THE UNIVERSITY OF TEXAS AT AUSTIN
McCombs School of Business

STA 372.5 **Tom Shively**

AUTOREGRESSIVE CONDITIONALLY HETEROSCEDASTICITY (ARCH) MODELS PART 1

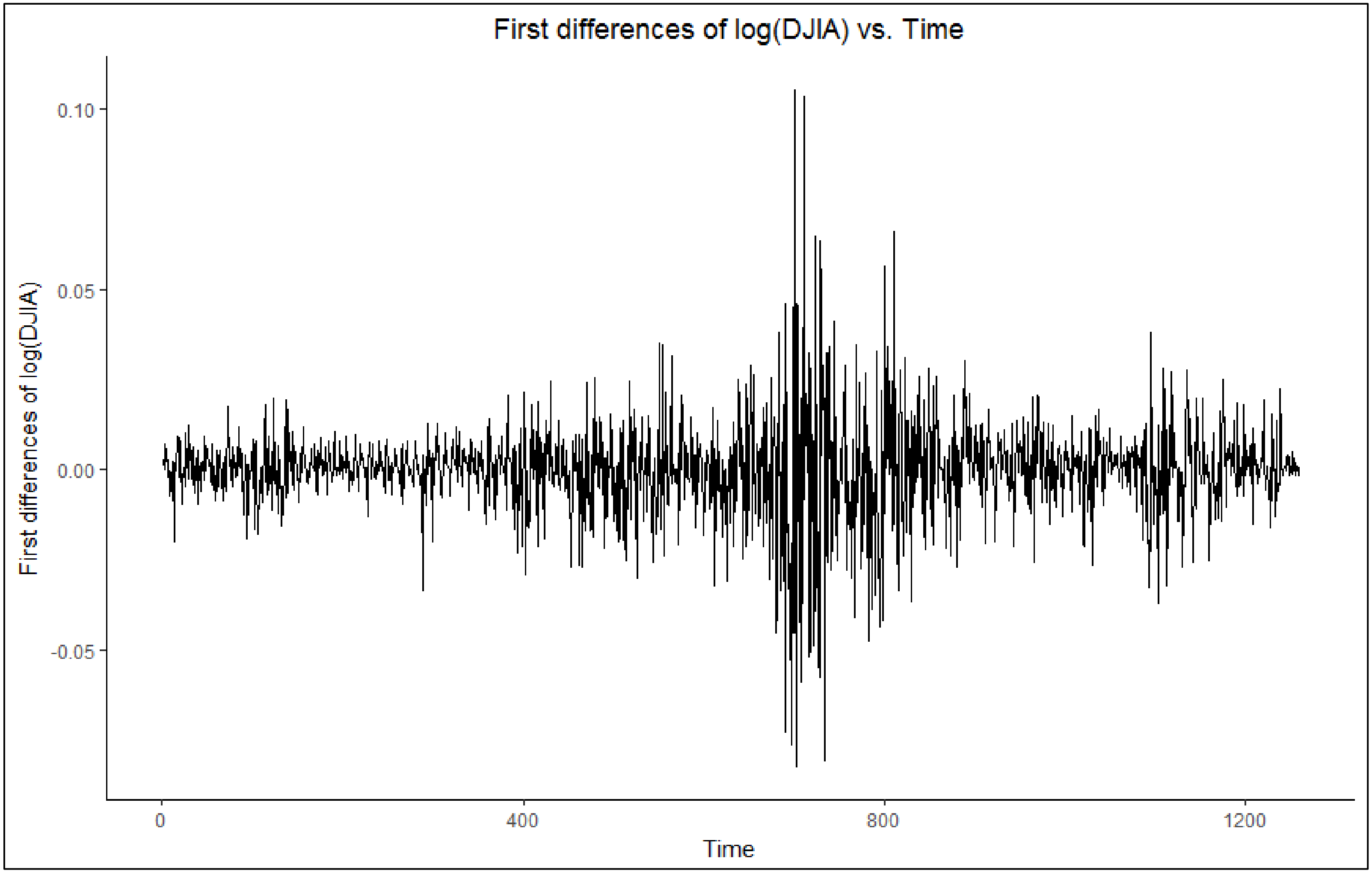
Example #1

Daily closing value of the Dow Jones Industrial Average:
January 3, 2006 – December 31, 2010
Source: https://research.stlouisfed.org/fred2/series/DJIA/downloaddata

```
-----------------------------------------------------------------------------------------------
#
#   Read and print DJIA data
#
data_table <- read.table("DJIA_2006-01-02_through_2010-12-31.dat", header=FALSE)
colnames(data_table) <- c("date", "time", "DJIA")
data_table$log_DJIA <- log(data_table$DJIA)
data_table$diff_log_DJIA[2:1259] <- data_table$log_DJIA[2:1259] - data_table$log_DJIA[1:1258]
data_table$diff_log_DJIA[1] <- NA
head(data_table)
tail(data_table)
-----------------------------------------------------------------------------------------------
```

```
        date time    DJIA log_DJIA diff_log_DJIA
1 2006-01-03    1 10847.4 9.291681            NA
2 2006-01-04    2 10880.2 9.294700  3.019204e-03
3 2006-01-05    3 10882.2 9.294884  1.838033e-04
4 2006-01-06    4 10959.3 9.301944  7.059984e-03
5 2006-01-09    5 11011.9 9.306732  4.788095e-03
6 2006-01-10    6 11011.6 9.306705 -2.724363e-05

           date time    DJIA log_DJIA diff_log_DJIA
1254 2010-12-23 1254 11573.5 9.356473  0.0012103922
1255 2010-12-27 1255 11555.0 9.354874 -0.0015997582
1256 2010-12-28 1256 11575.5 9.356646  0.0017725519
1257 2010-12-29 1257 11585.4 9.357501  0.0008548891
1258 2010-12-30 1258 11569.7 9.356145 -0.0013560730
1259 2010-12-31 1259 11577.5 9.356819  0.0006739476
```

```
-------------------------------------------------------------------------------------------------------
#
#   Construct plot of diff_log_DJIA vs. time
#
figure <- ggplot()
figure <- figure + geom_line(aes(x=data_table$time, y=data_table$diff_log_DJIA))
figure <- figure + ggtitle("First differences of log(DJIA) vs. Time")
figure <- figure + xlab("Time") + ylab("First differences of log(DJIA)")
figure <- figure + theme(plot.title = element_text(hjust = 0.5))
figure <- figure + theme(panel.grid.major = element_blank(), panel.grid.minor = element_blank(),
                         panel.background = element_blank(), axis.line = element_line(colour = "black"))
figure
-------------------------------------------------------------------------------------------------------
```



Example #2

Monthly return of Intel stock for January, 1973 – December, 2008 (432 months)
Source: Tsay, R. (2010). *Analysis of Financial Time Series*, third edition, John Wiley and Sons, Inc., New Jersey.

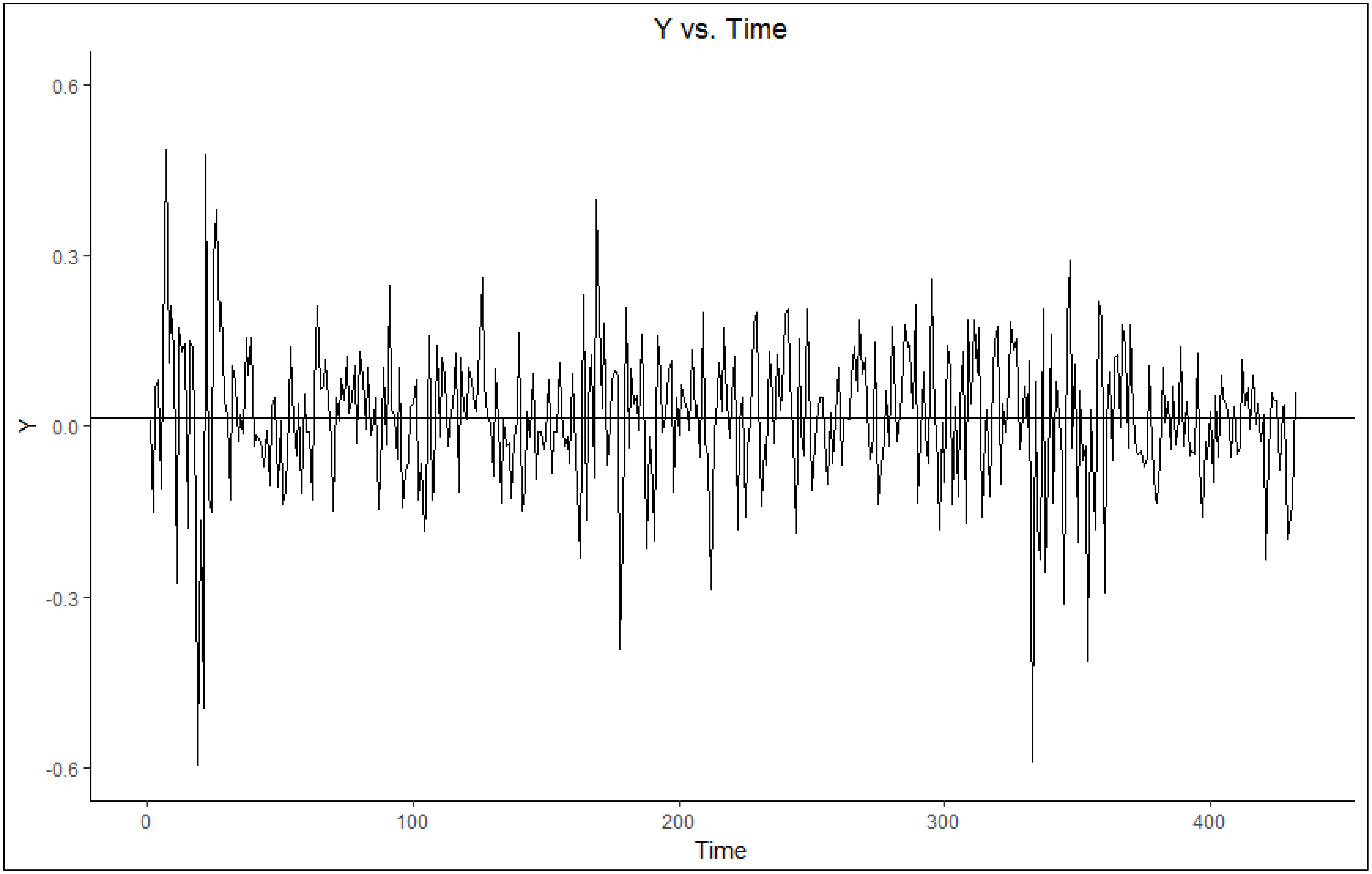
Let Y_t represent the continuously compounded return on Intel stock in month t.

```
-----------------------------------------------------------------------------------
#
#   Read and print Intel return data
#
data_table <- read.table("IntelReturns_Monthly_1973-2008_NoHeader.dat", header=FALSE)
colnames(data_table) <- c("time", "date", "Y")
head(data_table)
tail(data_table)
-----------------------------------------------------------------------------------
```

```
  time     date            Y
1    1 19730131  0.009999835
2    2 19730228 -0.150012753
3    3 19730330  0.067064079
4    4 19730430  0.082948635
5    5 19730531 -0.110348491
6    6 19730629  0.125162849

    time     date           Y
427  427 20080731  0.03251946
428  428 20080829  0.03628757
429  429 20080930 -0.19969928
430  430 20081031 -0.15560173
431  431 20081128 -0.13976219
432  432 20081231  0.06045425
```

```
---------------------------------------------------------------------------
#
#   Compute mean and standard deviation of Y
#
mean_Y <- mean(data_table$Y)
stdev_Y <- sd(data_table$Y)
mean_Y
pstdev_Y
---------------------------------------------------------------------------
```

```
[1] 0.0138819

[1] 0.1280031
```


Y vs. Time
0.6
0.3
0.0
-0.3
-0.6
Y
0
100
200
300
400
Time

Useful information

If η_t iid $N(\mu, \sigma^2)$ then for a constant a

$$E(a\eta_t) = aE(\eta_t)$$

$$= a\mu,$$

$$Var(a\eta_t) = a^2 Var(\eta_t)$$

$$= a^2\sigma^2$$

$$= (a\sigma)^2,$$

and

$$a\eta_t \text{ iid } N(a\mu, (a\sigma)^2).$$

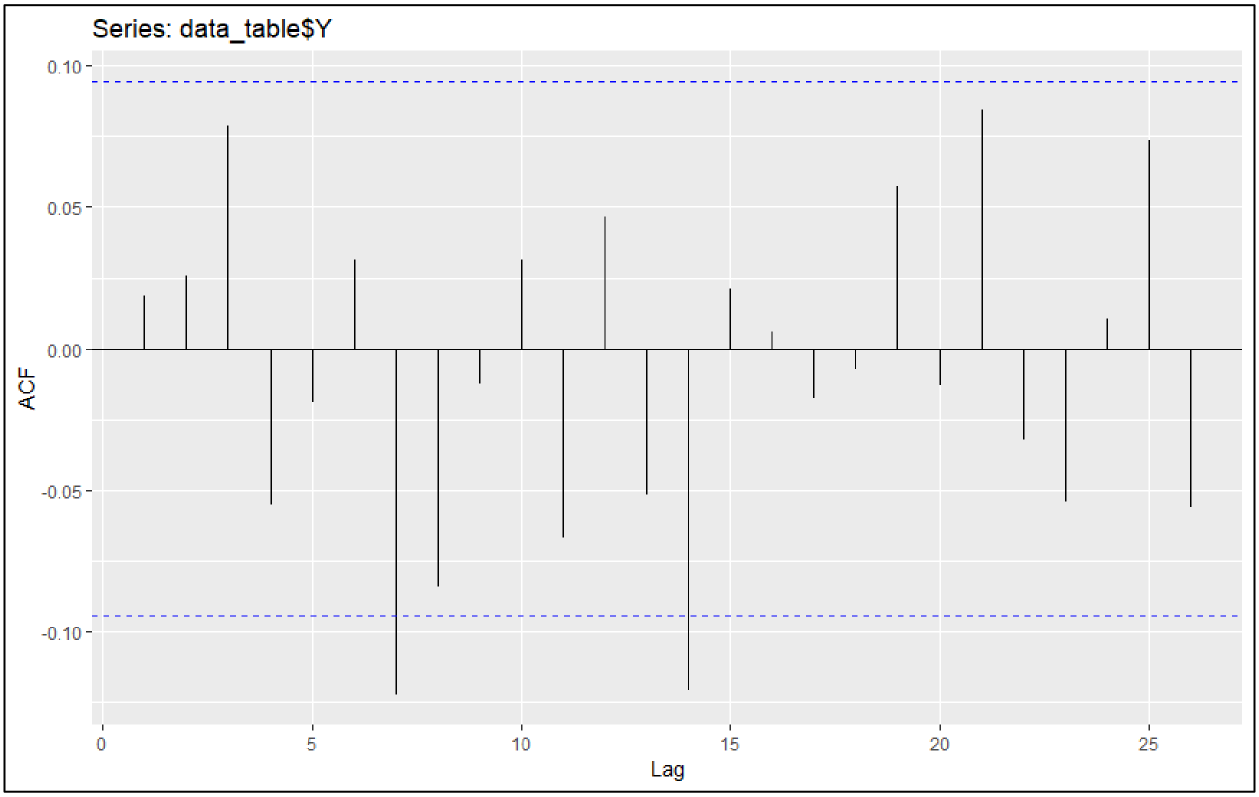
Recall that for a random variable Y

$$E(Y) = \mu$$

and

$$Var(Y) = E(Y - \mu)^2.$$

```
--------------------------------------------------------------------------
#
#   Compute ACF of Y
#
ggAcf(data_table$Y)
--------------------------------------------------------------------------
```

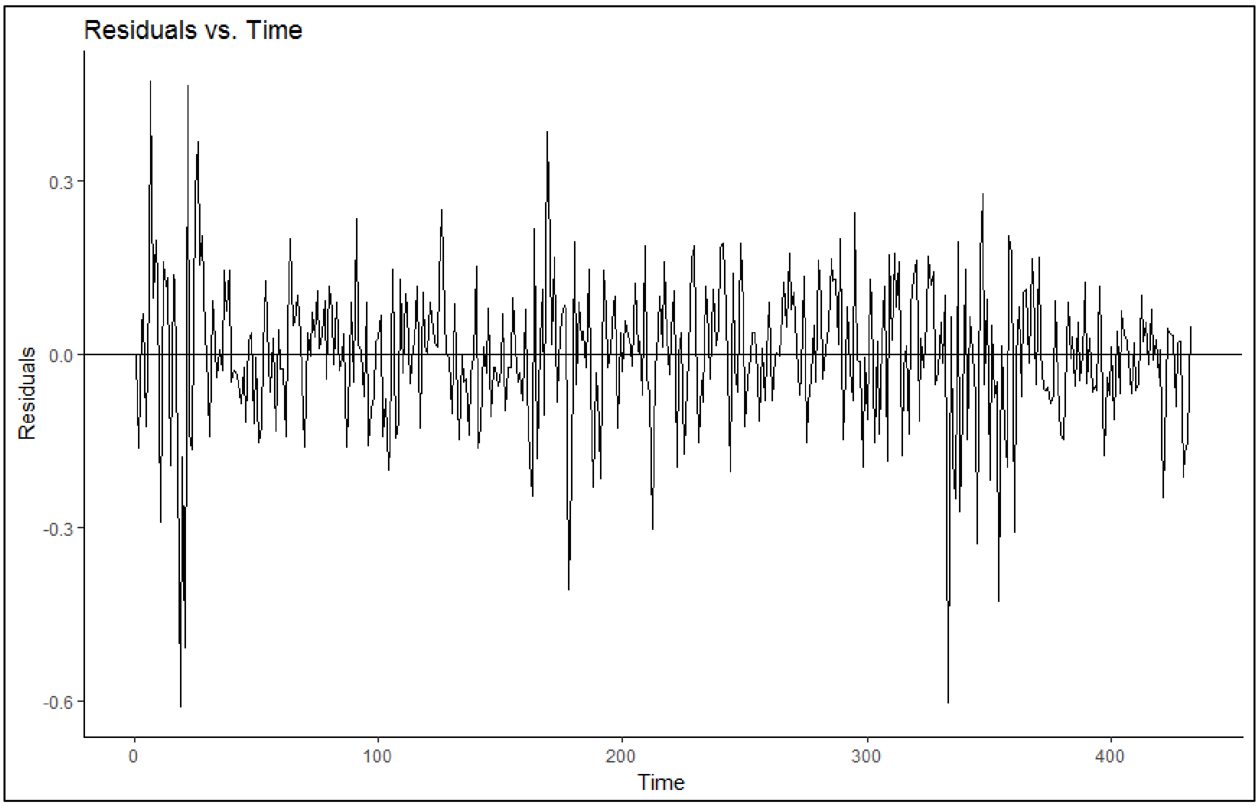



```
--------------------------------------------------------------------------
#
#   Estimate parameters in a ARIMA(0,0,0) model for Y(t)
#
y_time_series <- ts(data_table$Y)
result <- Arima(y_time_series, order=c(0, 0, 0), include.constant=TRUE)
result
--------------------------------------------------------------------------
```

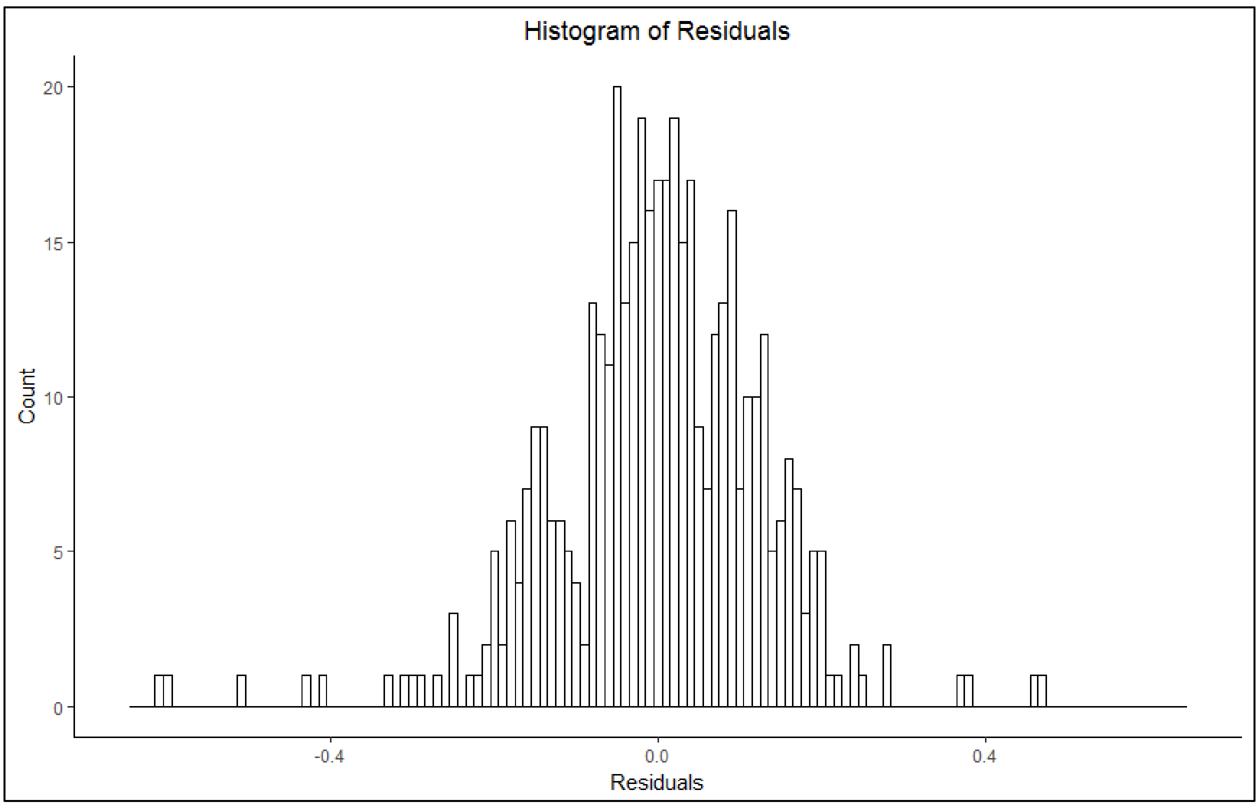
```
Series: y_time_series
ARIMA(0,0,0) with non-zero mean

Coefficients:
        mean
      0.0139
s.e.  0.0062

sigma^2 estimated as 0.01638:  log likelihood=275.58
AIC=-547.16   AICc=-547.14   BIC=-539.03
```

Estimate of σ is $\sqrt{0.01638} = 0.12798$

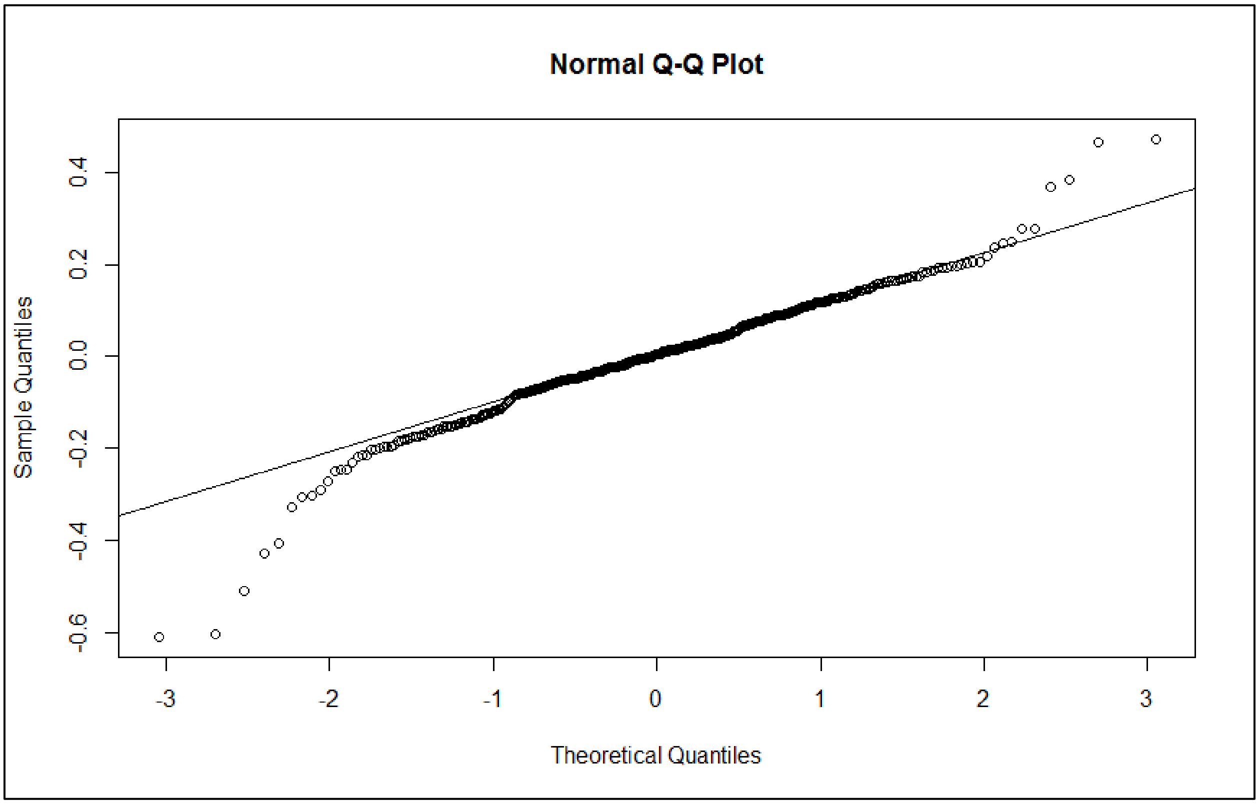
```
--------------------------------------------------------------------------------
#
#   Plot residuals
#
figure = ggplot()
figure <- figure + geom_line(aes(data_table$time, result$residuals))
figure <- figure + geom_hline(yintercept=0)
figure <- figure + scale_y_continuous()
figure <- figure + ggtitle("Residuals vs. Time") + xlab("Time") + ylab("Residuals")
figure <- figure + theme(panel.grid.major = element_blank(),
                         panel.grid.minor = element_blank(),
                         panel.background = element_blank())
figure <- figure + theme(axis.line.x = element_line(color="black", size = 0.5),
                         axis.line.y = element_line(color="black", size = 0.5))
figure
--------------------------------------------------------------------------------
```



```
-----------------------------------------------------------------------------
#
#   Construct histogram of residuals
#
figure <- ggplot()
figure <- figure + geom_histogram(aes(x=result$residuals), binwidth=0.01,
                                  color="black", fill="white")
figure <- figure + ggtitle("Histogram of Residuals")
figure <- figure + xlab("Residuals") + ylab("Count")
figure <- figure + xlim(-0.65, 0.65)
figure <- figure + theme(plot.title = element_text(hjust = 0.5))
figure <- figure + theme(panel.grid.major = element_blank(),
                         panel.grid.minor = element_blank(),
                         panel.background = element_blank(),
                         axis.line = element_line(colour = "black"))
figure
-----------------------------------------------------------------------------
```



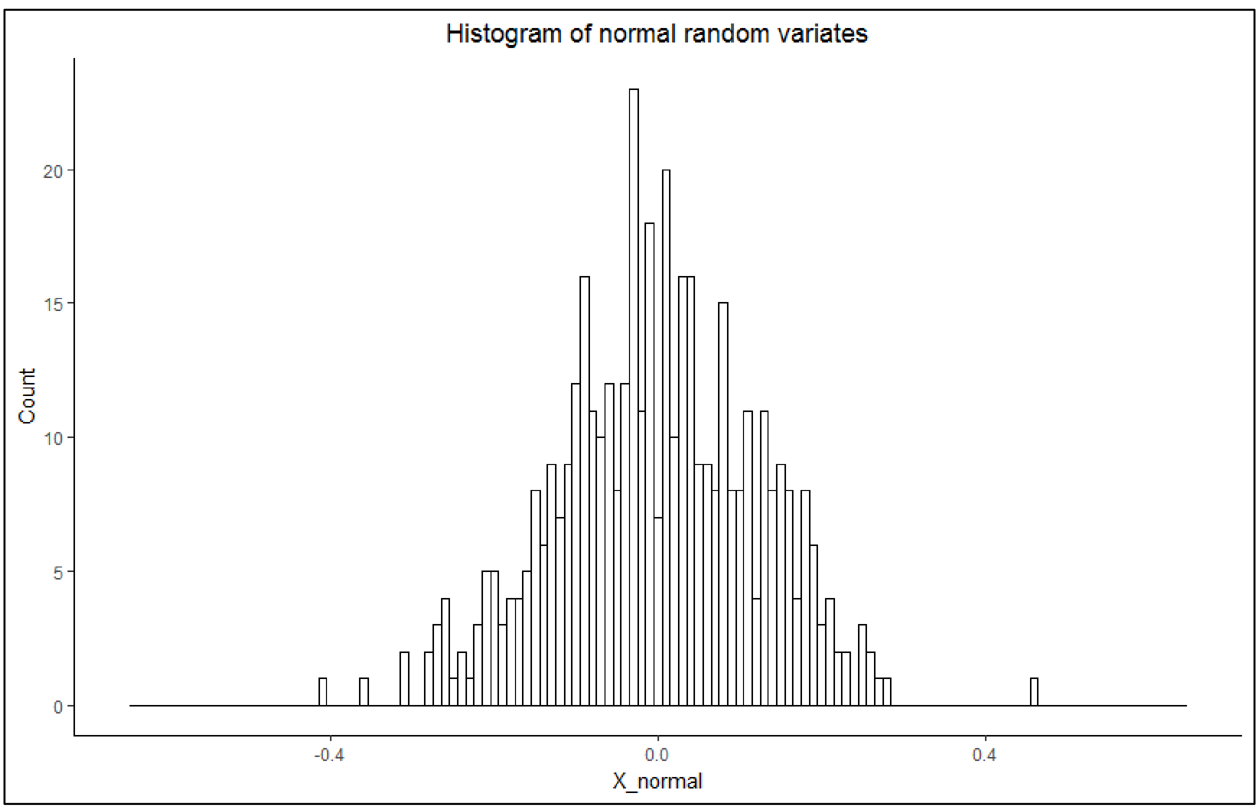
```
------------------------------------------------------------------------------
#
#   Construct a QQ-plot of residuals to check the normality assumption
#
qqnorm(result$residuals, datax=FALSE)
qqline(result$residuals)
------------------------------------------------------------------------------
```



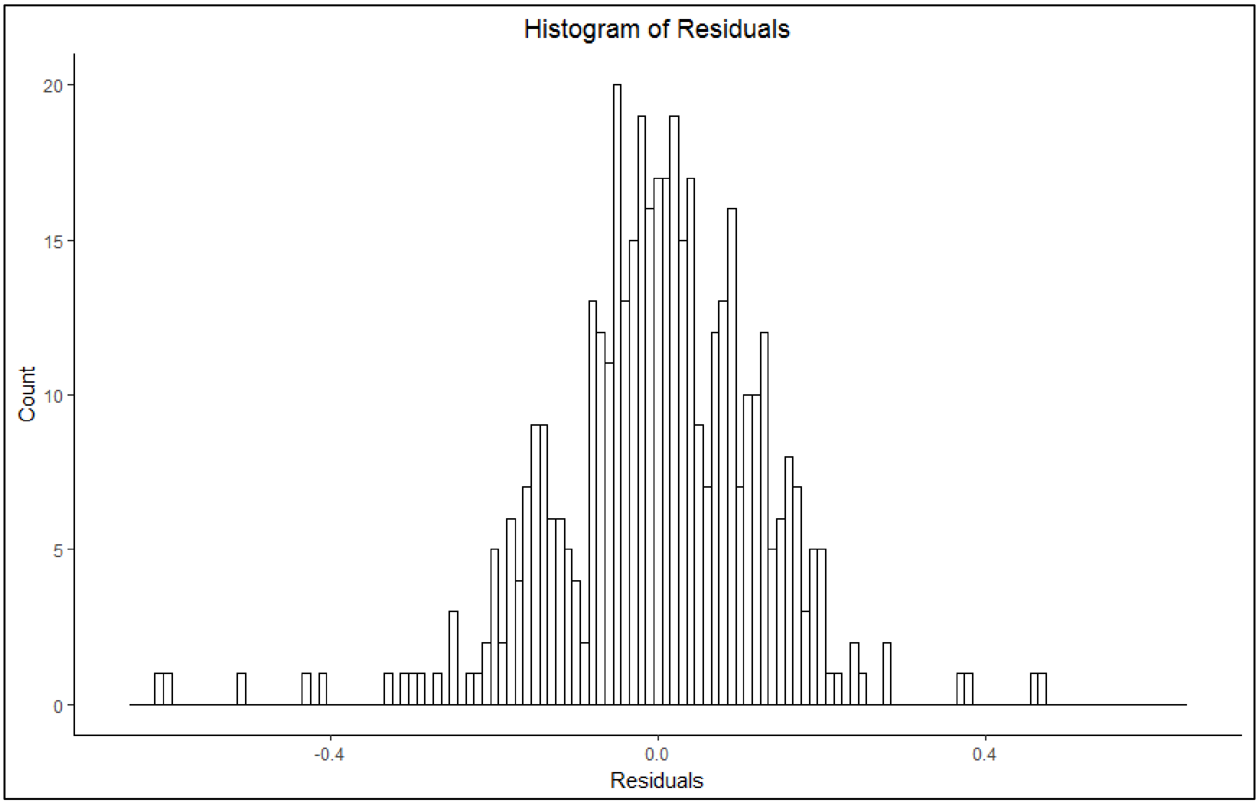

```
------------------------------------------------------------------------------
#
#   Run Anderson-Darling test on residuals to test the normality assumption
#
ad.test(result$residuals)
------------------------------------------------------------------------------
```

```
        Anderson-Darling normality test

data:  result$residuals
A = 2.3446, p-value = 6.069e-06
```

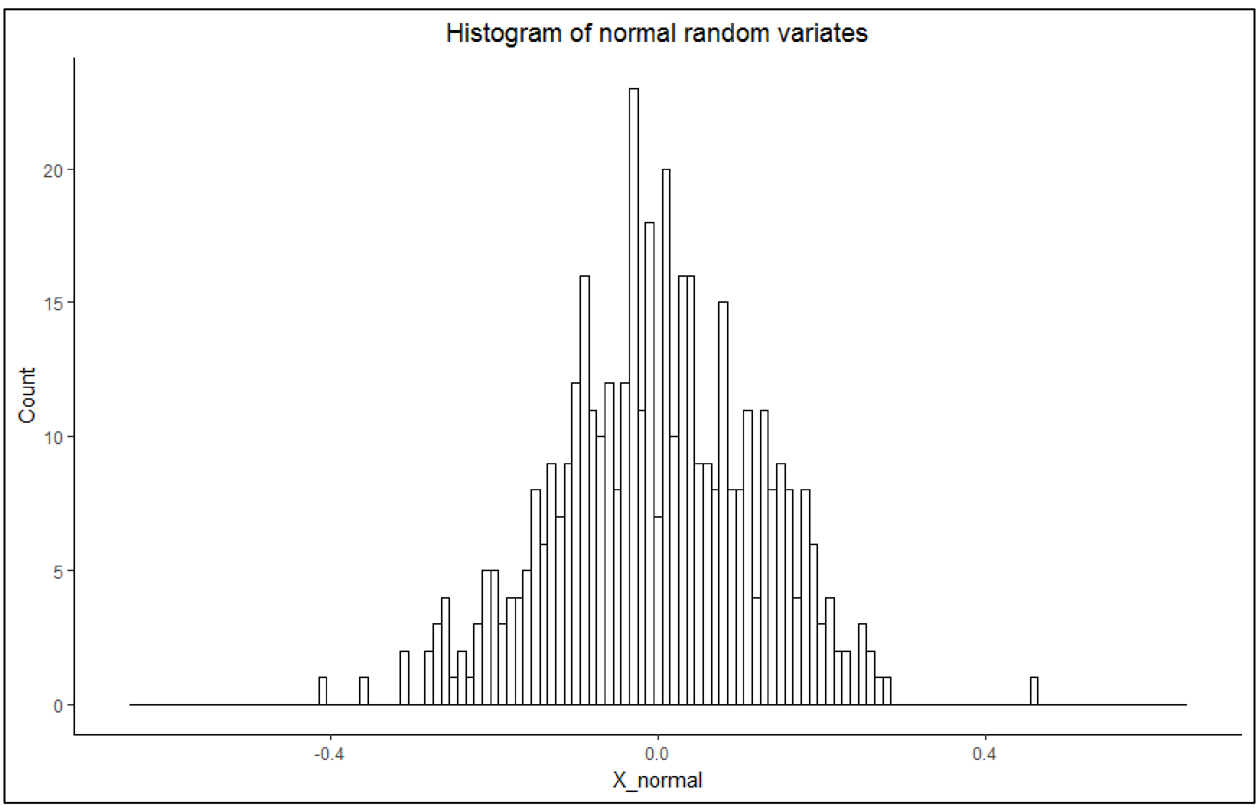
```
------------------------------------------------------------------------------
#
#   Generate nobs=432 normal random variates with mean = 0 and
#     standard deviation = standard errors of the residuals
#
nobs <- nrow(data_table)
set.seed(134769)
data_table$x_normal <- rnorm(nobs, mean=0, sd=sqrt(result$sigma2))
#
#   Construct histogram of x_normal
#
figure <- ggplot()
figure <- figure + geom_histogram(aes(x= data_table x_normal), binwidth=0.01,
                                  color="black", fill="white")
figure <- figure + ggtitle("Histogram of normal random variates")
figure <- figure + xlab("X_normal") + ylab("Count")
figure <- figure + xlim(-0.65, 0.65)
figure <- figure + theme(plot.title = element_text(hjust = 0.5))
figure <- figure + theme(panel.grid.major = element_blank(),
                         panel.grid.minor = element_blank(),
                         panel.background = element_blank(),
                         axis.line = element_line(colour = "black"))
figure
------------------------------------------------------------------------------
```



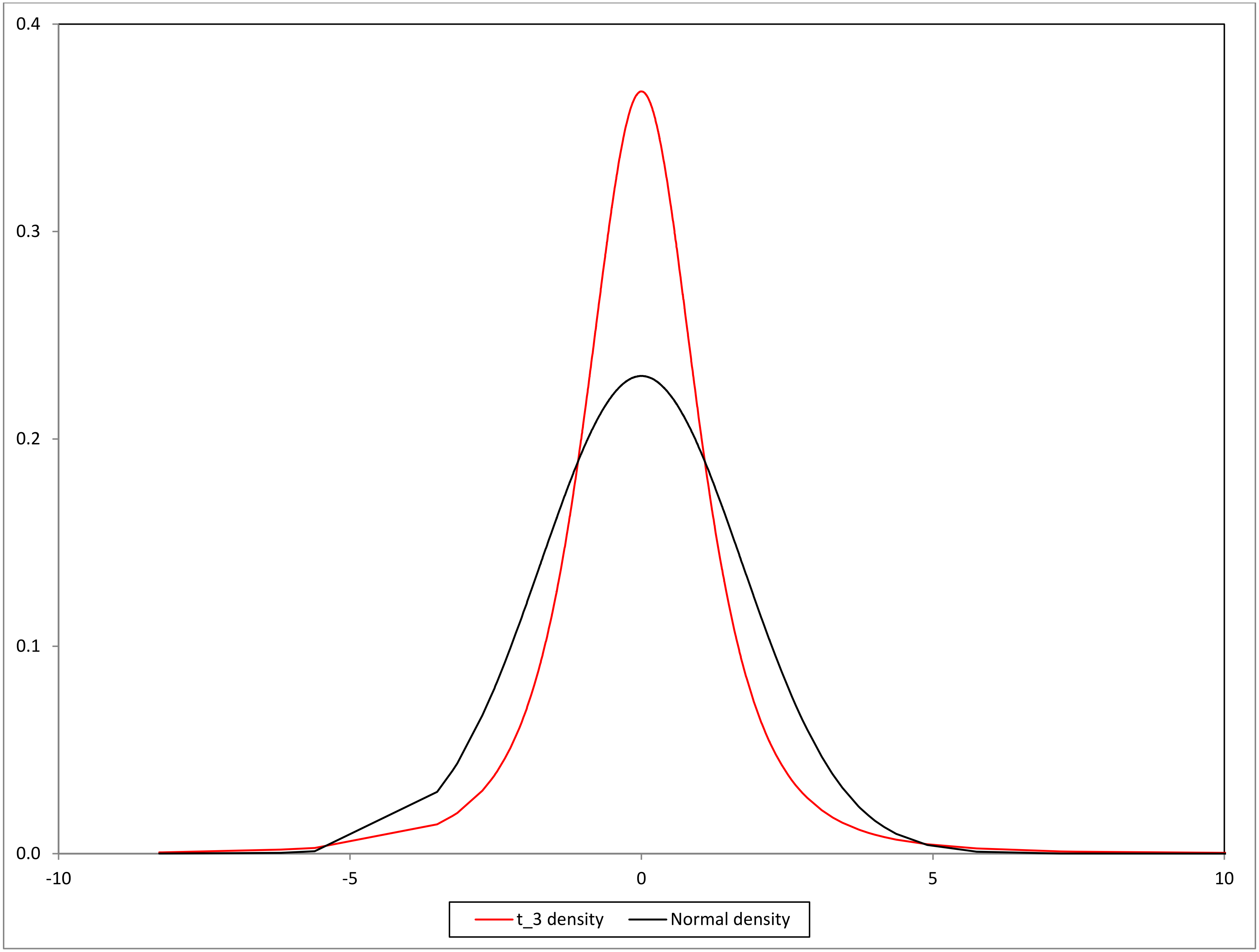
Histogram of residuals (same histogram as before)



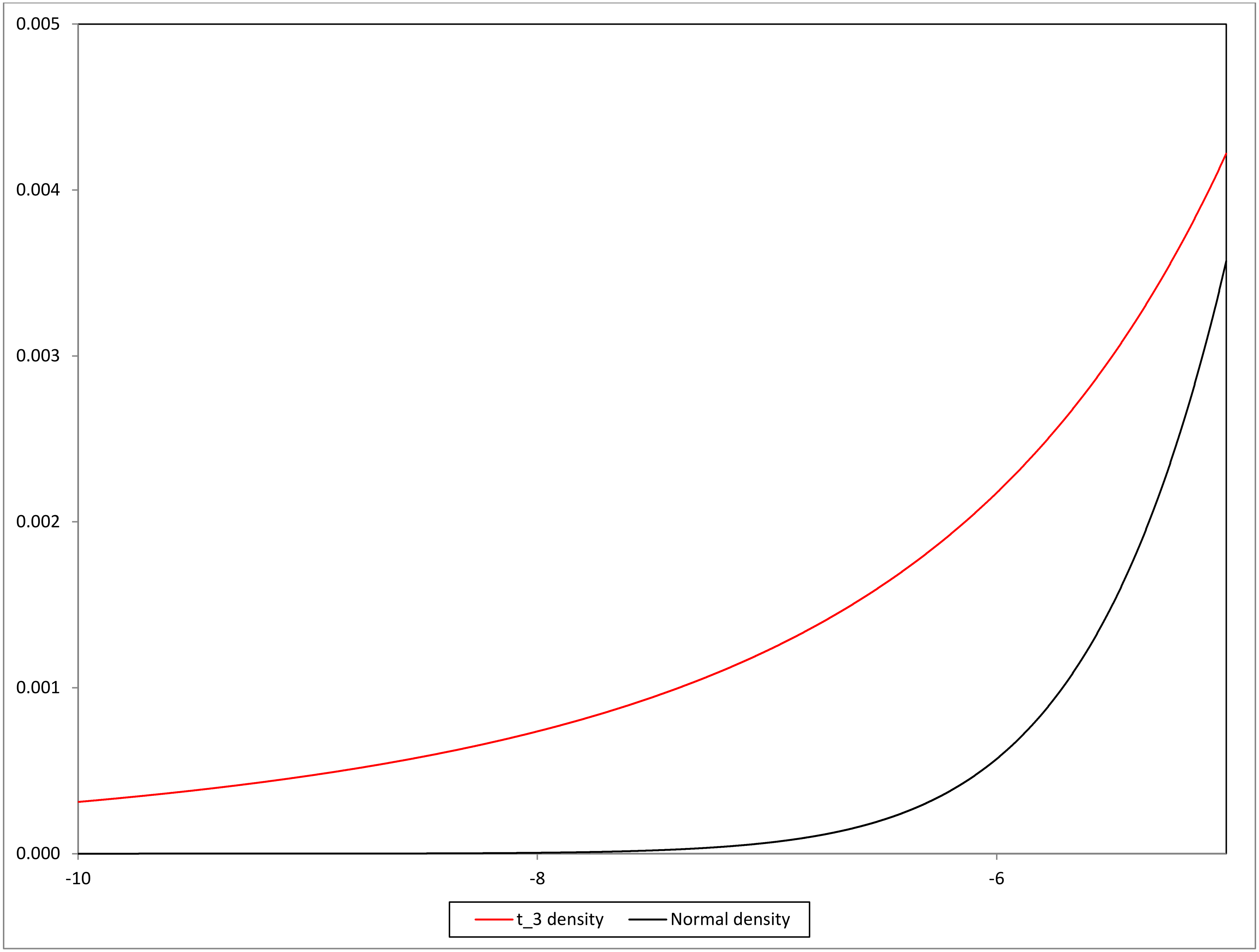
Histogram of normal random variable with mean zero and same standard deviation as for residuals (same histogram as before)



t_3 distribution and a normal distribution with the same mean and variance



Tail of a t_3 distribution and a normal distribution with the same mean and variance



ARCH(1) model for the errors ε_t

An ARCH model for the errors ε_t in a time series model is:

$$Y_t = \mu + \varepsilon_t$$

$$\varepsilon_t = \sigma_t \eta_t$$

where

$$\eta_t \text{ iid } N(0, 1)$$

and

$$\sigma_t^2 = \gamma_0 + \gamma_1 \varepsilon_{t-1}^2 .$$

This means the conditional variance of ε_t given ε_{t-1} is $\sigma_t^2 = \gamma_0 + \gamma_1 \varepsilon_{t-1}^2$, and the conditional distribution of $\varepsilon_t \mid \varepsilon_{t-1}$ is

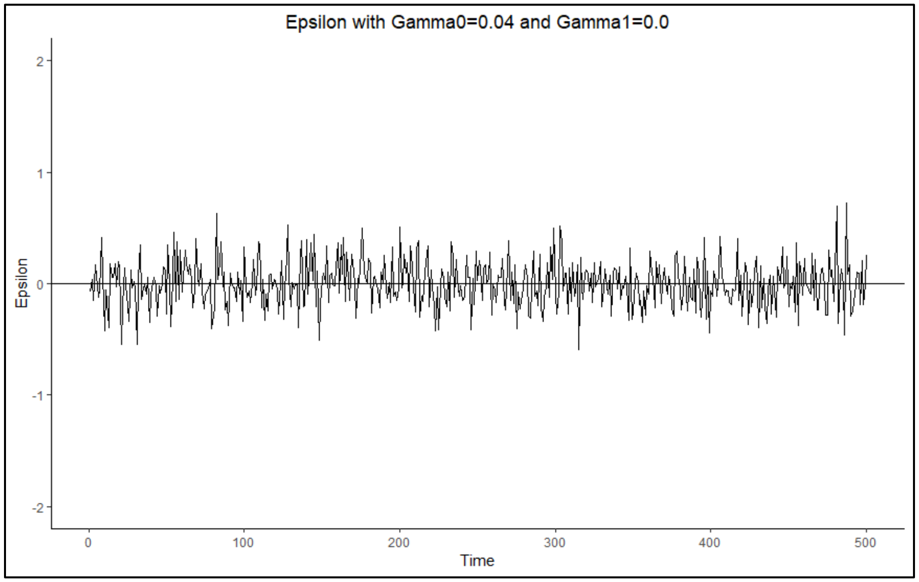
$$\varepsilon_t \mid \varepsilon_{t-1} \text{ iid } N(0, \sigma_t^2 = \gamma_0 + \gamma_1 \varepsilon_{t-1}^2),$$

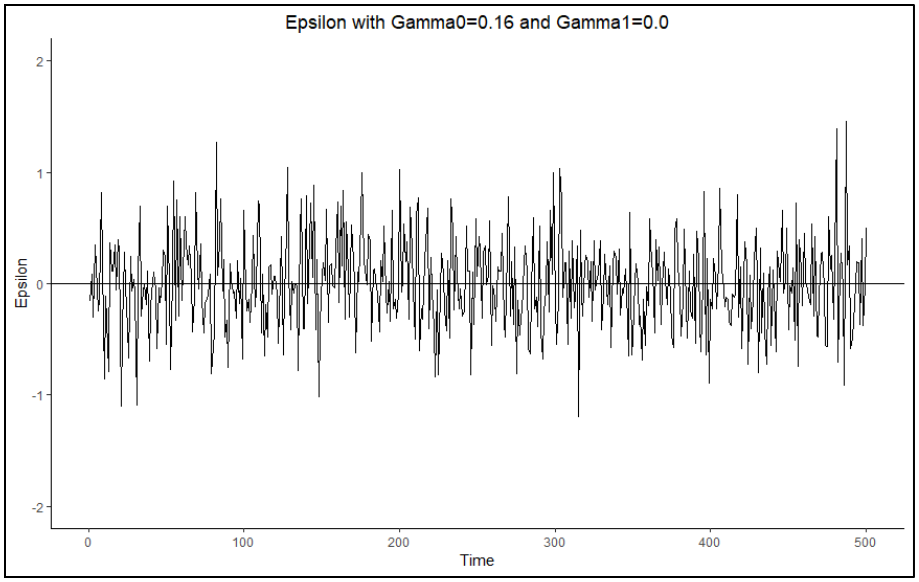
i.e. $\varepsilon_t \mid \varepsilon_{t-1}$ for $t = 2, \ldots, n$ are conditionally independent.

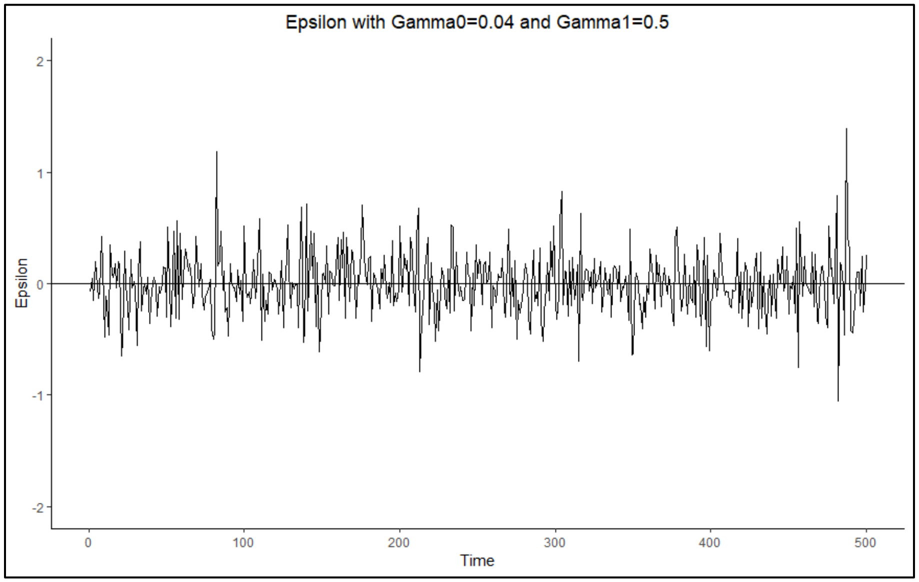
The random variables ε_t and ε_{t-1} are not independent (i.e. there is a relationship between ε_t and ε_{t-1}, although not a linear relationship).

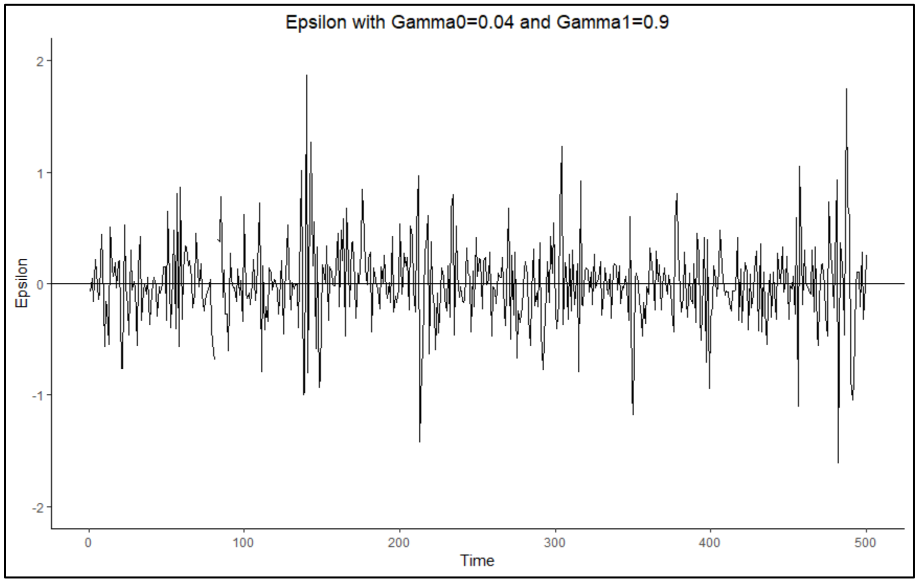
ARCH is an acronym for **A**uto**R**egressive **C**onditionally **H**eteroscedastic

Time series plots of epsilon values simulated from an ARCH(1) process with different γ_0 and γ_1 values

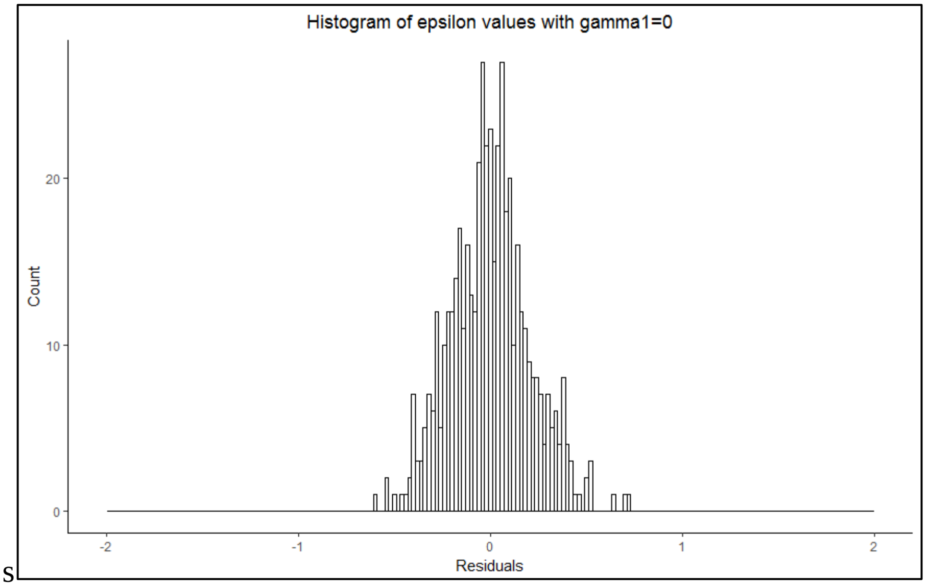




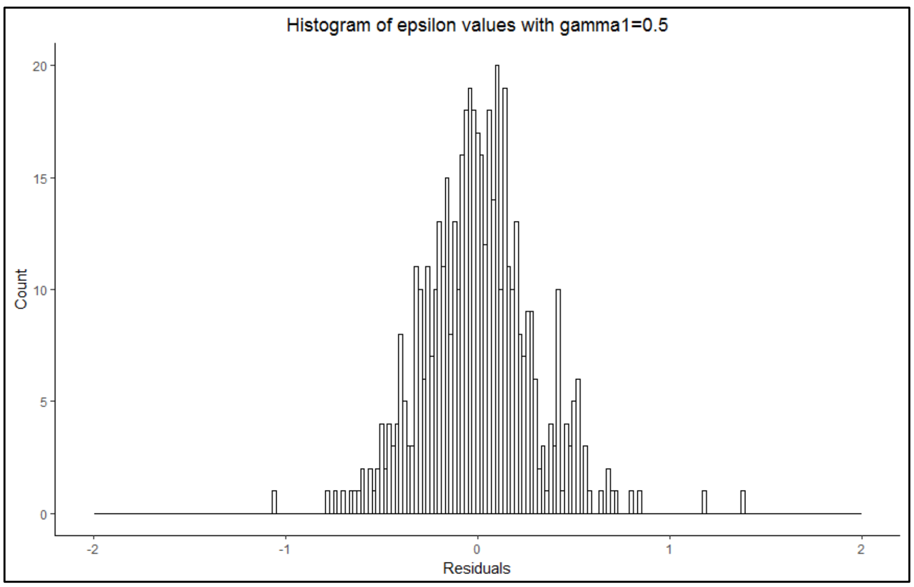


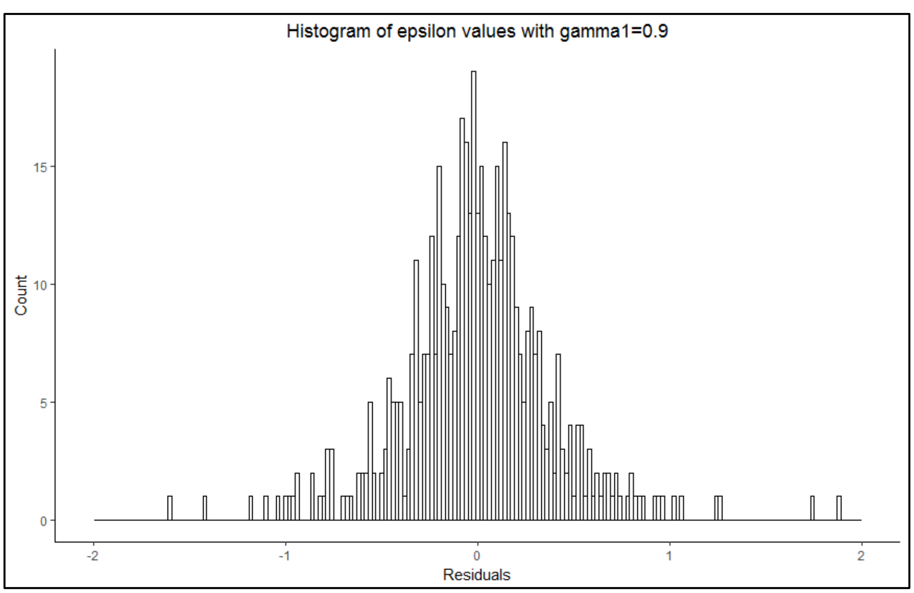


Epsilon values simulated from an ARCH(1) process with $\gamma_0 = 0.04$ and different γ_1 values



S





An ARCH(1) model gives an AR(1) process for ε_t^2

The defining equations are written as

$$\varepsilon_t^2 = \sigma_t^2 \eta_t^2 \qquad \text{since } \varepsilon_t = \sigma_t \eta_t$$

$$\gamma_0 + \gamma_1 \varepsilon_{t-1}^2 = \sigma_t^2$$

Subtracting gives

$$\varepsilon_t^2 - (\gamma_0 + \gamma_1 \varepsilon_{t-1}^2) = \sigma_t^2 \eta_t^2 - \sigma_t^2$$

which is equivalent to

$$\varepsilon_t^2 = (\gamma_0 + \gamma_1 \varepsilon_{t-1}^2) + u_t$$

where

$$u_t = \sigma_t^2 \eta_t^2 - \sigma_t^2 .$$

It can be shown that u_t is an uncorrelated series and $E(u_t) = 0$ - see Tsay (2010, page 120).

This means ε_t^2 is an AR(1) process.

This is where the term "**A**uto**R**egressive" comes from in the ARCH terminology:

Auto**R**egressive **C**onditionally **H**eteroscedastic – ARCH(1) − process

$\varepsilon_t \mid \varepsilon_{t-1}$ iid $N(0,\ \sigma_t^2 = \gamma_0 + \gamma_1 \varepsilon_{t-1}^2)$: **C**onditionally **H**eteroscedastic

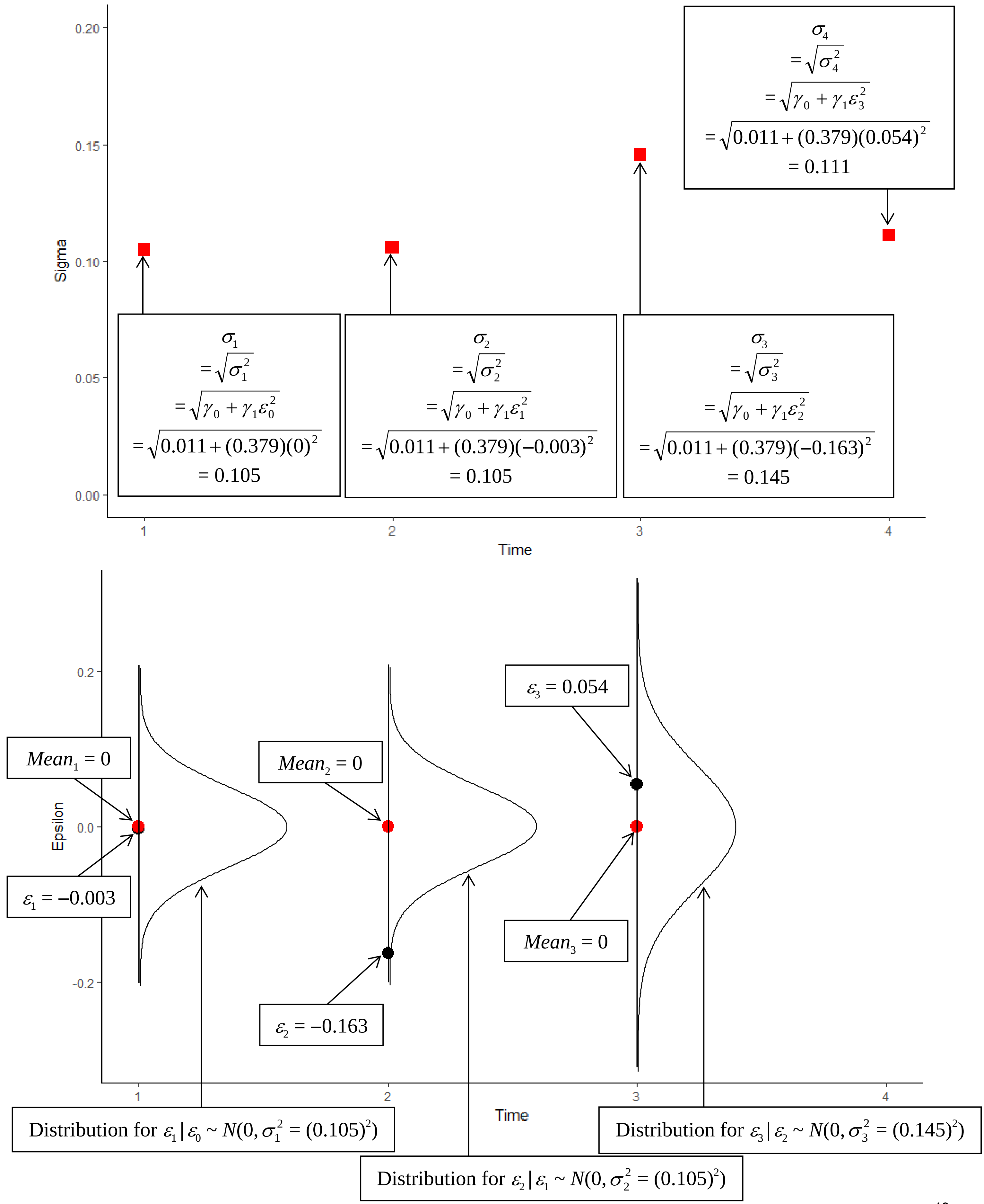
(ε_t is heteroscedastic when we condition on ε_{t-1})

$\varepsilon_t^2 = \gamma_0 + \gamma_1 \varepsilon_{t-1}^2 + u_t$: ε_t^2 is an **A**uto**R**egressive(1) process

Note that ε_t and ε_{t-1} are not independent because ε_t is directly related to ε_{t-1}:

$$\varepsilon_t = \sqrt{\gamma_0 + \gamma_1 \varepsilon_{t-1}^2 + u_t} .$$

Graphical interpretation of an ARCH(1) process with $\gamma_0 = 0.011$, $\gamma_1 = 0.379$ and $\varepsilon_0 = 0$



Estimate μ, γ_0 and γ_1 in an ARCH(1) process using Solver in Excel

<u>You are not responsible for knowing how to compute the log likelihood on this page</u>

Row	A	B	C	D	E	F	G	H	I	J
1	Time	Est_Mu	Y Y_t	Epsilon $\varepsilon_t =$ $Y_t - \mu$	SigmaSq $\sigma_t^2 = \gamma_0 + \gamma_1 \varepsilon_{t-1}^2$	log(SigmaSq) $\log(\sigma_t^2)$	Intermediate term #2 $\frac{\varepsilon_t^2}{\sigma_t^2}$	Determinant term $-0.5\sum_{t=1}^{n} \log(\sigma_t^2)$	Exponent term $-0.5\sum_{t=1}^{n} \frac{\varepsilon_t^2}{\sigma_t^2}$	Log_likelihood $-0.5n\log(2\pi)$ $-0.5\sum_{t=1}^{n} \log(\sigma_t^2)$ $-0.5\sum_{t=1}^{n} \frac{\varepsilon_t^2}{\sigma_t^2}$
2	0	0.013 μ		0.000				901.263	-216.000	288.279
3	1	Est_Gamma0	0.010 Y_1	-0.003 ε_1	0.011 $\sigma_1^2 = \gamma_0 + \gamma_1 \varepsilon_0^2$	-4.498 $\log(\sigma_1^2)$	0.001 $\frac{\varepsilon_1^2}{\sigma_1^2}$			
4	2	0.011 γ_0	-0.150 Y_2	-0.163 ε_2	0.011 $\sigma_2^2 = \gamma_0 + \gamma_1 \varepsilon_1^2$	-4.498 $\log(\sigma_2^2)$	2.377 $\frac{\varepsilon_2^2}{\sigma_2^2}$			
5	3	Est_Gamma1	0.067	0.054	0.021	-3.846	0.139			
6	4	0.387 γ_1	0.083	0.070	0.012	-4.400	0.403			
7	5		-0.110	-0.123	0.013	-4.340	1.160			
8	6		0.125	0.113	0.017	-4.075	0.746			
9	7		0.486	0.473	0.016	-4.133	13.948			
⋮	⋮		⋮	⋮	⋮	⋮	⋮			
431	429		-0.200	-0.212	0.011	-4.479	3.974			
432	430		-0.156	-0.168	0.029	-3.555	0.990			
433	431		-0.140	-0.152	0.022	-3.813	1.051			
444	432		0.060	0.048	0.020	-3.906	0.114			

Estimate μ, γ_0 and γ_1 using R

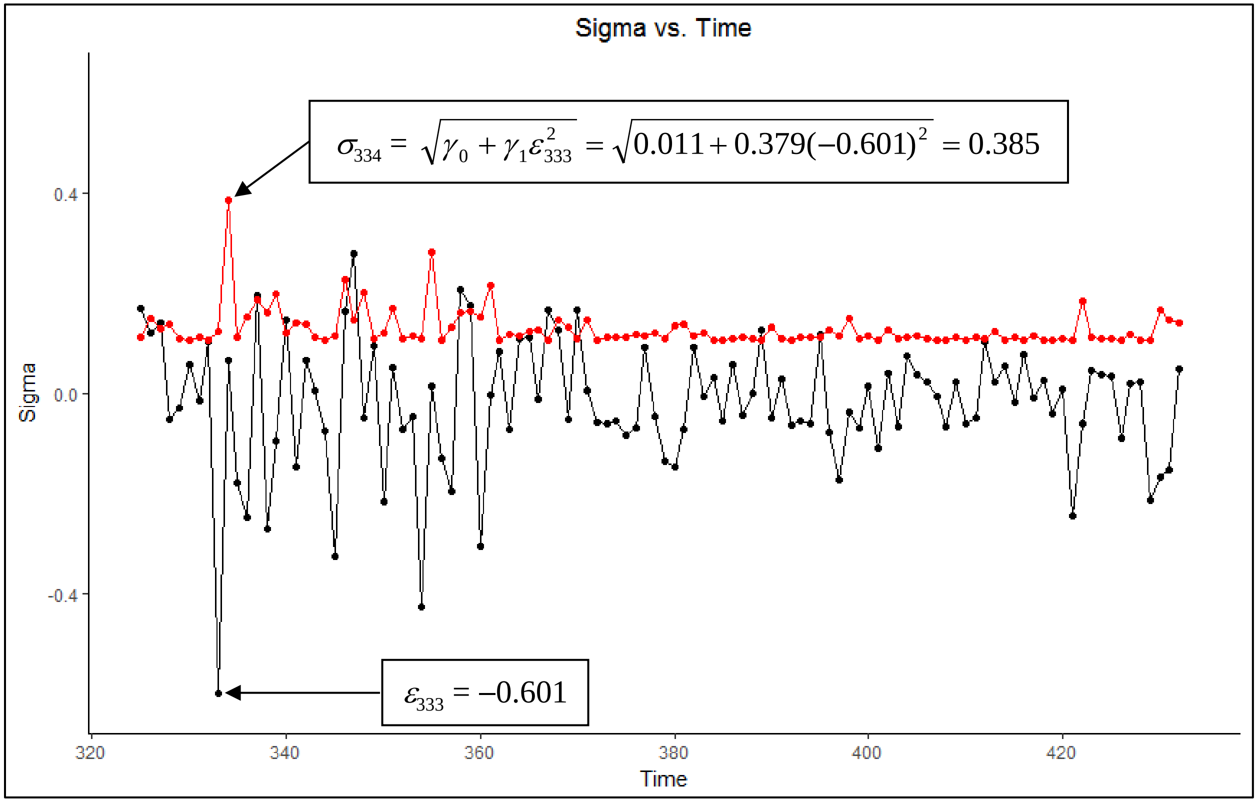
```
------------------------------------------------------------------------------------------
#
#   Estimate parameters using garchFit - See Tsay (2010), Chapter 3
#
result <- garchFit(data_table$Y~garch(1,0), data=data_table$Y, include.constant=T, trace=F)
summary(result)
------------------------------------------------------------------------------------------
```

```
Title:
 GARCH Modelling

Call:
 garchFit(formula = data_table$Y ~ garch(1, 0), data = data_table$Y,
    trace = F, include.constant = T)

Mean and Variance Equation:
 data ~ garch(1, 0)
<environment: 0x0000000019e30178>
 [data = data_table$Y]

Conditional Distribution:
 norm

Coefficient(s):
      mu     omega    alpha1
0.012637  0.011195  0.379492

Std. Errors:
 based on Hessian

Error Analysis:
        Estimate  Std. Error  t value Pr(>|t|)
mu      0.012637    0.005428    2.328  0.01990 *
omega   0.011195    0.001239    9.034  < 2e-16 ***
alpha1  0.379492    0.115534    3.285  0.00102 **
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Log Likelihood:
 288.0589    normalized:  0.6668031

Standardised Residuals Tests:
                                Statistic p-Value
 Jarque-Bera Test   R    Chi^2  137.919   0
 Shapiro-Wilk Test  R    W      0.9679248 4.024058e-08
 Ljung-Box Test     R    Q(10)  12.54002  0.2505382
 Ljung-Box Test     R    Q(15)  21.33508  0.1264607
 Ljung-Box Test     R    Q(20)  23.19679  0.2792354
 Ljung-Box Test     R^2  Q(10)  16.0159   0.09917815
 Ljung-Box Test     R^2  Q(15)  36.08022  0.001721296
 Ljung-Box Test     R^2  Q(20)  37.43683  0.01036728
 LM Arch Test       R    TR^2   26.57744  0.008884587

Information Criterion Statistics:
      AIC       BIC       SIC      HQIC
-1.319717 -1.291464 -1.319813 -1.308563
```

```
--------------------------------------------------------------------------------
head(result@residuals)
tail(result@residuals)
--------------------------------------------------------------------------------

[1] -0.00263673 -0.16264932  0.05442751  0.07031207 -0.12298506  0.11252628

[1]  0.01988290  0.02365101 -0.21233584 -0.16823830 -0.15239876  0.04781769


--------------------------------------------------------------------------------
head(result@sigma.t)
tail(result@sigma.t)
--------------------------------------------------------------------------------

[1] 0.1319058 0.1058191 0.1457204 0.1109920 0.1143292 0.1301345

[1] 0.1191125 0.1065132 0.1068051 0.1682409 0.1481088 0.1414528


---------------------------------------------------------------------------------
predict(result,5)
---------------------------------------------------------------------------------

  meanForecast meanError standardDeviation
1   0.01263656 0.1098306         0.1098306
2   0.01263656 0.1255897         0.1255897
3   0.01263656 0.1310751         0.1310751
4   0.01263656 0.1330976         0.1330976
5   0.01263656 0.1338571         0.1338571
```

Plot of ε_t and $\sigma_t \mid \varepsilon_{t-1} = \sqrt{\gamma_0 + \gamma_1 \varepsilon_{t-1}^2}$

```
---------------------------------------------------------------------------------------------------------
figure <- ggplot()
figure <- figure + geom_point(aes(data_table$time[325:432], result@residuals[325:432]))
figure <- figure + geom_line(aes(data_table$time[325:432], result@residuals[325:432]))
figure <- figure + geom_point(aes(data_table$time[325:432], result@sigma.t[325:432]), color="Red")
figure <- figure + geom_line(aes(data_table$time[325:432], result@sigma.t[325:432]), color="Red")
figure <- figure + xlim(325,432) + ylim(-0.62,0.62)
figure <- figure + ggtitle("Sigma vs. Time") + xlab("Time") + ylab("Sigma")
figure <- figure + theme(plot.title = element_text(hjust = 0.5))
figure <- figure + theme(panel.grid.major = element_blank(), panel.grid.minor = element_blank(),
                         panel.background = element_blank(), axis.line = element_line(colour = "black"))
figure
---------------------------------------------------------------------------------------------------------
```



Spreadsheet to compute the in-sample values $\sigma_t^2 \mid \varepsilon_{t-1} = \gamma_0 + \gamma_1 \varepsilon_{t-1}^2$ for t = 1, …, 432 and future values $\sigma_{432+k}^2 \mid \varepsilon_{432}$ for k = 1, …, 5 using the estimated values of μ, γ_0 and γ_1 from R

Row	A	B	C	D	E	F
1	Time t	Mu	Y_t Y_t	Epsilon $\varepsilon_t = Y_t - \mu$	Sigma^2 $\sigma_t^2 = \gamma_0 + \gamma_1 \varepsilon_{t-1}^2$	Sigma $\sigma_t = \sqrt{\sigma_t^2}$
2	1	0.012637 α	0.010 Y_1	-0.003 $\varepsilon_1 = Y_1 - \mu$		
3	2	Gamma0	-0.150 Y_2	-0.163 $\varepsilon_2 = Y_2 - \mu$	0.011 $\sigma_2^2 = \gamma_0 + \gamma_1 \varepsilon_1^2$	0.106 $\sigma_2 = \sqrt{\sigma_2^2}$
4	3	0.011195 γ_0	0.067	0.054	0.021	0.146
5	4	Gamma1	0.083	0.070	0.012	0.111
6	5	0.379492 γ_1	-0.110	-0.123	0.013	0.114
7	6		0.125	0.113	0.017	0.130
8	7		0.486	0.473	0.016	0.126
9	8		0.111	0.099	0.096	0.310
10	9		0.211	0.198	0.015	0.122
11	10		0.135	0.122	0.026	0.162
⋮	⋮		⋮	⋮	⋮	⋮
429	428		0.036	0.024	0.011	0.107
430	429		-0.200	-0.212	0.011	0.107
431	430		-0.156	-0.168	0.028	0.168
432	431		-0.140	-0.152	0.022	0.148
433	432		0.060	0.048	0.020	0.141
434	433				0.012 $\sigma_{433}^2 = \gamma_0 + \gamma_1 \varepsilon_{432}^2$	0.109830 $\sigma_{433} = \sqrt{\sigma_{433}^2}$
435	434				0.016 $\sigma_{434}^2 = \gamma_0 + \gamma_1 \sigma_{433}^2$	0.125589 $\sigma_{434} = \sqrt{\sigma_{434}^2}$
436	435				0.017 $\sigma_{435}^2 = \gamma_0 + \gamma_1 \sigma_{434}^2$	0.131075 $\sigma_{435} = \sqrt{\sigma_{435}^2}$
437	436				0.018 $\sigma_{436}^2 = \gamma_0 + \gamma_1 \sigma_{435}^2$	0.133097 $\sigma_{436} = \sqrt{\sigma_{436}^2}$
438	437				0.018 $\sigma_{437}^2 = \gamma_0 + \gamma_1 \sigma_{436}^2$	0.133857 $\sigma_{437} = \sqrt{\sigma_{437}^2}$

Future σ_t^2 values given information through period t = 432

Recall that if $X \sim N(\mu = 0, \sigma^2)$, then

$$\sigma^2 = Var(X) = E[(X - \mu)^2] = E(X^2).$$

One-period ahead value $\sigma_{432}^2(1) = \sigma_{433}^2 \mid \varepsilon_{432} = Var(\varepsilon_{433} \mid \varepsilon_{432})$

From before, we know that

$$\varepsilon_t^2 = (\gamma_0 + \gamma_1 \varepsilon_{t-1}^2) + u_t$$

where $E(u_t \mid \varepsilon_{t-1}) = 0$.

This means

$$\varepsilon_{433}^2 = \gamma_0 + \gamma_1 \varepsilon_{432}^2 + u_{433}$$

where $E(u_{433} \mid \varepsilon_{432}) = 0$.

Then

$$\begin{aligned}
\sigma_{432}^2(1) &= Var(\varepsilon_{433} \mid \varepsilon_{432}) \\
&= E(\varepsilon_{433}^2 \mid \varepsilon_{432}) \\
&= E(\gamma_0 + \gamma_1 \varepsilon_{432}^2 + u_{433} \mid \varepsilon_{432}) \\
&= \gamma_0 + \gamma_1 \varepsilon_{432}^2 + E(u_{433} \mid \varepsilon_{432}) \\
&= \gamma_0 + \gamma_1 \varepsilon_{432}^2
\end{aligned}$$

since $E(u_{433} \mid \varepsilon_{432}) = 0$.

Therefore,

$$\sigma_{432}^2(1) = Var(\varepsilon_{433} \mid \varepsilon_{432}) = \gamma_0 + \gamma_1 \varepsilon_{432}^2 = 0.0112 + 0.3795(0.048)^2 = (0.1099)^2.$$

Two-period ahead value $\sigma_{432}^2(2) = Var(\varepsilon_{434} \mid \varepsilon_{432})$

From before, we know that

$$\varepsilon_{434}^2 = \gamma_0 + \gamma_1 \varepsilon_{433}^2 + u_{434}$$

where $E(u_{434} \mid \varepsilon_{432}) = 0$.

Then

$$\begin{aligned}
\sigma_{432}^2(2) &= Var(\varepsilon_{434} \mid \varepsilon_{432}) \\
&= E(\varepsilon_{434}^2 \mid \varepsilon_{432}) \\
&= E(\gamma_0 + \gamma_1 \varepsilon_{433}^2 + u_{434} \mid \varepsilon_{432}) \\
&= \gamma_0 + \gamma_1 E(\varepsilon_{433}^2 \mid \varepsilon_{432}) + E(u_{434} \mid \varepsilon_{432}) \\
&= \gamma_0 + \gamma_1 \sigma_{432}^2(1)
\end{aligned}$$

since $\sigma_{432}^2(1) = E(\varepsilon_{433}^2 \mid \varepsilon_{432})$ and $E(u_{434} \mid \varepsilon_{432}) = 0$.

Therefore,

$$\sigma_{432}^2(2) = Var(\varepsilon_{434} \mid \varepsilon_{432}) = \gamma_0 + \gamma_1 \sigma_{432}^2(1) = 0.0112 + 0.3795(0.1098)^2 = (0.1256)^2.$$

***k*-period ahead value** $\sigma_{432}^2(k) = Var(\varepsilon_{432+k} \mid \varepsilon_{432})$

In general,

$$\varepsilon_{432+k}^2 = \gamma_0 + \gamma_1 \varepsilon_{432+(k-1)}^2 + u_{432+k}$$

where $E(u_{432+k} \mid \varepsilon_{432}) = 0$.

Then

$$\sigma_{432}^2(k) = Var(\varepsilon_{432+k} \mid \varepsilon_{432})$$

$$= E(\varepsilon_{432+k}^2 \mid \varepsilon_{432})$$

$$= E(\gamma_0 + \gamma_1 \varepsilon_{432+(k-1)}^2 + u_{432+k} \mid \varepsilon_{432})$$

$$= \gamma_0 + \gamma_1 E(\varepsilon_{432+k-1}^2 \mid \varepsilon_{432}) + E(u_{432+k} \mid \varepsilon_{432})$$

$$= \gamma_0 + \gamma_1 \sigma_{432}^2(k-1)$$

since $\sigma_{432}^2(k-1) = E(\varepsilon_{432+(k-1)}^2 \mid \varepsilon_{432})$ and $E(u_{432+k} \mid \varepsilon_{432}) = 0$.

Therefore,

$$\sigma_{432}^2(k) = Var(\varepsilon_{434} \mid \varepsilon_{432}) = \gamma_0 + \gamma_1 \sigma_{432}^2(k-1) .$$