Erasmus School of Economics

MOOC Econometrics

Lecture 6.2 on Time Series: Representation

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Stationarity

- Time series: y_t, where $t = 1, \ldots, n$ is time index.
- y_t stationary if

  $\rightarrow$ mean $E(y_t) = \mu$ is fixed (same for all t)

  $\rightarrow$ autocovariance $E((y_t - \mu)(y_{t-k} - \mu)) = \gamma_k$ (same for all t)
- Special case: $\gamma_k = 0$ for all $k = 1, 2, \ldots$

  $\rightarrow$ WHITE NOISE
- Recall Assumption A5 (Lectures 1 & 2): $E(\varepsilon_i \varepsilon_j) = 0$ for all $i \neq j$.
- White noise cannot be predicted from own past (by linear models).

  $\rightarrow$ Purpose: Time series model such that residuals are white noise.

Autoregressive model

- Notation for white noise (uncorrelated series) with mean zero: ε_t
- AR(1): $y_t = \alpha + \beta y_{t-1} + \varepsilon_t$
- Stationary if $-1 < \beta < 1$

  $y_t = \alpha + \beta y_{t-1} + \varepsilon_t = \alpha + \beta(\alpha + \beta y_{t-2} + \varepsilon_{t-1}) + \varepsilon_t$

  $= \alpha(1 + \beta) + \varepsilon_t + \beta\varepsilon_{t-1} + \beta^2 y_{t-2} = \ldots$

  $= \alpha \sum_{j=0}^{t-2} \beta^j + \sum_{j=0}^{t-2} \beta^j \varepsilon_{t-j} + \beta^{t-1} y_1$

  For $t \to \infty$ we get $\beta^{t-1} y_1 \to 0$ and $y_t = \alpha/(1-\beta) + \sum_{j=0}^{\infty} \beta^j \varepsilon_{t-j}$
- AR(2): $y_t = \alpha + \beta_1 y_{t-1} + \beta_2 y_{t-2} + \varepsilon_t$
- AR(p): $y_t = \alpha + \beta_1 y_{t-1} + \beta_2 y_{t-2} + \ldots + \beta_p y_{t-p} + \varepsilon_t$

Test question

- AR(1) $y_t = \alpha + \beta y_{t-1} + \varepsilon_t$,

  ε_t uncorrelated with y_{t-k} for all $k = 1, 2, \ldots$.

Test

If $\beta = 1$, then argue why y_t can not be stationary.

Answer:

- If $\alpha \neq 0$, then y_t can not have fixed mean:

  $E(\varepsilon_t) = 0$, so $\mu = E(y_t) = \alpha + E(y_{t-1}) + 0 = \alpha + \mu \neq \mu$
- And if $\alpha = 0$ then y_t can not have fixed variance:

  $y_t = y_{t-1} + \varepsilon_t$, so $(y_t - \mu) = (y_{t-1} - \mu) + \varepsilon_t$ (uncorrelated)

  $E((y_t - \mu)^2) = E((y_{t-1} - \mu)^2) + E(\varepsilon_t^2) > E((y_{t-1} - \mu)^2)$.

Moving average

- MA(1): $y_t = \alpha + \varepsilon_t + \gamma\varepsilon_{t-1}$
- As ε_t is uncorrelated with its own past and future, y_t is correlated with y_{t-1} but not with y_{t-k} for $k = 2, 3, \ldots$.
- MA(q): $y_t = \alpha + \varepsilon_t + \gamma_1\varepsilon_{t-1} + \ldots + \gamma_q\varepsilon_{t-q}$
- ARMA(1,1) : $y_t = \alpha + \beta y_{t-1} + \varepsilon_t + \gamma\varepsilon_{t-1}$
- ARMA(p,q): $y_t = \alpha + \beta_1 y_{t-1} + \ldots + \beta_p y_{t-p} + \varepsilon_t + \gamma_1\varepsilon_{t-1} + \ldots + \gamma_q\varepsilon_{t-q}$

Two autoregressive equations

- If two autoregressive processes are related, the univariate process becomes ARMA.

Test

Let $\varepsilon_{x,t}$ and $\varepsilon_{y,t}$ be two mutually independent white noise processes, and let $y_t = \gamma x_t + \varepsilon_{y,t}$ and $x_t = \delta x_{t-1} + \varepsilon_{x,t}$. Derive the orders p and q for the ARMA model for y_t (that does not include x_t).
Hint: Eliminate x_t by considering $y_t - \delta y_{t-1}$.

Answer:

- $y_t - \delta y_{t-1} = \gamma(x_t - \delta x_{t-1}) + \varepsilon_{y,t} - \delta\varepsilon_{y,t-1}$
  $y_t = \delta y_{t-1} + \gamma\varepsilon_{x,t} + \varepsilon_{y,t} - \delta\varepsilon_{y,t-1}$
- AR-order $p = 1$, and error $\omega_t = \gamma\varepsilon_{x,t} + \varepsilon_{y,t} - \delta\varepsilon_{y,t-1}$ is MA(1): $E(\omega_t\omega_{t-1}) = -\delta \text{var}(\varepsilon_{y,t-1})$, $E(\omega_t\omega_{t-2}) = E(\omega_t\omega_{t-3}) = \ldots = 0$

(Partial) Autocorrelation Function - (P)ACF

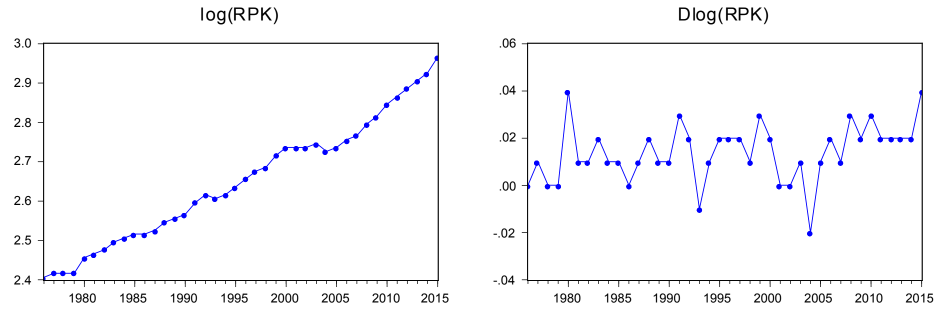
- k-th order sample autocorrelation coefficient:

  $\text{ACF}_k = \text{cor}(y_t, y_{t-k}) = \frac{\sum_{t=k+1}^{n}(y_t-\bar{y})(y_{t-k}-\bar{y})}{\sum_{t=k+1}^{n}(y_t-\bar{y})^2}$
- If y_t is MA(q), then $\text{ACF}_k \approx 0$ for all $k > q$.
- k-th order sample partial autocorrelation coefficient:

  PACF_k is the OLS coefficient b_k in regression model

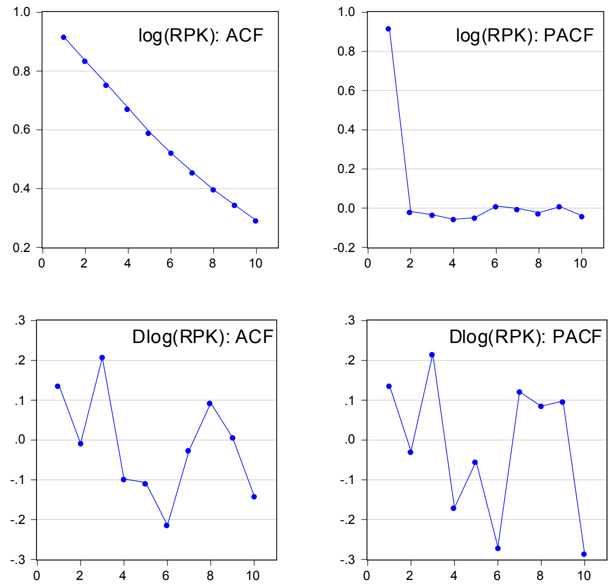
  $y_t = \alpha + \beta_1 y_{t-1} + \ldots + \beta_{k-1} y_{t-k+1} + \beta_k y_{t-k} + \varepsilon_t$
- If y_t is AR(p), then $\text{PACF}_k \approx 0$ for all $k > p$.
- 5% critical value: not significant if $-2/\sqrt{n} < \text{(P)ACF} < 2/\sqrt{n}$

Example: RPK of airline - time series



- log(RPK) is not stationary
- first difference of log(RPK) (yearly growth rate) is stationary

Example: RPK of airline - ACF and PACF



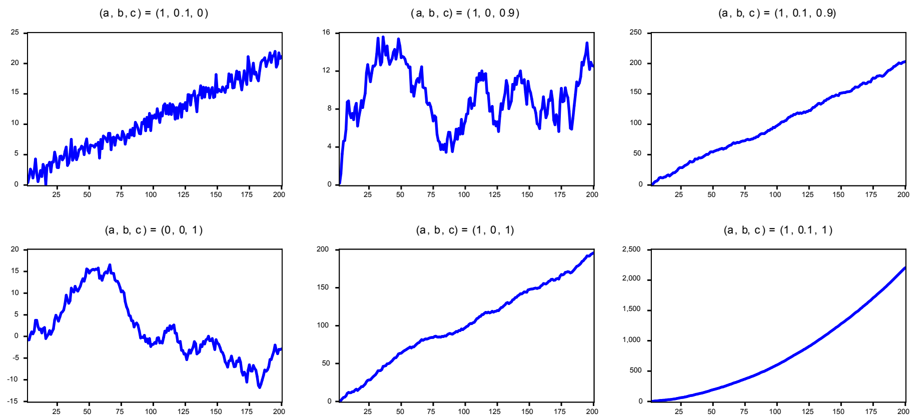
Trends: stochastic and deterministic

- $y_t = y_{t-1} + \varepsilon_t$: random walk, stochastic trend, no clear direction
- $y_t = \alpha + y_{t-1} + \varepsilon_t$ $(\alpha \neq 0)$: stochastic trend
- $y_t = \alpha + \beta t + y_{t-1} + \varepsilon_t$ $(\beta \neq 0)$: stochastic (explosive) trend
- $y_t = \alpha + \beta t + \varepsilon_t$ $(\beta \neq 0)$: deterministic trend
- $y_t = \alpha + \beta t + \gamma y_{t-1} + \varepsilon_t$ $(\beta \neq 0, |\gamma| < 1)$: deterministic trend
- Stochastic trend can be removed by taking first difference:

  Example: $y_t = \alpha + y_{t-1} + \varepsilon_t$, then $\Delta y_t = y_t - y_{t-1} = \alpha + \varepsilon_t$

Examples of deterministic and stochastic trends

- DGP: $y_t = a + bt + cy_{t-1} + \varepsilon_t$
- Stochastic trend: $c = 1$ (bottom row)



Cointegration

- Sometimes: x_t and y_t each have stochastic trend,

  but $y_t - cx_t$ is stationary for some value of c.
- Cointegration (common stochastic trend)

Test

Suppose that $z_t = z_{t-1} + \varepsilon_{z,t}$ is unobserved, whereas $x_t = \alpha_1 + \gamma_1 z_t + \varepsilon_{x,t}$ and $y_t = \alpha_2 + \gamma_2 z_t + \varepsilon_{y,t}$ are observed, where $\varepsilon_{z,t}, \varepsilon_{x,t}, \varepsilon_{y,t}$ are white noise processes. Show that x_t and y_t are cointegrated, and find the value of c for which $y_t - cx_t$ is stationary.

Answer:

- $\gamma_1 y_t - \gamma_2 x_t = (\gamma_1 \alpha_2 - \gamma_2 \alpha_1) + (\gamma_1 \varepsilon_{y,t} - \gamma_2 \varepsilon_{x,t})$,

  where $\varepsilon_t = \gamma_1 \varepsilon_{y,t} - \gamma_2 \varepsilon_{x,t}$ is white noise $\rightarrow$ stationary

  $\gamma_1 y_t - \gamma_2 x_t = \gamma_1 (y_t - \gamma_2 / \gamma_1 x_t)$, so $c = \gamma_2 / \gamma_1$.

TRAINING EXERCISE 6.2

- Train yourself by making the training exercise (see the website).
- After making this exercise, check your answers by studying the webcast solution (also available on the website).