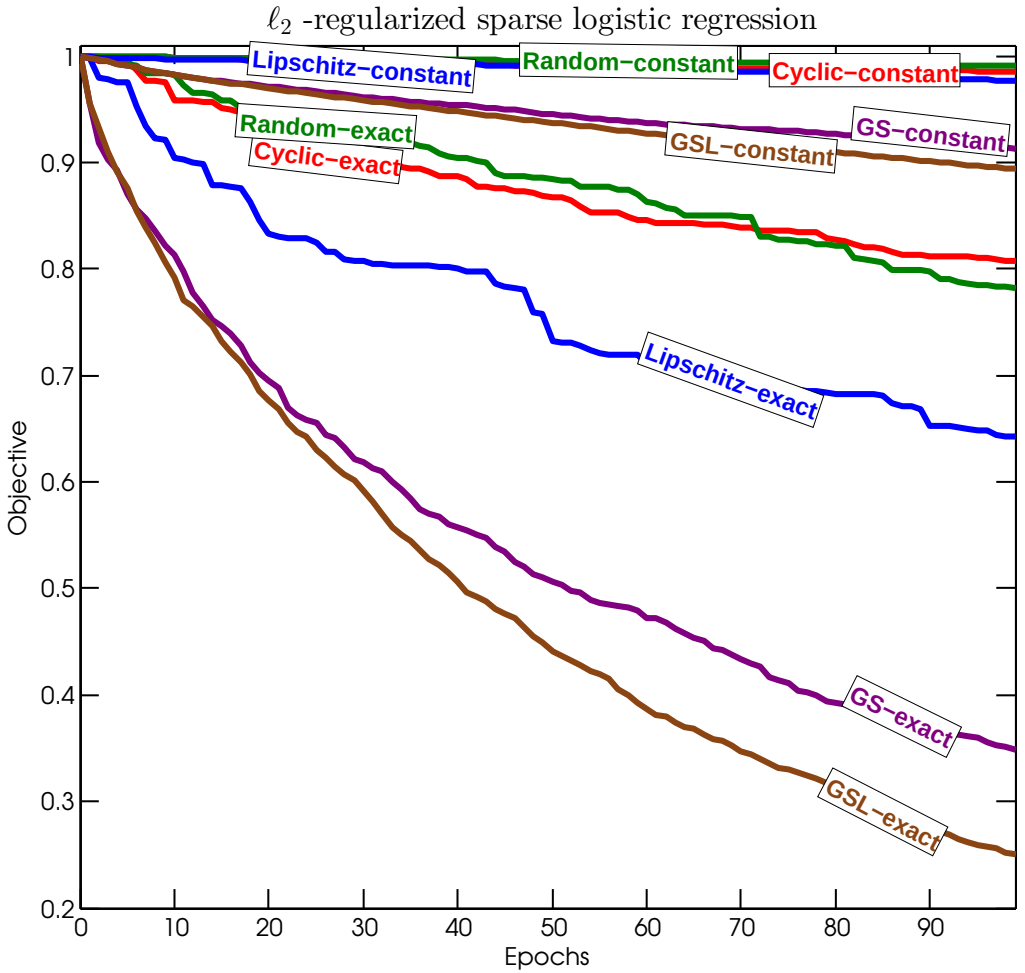
Experiment 1: Sparse ℓ_2-Regularized Least Squares

Least squares with ℓ_2-regularization and very sparse matrix.



Experiment 2: Sparse ℓ_2-Regularized Logistic

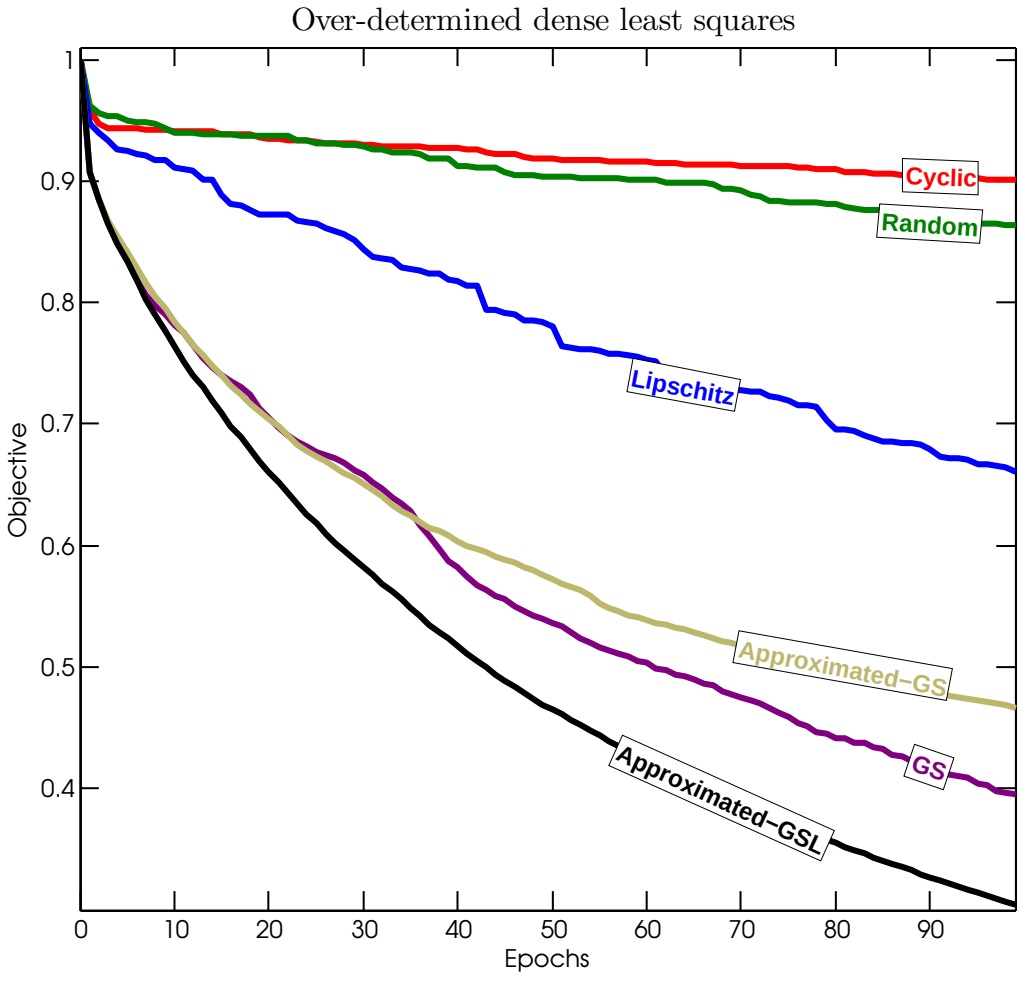
Logistic regression with ℓ_2-regularization and very sparse matrix.



Exact optimization makes a bigger difference than coordinate selection.

Experiment 3: Over-determined least squares

Least squares with dense matrix and nearest neighbour GS.



Approximate GS is still faster than random sampling.

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- But even approximate GS rules may outperform random.
- We've given a justification for line-search in certain scenarios.
- We proposed GSL rule, and approximate/proximal variants.
- Analysis extends to block updates.
- Could be used for accelerated/parallel methods [Fercocq & Richtárik, 2013], primal-dual methods [Shalev-Schwartz & Zhang, 2013], and without strong-convexity [Luo & Tseng, 1993].