

MOOC Econometrics

Lecture P.2 on Building Blocks:
Probability Distributions

Erik Kole

Erasmus University Rotterdam



Independence

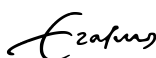
Independence

Two random variables x , y with joint density $f(v, w)$ and marginal densities $f_x(v)$, $f_y(w)$ are **independent** if and only if

$$f(v, w) = f_x(v) f_y(w) \quad \text{for all } v, w$$

Consequence: joint probability is product of marginal probabilities

$$\begin{aligned} P[a \leq x \leq b, c \leq y \leq d] &= \int_a^b \int_c^d f(v, w) dw dv \\ &= \int_a^b \int_c^d f_x(v) f_y(w) dw dv \stackrel{*}{=} \int_a^b f_x(v) dv \int_c^d f_y(w) dw \\ &= P[a \leq x \leq b] \cdot P[c \leq y \leq d] \end{aligned}$$



Moments

Let x be a random variable with pdf $f(v)$.

- Mean: first moment, $\mu = E[x]$
- Variance: second centered moment, $\sigma^2 = \text{var}[x] = E[(x - \mu)^2]$
- Centered moment of order k

$$\mu_k = E[(x - \mu)^k]$$

Often standardized by σ^k .

- Skewness: $\mu_3/\sigma^3 = E[(x - \mu)^3]/\sigma^3$ (asymmetry)
- Kurtosis: $\mu_4/\sigma^4 = E[(x - \mu)^4]/\sigma^4$ (tail fatness)



Independence and expectation

If x and y are independent, the expectation of a product is the product of expectations:

$$\begin{aligned} E[xy] &= E[x]E[y] \\ E[g(x)h(y)] &= E[g(x)]E[h(y)] \end{aligned}$$

Test

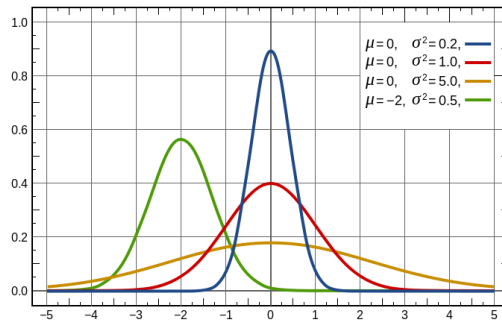
Calculate $\text{cov}[x, y]$ and $E[(x - \mu_x)^2(y - \mu_y)^2]$ when x and y are independent.

Answer

- $\text{cov}(x, y) = E[(x - \mu_x)(y - \mu_y)] = E[x - \mu_x]E[y - \mu_y] = 0$.
- $E[(x - \mu_x)^2(y - \mu_y)^2] = E[(x - \mu_x)^2]E[(y - \mu_y)^2] = \sigma_x^2\sigma_y^2$.

Normal distribution

$$x \sim N(\mu, \sigma^2): f(v) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(v-\mu)^2}{2\sigma^2}}$$



- $E[x] = \mu$ and $\text{var}[x] = \sigma^2$.
- Bell shaped, symmetric around μ .
- Skewness: 0; Kurtosis: 3.
- $y = (x - \mu)/\sigma$: standard normal distribution, $y \sim N(0, 1)$.

χ^2 (chi-square) distribution

Consider $y_i \sim NID(0, 1)$, $i = 1, 2, \dots, n$

$$z = y'y = \sum_{i=1}^n y_i^2: z \sim \chi^2(n)$$

- NID: normally and independently distributed
- "z follows a χ^2 -distribution with n degrees of freedom."
- $E[z] = E[\sum_{i=1}^n y_i^2] = \sum_{i=1}^n E[y_i^2] = \sum_{i=1}^n 1 = n$

Multivariate normal distribution

$$x \text{ multivariate normal: } x \sim N\left(\begin{matrix} \mu \\ (n \times 1) \end{matrix}, \begin{matrix} \Sigma \\ (n \times n) \end{matrix}\right)$$

- $E[x] = \mu$, $\text{var}[x] = \Sigma$.
- $x_i \sim N(\mu_i, \sigma_i^2)$.
- If $\sigma_{ij} = 0$, x_i and x_j are independent.
- Linear transformations of x remain normal:

$$\begin{matrix} y \\ (m \times 1) \end{matrix} = \begin{matrix} A \\ (m \times n) \end{matrix} \begin{matrix} x \\ (n \times 1) \end{matrix} + \begin{matrix} b \\ (m \times 1) \end{matrix}, \text{ then } y \sim N(A\mu + b, A\Sigma A')$$

Erasmus

Lecture P.2, Slide 6 of 12, Erasmus School of Economics

Sequence of random variables

Test

Let $x_i \sim NID(\mu, \sigma^2)$, $i = 1, 2, \dots, n$. Find the joint distribution of x , and the distributions of

$$y = \sum_{i=1}^n x_i \quad \text{and} \quad z = \sum_{i=1}^n (x_i - \mu)^2$$

Answer

- $x \sim N(\mu, \sigma^2 I)$, as $\sigma_{ij} = 0$ for all $i \neq j$.
- $y = \iota'x$ so $y \sim N(\mu_y, \sigma_y^2)$ with $\mu_y = \iota'\mu = \iota'\iota\mu = n\mu$ and $\sigma_y^2 = \iota'\sigma^2\iota = \iota'\iota\sigma^2 = \iota'\iota\sigma^2 = n\sigma^2$.
- z/σ^2 : sum of squared iid normals, so $z/\sigma^2 \sim \chi^2(n)$.

Erasmus

Lecture P.2, Slide 7 of 12, Erasmus School of Economics

Erasmus

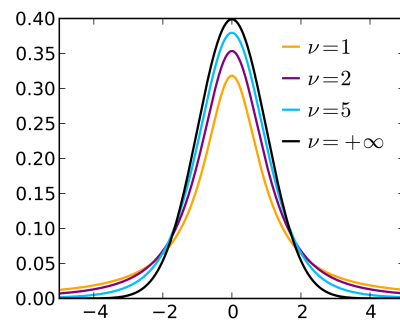
Lecture P.2, Slide 8 of 12, Erasmus School of Economics

Student's t distribution

Let $y \sim N(0, 1)$ and $z \sim \chi^2(\nu)$, y, z independent

$$\frac{y}{\sqrt{z/\nu}} \sim t(\nu)$$

- $t(\nu)$: Student's t distribution with ν degrees of freedom.
- fatter tails than a normal distribution
- converges to a standard normal distribution for $\nu \rightarrow \infty$



Lecture P.2, Slide 9 of 12, Erasmus School of Economics

Erasmus

F -distribution

Let $z_1 \sim \chi^2(d_1)$ and $z_2 \sim \chi^2(d_2)$, z_1 and z_2 independent

$$\frac{z_1/d_1}{z_2/d_2} \sim F(d_1, d_2)$$

- F -distribution with d_1 and d_2 degrees of freedom.

Erasmus

Lecture P.2, Slide 10 of 12, Erasmus School of Economics

t – and F -distributions

Test

Let $x_1, x_2 \stackrel{\text{iid}}{\sim} t(n)$. Find distributions for $x_1 + x_2$ and x_1^2 .

Answer

$x_1 = \frac{y_1}{\sqrt{z_1/n}}$, with y_1 and z_1 independent; similar for x_2

- $x_1 + x_2 = \frac{y_1}{\sqrt{z_1/n}} + \frac{y_2}{\sqrt{z_2/n}}$, no further simplification
- $x_1^2 = \frac{y_1^2}{z_1/n}$, and $y_1^2 \sim \chi^2(1)$, so $x_1^2 \sim F(1, n)$.

Erasmus

Lecture P.2, Slide 11 of 12, Erasmus School of Economics

Training Exercise P.2

- Train yourself by making the training exercise (see the website).
- After making this exercise, check your answers by studying the webcast solution (also available on the website).

Erasmus

Lecture P.2, Slide 12 of 12, Erasmus School of Economics