Erasmus School of Economics

# MOOC Econometrics

## Lecture M.2 on Building Blocks: Special Matrix Operations

Erik Kole

Erasmus University Rotterdam

## Transpose: Example

$$A = \begin{pmatrix} 25.5 & 1.23 \\ 40.8 & 1.89 \\ 30.2 & 1.55 \\ 4.3 & 1.18 \\ 10.7 & 1.68 \end{pmatrix} \qquad A' = \begin{pmatrix} 25.5 & 40.8 & 30.2 & 4.3 & 10.7 \\ 1.23 & 1.89 & 1.55 & 1.18 & 1.68 \end{pmatrix}$$

$$y = \begin{pmatrix} 15.1 \\ 7.9 \\ 4.3 \\ 12.8 \\ 10.5 \end{pmatrix} \qquad y' = \begin{pmatrix} 15.1 & 7.9 & 4.3 & 12.8 & 10.5 \end{pmatrix}$$



## Transposition: definition and properties

### Definition

$$\underset{(p\times q)}{A} = \begin{pmatrix} a_{11} & \cdots & \cdots & a_{1q} \\ \vdots & \ddots & a_{ij} & \vdots \\ a_{p1} & \cdots & \cdots & a_{pq} \end{pmatrix} \qquad \underset{(q\times p)}{B} = A' = \begin{pmatrix} a_{11} & \cdots & a_{p1} \\ \vdots & \ddots & \vdots \\ \vdots & a_{ij} & \vdots \\ a_{1q} & \cdots & a_{pq} \end{pmatrix}$$

General: $b_{ji} = a_{ij}$

- $C = B'$, then $c_{ij} = b_{ji} = a_{ij}$ for all $i, j$, so $(A')' = A$.
- Symmetric if $A = A'$, so $a_{ij} = a_{ji}$ for all $i, j$.
- $c$ scalar, then $c' = c$.



## Transposition and addition

Addition: $(\underset{(p\times q)}{A} + \underset{(p\times q)}{B})' = A' + B'$

### Proof

1. Let $1 \leq i \leq p, 1 \leq j \leq q$.
2. Define $C = A + B$, then $c_{ij} = a_{ij} + b_{ij}$.
3. Define $D = C'$, then $d_{ji} = c_{ij} = a_{ij} + b_{ij}$.
4. Define $E = A' + B'$, then $e_{ji} = (A')_{ji} + (B')_{ji} = a_{ij} + b_{ij}$.
5. Steps 3 and 4 show that $d_{ji} = e_{ji}$.

## Transposition and multiplication

Multiplication: $(\underset{(p\times q)}{A} \cdot \underset{(q\times r)}{B})' = B'A'$

### Proof

1. Let $1 \leq i \leq p, 1 \leq j \leq r$.
2. Define $C = AB$, then $c_{ij} = \sum_{k=1}^{q} a_{ik} b_{kj}$.
3. Define $D = C'$, then $d_{ji} = c_{ij}$.
4. Define $E = B'A'$, then $e_{ji} = \sum_{k=1}^{q} (B')_{jk} (A')_{ki} = \sum_{k=1}^{q} b_{kj} a_{ik}$.
   Transpose: $(B')_{j\bullet} = B_{\bullet j}$ and $(A')_{\bullet i} = A_{i\bullet}$.
5. Steps 3 and 4 show that $d_{ji} = e_{ji}$.



## Question

### Test

Consider the linear model $y = Xb + e$, with y $(n \times 1)$, $X$ $(n \times k)$, $b$ $(k \times 1)$, and $e$ $(n \times 1)$. For given $y$, $X$ and $b$, $e = y - Xb$. Find an expression without parentheses for the sum of squared residuals $e'e$.

### Answer

$$
\begin{aligned}
e'e &= (y - Xb)'(y - Xb) \\
&= (y' - (Xb)')(y - Xb) = (y' - b'X')(y - Xb) \\
&= y'y \boxed{- y'Xb - b'X'y} - b'X'Xb \\
&= y'y \boxed{- 2y'Xb} - b'X'Xb
\end{aligned}
$$

$b'X'y$ returns a scalar, so $b'X'y = y'Xb$.



## Trace

### Definition

For a square matrix $\underset{(p\times p)}{A}$ :

$$
\operatorname{tr}(A) = \sum_{i=1}^{p} a_{ii}, \qquad \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ a_{21} & a_{22} & \cdots & a_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ a_{p1} & a_{p2} & \cdots & a_{pp} \end{pmatrix}
$$

Trace of transpose: $\operatorname{tr}(A') = \operatorname{tr}(A)$

### Proof

$\operatorname{tr}(A') = \sum_{i=1}^{p} (A')_{ii} = \sum_{i=1}^{p} a_{ii} = \operatorname{tr}(A)$



## Trace and addition

Addition: $\operatorname{tr}(\underset{(p\times p)}{A} + \underset{(p\times p)}{B}) = \operatorname{tr}(A) + \operatorname{tr}(B)$

### Proof

$\operatorname{tr}(A + B) = \sum_{i=1}^{p} (A + B)_{ii} = \sum_{i=1}^{p} (a_{ii} + b_{ii}) = \sum_{i=1}^{p} a_{ii} + \sum_{i=1}^{p} b_{ii} = \operatorname{tr}(A) + \operatorname{tr}(B)$

## Trace and multiplication

Multiplication: $\mathrm{tr}(\underset{(p\times q)}{A} \cdot \underset{(q\times p)}{B}) = \mathrm{tr}(BA)$

### Proof

1. Define $\underset{(p\times p)}{C} = AB$ then $c_{ii} = \sum_{j=1}^{q} a_{ij} b_{ji}$.
2. $\mathrm{tr}(C) = \sum_{i=1}^{p} c_{ii} = \sum_{i=1}^{p} \sum_{j=1}^{q} a_{ij} b_{ji}$.
3. Define $\underset{(q\times q)}{D} = BA$, then $d_{jj} = \sum_{i=1}^{p} b_{ji} a_{ij}$.
4. $\mathrm{tr}(D) = \sum_{j=1}^{q} d_{jj} = \sum_{j=1}^{q} \sum_{i=1}^{p} b_{ji} a_{ij}$
5. Because $\sum_{j=1}^{q} \sum_{i=1}^{p} b_{ji} a_{ij} \overset{*}{=} \sum_{i=1}^{p} \sum_{j=1}^{q} a_{ij} b_{ji}$, $\mathrm{tr}(D) = \mathrm{tr}(C)$.



## Linear independence

$$A = \begin{pmatrix} 25.5 & 1.23 \\ 40.8 & 1.89 \\ 30.2 & 1.55 \\ 4.5 & 1.18 \\ 10.7 & 1.68 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 3 & 2 & 5 \\ 2 & 6 & 7 & 13 \\ 0 & 0 & 3 & 3 \end{pmatrix}$$

- $B_{\bullet 2} = 3B_{\bullet 1}$ and $B_{\bullet 4} = B_{\bullet 2} + B_{\bullet 3}$
- column rank A $= 2$
- column rank B $= 2$



## Question

### Test

What is the row rank of matrix B?

$$B = \begin{pmatrix} 1 & 3 & 2 & 5 \\ 2 & 6 & 7 & 13 \\ 0 & 0 & 3 & 3 \end{pmatrix}$$

### Answer: two

Rows 1 and 2 are independent, but $B_{3\bullet} = B_{2\bullet} - 2B_{1\bullet}$, so the row rank equals 2.



## Rank

- Rank of a matrix = # linearly independent rows = # linearly independent columns
- For any $\underset{(p\times q)}{A}$ : $\mathrm{rank}(A) \leq \min(p, q)$
- $\mathrm{rank}(\underset{(p\times q)}{A}) = q$: full column rank; $\mathrm{rank}(A) = p$: full row rank.
- $\mathrm{rank}(\underset{(p\times p)}{A}) = p$: full rank;
- Rank and transpose: $\mathrm{rank}(A') = \mathrm{rank}(A)$

## Rank and linear systems

System of linear equations: $\underset{(p\times q)}{A} \cdot \underset{(q\times 1)}{c} = \underset{(p\times 1)}{d}$, $d$ given, $c$ unknown.

If $\text{rank}(A) = q$, and $Ac = 0$ then $c = 0$.

**Proof**

- Because $\text{rank}(A) = q$, all columns of $A$ are linearly independent.
- No linear combination $c \neq 0$ of the columns of $A$ can produce $d = 0$, so $c = 0$

If $\text{rank}(A) < q$, we can find $c \neq 0$ such that $Ac = 0$.

**Proof**

Because $\text{rank}(A) < q$, we can find at least one column, say $j$, that we can construct as a linear combination of the other columns. Put this linear combination in the vector $c$, with $c_j = -1$. Then $Ac = 0$.



## Solving linear systems: example

$$A = \begin{pmatrix} 25.5 & 1.23 \\ 40.8 & 1.89 \\ 30.2 & 1.55 \\ 4.5 & 1.18 \\ 10.7 & 1.68 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 3 & 2 & 5 \\ 2 & 6 & 7 & 13 \\ 0 & 0 & 3 & 3 \end{pmatrix}$$

- $A$ is $5 \times 2$ and $\text{rank}(A) = 2$, so only $c = 0$ solves $Ac = 0$.
- $B$ is $3 \times 4$ and $\text{rank}(B) = 2$. For $c = \begin{pmatrix} 3 \\ -1 \\ 0 \\ 0 \end{pmatrix}$, and $c = \begin{pmatrix} 0 \\ 1 \\ 1 \\ -1 \end{pmatrix}$, $Bc = 0$.



## Rank and multiplication

General: $\text{rank}(\underset{(p\times q)}{A} \cdot \underset{(q\times r)}{B}) \leq \min(\text{rank}(A), \text{rank}(B))$

Useful in econometrics: for $\underset{(p\times q)}{A}$: $\text{rank}(A'A) = \text{rank}(A)$



## Inverse

**Definition**

The inverse of a $(p \times p)$ matrix $A$ is a matrix $B$ with properties

$$B \cdot A = A \cdot B = I$$

If $B$ exists, we write $B = A^{-1}$.

**Example**

$$A = \begin{pmatrix} 3 & 2 \\ 5 & 4 \end{pmatrix} \qquad A^{-1} = \begin{pmatrix} 2 & -1 \\ -2.5 & 1.5 \end{pmatrix}$$

A square matrix $A$ is invertible if and only if it has full rank.

## Inverse: properties

Inverse of the inverse: for invertible $A$: $\left(A^{-1}\right)^{-1} = A$

**Proof**

- Let $B = A^{-1}$, then $AB = BA = I$.
- This implies that $A$ is the inverse of $B$.

Inverse and transpose: for invertible $A$: $(A')^{-1} = \left(A^{-1}\right)'$

**Proof**

- Let $B = A^{-1}$, then $AB = BA = I$.
- Take transposes: $B'A' = A'B' = I' = I$.
- So $B' = \left(A^{-1}\right)'$ is the inverse of $A'$.



## Inverse and multiplication

Let $\underset{(p\times p)}{A}$ and $\underset{(p\times p)}{C}$ be invertible, then $(AC)^{-1} = C^{-1}A^{-1}$.

**Proof**

- To see if $(AC)^{-1} = C^{-1}A^{-1}$, check $AC \cdot (AC)^{-1} = I$.
- $A \cdot C \cdot C^{-1} \cdot A^{-1} = A \cdot I \cdot A^{-1} = A \cdot A^{-1} = I$.

Solving systems: for invertible $A$: $Ab = c$ implies $b = A^{-1}c$.

**Proof**

Multiply both sides by $A^{-1}$: $A^{-1}Ab = A^{-1}c$, and simplify: $Ib = b = A^{-1}c$.



## Question

**Test**

Let $A$ be a $p \times q$ matrix with $\text{rank}(A) = q$. What properties does $C = A'A$ have?

**Answer**

1. $C$ is symmetric: $C' = (A' \cdot A)' = A' \cdot (A')' = A' \cdot A = C$.
2. $C$ has full rank: $\text{rank}(C) = \text{rank}(A'A) = \text{rank}(A) = q$, $C$ is $q \times q$ so $C$ has full rank and is invertible.
3. $C^{-1}$ is symmetric: $C^{-1}$ exists and $\left(C^{-1}\right)' = (C')^{-1} = C^{-1}$ .



## Training Exercise M.2

- Train yourself by making the training exercise (see the website).
- After making this exercise, check your answers by studying the webcast solution (also available on the website).