Erasmus School of Economics

MOOC Econometrics

Lecture 2.4.1 on Multiple Regression: Evaluation - Statistical Properties

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Six DGP assumptions

A1 Linear model: $y = X\beta + \varepsilon$.

A2 Fixed regressors: X non-random.

A3 Random error terms with mean zero: $E(\varepsilon) = 0$.

A4 Homoskedastic error terms: $E(\varepsilon_i^2) = \sigma^2$ for all $i = 1, \ldots, n$.

A5 Uncorrelated error terms: $E(\varepsilon_i \varepsilon_j) = 0$ for all $i \neq j$.

A6 Parameters β and σ^2 are fixed and unknown.

Test

Prove that A4 and A5 imply that $E(\varepsilon\varepsilon') = \sigma^2 I$.

- Answer: Direct calculation of variance-covariance matrix.

A1,2,3,6: b unbiased, $E(b) = \beta$

Under A1, A2, A3, and A6, OLS is unbiased: $E(b) = \beta$.

Test

Express OLS estimator b in terms of ε.

- Answer: $b = (X'X)^{-1}X'y \overset{(A1)}{=} (X'X)^{-1}X'(X\beta + \varepsilon)$
  $= \beta + (X'X)^{-1}X'\varepsilon$.

- $E(b) \overset{(A6)}{=} \beta + E((X'X)^{-1}X'\varepsilon) \overset{(A2)}{=} \beta + (X'X)^{-1}X'E(\varepsilon)$
  $\overset{(A3)}{=} \beta + (X'X)^{-1}X'0 = \beta$.

A1-A6: $\mathrm{var}(b) = \sigma^2(X'X)^{-1}$

- Seen before: $b = \beta + (X'X)^{-1}X'\varepsilon$.

- $$\begin{aligned}\mathrm{var}(b) = \; & E\Big((b - Eb)(b - Eb)'\Big) \overset{(A1,2,3,6)}{=} \\ & E\Big((b-\beta)(b-\beta)'\Big) = \\ & E\Big((X'X)^{-1}X'\varepsilon\varepsilon'X(X'X)^{-1}\Big) \overset{(A2)}{=} \\ & (X'X)^{-1}X' \; E(\varepsilon\varepsilon') \; X(X'X)^{-1} \overset{(A4,5)}{=} \\ & (X'X)^{-1}X' \; \sigma^2 I \; X(X'X)^{-1} = \\ & \sigma^2(X'X)^{-1}X'X(X'X)^{-1} = \sigma^2(X'X)^{-1}.\end{aligned}$$

- Let a_{jh} be (j,h)-th element of $(k \times k)$ matrix $(X'X)^{-1}$, then $\mathrm{var}(b_j) = \sigma^2 a_{jj}$ and $\mathrm{cov}(b_j, b_h) = \sigma^2 a_{jh}$.

OLS estimator of σ^2

Under A1-A6, $s^2 = e'e/(n-k)$ is unbiased: $E(s^2) = \sigma^2$.

- Idea of proof: (a) Express e in ε.
  (b) Compute $E(ee')$.
  (c) Use 'trace trick' to get $E(e'e)$.
- (a) Previous lecture: $e = My$ where $M = I - X(X'X)^{-1}X'$ with $M' = M = M^2$ and $MX = 0$.

  Then $e = My \overset{(A1)}{=} M(X\beta + \varepsilon) = MX\beta + M\varepsilon = M\varepsilon$.
- (b) $E(ee') = E(M\varepsilon\varepsilon' M') \overset{(A2)}{=} ME(\varepsilon\varepsilon')M \overset{(A4,5)}{=} M\sigma^2 IM = \sigma^2 M$.
- (c) 'Trace trick': $E(e'e) = \text{trace}(E(ee')) = \sigma^2\text{trace}(M) = (n-k)\sigma^2$.

Details of 'trace trick' (optional)

- $\text{trace}(AB) = \text{trace}(BA)$, where 'trace' is sum of diagonal elements of square matrix (see Building Blocks).
- Trace trick:

$$
\begin{aligned}
E(e'e) &= E(\textstyle\sum_{i=1}^n e_i^2) = E(\text{trace}(ee')) = \text{trace}(E(ee')) \\
&= \text{trace}(\sigma^2 M) = \sigma^2\text{trace}(I_n - X(X'X)^{-1}X') \\
&= \sigma^2\text{trace}(I_n) - \sigma^2\text{trace}(X(X'X)^{-1}X') \\
&= n\sigma^2 - \sigma^2\text{trace}((X'X)^{-1}X'X) \\
&= n\sigma^2 - \sigma^2\text{trace}(I_k) = (n-k)\sigma^2.
\end{aligned}
$$

- As $E(e'e) = (n-k)\sigma^2$, it follows that $E(s^2) = \sigma^2$.

Efficiency of OLS

- A1-A6: OLS b is Best Linear Unbiased Estimator (BLUE).
- This is the so-called Gauss-Markov theorem.
- If $\hat{\beta} = Ay$ is linear estimator, A non-random $(k \times n)$ matrix,

  and if $\hat{\beta}$ is unbiased, $E(\hat{\beta}) = \beta$,

  then $\text{var}(\hat{\beta}) - \text{var}(b)$ is positive semi-definite (PSD).

  (see Building Blocks for PSD)
- As b has smallest variance of all linear unbiased estimators, OLS is efficient (in this class).

TRAINING EXERCISE 2.4.1

- Train yourself by making the training exercise (see the website).
- After making this exercise, check your answers by studying the webcast solution (also available on the website).