Erasmus School of Economics

MOOC Econometrics

Lecture M.3 on Building Blocks: Vectors and Differentiation

Erik Kole

Erasmus University Rotterdam

Linear Model

$$y = Xb + e$$

- y: $p \times 1$ vector, dependent variable
- X: $p \times q$ matrix, explanatory variables
- b: $q \times 1$ vector, coefficients
- e: $p \times 1$ vector, residuals, $e = y - Xb$

$$\begin{aligned} e'e = \sum_{i=1}^{p} e_i^2 &= (y - Xb)'(y - Xb) \\ &= y'y - 2y'Xb + b'X'Xb \end{aligned}$$

Partial Derivatives

$$\begin{aligned} f(x,y) \quad &\rightarrow \quad f_x(x; y) \text{ (y constant)} \quad \rightarrow \quad \frac{\partial f}{\partial x}(x,y) = \frac{df_x}{dx}(x; y) \\ &\rightarrow \quad f_y(y; x) \text{ (x constant)} \quad \rightarrow \quad \frac{\partial f}{\partial y}(x,y) = \frac{df_y}{dy}(y; x) \end{aligned}$$

Examples

- $f(x,y) = x + 2y \quad \rightarrow \frac{\partial f}{\partial x}(x,y) = 1, \quad \frac{\partial f}{\partial y}(x,y) = 2$
- $f(x,y) = x^2 y \quad \rightarrow \frac{\partial f}{\partial x}(x,y) = 2xy, \quad \frac{\partial f}{\partial y}(x,y) = x^2$

Question

Test

What are the partial derivatives of

$$f(x,y) = \ln\left(x^2 y + y^2 + 3xy\right), \; x > 0, y > 0$$

Answer

$$\begin{aligned} \frac{\partial f}{\partial x}(x,y) &= \frac{2xy + 3y}{x^2 y + y^2 + 3xy} \\ \frac{\partial f}{\partial y}(x,y) &= \frac{x^2 + 2y + 3x}{x^2 y + y^2 + 3xy} \end{aligned}$$

Function of a vector

$$f(\underset{(q\times 1)}{b}) = f(b_1,\dots,b_q) \quad \to \quad f_i(b_i) \text{ (all arguments } b_j, j\neq i \text{ constant)}$$

$$\to \quad \frac{\partial f}{\partial b_i}(b) = \frac{df_i}{db_i}(b_i)$$

gradient: $$\frac{\partial f}{\partial b}(b) = \begin{pmatrix} \frac{\partial f}{\partial b_1}(b) \\ \vdots \\ \frac{\partial f}{\partial b_q}(b) \end{pmatrix}$$

Example 1:

$$f(b) = a'b = \textstyle\sum_{i=1}^q a_i b_i \quad \to \quad \frac{\partial f}{\partial b_i}(b) = a_i \quad \to \quad \frac{\partial f}{\partial b}(b) = a$$

Example 2

$$f(\underset{(q\times 1)}{b}) = b' \underset{(q\times q)}{A} b = \sum_{j=1}^q \sum_{k=1}^q b_j a_{jk} b_k$$

$$\frac{\partial f}{\partial b_i}(b) = \sum_{k=1}^q a_{ik} b_k + \sum_{j=1}^q b_j a_{ji} = A_{i\bullet} b + b' A_{\bullet i} = (A+A')_{i\bullet} b$$

$$\frac{\partial f}{\partial b}(b) = \begin{pmatrix} (A+A')_{1\bullet} b \\ \vdots \\ (A+A')_{q\bullet} b \end{pmatrix} = (A+A')b$$

Question

Test

Let a and b be $q\times 1$ vectors and C be a symmetric $q\times q$ matrix. Find the gradients of the functions f and g of b defined as

$$f(b) = b'a \qquad g(b) = b'Cb.$$

Answer

- $\partial f/\partial b = a$, so a column vector.
- $\partial g/\partial b = (C+C')b = 2Cb$, because C is symmetric, so $C' = C$.

Second order partial derivatives

$$f(x,y) \quad \to \quad \frac{\partial f}{\partial x}(x,y) = f_y'(x) \quad \text{and} \quad \frac{\partial f}{\partial y}(x,y) = f_x'(y)$$

$$g(x,y) = \frac{\partial f}{\partial x}(x,y) \quad \to \quad \frac{\partial g}{\partial x}(x,y) = \frac{\partial^2 f}{\partial x \partial x}(x,y) \quad \text{and}$$

$$\frac{\partial g}{\partial y}(x,y) = \frac{\partial^2 f}{\partial y \partial x}(x,y)$$

$$h(x,y) = \frac{\partial f}{\partial y}(x,y) \quad \to \quad \frac{\partial h}{\partial x}(x,y) = \frac{\partial^2 f}{\partial x \partial y}(x,y) \quad \text{and}$$

$$\frac{\partial h}{\partial y}(x,y) = \frac{\partial^2 f}{\partial y \partial y}(x,y)$$

Second order partial derivatives: example

$$f(x,y) = x^2 y$$

$$\frac{\partial f}{\partial x}(x,y) = 2xy \quad \rightarrow \quad \frac{\partial^2 f}{\partial x \partial x}(x,y) = 2y \quad \text{and}$$
$$\frac{\partial^2 f}{\partial y \partial x}(x,y) = 2x$$
$$\frac{\partial f}{\partial y}(x,y) = x^2 \quad \rightarrow \quad \frac{\partial^2 f}{\partial x \partial y}(x,y) = 2x \quad \text{and}$$
$$\frac{\partial^2 f}{\partial y \partial y}(x,y) = 0$$

Hessian matrix

$$f(\underset{(q\times 1)}{b}) = f(b_1, \ldots, b_q)$$

$$\rightarrow \text{ gradient: } \underbrace{\frac{\partial f}{\partial b}(b)}_{(q\times 1)}$$

$$\rightarrow \text{ Hessian: } \underbrace{\frac{\partial^2 f}{\partial b \partial b'}(b)}_{(q\times q)} = \begin{pmatrix} \frac{\partial^2 f}{\partial b_1 \partial b_1}(b) & \cdots & \frac{\partial^2 f}{\partial b_1 \partial b_q}(b) \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial b_q \partial b_1}(b) & \cdots & \frac{\partial^2 f}{\partial b_q \partial b_q}(b) \end{pmatrix}$$

Generally, Hessian matrix symmetric.

Optimization

Extreme values of $f(\underset{(q\times 1)}{b})$

First Order Condition: if optimum at b^*, $\frac{\partial f}{\partial b}(b^*) = 0$.

Second Order Condition for minimum: Hessian at b^* positive definite

Second Order Condition for maximum: Hessian at b^* negative definite

Definiteness

Consider the matrix $\underset{(q\times q)}{A}$ and a vector $\underset{(q\times 1)}{x}$.

- If for all $x \neq 0 : x'Ax > 0$, A positive definite
- If for all $x \neq 0 : x'Ax < 0$, A negative definite
- If for all $x : x'Ax \geq 0$, A positive semi-definite
- If for all $x : x'Ax \leq 0$, A negative semi-definite
- Otherwise indefinite.

Definiteness of $A'A$?

Test

Let A be a $p \times q$ matrix with $\text{rank}(A) = q$. What definiteness property does $B = A'A$ have?

Answer

1. Define $c = Ax$, then $x'Bx = x'A'Ax = c'c = \sum_{i=1}^{p} c_i^2 \geq 0$.
2. $\text{rank}(A) = q$, so $Ax = 0$ only for $x = 0$, so for all $x \neq 0 : x'A'Ax = c'c > 0$ and hence positive definite.

Optimization example

$$\min_b f(b), \quad f(b) = a'b + b'C'Cb$$

with a, b $q \times 1$ vectors, and C a $p \times q$ matrix and rank $C = q$.

1. FOC: $\dfrac{\partial f}{\partial b}(b^*) = a + 2C'Cb^* = 0 \Rightarrow b^* = -\dfrac{1}{2}(C'C)^{-1}a$
2. SOC: $\dfrac{\partial^2 f}{\partial b' \partial b}(b) = 2C'C$, positive definite, so b^* is a minimum.

Training Exercise M.3

- Train yourself by making the training exercise (see the website).
- After making this exercise, check your answers by studying the webcast solution (also available on the website).