

MOOC Econometrics

Lecture 2.4.1 on Multiple Regression: Evaluation - Statistical Properties

Christiaan Heij

A1,2,3,6: b unbiased, $E(b) = \beta$

Under A1, A2, A3, and A6, OLS is unbiased: $E(b) = \beta$.

Test

Express OLS estimator b in terms of ε .

- Answer: $b = (X'X)^{-1}X'y \stackrel{(A1)}{=} (X'X)^{-1}X'(X\beta + \varepsilon)$
 $= \beta + (X'X)^{-1}X'\varepsilon.$
- $E(b) \stackrel{(A6)}{=} \beta + E((X'X)^{-1}X'\varepsilon) \stackrel{(A2)}{=} \beta + (X'X)^{-1}X'E(\varepsilon)$
 $\stackrel{(A3)}{=} \beta + (X'X)^{-1}X'0 = \beta.$

Six DGP assumptions

- A1 Linear model: $y = X\beta + \varepsilon$.
- A2 Fixed regressors: X non-random.
- A3 Random error terms with mean zero: $E(\varepsilon) = 0$.
- A4 Homoskedastic error terms: $E(\varepsilon_i^2) = \sigma^2$ for all $i = 1, \dots, n$.
- A5 Uncorrelated error terms: $E(\varepsilon_i \varepsilon_j) = 0$ for all $i \neq j$.
- A6 Parameters β and σ^2 are fixed and unknown.

Test

Prove that A4 and A5 imply that $E(\varepsilon\varepsilon') = \sigma^2 I$.

- Answer: Direct calculation of variance-covariance matrix.

Lecture 2.4.1, Slide 2 of 8, Erasmus School of Economics

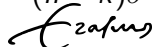
A1-A6: $\text{var}(b) = \sigma^2(X'X)^{-1}$

- Seen before: $b = \beta + (X'X)^{-1}X'\varepsilon$.
- $\text{var}(b) = E((b - Eb)(b - Eb)') \stackrel{(A1,2,3,6)}{=} E((b - \beta)(b - \beta)')$
 $= E((X'X)^{-1}X'\varepsilon\varepsilon'X(X'X)^{-1}) \stackrel{(A2)}{=} (X'X)^{-1}X'E(\varepsilon\varepsilon')X(X'X)^{-1} \stackrel{(A4,5)}{=} (X'X)^{-1}X'\sigma^2 I X(X'X)^{-1}$
 $= \sigma^2(X'X)^{-1}X'X(X'X)^{-1} = \sigma^2(X'X)^{-1}.$
- Let a_{jh} be (j, h) -th element of $(k \times k)$ matrix $(X'X)^{-1}$, then $\text{var}(b_j) = \sigma^2 a_{jj}$ and $\text{cov}(b_j, b_h) = \sigma^2 a_{jh}$.

OLS estimator of σ^2

Under A1-A6, $s^2 = e'e/(n - k)$ is unbiased: $E(s^2) = \sigma^2$.

- Idea of proof:
 - (a) Express e in ε .
 - (b) Compute $E(ee')$.
 - (c) Use 'trace trick' to get $E(e'e)$.
- (a) Previous lecture: $e = My$ where $M = I - X(X'X)^{-1}X'$ with $M' = M = M^2$ and $MX = 0$.
Then $e = My \stackrel{(A1)}{=} M(X\beta + \varepsilon) = MX\beta + M\varepsilon = M\varepsilon$.
- (b) $E(ee') = E(M\varepsilon\vare' M') \stackrel{(A2)}{=} ME(\varepsilon\vare')M \stackrel{(A4,5)}{=} M\sigma^2 IM = \sigma^2 M$.
- (c) 'Trace trick': $E(e'e) = \text{trace}(E(ee')) = \sigma^2 \text{trace}(M) = (n - k)\sigma^2$.



Lecture 2.4.1, Slide 5 of 8, Erasmus School of Economics

Details of 'trace trick' (optional)

- $\text{trace}(AB) = \text{trace}(BA)$, where 'trace' is sum of diagonal elements of square matrix (see Building Blocks).
- Trace trick:
$$\begin{aligned} E(e'e) &= E(\sum_{i=1}^n e_i^2) = E(\text{trace}(ee')) = \text{trace}(E(ee')) \\ &= \text{trace}(\sigma^2 M) = \sigma^2 \text{trace}(I_n - X(X'X)^{-1}X') \\ &= \sigma^2 \text{trace}(I_n) - \sigma^2 \text{trace}(X(X'X)^{-1}X') \\ &= n\sigma^2 - \sigma^2 \text{trace}((X'X)^{-1}X'X) \\ &= n\sigma^2 - \sigma^2 \text{trace}(I_k) = (n - k)\sigma^2. \end{aligned}$$
- As $E(e'e) = (n - k)\sigma^2$, it follows that $E(s^2) = \sigma^2$.



Lecture 2.4.1, Slide 6 of 8, Erasmus School of Economics

Efficiency of OLS

- A1-A6: OLS b is Best Linear Unbiased Estimator (BLUE).
- This is the so-called Gauss-Markov theorem.
- If $\hat{\beta} = Ay$ is linear estimator, A non-random ($k \times n$) matrix, and if $\hat{\beta}$ is unbiased, $E(\hat{\beta}) = \beta$, then $\text{var}(\hat{\beta}) - \text{var}(b)$ is positive semi-definite (PSD).
(see Building Blocks for PSD)
- As b has smallest variance of all linear unbiased estimators, OLS is efficient (in this class).



Lecture 2.4.1, Slide 7 of 8, Erasmus School of Economics

TRAINING EXERCISE 2.4.1

- Train yourself by making the training exercise (see the website).
- After making this exercise, check your answers by studying the webcast solution (also available on the website).



Lecture 2.4.1, Slide 8 of 8, Erasmus School of Economics