Erasmus
School of
Economics

MOOC Econometrics

Lecture S.2 on Building Blocks: Testing

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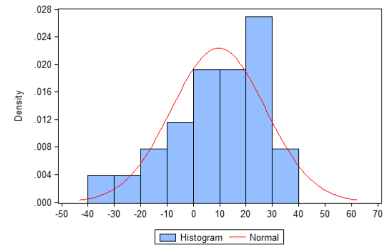
Sample mean and variance: example



x-axis: return (in %); y-axis: frequency

- stock return $y_i \sim NID(\mu, \sigma^2)$
- Sample mean: $m = 9.6\%$, sample standard deviation: $\sqrt{s^2} = 17.9\%$.

A friend claims: "The true mean μ equals zero".

Statistical Hypothesis

- Assertion about a parameter of a distribution.
- Examples: $\mu = 0$, $\mu \geq 0$, $\mu > 5$, $5 \leq \sigma \leq 15$.
- Test: one hypothesis (null) against another one (alternative)

$$\begin{cases} H_0: & \mu = 0 \\ H_1: & \mu \neq 0 \end{cases}$$

Statistical Test

- Statistical test: use observations to find support for H_0 against H_1.
- Two elements: test statistic: $t(y_1, y_2, \ldots, y_n)$, and critical region C.
- If t falls inside the critical region, we reject H_0 in favor of H_1; otherwise we do not reject H_0.
  "do not reject" $\neq$ accept.

Example

- $y_i \sim N(\mu, \sigma^2)$, σ^2 is known.
- $H_0 : \mu = 0$ versus $H_1 : \mu \neq 0$
- Test statistic: sample mean m.
- Critical region $C = \{(-\infty, -c]$ and $[c, \infty)\}$, with c constant.
- Reject H_0 when $m \leq -c$ or $m \geq c$.

Test Result

	H_0 true	H_0 false
t in C, so reject H_0	Type I Error, False Positive	Correct Decision True Positive
t not in C, so do not reject H_0	Correct Decision True Negative	Type II Error False Negative

- $P(t \text{ in } C|H_0 \text{ true})$: size of the test.
- $P(t \text{ in } C|H_0 \text{ false})$: power of the test.
- Preference for small size and large power.
- Smaller C means lower size, but also lower power.

Example: size

Test

You observe a set of n observations $y_i \sim NID(\mu, \sigma^2)$, with σ^2 known. You test $H_0 : \mu = 0$ versus $H_1 : \mu \neq 0$ based on the sample mean m with critical region $C = \{(-\infty, -c] \text{ and } [c, \infty)\}$, and c constant. What is the size of the test?

Answer

- $m \sim N(\mu, \sigma^2/n)$, so $\dfrac{m-\mu}{\sigma/\sqrt{n}} \sim N(0,1)$ with cdf Φ.
- size, so H_0 is true, and $\mu = 0$.
- $P(m \text{ in } C|\mu = 0) = P(m \leq -c|\mu = 0) + P(m \geq c|\mu = 0) =$
$P\left(\dfrac{m}{\sigma/\sqrt{n}} \leq \dfrac{-c}{\sigma/\sqrt{n}}\right) + P\left(\dfrac{m}{\sigma/\sqrt{n}} \geq \dfrac{c}{\sigma/\sqrt{n}}\right) = 2\Phi\left(\dfrac{-c}{\sigma/\sqrt{n}}\right)$

Some remarks

- Common to use test statistics that follow a standard distribution. So, $\dfrac{m-\mu}{\sigma/\sqrt{n}}$ instead of m.
- $H_0 : \mu = 0$ versus $H_1 : \mu \neq 0$:
two-sided test: reject H_0 if $m < -c$ or $m > c$.

  $H_0 : \mu = 0$ versus $H_1 : \mu > 0$ or $H_0 : \mu \leq 0$ versus $H_1 : \mu > 0$:
one-sided test: reject H_0 if $m > c$.
- Commonly we fix the size of the test and determine c, for example, 1, 5 or 10%.

p-value

p-value: minimum size that leads to rejection of H_0.

Test

Suppose you test $H_0 : \mu = 0$ versus $H_1 : \mu \neq 0$ and find $m/(\sigma/\sqrt{n}) = 2.1$. If $c/(\sigma/\sqrt{n}) = 2.1$, size for a two-sided test equals $2\Phi(-2.1) = 0.036$. For which sizes do you reject?

Answer

Rejection for all sizes larger than 3.6%, so for 5% and 10%, but not for sizes smaller than 3.6%, so not for 1%.

t-statistic

In reality: σ^2 unknown

- Standardized mean $\dfrac{m-\mu}{\sigma/\sqrt{n}}$ cannot be calculated.
- Use estimator s^2: t-statistic $t = \dfrac{m-\mu}{s/\sqrt{n}}$.
- $t = \dfrac{\sqrt{n}(m-\mu)}{s} = \dfrac{\sqrt{n}(m-\mu)/\sigma}{\sqrt{\dfrac{(n-1)s^2}{\sigma^2}/(n-1)}} \sim t(n-1)$

  because $\sqrt{n}(m-\mu)/\sigma \sim N(0,1)$,
  $(n-1)s^2/\sigma^2 \sim \chi^2(n-1)$, and independent.

Unknown versus known variance

Setting

You observe a set of $n = 15$ observations $y_i \sim NID(\mu, \sigma^2)$, with σ^2 unknown. You test $H_0 : \mu = 0$ versus $H_1 : \mu \neq 0$. The t-statistic equals 2.1.

- t-statistic $\sim t(14)$.
- Use Ψ_ν as cdf of $t(\nu)$, p-value equals $2\Psi_{14}(-2.1) = 0.054$. Reject H_0 at 10% level, not at 5% level.
- Compare $2\Phi(-2.1) = 0.036$ for σ^2 known.
- Unknown variance leads to additional uncertainty, so higher p-values. For fixed size, we need larger critical values to reject H_0.

Tests for variances

$H_0 : \sigma^2 = \sigma_0^2$ versus $H_1 : \sigma^2 > \sigma_0^2$.

- Test statistic: $(n-1)s^2/\sigma_0^2 \sim \chi^2(n-1)$ (does not depend on μ!)
- Reject if test statistic exceeds critical value c.

Two independent samples $i = 1, 2$ with n_i observations $y_{ij} \sim NID(\mu_i, \sigma_i^2)$
$H_0 : \sigma_1^2 = \sigma_2^2$ versus $H_0 : \sigma_1^2 \neq \sigma_2^2$

- Estimators: $(n_i - 1)s_i^2/\sigma_i^2 \sim \chi^2(n_i - 1)$, independent.
- $\dfrac{s_2^2/\sigma_2^2}{s_1^2/\sigma_2^1} \sim F(n_2 - 1, n_1 - 1)$.
- Under H_0, $\sigma_1^2 = \sigma_2^2$ so test-statistic s_2^2/s_2^1.

The stock market example

26 observations, $m = 9.6\%$, $s = 17.9\%$

- $H_0 : \mu = 0$ versus $H_1 : \mu > 0$.
- $t = \sqrt{26} \cdot 9.6/17.9 = 2.75$.
- one-sided p-value $1 - \Psi_{25}(2.75) = 0.0054$.
- We reject H_0 in favor of H_1.

Test

Test $H_0 : \sigma = 0.25$ versus $H_1 : \sigma < 0.25$. The 5% critical value equals 14.61. What do you decide?

- Test statistic: $(n-1)s^2/\sigma^2 = 25 \cdot 0.179^2/0.25^2 = 12.74$.
- Critical region: $C = (0, 14.61]$, so reject H_0.
- p-value based on $\chi^2(25)$ is 0.021.

Training Exercise S.2

- Train yourself by making the training exercise (see the website).

- After making this exercise, check your answers by studying the webcast solution (also available on the website).