Game Theory

Lecture notes for MATH11090 & MATH09002

Peter Richtárik

University of Edinburgh

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Course Organization

These lecture notes are for

- **GT:** Game Theory (MATH11090)
- **DPG:** Discrete Programming and Game Theory (MATH09002)

Basic info:

- Dates: October: 26, November: 2, 9, 16, 23 (5 lectures; Tuesdays)
- Times: 2pm-3:50pm
- Location: JCMB ThA
- Assessment:
  - Continuous: 15% for GT, 7.5% for DPG, there are 2 assignments
    - Nov 5: in class write-up 9:00–9:50, JCMB 1501 (GT), 2pm deadline (DPG)
    - Nov 19: in class write-up 11:10-12:00, JCMB 1501 (GT), 2pm deadline (DPG)
  - Exam: 85% GT, 42.5% DPG (in Semester 2)

All materials will be posted at

http://www.maths.ed.ac.uk/~prichtar/teaching/GT

What is Game Theory?

Game theory is

- a **mathematical** theory
- studying situations of **conflict** and **cooperation**
- between **rational** decision-makers (players)

Players: people, companies, nature, genes, computers, . . .

Rationality:

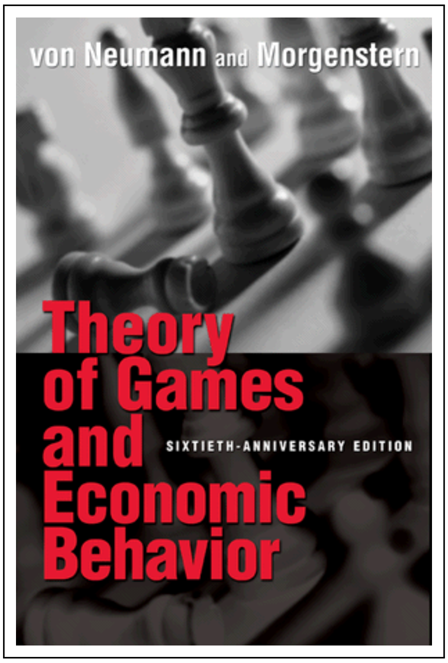
- players choose their strategies to **maximize** their individual **payoff** (utility, profit, . . .)
- players know that the others players do the same

Classification of Games

- How many players are there?
  - 1-player (decision problems)
  - 2-player
  - N-player
- Is cooperation allowed?
  - Cooperative
  - Noncooperative
- Is the sum of the payoffs always zero?
  - Zero-sum
  - Non-zero-sum
- Are the rules of the game the same for all the players?
  - Symmetric
  - Nonsymmetric
- Is the number of strategies finite?
  - Finite
  - Infinite
- Is the model dynamic or static?
  - Extensive (tree) form
  - Strategic (normal) form
- Is the game played only once?
  - Static
  - Repeated

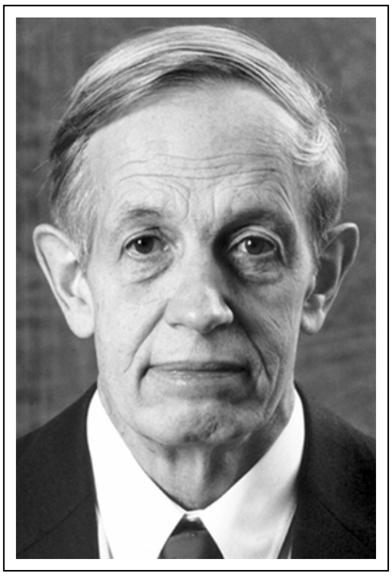
Important Historical Figures

- J. von Neumann and O. Morgenstern [1944], *Theory of Games and Economic Behavior,* Princeton University Press
- J. Nash [1951], *Non-cooperative games,* Annals of Mathematics, 54, 286-295
- J. Harsanyi [1967-8], *Games with Incomplete Information Played by Bayesian Players,* Management Science, 14, 159-82,320-34, 486-502
- R. Selten [1965], *Spieltheoretische Behandlung eines Oligopolmodells mit Nachfragentragheit,* Zeitschrift fur die gesamte Staatswissenschaft, 12, 201-324

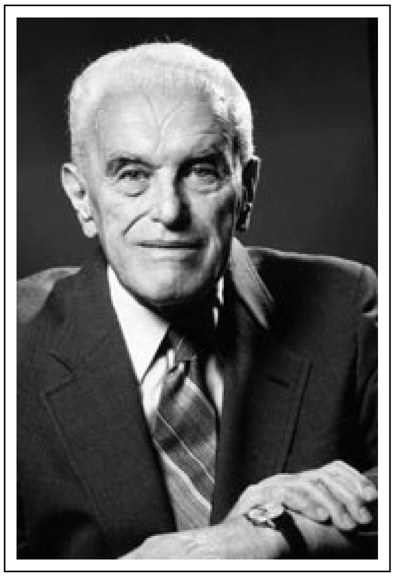


3 × John

John von Neumann
(1903–1957)



John Nash
(1928)

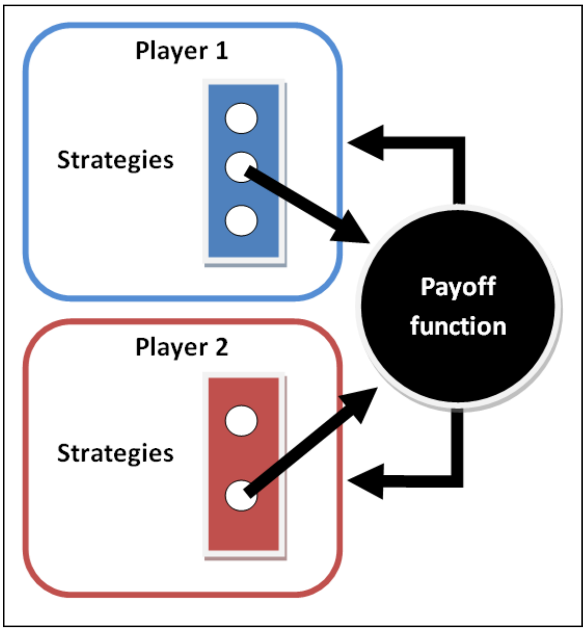


John Harsanyi
(1920–2000)

Games in Strategic Form: Nontechnical Description

There are several agents **(players)** taking part in a certain process **(game)**

- Each player has a portfolio of possible behaviours **(strategies)**
- All players **simultaneously** choose a strategy
- No player knows what strategy will be chosen by the other players
- These choices uniquely determine a certain result **(payoff)** for each player
- All players know
  - all possible strategies of the other players and
  - the way all combinations of these strategies determine the payoffs
- Each player wants to **maximize** her payoff



Questions You as a Player Might Want to Ask

You are one of the players.

- How would you **pick** your strategy?
- Would you be **happy** with your choice after you'll have learnt what the others have done?
- Could you all have done **better?**

Game: Prisoners' Dilemma

Two suspects in a serious crime are questioned **independently**, each can decide either to cooperate (C) or to defect (D)
$\Rightarrow$ 4 possible outcomes (**payoffs**) for each **player.**

	C	D
C	-1 -1	-5 0
D	0 -5	-3 -3

Reading the table: If the row player chooses C and the column player D, then the first gets 5 years in prison & the second 0.

- Whatever the other player does, each is better off by defecting $\Rightarrow$ strategy D is **strictly dominant** for both.
- The "solution" (D, D) is NOT **Pareto optimal: it is possible to increase payoff to at last one player without decreasing payoff of the others** (look at (C, C)).
- (D, D) is a **Nash equilibrium: no player has incentive to unilaterally deviate.** There is no other NE.

Game: Matching Pennies

Two players **simultaneously** each place a coin (a penny) on the table, either heads up (H) or tails up (D)

- If there is a match, RP wins both coins, otherwise CP wins
- This is an example of a **zero-sum game.**

	H	T
H	$+1$ -1	-1 $+1$
T	-1 $+1$	$+1$ -1

How should they play?

- If CP plays H then RP should play H. If RP plays H, CP should play T. If CP plays T, RP should play T. If RP plays T, CP should play H. $\Rightarrow$ there is **no Nash equilibrium.**
- **No dominant strategies**
- **All** pairs of strategies **are Pareto optimal** since the sum of payoffs is always 0 ("zero-sum" game).

Game: Battle of the Sexes

Husband (RP) and wife (CP) are trying to **coordinate:** decide whether to watch football (F) or a soap opera (S) on TV.

- Watching together $\Rightarrow$ payoff of 2 to both.
- Watching their preferred programme $\Rightarrow$ payoff 1.

	F		S	
F	3	2	1	1
S	0	0	2	3

How should they coordinate?

- No dominant strategies.
- **Two Nash equilibria:** (F, F) and (S, S).
- Both NE are Pareto optimal.

How can they decide between the NE?

Game: War of Attrition ("waiting game")

Two players **compete for a resource** which has value v to both.

- Both players chose a time $t_i \geq 0$ until which they are willing to persist in the contest
- Payoffs decrease linearly with time at rate α, equally to both
- The resource is won by the one who quits last (a tie $\Rightarrow$ no reward)

$$\pi_1(t_1, t_2) = \begin{cases} v - \alpha t_2 & t_1 > t_2 \\ -\alpha t_2 & t_1 \leq t_2 \end{cases}$$

$$\pi_2(t_1, t_2) = \begin{cases} v - \alpha t_1 & t_2 > t_1 \\ -\alpha t_1 & t_2 \leq t_1 \end{cases}$$

Are there any Nash equilibria?

Games in Strategic Form

A game in **strategic (normal, static) form** is specified by

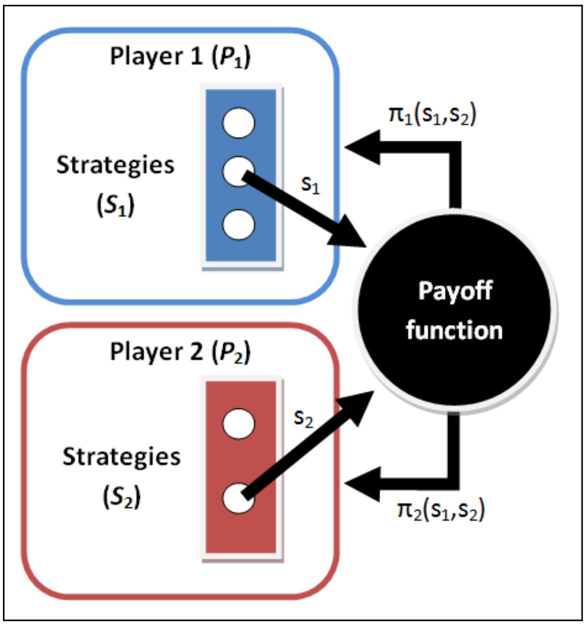
- a set of **players** $P = \{P_1, \ldots, P_N\}$
- sets of **pure strategies** $S_1, S_2, \ldots, S_N$ of each player

$$S = S_1 \times S_2 \times \cdots \times S_N$$

- **payoff (utility) functions** $\pi_i : S \to R$ for each player P_i

Playing with Pure Strategies:

- All players **simultaneously** select pure strategies from their strategy sets: P_i selects $s_i \in S_i$
- Player P_i gets **payoff** $\pi_i(s_1, s_2, \ldots, s_N)$



Playing with Mixed Strategies

Definition

When players play their pure strategies in a randomized way we say they use a **mixed (randomized) strategy.**

- A mixed strategy s_i is a **rand. variable** with values in S_i
- **Σ_i = set of mixed strategies over S_i**
- $S_i \subset \Sigma_i$ since a probability distribution giving all weight to a single pure strategy is a special case of a mixed strategy

Game play in mixed strategies

- All players P_i **simultaneously** and **independently** choose a random $s_i \in S_i$ ($s = (s_1, \ldots, s_N)$ is thus a random vector)
- The **expected payoff** of player P_i is given by $E(\pi_i(s_1, \ldots, s_N))$. At the expense of some notation abuse, we will use the following simplified notation:

$$\pi_i(s_1, \ldots, s_N) \stackrel{\text{def}}{=} E(\pi_i(s_1, \ldots, s_N))$$

Why to Mix Strategies?

Mixed strategies allow

- a player to hide his actual strategy behind randomness in a **repeated game**
- us to study **repeated games** in static form
- for a generalization of the **NE** concept in which a **Nash Equilibrium always exists (Nash's Theorem 1951)**
  - Matching Coins: no pure strategy NE; has mixed strategy NE (play H and T with probability $\frac{1}{2}$ each)

Finite Games, Zero-Sum Games and Matrix Games

Definition

A game is

- **finite** if the strategy sets S_i are finite
- **two-person** if $N = 2$
- **zero-sum** if $\sum_i \pi_i(s_1, \ldots, s_N) = 0$ for all $s_1 \in S_1, \ldots, s_N \in S_N$
- a **matrix game** if it is finite, two-person and zero-sum

Examples:

- Prisoners' Dilemma: a finite 2-person game
- Matching Pennies: a matrix game
- Battle of the Sexes: a finite 2-person game
- War of Attrition: an infinite 2-person game (a type of auction)

!!!Agreement: From now on we will state all definition and theorems for $N = 2$ but, unless implied or explicitly stated, they hold in general

Payoff Representation of Finite Games

If $|S_1| = m$ and $|S_2| = n$, with

$$S_1 = \{s_1^1, s_1^2, \ldots, s_1^m\}, \quad \text{and} \quad S_2 = \{s_2^1, s_2^2, \ldots, s_2^n\},$$

then **payoffs** can be **represented** by a **pair of $m \times n$ matrices** A and B such that

$$A_{ij} = \pi_1(s_1^i, s_2^j) \qquad B_{ij} = \pi_2(s_1^i, s_2^j).$$

Example: In Prisoners' Dilemma we have $m = n = 2$, $S_1 = \{s_1^1 = C, s_1^2 = D\}$, $S_2 = \{s_2^1 = C, s_2^2 = D\}$ and

$$A = \begin{pmatrix} -1 & -5 \\ 0 & -3 \end{pmatrix} \qquad B = \begin{pmatrix} -1 & 0 \\ -5 & -3 \end{pmatrix}$$

For a **matrix game**, one matrix is enough since $A = -B$. For example, in Matching Pennies it is enough to store B:

$$B = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$$

Representation of Strategies in Finite Games

Pure Strategies can be represented by **unit coordinate vectors** since

$$\pi_1(s_1^i, s_2^j) = A_{ij} = e_i^T A e_j \qquad \pi_2(s_1^i, s_2^j) = B_{ij} = e_i^T B e_j$$

Mixed Strategies can be represented as **vectors of probabilities:**
If for $(s_1, s_2) \in \Sigma_1 \times \Sigma_2$ we define

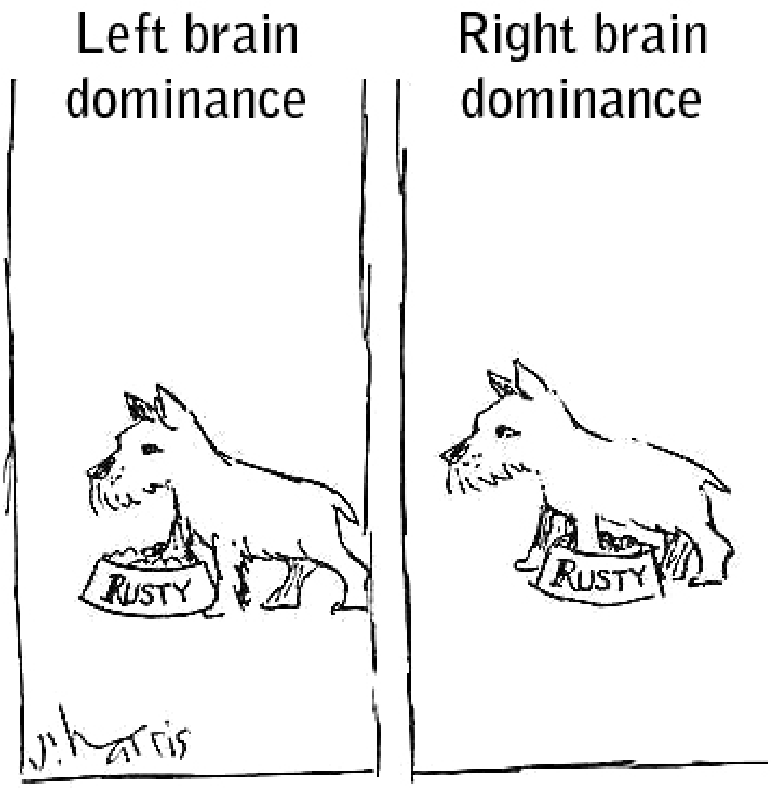
$$p_i = Pr(s_1 = s_1^i) \quad \text{and} \quad q_j = Pr(s_2 = s_2^j),$$

then

$$\pi_1(s_1, s_2) = E(\pi_1(s_1, s_2)) = \sum_{i=1}^{m} \sum_{j=1}^{n} p_i q_j \pi_1(s_1^i, s_2^j) = p^T A q$$

$$\pi_2(s_1, s_2) = E(\pi_2(s_1, s_2)) = \sum_{i=1}^{m} \sum_{j=1}^{n} p_i q_j \pi_2(s_1^i, s_2^j) = p^T B q$$

Dominance



Dominance

Definition

A pure strategy $s_1 \in S_1$ is **strictly dominated** by $s_1^* \in S_1$ if

$$\pi_1(s_1, s_2) \overset{(*1)}{<} \pi_1(s_1^*, s_2) \quad \text{for all} \quad s_2 \in S_2,$$

that is, if the payoffs of P_1 under this strategy are always strictly worse than under s_1^*.

A pure strategy $s_1 \in S_1$ is **weakly dominated** by $s_1^* \in S_1$ if the inequality $(*1)$ holds weakly ($\leq$) for all $s_2 \in S_2$ and strictly ($<$) for at least one $s_2 \in S_2$.

- We have defined the concepts for player P_1 only, the definition is analogous for P_2: swap 1 and 2 in all the subscripts.
- If a strategy is strictly dominated, it is also weakly dominated.

Example: Cooperation is a strictly dominated strategy (by defection) in Prisoners' Dilemma

Pareto Optimality: A Way to Measure Social Optimum

Definition

A pair of **mixed** strategies $(s_1^*, s_2^*) \in \Sigma_1 \times \Sigma_2$ is **Pareto Optimal** if

$$\pi_1(s_1, s_2^*) > \pi_1(s_1^*, s_2^*) \Rightarrow \pi_2(s_1, s_2^*) < \pi_2(s_1^*, s_2^*) \text{ for all } s_1 \in \Sigma_1$$

$$\pi_2(s_1^*, s_2) > \pi_2(s_1^*, s_2^*) \Rightarrow \pi_1(s_1^*, s_2) < \pi_1(s_1^*, s_2^*) \text{ for all } s_2 \in \Sigma_2$$

In words: A pair of strategies is Pareto Optimal if it is not possible to increase anyone's payoff without decreasing the payoff of someone else.

If we are reluctant to **directly compare payoffs** of different players, Pareto solutions are a notion of **social optimality.**

Definition of Nash Equilibrium

Definition

A pair of **mixed** strategies $(s_1^*, s_2^*) \in \Sigma_1 \times \Sigma_2$ is a **Nash equilibrium** if the following **best response inequalities** hold

$$\underbrace{\pi_1(s_1, s_2^*) \leq \pi_1(s_1^*, s_2^*) \text{ for all } s_1 \in \Sigma_1}_{s_1^* \text{ is the best response to } s_2^*}$$

$$\underbrace{\pi_2(s_1^*, s_2) \leq \pi_2(s_1^*, s_2^*) \text{ for all } s_2 \in \Sigma_2}_{s_2^* \text{ is the best response to } s_1^*}$$

In words: A pair of strategies is a NE if no player can increase his payoff by **unilaterally** changing his strategy

Existence of Nash Equilibrium: Nash's Theorem

Matching Pennies: **pure** strategy Nash equilibrium does not have to exist

Theorem (Nash 1950)

Every game *in strategic form has a* **mixed** *strategy Nash equilibrium.*

Proof.

Involves the use of Brouwer's or Kakutani's fixed point theorem. □

Note:

- A NE can be pure, since this is a special case of a mixed strategy ($S_i \subset \Sigma_i$)
- (D, D) is a pure strategy NE in Prisoners' Dilemma (how do we know???)

Finding Pure Nash Equilibria via Pure Best Response

Checking whether $(s_1^*, s_2^*) \in \Sigma_1 \times \Sigma_2$ is a NE **using the definition** means

- checking the best response inequalities
- for all **mixed** strategies.

If looking for **pure NE,** there is a simplification:

Check the inequalities for **pure** strategies only.

This **validates the method** we used to looking for NE in the examples (e.g., Prisoners' Dilemma)

Finding Pure Nash Equilibria via Pure Best Response

Theorem (Pure Best Response)

Assume a pair of **pure** *strategies* $(s_1^*, s_2^*) \in S_1 \times S_2$ *satisfies the* **pure best response inequalities**

$$\pi_1(s_1, s_2^*) \leq \pi_1(s_1^*, s_2^*) \quad \textit{for all} \quad \underbrace{s_1 \in S_1}_{\textit{pure only}}$$

$$\pi_2(s_1^*, s_2) \leq \pi_2(s_1^*, s_2^*) \quad \textit{for all} \quad \underbrace{s_2 \in S_2}_{\textit{pure only}}$$

Then it is a Nash equilibrium.

Pure Best Response Theorem: The Proof

Proof.

We want to show that if $s_1^* \in S_1$ is a pure best response to $s_2^* \in S_2$, then it is also the mixed best response (and the same with 1 and 2 swapped). This would imply that the pair of pure strategies (s_1^*, s_2^*) is a NE. Pick any $s_1 \in \Sigma_1$. Let $p(s)$ be the probability density function of s_1. Then by using the first pure best response inequality and monotonicity of expectation (integral) we get

$$\begin{aligned}
\pi_1(s_1, s_2^*) = E(\pi_1(s_1, s_2^*)) &= \int_{s \in S_1} \pi_1(s, s_2^*) p(s)\, ds \\
&\leq \int_{s \in S_1} \pi_1(s_1^*, s_2^*) p(s)\, ds \\
&= \pi_1(s_1^*, s_2^*) \underbrace{\int_{s \in S_1} p(s)\, ds}_{=1} \\
&= \pi_1(s_1^*, s_2^*).
\end{aligned}$$

□

Example: Find Pure Nash Equilibria

Consider the 2-person finite game represented by the payoff matrix:

	1		2		3	
A	1	$\boxed{3}$	$\boxed{4}$	2	2	2
B	$\boxed{4}$	0	0	$\boxed{3}$	4	1
C	2	5	3	4	$\boxed{5}$	$\boxed{6}$

Result: The only strategy pair which is the **best response to each other** is $(C, 3)$: this is a pure NE

Support of a Mixed Strategy

In the definition below we assume that we have a finite game.

Definition (Support of a Mixed Strategy)

Let $s_1 \in \Sigma_1$ and define

$$Supp(s_1) = \{s \in S_1 \ : \ Pr(s_1 = s) \neq 0\}.$$

The set $Supp(s_1)$ is called the **support** of the mixed strategy (random variable) s_1.

Example: Support of the mixed strategy s_1 in the game on the previous slide which assigns probability $1/3$ to the pure strategy A and $2/3$ to the pure strategy C is $\{A, C\}$.

Finding Mixed Nash Equilibria of Finite Games

The following theorem gives

- **necessary conditions**
- of **combinatorial** nature
- characterizing **mixed Nash equilibria**
- of **finite** games

Theorem (Equality of Payoffs)

Assume G is a **finite game** *and let $(s_1^*, s_2^*) \in \Sigma_1 \times \Sigma_2$ be a Nash equilibrium. Then*

- $\pi_1(s, s_2^*) = \pi_1(s_1^*, s_2^*)$ *for all* $s \in Supp(s_1^*)$ $\quad (EP1)$
- $\pi_2(s_1^*, s) = \pi_2(s_1^*, s_2^*)$ *for all* $s \in Supp(s_2^*)$ $\quad (EP2)$

In words: All pure strategies forming the support of a Nash strategy attain identical payoff when used against the Nash strategy of the opponent.

Equality of Payoffs: The Proof

Proof.

We only show necessity of (EP1), (EP2) can be proved in the same way. For $s \in X \stackrel{\text{def}}{=} Supp(s_1^*)$ define $p(s) = Pr(s_1^* = s)$ and for $t \in S_2$ let $q(t) = Pr(s_2^* = t)$. Then

$$\begin{aligned}
\pi_1(s_1^*, s_2^*) &= \sum_{s \in X} \sum_{t \in S_2} p(s) q(t) \pi_1(s, t) \\
&= \sum_{s \in X} p(s) \sum_{t \in S_2} q(t) \pi_1(s, t) \\
&= \sum_{s \in X} p(s) \pi_1(s, s_2^*) \\
&\stackrel{\text{Def of NE}}{\leq} \sum_{s \in X} p(s) \pi_1(s_1^*, s_2^*) \\
&= \pi_1(s_1^*, s_2^*)
\end{aligned}$$

Since $p(s) > 0$ for all $s \in X$, the inequality in the above chain would be sharp in case we did not have $\pi_1(s, s_2^*) = \pi_1(s_1^*, s_2^*)$ for all $s \in X$, which would be a contradiction. $\square$

Example: Find Mixed NE in Matching Pennies

	H		T	
H	$+1$	-1	-1	$+1$
T	-1	$+1$	$+1$	-1

Let (s_1^*, s_2^*) be a NE and assume P_2 plays H with probability q and T with probability $1-q$.

- Suppose that the NE strategy of P_1 is **not pure.**
- We do this to **fix the support of s_1^***

Then by the Equality of Payoffs theorem we have

$$\pi_1(H, s_2^*) = \pi_1(T, s_2^*)$$

$$q\pi_1(H,H) + (1-q)\pi_1(H,T) = q\pi_1(T,H) + (1-q)\pi_1(T,T)$$
$$q - (1-q) = -q + (1-q) \quad \Rightarrow \quad q = \tfrac{1}{2}$$

Using the same argument with P_1 and P_2 swapped gives the following NE: Both players should play their pure strategies with probability $1/2$.