Erasmus School of Economics

MOOC Econometrics

Lecture S.1 on Building Blocks: Estimation

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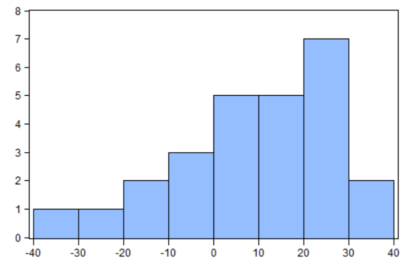
Statistics

- Probability theory: parameters of distributions are known
- Statistics: parameters are not known.
  Instead: observations



x-axis: return (in %); y-axis: frequency

- 26 yearly returns on the stock market.
- Sample mean: 9.6%

Statistical concepts

- $y_i \sim NID(\mu, \sigma^2)$, $i = 1, 2, \ldots, n$; μ and σ^2 unknown
- Statistic: function $g(y) = g(y_1, y_2, \ldots, y_n)$
- Estimator: statistic related to a parameter (so μ or σ^2)

  Sample mean: $$m = \frac{1}{n}\sum_{i=1}^{n} y_i$$

  General: estimator $\hat{\theta}$ for parameter θ.
- Estimate: value of an estimator for a particular sample.
  Example: $m = 9.6\%$.

Properties of sample mean m

Test

Find the distribution of the vector y and of $m = \frac{1}{n}\sum_{i=1}^{n} y_i = \frac{1}{n}\iota' y$.

Answer

- $y \sim N(\mu\iota, \sigma^2 I)$, multivariate normal as $y_i \sim NID(\mu, \sigma^2)$.
- $m \sim N(\mu, \frac{1}{n}\sigma^2)$, sum of normals is normal, $\iota' I \iota = \iota'\iota = n$, so

$$E[m] = E\left[\frac{1}{n}\iota' y\right] = \frac{1}{n}\iota'\iota\mu = \mu$$

$$\text{var}[m] = \text{var}\left[\frac{1}{n}\iota' y\right] = \frac{1}{n^2}\iota'\sigma^2 I\iota = \frac{1}{n^2}\iota'\iota\sigma^2 = \frac{1}{n}\sigma^2$$

Bias and efficiency

Estimator $\hat{\theta}$ for parameter θ.

- If $E\left[\hat{\theta}\right] = \theta$, $\hat{\theta}$ unbiased.
- If $E\left[\hat{\theta}\right] \neq \theta$, $\hat{\theta}$ biased, bias $E\left[\hat{\theta}\right] - \theta$.
- Since $E[m] = \mu$, sample mean is unbiased.
- Standard error: $\sqrt{\mathrm{var}\left[\hat{\theta}\right]}$.
- Efficient estimator: $\mathrm{var}\left[\hat{\theta}\right]$ lowest over a set of estimators.

Towards an estimator for σ^2

Definition: $\sigma^2 = \mathrm{var}[y_i] = E\left[(y_i - \mu)^2\right]$

- μ is unknown, use m instead
- Define $z_i = y_i - m$
- z is a linear transformation of y:

$$z = y - \iota m = y - \iota \cdot \frac{1}{n}\iota' y = \left(I - \frac{1}{n}\iota\iota'\right) y$$

z as a linear transformation of y

Properties of M

Write $z = My$ with $M = I - \frac{1}{n}\iota\iota'$. What properties does M have?

$$M = \frac{1}{n}\begin{pmatrix} n-1 & -1 & \cdots & -1 \\ -1 & n-1 & \ddots & -1 \\ \vdots & \ddots & \ddots & \vdots \\ -1 & -1 & \cdots & n-1 \end{pmatrix}$$

- M is symmetric, $(I - \frac{1}{n}\iota\iota')' = I' - \frac{1}{n}(\iota\iota')' = I - \frac{1}{n}\iota\iota'$.
- $\mathrm{tr}(M) = \frac{1}{n}\sum_{i=1}^{n} m_{ii} = \frac{1}{n}\sum_{i=1}^{n}(n-1) = n-1$.
- $M^2 = (I - \frac{1}{n}\iota\iota')^2 = I^2 - 2\frac{1}{n}\iota\iota' I + \frac{1}{n^2}\iota\iota'\iota\iota' = I - 2\frac{1}{n}\iota\iota' + \frac{n}{n^2}\iota\iota' = I - \frac{1}{n}\iota\iota' = M$.

Distribution of z

$$y \sim N(\mu\iota, \sigma^2 I), \qquad z = My, \qquad M = I - \frac{1}{n}\iota\iota'$$

- $z \sim N(\mu_z, \Sigma_z)$, with
  $\mu_z = 0$, because $E[z_i] = E[y_i - m] = \mu - \mu = 0$;
  $\Sigma_z = M\sigma^2 I M' = \sigma^2 M$.
- Unbiased estimator: so we need $E\left[\sum_{i=1}^{n} z_i^2\right] = E[z'z]$.

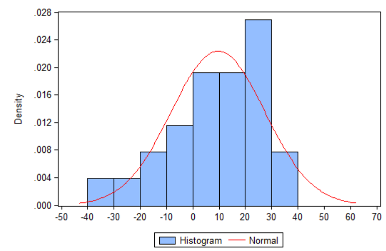
$$\begin{aligned} E[z'z] &= E\left[\mathrm{tr}(z \cdot z')\right] = \mathrm{tr}\left(E[z \cdot z']\right) = \mathrm{tr}(\Sigma_z) \\ &= \sigma^2\,\mathrm{tr}(M) = (n-1)\sigma^2. \end{aligned}$$

Sample variance

Sample variance: $s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(y_i - m)^2$

- Unbiased: $E[s^2] = \frac{1}{n-1}E\left[\sum_{i=1}^n (y_i - m)^2\right] = \frac{1}{n-1}E[z'z] = \sigma^2$.
- $E\left[\frac{1}{n}\sum_{i=1}^n (y_i - m)^2\right] = \frac{n-1}{n}\sigma^2$: biased,
  bias: $\frac{n-1}{n}\sigma^2 - \sigma^2 = -\frac{1}{n}\sigma^2$.
- $\sum_{i=1}^n ((y_i - m)/\sigma)^2 = \frac{z'z}{\sigma^2} = \frac{(n-1)s^2}{\sigma^2} \sim \chi^2(n-1)$
- m and s^2 are independent.

Sample mean and variance: example



x-axis: return (in %); y-axis: frequency

- Normal distribution fits reasonably well.
- Sample mean: $m = 9.6\%$, sample standard deviation: $\sqrt{s^2} = 17.9\%$.
- Standard error (se): $\sqrt{\text{var}[m]} = 17.9/\sqrt{26} = 3.5\%$.
- Rule-of-thumb: 95% confidence interval for m is $(m - 2se, m + 2se)$, so $(2.6, 15.6)$.

Consistency

- Consistent: estimator $\hat{\theta}$ ever more concentrated at θ.
- Sufficient conditions: if $E[\hat{\theta}] \to \theta$ and $\text{var}[\hat{\theta}] \to 0$ for increasing n, $\hat{\theta}$ is consistent.

Test

Are the sample average and sample variance consistent? (Hint: variance of $\chi^2(k)$ equals $2k$).

Answer: both

- $\text{E}[m] = \mu$ and $\text{E}[s^2] = \sigma^2$, so first part fine.
- $\text{var}[m] = \sigma^2/n$
- $(n-1)s^2/\sigma^2 \sim \chi^2(n-1)$, so
  $\text{var}[(n-1)s^2/\sigma^2] = (n-1)^2\,\text{var}[s^2]/\sigma^4 = 2(n-1)$, so
  $\text{var}[s^2] = 2\sigma^4/(n-1)$.
- If n increases, $\text{var}[m] \to 0$ and $\text{var}[s^2] \to 0$, so second part also fine.

Training Exercise S.1

- Train yourself by making the training exercise (see the website).
- After making this exercise, check your answers by studying the webcast solution (also available on the website).