XRTM:
Implementation optimized equations

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Contents

Table 1: Definitions of common variables

Variable	Definition
i	matrix row index
j	matrix column index
N	number of quadrature points
k	layer index
K	number of layers
l	level index (number of levels is $K+1$)
l	associated Legendre polynomial index (degree of the polynomial)
L	number of associated Legendre polynomials (maximum degree is $L-1$)
m	Fourier series expansion (azimuthal) term index and also the order of an associated Legendre polynomial)
M	number of Fourier series expansion (azimuthal) terms (maximum order is $M-1$)
x_k	optical thickness of layer k
τ_l	optical depth from TOA to level l
υ_k	optical depth from the top of layer k
ω	single scattering albedo
β	phase function Legendre expansion coefficient
λ_k	average secant of solar zenith angle for layer k ($1/\mu_0$ for plane-parallel)
$\mathcal{X}_{\mathrm{b},k}$	transmission of the solar beam in layer k
$\mathcal{T}_{\mathrm{b},l}$	transmission of the solar beam from TOA to level l
b_l	planck radiance at level l

1 Common

$$\mathcal{X}_{\mathrm{b},k} = e^{x_k \lambda_k} \tag{1}$$

$$\mathbf{M} = \mathrm{diag}[\mu_1 \dots \mu_N] \tag{2}$$

$$\mathbf{W} = \mathrm{diag}[a_1 \dots a_N] \tag{3}$$

2 Delta-M scaling

2.1 Forward

$$a_l = 2l + 1 \tag{4}$$

$$\beta_l' = \frac{\beta_l - a_l f}{1 - f} \tag{5}$$

$$\omega' = \frac{1 - f}{1 - \omega f}\omega \tag{6}$$

$$x' = (1 - \omega f)x \tag{7}$$

2.2 Tangent linear

$$\mathcal{L}(\beta_l') = \frac{\mathcal{L}(\beta_l) - a_l\mathcal{L}(f)}{1-f} + (\beta_l - a_l f)\frac{\mathcal{L}(f)}{(1-f)^2} \tag{8}$$

$$\mathcal{L}(\omega') = \frac{\mathcal{L}(\omega)(1-f) - \omega\mathcal{L}(f)}{(1-\omega f)} + \omega(1-f)\frac{\mathcal{L}(\omega)f + \omega\mathcal{L}(f)}{(1-\omega f)^2} \tag{9}$$

$$\mathcal{L}(x') = -\left[\mathcal{L}(\omega)f + \omega\mathcal{L}(f)\right]x + (1-\omega f)\mathcal{L}(x) \tag{10}$$

2.3 Adjoint of tangent linear

2.3.1 β_l'

$$t_l = \frac{\mathcal{A}(\beta_l')}{1-f} \tag{11}$$

$$\mathcal{A}(\beta_l) = \mathcal{A}(\beta_l) + t_l \tag{12}$$

$$\mathcal{A}(f) = \mathcal{A}(f) - a_l t_l \tag{13}$$

$$\mathcal{A}(f) = \mathcal{A}(f) + \mathcal{A}(\beta_l')\frac{\beta_l - a_l f}{(1-f)^2} \tag{14}$$

2.3.2 ω'

$$t = \frac{\mathcal{A}(\omega')}{1-\omega f} \tag{15}$$

$$\mathcal{A}(\omega) = \mathcal{A}(\omega) + t(1-f) \tag{16}$$

$$\mathcal{A}(f) = \mathcal{A}(f) - t\omega \tag{17}$$

$$t = \mathcal{A}(\omega')\frac{\omega(1-f)}{(1-\omega f)^2} \tag{18}$$

$$\mathcal{A}(\omega) = \mathcal{A}(\omega) + tf \tag{19}$$

$$\mathcal{A}(f) = \mathcal{A}(f) + t\omega \tag{20}$$

2.3.3 x'

$$t = -\mathcal{A}(x')x \tag{21}$$

$$\mathcal{A}(\omega) = \mathcal{A}(\omega) + tf \tag{22}$$

$$\mathcal{A}(f) = \mathcal{A}(f) + t\omega \tag{23}$$

$$\mathcal{A}(x) = \mathcal{A}(x) + \mathcal{A}(x')(1-\omega f) \tag{24}$$

3 Single scattering

3.1 Forward

3.1.1 Up

$$a = \frac{1 - e^{-(\tau_l - \tau)/\mu} e^{-(\tau_l - \tau)\lambda_l}}{1/\mu + \lambda_l} \tag{25}$$

$$b = \mathcal{T}_{b,l-1} e^{-(\tau - \tau_{l-1})\lambda_l} \omega_l P_l(\mu, \mu_0) a \tag{26}$$

$$I_{\text{ss}}^{\uparrow}(\tau) = \frac{F_0}{\mu} \sum_{l=L-1}^{0} I_{\text{ss},l+1} e^{-(\tau_l - \tau)/\mu} + b \tag{27}$$

3.1.2 Down

$$c = \frac{e^{-(\tau - \tau_{l-1})/\mu} - e^{-(\tau - \tau_{l-1})\lambda_l}}{\lambda_l - 1/\mu} \tag{28}$$

$$d = \mathcal{T}_{b,l-1} \omega_l P_l(\mu, \mu_0) c \tag{29}$$

$$I_{\text{ss}}^{\downarrow}(\tau) = \frac{F_0}{\mu} \sum_{l=1}^{L} I_{\text{ss},l-1} e^{-(\tau - \tau_{l-1})/\mu} + d \tag{30}$$

3.2 Tangent linear

3.2.1 Up

$$\mathcal{L}(a) = \frac{\left[\mathcal{L}(\tau_l) - \mathcal{L}(\tau)\right]/\mu e^{-(\tau_l - \tau)/\mu} e^{-(\tau_l - \tau)\lambda_l} - e^{-(\tau_l - \tau)/\mu} \left\{-[\,\mathcal{L}(\tau_l) - \mathcal{L}(\tau)\,]\lambda_l - (\tau_l - \tau)\mathcal{L}(\lambda_l)\right\} e^{-(\tau_l - \tau)\lambda_l} - a\mathcal{L}(\lambda_l)}{1/\mu + \lambda_l} \tag{31}$$

$$\begin{aligned}
\mathcal{L}(b) &= \mathcal{L}(\mathcal{T}_{b,l-1}) e^{-(\tau - \tau_{l-1})\lambda_l} \omega_l P_l(\mu, \mu_0) a \\
&+ -\mathcal{T}_{b,l-1} \left\{\left[\mathcal{L}(\tau) - \mathcal{L}(\tau_{l-1})\right]\lambda_l + (\tau - \tau_{l-1})\mathcal{L}(\lambda_l)\right\} e^{-(\tau - \tau_{l-1})\lambda_l} \omega_l P_l(\mu, \mu_0) a \\
&+ \mathcal{T}_{b,l-1} e^{-(\tau - \tau_{l-1})\lambda_l} \mathcal{L}(\omega_l) P_l(\mu, \mu_0) a \\
&+ \mathcal{T}_{b,l-1} e^{-(\tau - \tau_{l-1})\lambda_l} \omega_l \mathcal{L}\left[P_l(\mu, \mu_0)\right] a \\
&+ \mathcal{T}_{b,l-1} e^{-(\tau - \tau_{l-1})\lambda_l} \omega_l P_l(\mu, \mu_0) \mathcal{L}(a)
\end{aligned} \tag{32}$$

$$K_{\text{ss}}^{\uparrow}(\tau) = \frac{F_0}{\mu} \sum_{l=L-1}^{0} K_{\text{ss},l+1}^{\uparrow} e^{-(\tau_l - \tau)/\mu} - I_{\text{ss},l+1}^{\uparrow} \left[\mathcal{L}(\tau_l) - \mathcal{L}(\tau)\right]/\mu e^{-(\tau_l - \tau)/\mu} + b \tag{33}$$

3.2.2 Down

$$\mathcal{L}(c) = \frac{-\left[\mathcal{L}(\tau) - \mathcal{L}(\tau_{l-1})\right]/\mu e^{-(\tau-\tau_{l-1})/\mu} - \left\{-\left[(\mathcal{L}(\tau) - \mathcal{L}(\tau_{l-1})\right]\lambda_l - (\tau - \tau_{l-1})\mathcal{L}(\lambda_l)\right\} e^{-(\tau-\tau_{l-1})\lambda_l} - c\mathcal{L}(\lambda_l)}{\lambda_l - 1/\mu} \tag{34}$$

$$\begin{aligned} \mathcal{L}(d) &= \mathcal{L}(\mathcal{T}_{b,l-1})\omega_l P_l(\mu,\mu_0)c \\ &+ \mathcal{T}_{b,l-1}\mathcal{L}(\omega_l)P_l(\mu,\mu_0)c \\ &+ \mathcal{T}_{b,l-1}\omega_l\mathcal{L}\left[P_l(\mu,\mu_0)\right]c \\ &+ \mathcal{T}_{b,l-1}\omega_l P_l(\mu,\mu_0)\mathcal{L}(c) \end{aligned} \tag{35}$$

$$K^{\downarrow}_{\mathrm{ss}}(\tau) = \frac{F_0}{\mu}\sum_{l=1}^{L} K^{\downarrow}_{\mathrm{ss},l-1}e^{-(\tau-\tau_{l-1})/\mu} - I^{\downarrow}_{\mathrm{ss},l-1}\left[\mathcal{L}(\tau) - \mathcal{L}(\tau_{l-1})\right]/\mu e^{-(\tau-\tau_{l-1})/\mu} + d \tag{36}$$

3.3 Adjoint of tangent linear

3.3.1 Up

$$\mathcal{A}(I^{\uparrow}_{\mathrm{ss,l+1}}) = \mathcal{A}(I^{\uparrow}_{\mathrm{ss,l+1}}) + \mathcal{A}\left[I^{\uparrow}_{\mathrm{ss}}(\tau)\right]\frac{F_0}{\mu}e^{-(\tau_l-\tau)/\mu} \tag{37}$$

$$\mathcal{A}(\tau_l) = \mathcal{A}(\tau_l) - \mathcal{A}\left[I^{\uparrow}_{\mathrm{ss}}(\tau)\right]I^{\uparrow}_{\mathrm{ss,l+1}}\frac{F_0}{\mu^2}e^{-(\tau_l-\tau)/\mu} \tag{38}$$

$$\mathcal{A}(\tau) = \mathcal{A}(\tau) + \mathcal{A}\left[I^{\uparrow}_{\mathrm{ss}}(\tau)\right]I^{\uparrow}_{\mathrm{ss,l+1}}\frac{F_0}{\mu^2}e^{-(\tau_l-\tau)/\mu} \tag{39}$$

$$\mathcal{A}(b) = \mathcal{A}(b) + \mathcal{A}\left[I^{\uparrow}_{\mathrm{ss}}(\tau)\right]\frac{F_0}{\mu} \tag{40}$$

$$\mathcal{A}(\mathcal{T}_{b,l-1}) = \mathcal{A}(\mathcal{T}_{b,l-1}) + \mathcal{A}(b)e^{-(\tau-\tau_{l-1})\lambda_l}\omega_l P_l(\mu,\mu_0)a \tag{41}$$

$$\mathcal{A}(\tau) = \mathcal{A}(\tau) - \mathcal{A}(b)\mathcal{T}_{b,l-1}\lambda_l e^{-(\tau-\tau_{l-1})\lambda_l}\omega_l P_l(\mu,\mu_0)a \tag{42}$$

$$\mathcal{A}(\tau_{l-1}) = \mathcal{A}(\tau_{l-1}) + \mathcal{A}(b)\mathcal{T}_{b,l-1}\lambda_l e^{-(\tau-\tau_{l-1})\lambda_l}\omega_l P_l(\mu,\mu_0)a \tag{43}$$

$$\mathcal{A}(\lambda_l) = \mathcal{A}(\lambda_l) - \mathcal{A}(b)\mathcal{T}_{b,l-1}(\tau - \tau_{l-1})e^{-(\tau-\tau_{l-1})\lambda_l}\omega_l P_l(\mu,\mu_0)a \tag{44}$$

$$\mathcal{A}(\omega_l) = \mathcal{A}(\omega_l) + \mathcal{A}(b)\mathcal{T}_{b,l-1}e^{-(\tau-\tau_{l-1})\lambda_l}P_l(\mu,\mu_0)a \tag{45}$$

$$\mathcal{A}\left[P_l(\mu,\mu_0)\right] = \mathcal{A}\left[P_l(\mu,\mu_0)\right] + \mathcal{A}(b)\mathcal{T}_{b,l-1}e^{-(\tau-\tau_{l-1})\lambda_l}\omega_l a \tag{46}$$

$$\mathcal{A}(a) = \mathcal{A}(a) + \mathcal{A}(b)\mathcal{T}_{b,l-1}e^{-(\tau-\tau_{l-1})\lambda_l}\omega_l P_l(\mu,\mu_0) \tag{47}$$

$$t = \frac{\mathcal{A}(a)}{1/\mu + \lambda_l} \tag{48}$$

$$\mathcal{A}(\tau_l) = \mathcal{A}(\tau_l) + t\frac{1}{\mu}e^{-(\tau_l-\tau)/\mu}e^{-(\tau_l-\tau)\lambda_l} \tag{49}$$

$$\mathcal{A}(\tau) = \mathcal{A}(\tau) - t\frac{1}{\mu}e^{-(\tau_l-\tau)/\mu}e^{-(\tau_l-\tau)\lambda_l} \tag{50}$$

$$\mathcal{A}(\tau_l) = \mathcal{A}(\tau_l) + te^{-(\tau_l-\tau)/\mu}\lambda_l e^{-(\tau_l-\tau)\lambda_l} \tag{51}$$

$$\mathcal{A}(\tau) = \mathcal{A}(\tau) - te^{-(\tau_l-\tau)/\mu}\lambda_l e^{-(\tau_l-\tau)\lambda_l} \tag{52}$$

$$\mathcal{A}(\lambda_l) = \mathcal{A}(\lambda_l) + te^{-(\tau_l-\tau)/\mu}(\tau_l - \tau)e^{-(\tau_l-\tau)\lambda_l} \tag{53}$$

$$\mathcal{A}(\lambda_l) = \mathcal{A}(\lambda_l) - ta \tag{54}$$

3.3.2 Down

$$\mathcal{A}(I^{\downarrow}_{\text{ss,l-1}}) = \mathcal{A}(I^{\downarrow}_{\text{ss,l-1}}) + \mathcal{A}\left[I^{\downarrow}_{\text{ss}}(\tau)\right]\frac{F_0}{\mu}e^{-(\tau-\tau_{l-1})/\mu} \tag{55}$$

$$\mathcal{A}(\tau) = \mathcal{A}(\tau) - \mathcal{A}\left[I^{\downarrow}_{\text{ss}}(\tau)\right]I^{\downarrow}_{\text{ss,l-1}}\frac{F_0}{\mu^2}e^{-(\tau-\tau_{l-1})/\mu} \tag{56}$$

$$\mathcal{A}(\tau_{l-1}) = \mathcal{A}(\tau_{l-1}) + \mathcal{A}\left[I^{\downarrow}_{\text{ss}}(\tau)\right]I^{\downarrow}_{\text{ss,l-1}}\frac{F_0}{\mu^2}e^{-(\tau-\tau_{l-1})/\mu} \tag{57}$$

$$\mathcal{A}(d) = \mathcal{A}(d) + \mathcal{A}\left[I^{\downarrow}_{\text{ss}}(\tau)\right]\frac{F_0}{\mu} \tag{58}$$

$$\mathcal{A}(\mathcal{T}_{b,l-1}) = \mathcal{A}(\mathcal{T}_{b,l-1}) + \mathcal{A}(d)\omega_l P_l(\mu,\mu_0)c \tag{59}$$

$$\mathcal{A}(\omega_l) = \mathcal{A}(\omega_l) + \mathcal{A}(d)\mathcal{T}_{b,l-1}P_l(\mu,\mu_0)c \tag{60}$$

$$\mathcal{A}\left[P_l(\mu,\mu_0)\right] = \mathcal{A}\left[P_l(\mu,\mu_0)\right] + \mathcal{A}(d)\mathcal{T}_{b,l-1}\omega_l c \tag{61}$$

$$\mathcal{A}(c) = \mathcal{A}(c) + \mathcal{A}(d)\mathcal{T}_{b,l-1}\omega_l P_l(\mu,\mu_0) \tag{62}$$

$$t = \frac{\mathcal{A}(c)}{\lambda_l - 1/\mu} \tag{63}$$

$$\mathcal{A}(\tau) = \mathcal{A}(\tau) - t\frac{1}{\mu}e^{-(\tau-\tau_{l-1})/\mu} \tag{64}$$

$$\mathcal{A}(\tau_{l-1}) = \mathcal{A}(\tau_{l-1}) + t\frac{1}{\mu}e^{-(\tau-\tau_{l-1})/\mu} \tag{65}$$

$$\mathcal{A}(\tau) = \mathcal{A}(\tau) + t\lambda_l e^{-(\tau-\tau_{l-1})\lambda_l} \tag{66}$$

$$\mathcal{A}(\tau_{l-1}) = \mathcal{A}(\tau_{l-1}) - t\lambda_l e^{-(\tau-\tau_{l-1})\lambda_l} \tag{67}$$

$$\mathcal{A}(\lambda_l) = \mathcal{A}(\lambda_l) + t(\tau - \tau_{l-1}) e^{-(\tau-\tau_{l-1})\lambda_l} \tag{68}$$

$$\mathcal{A}(\lambda_l) = \mathcal{A}(\lambda_l) - tc \tag{69}$$

4 Phase matrices

4.1 Scalar

4.1.1 Forward

4.1.2 Tangent linear

4.1.3 Adjoint of tangent linear

4.2 Vector

4.2.1 Forward

$$\mathbf{B}_l = \begin{bmatrix} a_{1,l} & -b_{1,l} & 0 & 0 \\ -b_{1,l} & a_{2,l} & 0 & 0 \\ 0 & 0 & a_{3,l} & b_{2,l} \\ 0 & 0 & -b_{2,l} & a_{4,l} \end{bmatrix} \tag{70}$$

$$\mathbf{P}^{++} = \sum_{l=m}^{2n-1} \mathbf{\Pi}_l \mathbf{B}_l \mathbf{\Pi}_l^T \tag{71}$$

$$f(x) = \begin{cases} 1 & \text{if } \mathrm{mod}(x-m) = 0 \\ -1 & \text{otherwise} \end{cases} \tag{72}$$

$$\mathbf{P}^{-+} = \sum_{l=m}^{2n-1} f(l) \mathbf{\Pi}_l \mathbf{B}_l \mathbf{D} \mathbf{\Pi}_l^T \tag{73}$$

4.2.2 Tangent linear

$$\mathcal{L}(\mathbf{B}_l) = \begin{bmatrix} \mathcal{L}(a_{1,l}) & -\mathcal{L}(b_{1,l}) & 0 & 0 \\ -\mathcal{L}(b_{1,l}) & \mathcal{L}(a_{2,l}) & 0 & 0 \\ 0 & 0 & \mathcal{L}(a_{3,l}) & \mathcal{L}(b_{2,l}) \\ 0 & 0 & -\mathcal{L}(b_{2,l}) & \mathcal{L}(a_{4,l}) \end{bmatrix} \tag{74}$$

$$\mathcal{L}(\mathbf{P}^{++}) = \sum_{l=m}^{2n-1} \mathbf{\Pi}_l \mathcal{L}(\mathbf{B}_l) \mathbf{\Pi}_l^T \tag{75}$$

$$\mathcal{L}(\mathbf{P}^{-+}) = \sum_{l=m}^{2n-1} f(l) \mathbf{\Pi}_l \mathcal{L}(\mathbf{B}_l) \mathbf{D} \mathbf{\Pi}_l^T \tag{76}$$

4.2.3 Adjoint of tangent linear

$$\mathcal{A}(\mathbf{B}_l) = \sum_{l=m}^{2n-1} \mathbf{\Pi}_l^T \mathcal{A}(\mathbf{P}^{++}) \mathbf{\Pi}_l \tag{77}$$

$$\mathcal{A}(\mathbf{B}_l) = \mathcal{A}(\mathbf{B}_l) + \sum_{l=m}^{2n-1} f(l) \mathbf{\Pi}_l^T \mathcal{A}(\mathbf{P}^{-+}) \mathbf{\Pi}_l \mathbf{D} \tag{78}$$

$$\begin{aligned}
\mathcal{A}(a_{1,l}) &= \mathcal{A}(B_{l,1,1}) && (79)\\
\mathcal{A}(a_{2,l}) &= \mathcal{A}(B_{l,2,2}) && (80)\\
\mathcal{A}(a_{3,l}) &= \mathcal{A}(B_{l,3,3}) && (81)\\
\mathcal{A}(a_{4,l}) &= \mathcal{A}(B_{l,4,4}) && (82)\\
\mathcal{B}(b_{1,l}) &= -\mathcal{A}(B_{l,1,2}) - \mathcal{A}(B_{l,2,1}) && (83)\\
\mathcal{B}(b_{2,l}) &= \mathcal{A}(B_{l,3,4}) - \mathcal{A}(B_{l,4,3}) && (84)
\end{aligned}$$

5 Local r and t

5.1 Forward

$$\mathbf{r} = -(1+\delta_{0,m}) \frac{\omega}{4} \mathbf{M}^{-1} \mathbf{P}^{-} \mathbf{W} \tag{85}$$

$$\mathbf{t} = -\mathbf{M}^{-1} + (1+\delta_{0,m}) \frac{\omega}{4} \mathbf{M}^{-1} \mathbf{P}^{+} \mathbf{W} \tag{86}$$

5.2 Tangent linear

$$\mathcal{L}(\mathbf{r}) = -(1+\delta_{0,m}) \frac{1}{4} \mathbf{M}^{-1} \left[\mathcal{L}(\omega) \mathbf{P}^{-} + \omega \mathcal{L}(\mathbf{P}^{-}) \right] \mathbf{W} \tag{87}$$

$$\mathcal{L}(\mathbf{t}) = (1+\delta_{0,m}) \frac{1}{4} \mathbf{M}^{-1} \left[\mathcal{L}(\omega) \mathbf{P}^{+} + \omega \mathcal{L}(\mathbf{P}^{+}) \right] \mathbf{W} \tag{88}$$

5.3 Adjoint of tangent linear

$$\mathbf{t} = -(1+\delta_{0,m}) \frac{1}{4} \mathbf{M}^{-1} \mathcal{A}(\mathbf{r}) \mathbf{W} \tag{89}$$

$$\mathcal{A}(\omega) = \sum_{i}^{n} \sum_{j}^{n} \mathbf{t}_{ij} (\mathbf{P}^{-})_{ij} \tag{90}$$

$$\mathcal{A}(\mathbf{P}^{-}) = \omega \mathbf{t} \tag{91}$$

$$\mathbf{t} = -(1+\delta_{0,m}) \frac{1}{4} \mathbf{M}^{-1} \mathcal{A}(\mathbf{t}) \mathbf{W} \tag{92}$$

$$\mathcal{A}(\omega) = \mathcal{A}(\omega) + \sum_{i}^{n} \sum_{j}^{n} \mathbf{t}_{ij} (\mathbf{P}^{+})_{ij} \tag{93}$$

$$\mathcal{A}(\mathbf{P}^{+}) = \omega \mathbf{t} \tag{94}$$

6 Doubling

6.1 Forward

$$\mathcal{T}_0 = e^{-d\tau\lambda} \tag{95}$$

$$\mathcal{L}(\mathcal{T}_0) = [-\mathcal{L}(d\tau)\lambda]\,\mathcal{T}_0 \tag{96}$$

$$f_0 = (b_{l+1}/b_l - 1)/n_{\mathrm{doub}}^2 \tag{97}$$

$$\mathcal{L}(f_0) = (\mathcal{L}(b_{l+1}) - b_{l+1}\mathcal{L}(b_l)/b_l)/b_l/n_{\mathrm{doub}}^2 \tag{98}$$

$$\mathbf{P}_n = (\mathbf{E} - \mathbf{R}_n\mathbf{R}_n)^{-1} \tag{99}$$

$$\mathbf{A}_n = \mathbf{T}_n\mathbf{P}_n \tag{100}$$

$$\mathbf{B}_n = \mathbf{R}_n\mathbf{T}_n \tag{101}$$

$$\mathbf{a}_n = \mathbf{R}_n\mathbf{Se}_n^- + \mathbf{Se}_n^+\mathcal{T}_n \tag{102}$$

$$\mathbf{b}_n = \mathbf{R}_n\mathbf{Se}_n^+\mathcal{T}_n + \mathbf{Se}_n^- \tag{103}$$

$$\mathbf{c}_n = \mathbf{R}_n\mathbf{L}_n^- + \mathbf{L}_n^+ \tag{104}$$

$$\mathbf{d}_n = \mathbf{R}_n\mathbf{L}_n^+ + \mathbf{L}_n^- \tag{105}$$

$$\mathbf{e}_n = \mathbf{R}_n\mathbf{Sl}_n^- + \mathbf{Sl}_n^+ + \mathbf{L}_n^+ f \tag{106}$$

$$\mathbf{f}_n = \mathbf{R}_n(\mathbf{Sl}_n^+ + \mathbf{L}_n^+ f) + \mathbf{Sl}_n^- \tag{107}$$

$$\mathcal{T}_{n+1} = \mathcal{T}_n^2 \tag{108}$$

$$f_{n+1} = 2f_n \tag{109}$$

$$\mathbf{Se}_{n+1}^+ = \mathbf{A}_n\mathbf{a}_n + \mathbf{Se}_n^+ \tag{110}$$

$$\mathbf{Se}_{n+1}^- = \mathbf{A}_n\mathbf{b}_n + \mathbf{Se}_n^-\mathcal{T}_n \tag{111}$$

$$\mathbf{L}_{n+1}^+ = \mathbf{A}_n\mathbf{c}_n + \mathbf{L}_n^+ \tag{112}$$

$$\mathbf{L}^-_{n+1} = \mathbf{A}_n\mathbf{d}_n + \mathbf{L}^-_n \tag{113}$$

$$\mathbf{Sl}^+_{n+1} = \mathbf{A}_n\mathbf{e}_n + \mathbf{Sl}^+_n \tag{114}$$

$$\mathbf{Sl}^-_{n+1} = \mathbf{A}_n\mathbf{f}_n + \mathbf{Sl}^-_n + \mathbf{L}^-_n f \tag{115}$$

$$\mathbf{R}_{n+1} = \mathbf{R}_n + \mathbf{A}_n\mathbf{B}_n \tag{116}$$

$$\mathbf{T}_{n+1} = \mathbf{A}_n\mathbf{T}_n \tag{117}$$

6.2 Tangent linear

$$\mathcal{L}(\mathcal{T}_{n+1}) = 2\mathcal{L}(\mathcal{T}_n)\mathcal{T}_n \tag{118}$$

$$\mathcal{L}(f_{n+1}) = 2\mathcal{L}(f_n) \tag{119}$$

$$\mathcal{L}(\mathbf{A}_n) = \mathcal{L}(\mathbf{T}_n)\mathbf{P}_n + \mathbf{A}_n(\mathcal{L}(\mathbf{R}_n)\mathbf{R}_n + \mathbf{R}_n\mathcal{L}(\mathbf{R}_n))\mathbf{P}_n \tag{120}$$

$$\mathcal{L}(\mathbf{Se}^+_{n+1}) = \mathcal{L}(\mathbf{A}_n)\mathbf{a} + \mathbf{A}_n(\mathcal{L}(\mathbf{R}_n)\mathbf{Se}^-_n + \mathbf{R}_n\mathcal{L}(\mathbf{Se}^-_n) + \mathcal{L}(\mathbf{Se}^+_n)\mathcal{T}_n + \mathbf{Se}^+_n\mathcal{L}(\mathcal{T}_n)) + \mathcal{L}(\mathbf{Se}^+_n) \tag{121}$$

$$\mathcal{L}(\mathbf{Se}^-_{n+1}) = \mathcal{L}(\mathbf{A}_n)\mathbf{b}_n + \mathbf{A}_n[\mathcal{L}(\mathbf{R}_n)\mathbf{Se}^+_n\mathcal{T}_n + \mathbf{R}_n(\mathcal{L}(\mathbf{Se}^+_n)\mathcal{T}_n + \mathbf{Se}^+_n\mathcal{L}(\mathcal{T}_n)) + \mathcal{L}(\mathbf{Se}^-_n)] + \mathcal{L}(\mathbf{Se}^-_n)\mathcal{T}_n + \mathbf{Se}^-_n\mathcal{L}(\mathcal{T}_n) \tag{122}$$

$$\mathcal{L}(\mathbf{L}^+_{n+1}) = \mathbf{A}_n\left[\mathcal{L}(\mathbf{R}_n\mathcal{L}(\mathbf{L}^-_n) + \mathcal{L}(\mathbf{L}^+_n)\right] + \mathcal{L}(\mathbf{L}^+_n) \tag{123}$$

$$\mathcal{L}(\mathbf{L}^-_{n+1}) = \mathbf{A}_n\left[\mathcal{L}(\mathbf{R}_n\mathcal{L}(\mathbf{L}^+_n) + \mathcal{L}(\mathbf{L}^-_n)\right] + \mathcal{L}(\mathbf{L}^-_n) \tag{124}$$

$$\mathcal{L}(\mathbf{L}^+_{n+1}) = \mathcal{L}(\mathbf{A}_n)\mathbf{c}_n + \mathbf{A}_n\left[\mathcal{L}(\mathbf{R}_n)\mathbf{L}^-_n + \mathbf{R}_n\mathcal{L}(\mathbf{L}^-_n) + \mathcal{L}(\mathbf{L}^+_n)\right] + \mathcal{L}(\mathbf{L}^+_n) \tag{125}$$

$$\mathcal{L}(\mathbf{L}^-_{n+1}) = \mathcal{L}(\mathbf{A}_n)\mathbf{d}_n + \mathbf{A}_n\left[\mathcal{L}(\mathbf{R}_n)\mathbf{L}^+_n + \mathbf{R}_n\mathcal{L}(\mathbf{L}^+_n) + \mathcal{L}(\mathbf{L}^-_n)\right] + \mathcal{L}(\mathbf{L}^-_n) \tag{126}$$

$$\mathcal{L}(\mathbf{Sl}^+_{n+1}) = \mathbf{A}_n\left[\mathbf{R}_n\mathcal{L}(\mathbf{Sl}^-_n) + \mathcal{L}(\mathbf{Sl}^+_n) + \mathcal{L}(\mathbf{L}^+_n)f + \mathbf{L}^+_n\mathcal{L}(f)\right] + \mathcal{L}(\mathbf{Sl}^+_n) \tag{127}$$

$$\mathcal{L}(\mathbf{Sl}^-_{n+1}) = \mathbf{A}_n\left[\mathbf{R}_n(\mathcal{L}(\mathbf{Sl}^+_n) + \mathcal{L}(\mathbf{L}^+_n)f + \mathbf{L}^+_n\mathcal{L}(f)) + \mathcal{L}(\mathbf{Sl}^-_n)\right] + \mathcal{L}(\mathbf{Sl}^-_n) + \mathcal{L}(\mathbf{L}^-_n)f + \mathbf{L}^-_n\mathcal{L}(f) \tag{128}$$

$$\mathcal{L}(\mathbf{Sl}^+_{n+1}) = \mathcal{L}(\mathbf{A}_n)\mathbf{e}_n + \mathbf{A}_n\left[\mathcal{L}(\mathbf{R}_n)\mathbf{Sl}^-_n + \mathbf{R}_n\mathcal{L}(\mathbf{Sl}^-_n) + \mathcal{L}(\mathbf{Sl}^+_n) + \mathcal{L}(\mathbf{L}^+_n)f + \mathbf{L}^+_n\mathcal{L}(f)\right] + \mathcal{L}(\mathbf{Sl}^+_n) \tag{129}$$

$$\mathcal{L}(\mathbf{Sl}^-_{n+1}) = \mathcal{L}(\mathbf{A}_n)\mathbf{f}_n + \mathbf{A}_n\left[\mathcal{L}(\mathbf{R}_n)(\mathbf{Sl}^+_n + \mathbf{L}^+_n f) + \mathbf{R}_n(\mathcal{L}(\mathbf{Sl}^+_n) + \mathcal{L}(\mathbf{L}^+_n)f + \mathbf{L}^+_n\mathcal{L}(f)) + \mathcal{L}(\mathbf{Sl}^-_n)\right] + \mathcal{L}(\mathbf{Sl}^-_n) + \mathcal{L}(\mathbf{L}^-_n)f + \mathbf{L}^-_n\mathcal{L}(f) \tag{130}$$

$$\mathcal{L}(\mathbf{R}_{n+1}) = \mathcal{L}(\mathbf{R}_n) + \mathcal{L}(\mathbf{A}_n)\mathbf{B}_n + \mathbf{A}_n(\mathcal{L}(\mathbf{R}_n)\mathbf{T}_n + \mathbf{R}_n\mathcal{L}(\mathbf{T}_n)) \tag{131}$$

$$\mathcal{L}(\mathbf{T}_{n+1}) = \mathcal{L}(\mathbf{A}_n)\mathbf{T}_n + \mathbf{A}_n\mathcal{L}(\mathbf{T}_n) \tag{132}$$

6.3 Adjoint of tangent linear

7 Eigen problem

7.1 Tangent linear

$$\mathbf{A} = \begin{bmatrix} 2\xi_i\chi_{1i} & \xi_i^2 - \Gamma_{11} & -\Gamma_{12} & \cdots & -\Gamma_{1n} \\ 2\xi_i\chi_{2i} & -\Gamma_{21} & \xi_i^2 - \Gamma_{22} & \cdots & -\Gamma_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 2\xi_i\chi_{ni} & -\Gamma_{n1} & -\Gamma_{n2} & \cdots & \xi_i^2 - \Gamma_{nn} \\ 0 & \chi_{1i} & \chi_{1i} & \cdots & \chi_{ni} \end{bmatrix} \tag{133}$$

$$\mathbf{b} = \begin{bmatrix} \mathbf{\Delta}\boldsymbol{\chi}_i \\ 0 \end{bmatrix} = \begin{bmatrix} b_i \\ b_i \\ \vdots \\ b_n \\ 0 \end{bmatrix} = \begin{bmatrix} \sum_j^n \mathcal{L}(\Gamma_{1j})\chi_{j,i} \\ \sum_j^n \mathcal{L}(\Gamma_{2j})\chi_{j,i} \\ \vdots \\ \sum_j^n \mathcal{L}(\Gamma_{nj})\chi_{j,i} \\ 0 \end{bmatrix} \tag{134}$$

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b} = \begin{bmatrix} \mathcal{L}(\xi_i) \\ \mathcal{L}(\chi_{1i}) \\ \mathcal{L}(\chi_{2i}) \\ \vdots \\ \mathcal{L}(\chi_{ni}) \end{bmatrix} = \mathbf{\Gamma}\boldsymbol{\chi}_i \tag{135}$$

7.2 Adjoint of tangent linear

$$\mathcal{A}(\mathbf{b}) = \mathbf{A}^{-T}\mathcal{A}(\mathbf{x}) \tag{136}$$

$$\mathcal{A}(\mathbf{\Delta}) = \mathcal{A}(\mathbf{b})\boldsymbol{\chi}_i^T \tag{137}$$

7.3 Reduction of order

7.3.1 Forward

7.3.2 Tangent linear

7.3.3 Adjoint of tangent linear

7.4 Inversion of the reduction of order

7.4.1 Forward

$$\nu_i = \sqrt{\xi_i} \tag{138}$$

$$\mathbf{a} = \mathrm{diag}(\nu_i) \tag{139}$$

$$\mathbf{b} = (\mathbf{t} + \mathbf{r})\boldsymbol{\chi}\mathbf{a}^{-1} \tag{140}$$

$$\mathbf{X}_+ = \frac{1}{2}(\boldsymbol{\chi} + \mathbf{b}) \tag{141}$$

$$\mathbf{X}_- = \frac{1}{2}(\boldsymbol{\chi} - \mathbf{b}) \tag{142}$$

7.4.2 Tangent linear

$$\mathcal{L}(\nu_i) = \mathcal{L}(\xi_i) \tag{143}$$

$$\mathbf{c} = \mathrm{diag}[\mathcal{L}(\nu_i)] \tag{144}$$

$$\mathbf{d} = \{[\mathcal{L}(\mathbf{t}) + \mathcal{L}(\mathbf{r})]\,\boldsymbol{\chi} + (\mathbf{t} + \mathbf{r})\mathcal{L}(\boldsymbol{\chi}) - \mathbf{b}\mathbf{c}\}\,\mathbf{a}^{-1} \tag{145}$$

$$\mathcal{L}(\mathbf{X}_+) = \frac{1}{2}(\mathcal{L}(\boldsymbol{\chi}) + \mathbf{d}) \tag{146}$$

$$\mathcal{L}(\mathbf{X}_-) = \frac{1}{2}(\mathcal{L}(\boldsymbol{\chi}) - \mathbf{d}) \tag{147}$$

7.4.3 Adjoint of tangent linear

$$\mathcal{A}(\boldsymbol{\chi}) = \frac{1}{2}(\mathcal{A}(\mathbf{X}_+) + \mathcal{A}(\mathbf{X}_-)) \tag{148}$$

$$\mathcal{A}(\mathbf{d}) = \frac{1}{2}(\mathcal{A}(\mathbf{X}_+) - \mathcal{A}(\mathbf{X}_-)) \tag{149}$$

$$\mathbf{t} = \mathcal{A}(\mathbf{d})\mathbf{a}^{-T} \tag{150}$$

$$\mathcal{A}(\mathbf{t} + \mathbf{r}) = \mathbf{t}\boldsymbol{\chi}^T \tag{151}$$

$$\mathcal{A}(\boldsymbol{\chi}) = \mathcal{A}(\boldsymbol{\chi}) + (\mathbf{t} + \mathbf{r})^T\mathbf{t} \tag{152}$$

$$\mathcal{A}(\mathbf{c}) = -\mathbf{b}^T\mathbf{t} \tag{153}$$

$$\mathcal{A}(\nu_i) = \mathcal{A}(\nu_i) + \mathcal{A}(\mathbf{c}_{ii}) \tag{154}$$

$$\mathcal{A}(\xi_i) = \mathcal{A}(\nu_i) \tag{155}$$

8 Global R and T from Eigenvalues/matrix

8.0.4 Forward

$$\mathbf{\Lambda} = \mathrm{diag}(e^{-\nu_i x}) \tag{156}$$

$$\mathbf{a} = \mathbf{X}_{+}\mathbf{\Lambda} \tag{157}$$

$$\mathbf{b} = \mathbf{a}\mathbf{X}_{-}^{-1} \tag{158}$$

$$\mathbf{c} = \mathbf{X}_{-} - \mathbf{b}\mathbf{a} \tag{159}$$

$$\mathbf{d} = \mathbf{X}_{+}\mathbf{c}^{-1} \tag{160}$$

$$\mathbf{e} = \mathbf{X}_{-}\mathbf{\Lambda}\mathbf{c}^{-1} \tag{161}$$

$$\mathbf{R} = \mathbf{e}\mathbf{b} - \mathbf{d} \tag{162}$$

$$\mathbf{T} = \mathbf{e} - \mathbf{d}\mathbf{b} \tag{163}$$

8.0.5 Tangent linear

$$\mathbf{\Lambda} = \mathrm{diag}(e^{-\nu_i x}) \tag{164}$$

$$\mathbf{f} = \mathcal{L}(\mathbf{a}) = \mathcal{L}(\mathbf{X}_{+})\mathbf{\Lambda} + \mathbf{X}_{+}\mathcal{L}(\mathbf{\Lambda}) \tag{165}$$

$$\mathbf{g} = \mathcal{L}(\mathbf{b}) = \left[\mathbf{f} - \mathbf{b}\mathcal{L}(\mathbf{X}_{-})\right]\mathbf{X}_{-}^{-1} \tag{166}$$

$$\mathbf{h} = \mathcal{L}(\mathbf{c}) = \mathcal{L}(\mathbf{X}_{-}) - \mathbf{g}\mathbf{a} - \mathbf{b}\mathbf{f} \tag{167}$$

$$\mathbf{p} = \mathcal{L}(\mathbf{d}) = \left[\mathcal{L}(\mathbf{X}_{+}) - \mathbf{d}\mathbf{h}\right]\mathbf{c}^{-1} \tag{168}$$

$$\mathbf{q} = \mathcal{L}(\mathbf{e}) = \left[\mathcal{L}(\mathbf{X}_{-})\mathbf{\Lambda} + \mathbf{X}_{-}\mathcal{L}(\mathbf{\Lambda}) - \mathbf{e}\mathbf{h}\right]\mathbf{c}^{-1} \tag{169}$$

$$\mathcal{L}(\mathbf{R}) = \mathbf{q}\mathbf{b} + \mathbf{e}\mathbf{g} - \mathbf{p} \tag{170}$$

$$\mathcal{L}(\mathbf{T}) = \mathbf{q} - \mathbf{p}\mathbf{b} - \mathbf{d}\mathbf{g} \tag{171}$$

8.0.6 Adjoint of tangent linear

$$\mathcal{A}(\mathbf{q}) = \mathcal{A}(\mathbf{T}) \tag{172}$$

$$\mathcal{A}(\mathbf{p}) = -\mathcal{A}(\mathbf{T})\mathbf{b}^T \tag{173}$$

$$\mathcal{A}(\mathbf{g}) = -\mathbf{d}^T\mathcal{A}(\mathbf{T}) \tag{174}$$

$$\mathcal{A}(\mathbf{q}) = \mathcal{A}(\mathbf{q}) + \mathcal{A}(\mathbf{R})\mathbf{b}^T \tag{175}$$

$$\mathcal{A}(\mathbf{g}) = \mathcal{A}(\mathbf{g}) + \mathbf{e}^T\mathcal{A}(\mathbf{R}) \tag{176}$$

$$\mathcal{A}(\mathbf{p}) = \mathcal{A}(\mathbf{p}) - \mathcal{A}(\mathbf{R}) \tag{177}$$

$$\mathbf{t} = \mathcal{A}(\mathbf{q})\mathbf{c}^{-T} \tag{178}$$

$$\mathcal{A}(\mathbf{X}_{-}) = \mathbf{t}\boldsymbol{\Lambda}^T \tag{179}$$

$$\mathcal{A}(\boldsymbol{\Lambda}) = \mathbf{X}_{-}^T\mathbf{t} \tag{180}$$

$$\mathbf{t} = \mathcal{A}(\mathbf{p})\mathbf{c}^{-T} \tag{181}$$

$$\mathcal{A}(\mathbf{X}_{+}) = \mathbf{t} \tag{182}$$

$$\mathcal{A}(\mathbf{h}) = \mathcal{A}(\mathbf{h}) - \mathbf{d}^T\mathbf{t} \tag{183}$$

$$\mathcal{A}(\mathbf{X}_{-}) = \mathcal{A}(\mathbf{X}_{-}) + \mathcal{A}(\mathbf{h}) \tag{184}$$

$$\mathcal{A}(\mathbf{g}) = \mathcal{A}(\mathbf{g}) - \mathcal{A}(\mathbf{h})\mathbf{a}^T \tag{185}$$

$$\mathcal{A}(\mathbf{f}) = -\mathbf{b}^T\mathcal{A}(\mathbf{h}) \tag{186}$$

$$\mathbf{t} = \mathcal{A}(\mathbf{g})\mathbf{X}_{-}^{-T} \tag{187}$$

$$\mathcal{A}(\mathbf{f}) = \mathcal{A}(\mathbf{f}) + \mathbf{t} \tag{188}$$

$$\mathcal{A}(\mathbf{X}_{-}) = \mathcal{A}(\mathbf{X}_{-}) - \mathbf{b}^T\mathbf{t} \tag{189}$$

$$\mathcal{A}(\mathbf{X}_{+}) = \mathcal{A}(\mathbf{X}_{+}) + \mathcal{A}(\mathbf{f})\boldsymbol{\Lambda}^T \tag{190}$$

$$\mathcal{A}(\boldsymbol{\Lambda}) = \mathcal{A}(\boldsymbol{\Lambda}) + \mathbf{X}_{+}^T\mathcal{A}(\mathbf{f}) \tag{191}$$

9 Pade approximation

9.1 Forward

9.2 Tangent linear

9.3 Adjoint of tangent linear

10 Solar source

10.1 Local solar source, classical, full order

10.1.1 Forward

$$\mathbf{B} = \lambda \mathbf{E} - \mathbf{A} \tag{192}$$

$$\mathbf{C}^{\mp} = \frac{F_0}{4\pi} \mathbf{M} \tag{193}$$

$$\mathbf{D}^{\mp} = \mathbf{C}^{\mp} \mathbf{P}_{\circ}^{\mp} \tag{194}$$

$$\mathbf{F}^{\pm} = \mathbf{B}^{-1} \omega \mathbf{D}^{\mp} \tag{195}$$

10.1.2 Tangent linear

$$\mathcal{L}(\mathbf{F}^{\pm}) = \mathbf{B}^{-1} \left[-\mathcal{L}(\mathbf{B}) \mathbf{F}^{\pm} + \mathcal{L}(\omega) \mathbf{D}^{\mp} + \omega \mathbf{C}^{\mp} \mathcal{L}(\mathbf{P}_{\circ}^{\mp}) \right] \tag{196}$$

10.1.3 Adjoint of tangent linear

10.2 Local solar source, classical, reduced order

10.2.1 Forward

$$\mathbf{a} = \frac{F_0 \omega}{4\pi} \mathbf{M}^{-1} \tag{197}$$

$$\mathbf{b} = \mathbf{a} \mathbf{P}_{\circ}^{+} \tag{198}$$

$$\mathbf{c} = \mathbf{a} \mathbf{P}_{\circ}^{-} \tag{199}$$

$$\mathbf{d} = \mathbf{b} + \mathbf{c} \tag{200}$$

$$\mathbf{e} = \mathbf{b} - \mathbf{c} \tag{201}$$

$$\mathbf{f} = \left(\boldsymbol{\Gamma} - \lambda^2 \mathbf{E} \right)^{-1} \tag{202}$$

$$\mathbf{g} = \left[-(\mathbf{t} - \mathbf{r}) \mathbf{e} - \lambda \mathbf{d} \right] \tag{203}$$

$$\mathbf{p} = \mathbf{f}\mathbf{g} \tag{204}$$

$$\mathbf{h} = (\mathbf{t} + \mathbf{r})\mathbf{p} + \mathbf{e} \tag{205}$$

$$\mathbf{F}^{+} = \frac{1}{2}(\frac{1}{\lambda}\mathbf{h} + \mathbf{p}) \tag{206}$$

$$\mathbf{F}^{-} = \mathbf{F}^{+} - \mathbf{p} \tag{207}$$

10.2.2 Tangent linear

$$\mathcal{L}(\mathbf{a}) = \frac{F_0\mathcal{L}(\omega)}{4\pi}\mathbf{M}^{-1} \tag{208}$$

$$\mathcal{L}(\mathbf{b}) = \mathcal{L}(\mathbf{a})\mathbf{P}_{\circ}^{+} + \mathbf{a}\mathcal{L}(\mathbf{P}_{\circ}^{+}) \tag{209}$$

$$\mathcal{L}(\mathbf{c}) = \mathcal{L}(\mathbf{a})\mathbf{P}_{\circ}^{-} + \mathbf{a}\mathcal{L}(\mathbf{P}_{\circ}^{-}) \tag{210}$$

$$\mathcal{L}(\mathbf{d}) = \mathcal{L}(\mathbf{b}) + \mathcal{L}(\mathbf{c}) \tag{211}$$

$$\mathcal{L}(\mathbf{e}) = \mathcal{L}(\mathbf{b}) - \mathcal{L}(\mathbf{c}) \tag{212}$$

$$\mathcal{L}(\mathbf{p}) = \mathbf{f}\left\{-\left[\mathcal{L}(\mathbf{\Gamma}) - 2\mathcal{L}(\lambda)\lambda\mathbf{E}\right]\mathbf{p} - \left[\mathcal{L}(\mathbf{t}) - \mathcal{L}(\mathbf{r})\right]\mathbf{e} - (\mathbf{t} - \mathbf{r})\mathcal{L}(\mathbf{e}) - \mathcal{L}(\lambda)\mathbf{d} - \lambda\mathcal{L}(\mathbf{d})\right\} \tag{213}$$

$$\mathcal{L}(\mathbf{h}) = \left[\mathcal{L}(\mathbf{t}) + \mathcal{L}(\mathbf{r})\right]\mathbf{p} + (\mathbf{t} + \mathbf{r})\mathcal{L}(\mathbf{p}) + \mathcal{L}(\mathbf{e}) \tag{214}$$

$$\mathcal{L}(\mathbf{F}^{+}) = \frac{1}{2}\left[-\frac{\mathcal{L}(\lambda)}{\lambda^2}\mathbf{h} + \frac{1}{\lambda}\mathcal{L}(\mathbf{h}) + \mathcal{L}(\mathbf{p})\right] \tag{215}$$

$$\mathcal{L}(\mathbf{F}^{-}) = \mathcal{L}(\mathbf{F}^{+}) - \mathcal{L}(\mathbf{p}) \tag{216}$$

10.2.3 Adjoint of tangent linear

$$\mathcal{A}(\mathbf{F}^{+}) = \mathcal{A}(\mathbf{F}^{+}) + \mathcal{A}(\mathbf{F}^{-}) \tag{217}$$

$$\mathcal{A}(\mathbf{p}) = -\mathcal{A}(\mathbf{F}^{-}) \tag{218}$$

$$\mathcal{A}(\lambda) = -\frac{1}{2\lambda^2}\mathbf{h}^T\mathcal{A}(\mathbf{F}^{+}) \tag{219}$$

$$\mathcal{A}(\mathbf{h}) = \frac{1}{2\lambda}\mathcal{A}(\mathbf{F}^{+}) \tag{220}$$

$$\mathcal{A}(\mathbf{p}) = \mathcal{A}(\mathbf{p}) + \frac{1}{2}\mathcal{A}(\mathbf{F}^{+}) \tag{221}$$

$$\mathcal{A}(\mathbf{t}+\mathbf{r}) = \mathcal{A}(\mathbf{h})\mathbf{p}^T \tag{222}$$

$$\mathcal{A}(\mathbf{p}) = \mathcal{A}(\mathbf{p}) + (\mathbf{t}+\mathbf{r})^T\mathcal{A}(\mathbf{h}) \tag{223}$$

$$\mathcal{A}(\mathbf{e}) = \mathcal{A}(\mathbf{h}) \tag{224}$$

$$\mathbf{t} = \mathbf{f}^T\mathcal{A}(\mathbf{p}) \tag{225}$$

$$\mathbf{t}_2 = -\mathbf{t}\mathbf{p}^T \tag{226}$$

$$\mathcal{A}(\boldsymbol{\Gamma}) = \mathbf{t}_2 \tag{227}$$

$$\mathcal{A}(\lambda) = \mathcal{A}(\lambda) - 2\lambda\mathbf{t}_2 \tag{228}$$

$$\mathcal{A}(\mathbf{t}+\mathbf{r}) = \mathcal{A}(\mathbf{t}+\mathbf{r}) - \mathbf{t}\mathbf{e}^T \tag{229}$$

$$\mathcal{A}(\mathbf{e}) = \mathcal{A}(\mathbf{e}) - (\mathbf{t}+\mathbf{r})^T\mathbf{t} \tag{230}$$

$$\mathcal{A}(\lambda) = \mathcal{A}(\lambda) - \mathbf{t}\mathbf{d}^T \tag{231}$$

$$\mathcal{A}(\mathbf{d}) = \lambda\mathbf{t} \tag{232}$$

$$\mathcal{A}(\mathbf{b}) = \mathcal{A}(\mathbf{e}) \tag{233}$$

$$\mathcal{A}(\mathbf{c}) = -\mathcal{A}(\mathbf{e}) \tag{234}$$

$$\mathcal{A}(\mathbf{b}) = \mathcal{A}(\mathbf{b}) + \mathcal{A}(\mathbf{d}) \tag{235}$$

$$\mathcal{A}(\mathbf{c}) = \mathcal{A}(\mathbf{c}) + \mathcal{A}(\mathbf{d}) \tag{236}$$

$$\mathcal{A}(\mathbf{a}) = \mathcal{A}(\mathbf{c})(\mathbf{P}_\circ^-)^T \tag{237}$$

$$\mathcal{A}(\mathbf{P}_\circ^-) = \mathbf{a}^T\mathcal{A}(\mathbf{c}) \tag{238}$$

$$\mathcal{A}(\mathbf{a}) = \mathcal{A}(\mathbf{a}) + \mathcal{A}(\mathbf{b})(\mathbf{P}_\circ^+)^T \tag{239}$$

$$\mathcal{A}(\mathbf{P}_\circ^+) = \mathbf{a}^T \mathcal{A}(\mathbf{b}) \tag{240}$$

$$\mathcal{A}(\omega) = \frac{F_0}{4\pi} \sum_i^n (\mathbf{M}^{-1}\mathcal{A}(\mathbf{a}))_i \tag{241}$$

10.3 Local solar source, Green's function

10.3.1 Forward

$$a = \frac{F_0}{4\pi} \tag{242}$$

$$b_i^-(\upsilon) = \frac{e^{-\upsilon\nu_i} - e^{-\upsilon\lambda}}{\lambda - \nu_i} \tag{243}$$

$$b_i^+(\upsilon) = \frac{e^{-\upsilon\lambda} - e^{-x\lambda}e^{-(x-\upsilon)\nu_i}}{\lambda + \nu_i} \tag{244}$$

$$c_i = \mu_j w_j \tag{245}$$

$$d_i = \sum_{j=1}^N c_j \left[X_{+,ji}X_{+,ji} - X_{-,ji}X_{-,ji} \right] \tag{246}$$

$$e_i = \frac{a\omega}{d_i} \tag{247}$$

$$f_i^-(\upsilon) = e_i b_i^-(\upsilon) \tag{248}$$

$$f_i^+(\upsilon) = e_i b_i^+(\upsilon) \tag{249}$$

$$g_i = \sum_{j=1}^N w_j (P_j^+ X_{-,ji} - P_j^- X_{+,ji}) \tag{250}$$

$$h_i = \sum_{j=1}^N w_j (P_j^+ X_{+,ji} - P_j^- X_{-,ji}) \tag{251}$$

$$q_i(\upsilon) = f_i^-(\upsilon) g_i \tag{252}$$

$$r_i(\upsilon) = f_i^+(\upsilon) h_i \tag{253}$$

$$\mathbf{F}^+(0) = \mathbf{X}_- \mathbf{r}(0) \tag{254}$$

$$\mathbf{F}^+(x) = \mathbf{X}_+ \mathbf{q}(x) \tag{255}$$

$$\mathbf{F}^{-}(0) = -\mathbf{X}_{+}\mathbf{r}(0) \tag{256}$$

$$\mathbf{F}^{-}(x) = -\mathbf{X}_{-}\mathbf{q}(x) \tag{257}$$

10.3.2 Tangent linear

$$\mathcal{L}\left[b_i^{-}(\upsilon)\right] = \frac{\left[-\mathcal{L}(\upsilon)\nu_i - \upsilon\mathcal{L}(\nu_i)\right]e^{-\upsilon\nu_i} - \left[-\mathcal{L}(\upsilon)\lambda - \upsilon\mathcal{L}(\lambda)\right]e^{-\upsilon\lambda} - b_i^{-}(\upsilon)\left[\mathcal{L}(\lambda) - \mathcal{L}(\nu_i)\right]}{\lambda - \nu_i} \tag{258}$$

$$\mathcal{L}\left[b_i^{+}(\upsilon)\right] = \frac{\left[-\mathcal{L}(\upsilon)\lambda - \upsilon\mathcal{L}(\lambda)\right]e^{-\upsilon\lambda} - \left[-\mathcal{L}(x)\lambda - x\mathcal{L}(\lambda)\right]e^{-x\lambda}e^{-(x-\upsilon)\nu_i} - e^{-x\lambda}\left\{-\left[\mathcal{L}(x) - \mathcal{L}(\upsilon)\right]\nu_i - (x-\upsilon)\mathcal{L}(\nu_i)\right\}e^{-(x-\upsilon)\nu_i} - b_i^{+}(\upsilon)\left[\mathcal{L}(\lambda) + \mathcal{L}(\nu_i)\right]}{\lambda + \nu_i}$$

$$\mathcal{L}(d_i) = \sum_{j=1}^{N} c_j 2\left[\mathcal{L}(X_{+,ji})X_{+,ji} - \mathcal{L}(X_{-,ji})X_{-,ji}\right] \tag{260}$$

$$\mathcal{L}(e_i) = \frac{a\mathcal{L}(\omega) - e_i\mathcal{L}(d_i)}{d_i} \tag{261}$$

$$\mathcal{L}\left[f_i^{-}(\upsilon)\right] = \mathcal{L}(e_i)b_i^{-}(\upsilon) + e_i\mathcal{L}\left[b_i^{-}(\upsilon)\right] \tag{262}$$

$$\mathcal{L}\left[f_i^{+}(\upsilon)\right] = \mathcal{L}(e_i)b_i^{+}(\upsilon) + e_i\mathcal{L}\left[b_i^{+}(\upsilon)\right] \tag{263}$$

$$\mathcal{L}(g_i) = \sum_{j=1}^{N} w_j\left[\mathcal{L}(P_j^{+})X_{-,ji} + P_j^{+}\mathcal{L}(X_{-,ji}) - \mathcal{L}(P_j^{-})X_{+,ji} - P_j^{-}\mathcal{L}(X_{+,ji})\right] \tag{264}$$

$$\mathcal{L}(h_i) = \sum_{j=1}^{N} w_j\left[\mathcal{L}(P_j^{+})X_{+,ji} + P_j^{+}\mathcal{L}(X_{+,ji}) - \mathcal{L}(P_j^{-})X_{-,ji} - P_j^{-}\mathcal{L}(X_{-,ji})\right] \tag{265}$$

$$\mathcal{L}\left[q_i(\upsilon)\right] = \mathcal{L}\left[d_i^{-}(\upsilon)\right]g_i + d_i^{-}(\upsilon)\mathcal{L}(g_i) \tag{266}$$

$$\mathcal{L}\left[r_i(\upsilon)\right] = \mathcal{L}\left[d_i^{+}(\upsilon)\right]h_i + d_i^{+}(\upsilon)\mathcal{L}(h_i) \tag{267}$$

$$\mathcal{L}\left[\mathbf{F}^{+}(0)\right] = \mathcal{L}(\mathbf{X}_{-})\mathbf{r}(0) + \mathbf{X}_{-}\mathcal{L}\left[\mathbf{r}(0)\right] \tag{268}$$

$$\mathcal{L}\left[\mathbf{F}^{+}(x)\right] = \mathcal{L}(\mathbf{X}_{+})\mathbf{q}(x) + \mathbf{X}_{+}\mathcal{L}\left[\mathbf{q}(x)\right] \tag{269}$$

$$\mathcal{L}\left[\mathbf{F}^{-}(0)\right] = -\mathcal{L}(\mathbf{X}_{+})\mathbf{r}(0) - \mathbf{X}_{+}\mathcal{L}\left[\mathbf{r}(0)\right] \tag{270}$$

$$\mathcal{L}\left[\mathbf{F}^{-}(x)\right] = -\mathcal{L}(\mathbf{X}_{-})\mathbf{q}(x) - \mathbf{X}_{-}\mathcal{L}\left[\mathbf{q}(x)\right] \tag{271}$$

10.3.3 Adjoint of tangent linear

10.4 Global solar source

10.4.1 Forward

$$\mathbf{S}^+ = \mathbf{F}^+ - \mathbf{T}^+\mathbf{F}^+\mathcal{X}_\mathrm{b} - \mathbf{R}^-\mathbf{F}^- \tag{272}$$

$$\mathbf{S}^- = \mathbf{F}^-\mathcal{X}_\mathrm{b} - \mathbf{T}^-\mathbf{F}^- - \mathbf{R}^+\mathbf{F}^+\mathcal{X}_\mathrm{b} \tag{273}$$

10.4.2 Tangent linear

$$\mathcal{L}(\mathcal{X}_\mathrm{b}) = [-\mathcal{L}(x)\lambda]\,\mathcal{X}_\mathrm{b} \tag{274}$$

$$\mathbf{A} = \mathcal{L}(\mathbf{F}^+)\mathcal{X}_\mathrm{b} + \mathbf{F}^+\mathcal{L}(\mathcal{X}_\mathrm{b}) \tag{275}$$

$$\mathcal{L}(\mathbf{S}^+) = \mathcal{L}(\mathbf{F}^+) - \mathcal{L}(\mathbf{T}^+)\mathbf{F}^+\mathcal{X}_\mathrm{b} - \mathbf{T}^+\mathbf{A} - \mathcal{L}(\mathbf{R}^-)\mathbf{F}^- - \mathbf{R}^-\mathcal{L}(\mathbf{F}^-) \tag{276}$$

$$\mathcal{L}(\mathbf{S}^-) = \mathcal{L}(\mathbf{F}^-)\mathcal{X}_\mathrm{b} + \mathbf{F}^-\mathcal{L}(\mathcal{X}_\mathrm{b}) - \mathcal{L}(\mathbf{T}^-)\mathbf{F}^- - \mathbf{T}^-\mathcal{L}(\mathbf{F}^-) - \mathcal{L}(\mathbf{R}^+)\mathbf{F}^+\mathcal{X}_\mathrm{b} - \mathbf{R}^+\mathbf{A} \tag{277}$$

10.4.3 Adjoint of tangent linear

$$\mathcal{A}(\mathbf{F}^-) = \mathcal{A}(\mathbf{S}^-)\mathcal{X}_\mathrm{b} \tag{278}$$

$$\mathcal{A}(t) = (\mathbf{F}^-)^T\mathcal{A}(\mathbf{S}^-) \tag{279}$$

$$\mathcal{A}(\mathbf{T}^-) = -\mathcal{A}(\mathbf{S}^-)(\mathbf{F}^-)^T \tag{280}$$

$$\mathcal{A}(\mathbf{F}^-) = \mathcal{A}(\mathbf{F}^-) - (\mathbf{T}^-)^T\mathcal{A}(\mathbf{S}^-) \tag{281}$$

$$\mathcal{A}(\mathbf{R}^+) = -\mathcal{A}(\mathbf{S}^-)(\mathbf{F}^+)^T\mathcal{X}_\mathrm{b} \tag{282}$$

$$\mathcal{A}(\mathbf{A}) = -(\mathbf{R}^+)^T\mathcal{A}(\mathbf{S}^-) \tag{283}$$

$$\mathcal{A}(\mathbf{F}^+) = \mathcal{A}(\mathbf{S}^+) \tag{284}$$

$$\mathcal{A}(\mathbf{W}^+) = -\mathcal{A}(\mathbf{S}^+)(\mathbf{F}^+)^T\mathcal{X}_\mathrm{b} \tag{285}$$

$$\mathcal{A}(\mathbf{A}) = \mathcal{A}(\mathbf{A}) - (\mathbf{T}^+)^T\mathcal{A}(\mathbf{S}^+) \tag{286}$$

$$\mathcal{A}(\mathbf{R}^-) = -\mathcal{A}(\mathbf{S}^+)(\mathbf{F}^-)^T \tag{287}$$

$$\mathcal{A}(\mathbf{F}^-) = \mathcal{A}(\mathbf{F}^-) - (\mathbf{R}^-)^T \mathcal{A}(\mathbf{S}^+) \tag{288}$$

$$\mathcal{A}(\mathbf{F}^+) = \mathcal{A}(\mathbf{F}^+) + \mathcal{A}(\mathbf{A})\mathcal{X}_\mathrm{b} \tag{289}$$

$$\mathcal{A}(\mathcal{X}_\mathrm{b}) = \mathcal{A}(\mathcal{X}_\mathrm{b}) + (\mathbf{F}^+)^T \mathcal{A}(\mathbf{A}) \tag{290}$$

10.5 Scale global solar source

10.5.1 Forward

$$\mathbf{S}^{+\prime} = \mathcal{T}_{\mathrm{b},l}\mathbf{S}^+ \tag{291}$$

$$\mathbf{S}^{-\prime} = \mathcal{T}_{\mathrm{b},l}\mathbf{S}^- \tag{292}$$

10.5.2 Tangent linear

$$\mathcal{L}(\mathbf{S}^{+\prime}) = \mathcal{L}(\mathcal{T}_{\mathrm{b},l})\mathbf{S}^+ + \mathcal{T}_{\mathrm{b},l}\mathcal{L}(\mathbf{S}^+) \tag{293}$$

$$\mathcal{L}(\mathbf{S}^{-\prime}) = \mathcal{L}(\mathcal{T}_{\mathrm{b},l})\mathbf{S}^- + \mathcal{T}_{\mathrm{b},l}\mathcal{L}(\mathbf{S}^-) \tag{294}$$

10.5.3 Adjoint of tangent linear

$$\mathcal{A}(\mathcal{T}_{\mathrm{b},l}) = (\mathbf{S}^+)^T \mathcal{A}(\mathbf{S}^{+\prime}) \tag{295}$$

$$\mathcal{A}(\mathbf{S}^+) = \mathcal{T}_{\mathrm{b},l}\mathcal{A}(\mathbf{S}^{+\prime}) \tag{296}$$

$$\mathcal{A}(\mathcal{T}_{\mathrm{b},l}) = \mathcal{A}(\mathcal{T}_{\mathrm{b},l}) + (\mathbf{S}^-)^T \mathcal{A}(\mathbf{S}^{-\prime}) \tag{297}$$

$$\mathcal{A}(\mathbf{S}^-) = \mathcal{T}_{\mathrm{b},l}\mathcal{A}(\mathbf{S}^{-\prime}) \tag{298}$$

11 Thermal source

11.1 Local thermal source

11.1.1 Forward

$$\mathbf{a} = \mathbf{A}^{-1} \begin{bmatrix} +1/\mu_0 \\ \vdots \\ +1/\mu_N \\ -1/\mu_0 \\ \vdots \\ -1/\mu_N \end{bmatrix} \tag{299}$$

$$\mathbf{b} = (1 - \omega)\mathbf{a} \tag{300}$$

$$\mathbf{c} = \mathbf{A}^{-1}\mathbf{b} \tag{301}$$

$$a = (b_0 - b_1)/x \tag{302}$$

$$\mathbf{F0}^{+} = b_0\mathbf{b}^{+} - a\mathbf{c}^{+} \tag{303}$$

$$\mathbf{F0}^{-} = b_0\mathbf{b}^{-} - a\mathbf{c}^{-} \tag{304}$$

$$\mathbf{F1}^{+} = b_1\mathbf{b}^{+} - a\mathbf{c}^{+} \tag{305}$$

$$\mathbf{F1}^{-} = b_1\mathbf{b}^{-} - a\mathbf{c}^{-} \tag{306}$$

11.1.2 Tangent linear

$$\mathcal{L}(\mathbf{a}) = -\mathbf{A}^{-1}\mathcal{L}(\mathbf{A})\mathbf{a} \tag{307}$$

$$\mathcal{L}(\mathbf{b}) = -\mathcal{L}(\omega)\mathbf{a} + (1-\omega)\mathcal{L}(\mathbf{a}) \tag{308}$$

$$\mathcal{L}(\mathbf{c}) = \mathbf{A}^{-1}\left[-\mathcal{L}(\mathbf{A})\mathbf{c} + \mathcal{L}(\mathbf{b})\right] \tag{309}$$

$$\mathcal{L}(a) = \frac{[\mathcal{L}(b_0) - \mathcal{L}(b_1) - a\mathcal{L}(x)]}{x} \tag{310}$$

$$\mathcal{L}(\mathbf{F0}^{+}) = \mathcal{L}(b_0)\mathbf{b}^{+} + b_0\mathcal{L}(\mathbf{b}^{+}) - \mathcal{L}(a)\mathbf{c}^{+} - a\mathcal{L}(\mathbf{c}^{+}) \tag{311}$$

$$\mathcal{L}(\mathbf{F0}^{-}) = \mathcal{L}(b_0)\mathbf{b}^{-} + b_0\mathcal{L}(\mathbf{b}^{-}) - \mathcal{L}(a)\mathbf{c}^{-} - a\mathcal{L}(\mathbf{c}^{-}) \tag{312}$$

$$\mathcal{L}(\mathbf{F1}^{+}) = \mathcal{L}(b_1)\mathbf{b}^{+} + b_1\mathcal{L}(\mathbf{b}^{+}) - \mathcal{L}(a)\mathbf{c}^{+} - a\mathcal{L}(\mathbf{c}^{+}) \tag{313}$$

$$\mathcal{L}(\mathbf{F1}^{-}) = \mathcal{L}(b_1)\mathbf{b}^{-} + b_1\mathcal{L}(\mathbf{b}^{-}) - \mathcal{L}(a)\mathbf{c}^{-} - a\mathcal{L}(\mathbf{c}^{-}) \tag{314}$$

11.1.3 Adjoint of tangent linear

11.2 Global thermal source

11.2.1 Forward

$$\mathbf{Sl}^{+} = \mathbf{F0}^{+} - \mathbf{T}^{+}\mathbf{F1}^{+} - \mathbf{R}^{-}\mathbf{F0}^{-} \tag{315}$$

$$\mathbf{Sl}^{-} = \mathbf{F1}^{-} - \mathbf{R}^{+}\mathbf{F1}^{+} - \mathbf{T}^{-}\mathbf{F0}^{-} \tag{316}$$

11.2.2 Tangent linear

$$\mathbf{Sl}^{+} = \mathcal{L}(\mathbf{F0}^{+}) - \mathcal{L}(\mathbf{T}^{+})\mathbf{F1}^{+} - \mathbf{T}^{+}\mathcal{L}(\mathbf{F1}^{+}) - \mathcal{L}(\mathbf{R}^{-})\mathbf{F0}^{-} - \mathbf{R}^{-}\mathcal{L}(\mathbf{F0}^{-}) \tag{317}$$

$$\mathbf{Sl}^{-} = \mathcal{L}(\mathbf{F1}^{-}) - \mathcal{L}(\mathbf{R}^{+})\mathbf{F1}^{+} - \mathbf{R}^{+}\mathcal{L}(\mathbf{F1}^{+}) - \mathcal{L}(\mathbf{T}^{-})\mathbf{F0}^{-} - \mathbf{T}^{-}\mathcal{L}(\mathbf{F0}^{-}) \tag{318}$$

11.2.3 Adjoint of tangent linear

12 Adding

$$\mathcal{X}_{12} = e^{-x_{12}\lambda_{12}} \tag{319}$$

12.1 Upward ($\mathbf{R}_{13}$, $\mathbf{T}_{31}$, $\mathbf{S}_{31}$)

12.1.1 Forward

$$\mathbf{P}_{31} = (\mathbf{E} - \mathbf{R}_{23}\mathbf{R}_{21})^{-1} \tag{320}$$

$$\mathbf{A}_{31} = \mathbf{T}_{21}\mathbf{P}_{31} \tag{321}$$

$$\mathbf{B}_{13} = \mathbf{R}_{23}\mathbf{T}_{12} \tag{322}$$

$$\mathbf{C}_{31} = \mathbf{S}_{32}\mathcal{X}_{12} + \mathbf{R}_{23}\mathbf{S}_{12} \tag{323}$$

$$\mathbf{S}_{31} = \mathbf{S}_{21} + \mathbf{A}_{31}\mathbf{C}_{31} \tag{324}$$

$$\mathbf{a}_{31} = \mathbf{A}_{31}(\mathbf{R}_{23}\mathbf{a}_{12} + \mathbf{a}_{32}) + \mathbf{a}_{21} \tag{325}$$

$$\mathbf{Sl}_{31} = \mathbf{A}_{31}(\mathbf{R}_{23}\mathbf{Sl}_{12} + \mathbf{Sl}_{32} + \mathbf{a}_{31}f) + \mathbf{Sl}_{21} \tag{326}$$

$$\mathbf{R}_{13} = \mathbf{R}_{12} + \mathbf{A}_{31}\mathbf{B}_{13} \tag{327}$$

$$\mathbf{T}_{31} = \mathbf{A}_{31}\mathbf{T}_{32} \tag{328}$$

12.1.2 Tangent linear

$$\mathcal{L}(\mathcal{X}_{12}) = \left[-\mathcal{L}(\tau_{12})\lambda_{12}\right]\mathcal{X}_{12} \tag{329}$$

12.1.2.1 unlinearized + particular linearized

$$\mathcal{L}(\mathbf{S}_{31}) = \mathbf{A}_{31}\mathcal{L}(\mathbf{S}_{32})\mathcal{X}_{12} \tag{330}$$

12.1.2.2 particular linearized + unlinearized

$$\mathcal{L}(\mathbf{S}_{31}) = \mathcal{L}(\mathbf{S}_{21}) + \mathbf{A}_{31}\mathbf{R}_{23}\mathcal{L}(\mathbf{S}_{12}) \tag{331}$$

12.1.2.3 particular linearized + particular linearized

$$\mathcal{L}(\mathbf{S}_{31}) = \mathcal{L}(\mathbf{S}_{21}) + \mathbf{A}_{31}(\mathcal{L}(\mathbf{S}_{32})\mathcal{X}_{12} + \mathbf{R}_{23}\mathcal{L}(\mathbf{S}_{12})) \tag{332}$$

12.1.2.4 unlinearized + homogeneous linearized

$$\mathbf{D}_{31} = \mathbf{A}_{31}\mathcal{L}(\mathbf{R}_{23}) \tag{333}$$

$$\mathbf{E}_{31} = \mathbf{D}_{31}\mathbf{R}_{21}\mathbf{P}_{31} \tag{334}$$

$$\mathcal{L}(\mathbf{S}_{31}) = \mathbf{E}_{31}\mathbf{C}_{31} + \mathbf{A}_{31}(\mathcal{L}(\mathbf{S}_{32})\mathcal{X}_{12} + \mathcal{L}(\mathbf{R}_{23})\mathbf{S}_{12}) \tag{335}$$

$$\mathcal{L}(\mathbf{R}_{13}) = \mathbf{E}_{31}\mathbf{B}_{13} + \mathbf{D}_{31}\mathbf{T}_{12} \tag{336}$$

$$\mathcal{L}(\mathbf{T}_{31}) = \mathbf{E}_{31}\mathbf{T}_{32} + \mathbf{A}_{31}\mathcal{L}(\mathbf{T}_{32}) \tag{337}$$

12.1.2.5 homogeneous linearized + unlinearized

$$\mathbf{D}_{31} = \mathbf{A}_{31}\mathbf{R}_{23} \tag{338}$$

$$\mathbf{E}_{31} = \left[\mathcal{L}(\mathbf{T}_{21}) + \mathbf{D}_{31}\mathcal{L}(\mathbf{R}_{21})\right]\mathbf{P}_{31} \tag{339}$$

$$\mathcal{L}(\mathbf{S}_{31}) = \mathcal{L}(\mathbf{S}_{21}) + \mathbf{E}_{31}\mathbf{C}_{31} + \mathbf{A}_{31}(\mathbf{S}_{32}\mathcal{L}(\mathcal{X}_{12}) + \mathbf{R}_{23}\mathcal{L}(\mathbf{S}_{12})) \tag{340}$$

$$\mathcal{L}(\mathbf{R}_{13}) = \mathcal{L}(\mathbf{R}_{12}) + \mathbf{E}_{31}\mathbf{B}_{13} + \mathbf{D}_{31}\mathcal{L}(\mathbf{T}_{12}) \tag{341}$$

$$\mathcal{L}(\mathbf{T}_{31}) = \mathbf{E}_{31}\mathbf{T}_{32} \tag{342}$$

12.1.2.6 particular linearized + homogeneous linearized

$$\mathbf{D}_{31} = \mathbf{A}_{31}\mathcal{L}(\mathbf{R}_{23}) \tag{343}$$

$$\mathbf{E}_{31} = \mathbf{D}_{31}\mathbf{R}_{21}\mathbf{P}_{31} \tag{344}$$

$$\mathcal{L}(\mathbf{S}_{31}) = \mathcal{L}(\mathbf{S}_{21}) + \mathbf{E}_{31}\mathbf{C}_{31} + \mathbf{A}_{31}(\mathcal{L}(\mathbf{S}_{32})\mathcal{X}_{12} + \mathcal{L}(\mathbf{R}_{23})\mathbf{S}_{12} + \mathbf{R}_{23}\mathcal{L}(\mathbf{S}_{12})) \tag{345}$$

$$\mathcal{L}(\mathbf{R}_{13}) = \mathbf{E}_{31}\mathbf{B}_{13} + \mathbf{D}_{31}\mathbf{T}_{12} \tag{346}$$

$$\mathcal{L}(\mathbf{T}_{31}) = \mathbf{E}_{31}\mathbf{T}_{32} + \mathbf{A}_{31}\mathcal{L}(\mathbf{T}_{32}) \tag{347}$$

12.1.2.7 homogeneous linearized + particular linearized

$$\mathbf{D}_{31} = \mathbf{A}_{31}\mathbf{R}_{23} \tag{348}$$

$$\mathbf{E}_{31} = (\mathcal{L}(\mathbf{T}_{21}) + \mathbf{D}_{31}\mathcal{L}(\mathbf{R}_{21}))\mathbf{P}_{31} \tag{349}$$

$$\mathcal{L}(\mathbf{S}_{31}) = \mathcal{L}(\mathbf{S}_{21}) + \mathbf{E}_{31}\mathbf{C}_{31} + \mathbf{A}_{31}(\mathcal{L}(\mathbf{S}_{32})\mathcal{X}_{12} + \mathbf{S}_{32}\mathcal{L}(\mathcal{X}_{12}) + \mathbf{R}_{23}\mathcal{L}(\mathbf{S}_{12})) \tag{350}$$

$$\mathcal{L}(\mathbf{R}_{13}) = \mathcal{L}(\mathbf{R}_{12}) + \mathbf{E}_{31}\mathbf{B}_{13} + \mathbf{D}_{31}\mathcal{L}(\mathbf{T}_{12}) \tag{351}$$

$$\mathcal{L}(\mathbf{T}_{31}) = \mathbf{E}_{31}\mathbf{T}_{32} \tag{352}$$

12.1.2.8 homogeneous linearized + homogeneous linearized

$$\mathbf{D}_{31} = [\mathcal{L}(\mathbf{T}_{21}) + \mathbf{A}_{31}(\mathcal{L}(\mathbf{R}_{23})\mathbf{R}_{21} + \mathbf{R}_{23}\mathcal{L}(\mathbf{R}_{21}))]\,\mathbf{P}_{31} \tag{353}$$

$$\mathcal{L}(\mathbf{S}_{31}) = \mathcal{L}(\mathbf{S}_{21}) + \mathbf{D}_{31}\mathbf{C}_{31} + \mathbf{A}_{31}(\mathcal{L}(\mathbf{S}_{32})\mathcal{X}_{12} + \mathbf{S}_{32}\mathcal{L}(\mathcal{X}_{12}) + \mathcal{L}(\mathbf{R}_{23})\mathbf{S}_{12} + \mathbf{R}_{23}\mathcal{L}(\mathbf{S}_{12})) \tag{354}$$

$$\mathcal{L}(\mathbf{R}_{13}) = \mathcal{L}(\mathbf{R}_{12}) + \mathbf{D}_{31}\mathbf{B}_{13} + \mathbf{A}_{31}(\mathcal{L}(\mathbf{R}_{23})\mathbf{T}_{12} + \mathbf{R}_{23}\mathcal{L}(\mathbf{T}_{12})) \tag{355}$$

$$\mathcal{L}(\mathbf{T}_{31}) = \mathbf{D}_{31}\mathbf{T}_{32} + \mathbf{A}_{31}\mathcal{L}(\mathbf{T}_{32}) \tag{356}$$

12.1.3 Adjoint of tangent linear

12.1.3.1 unlinearized + particular linearized

$$\mathcal{A}(\mathbf{S}_{32}) = \mathbf{A}_{31}^T\mathcal{A}(\mathbf{S}_{31})\mathcal{X}_{12} \tag{357}$$

12.1.3.2 particular linearized + unlinearized

$$\mathcal{A}(\mathbf{S}_{21}) = \mathcal{A}(\mathbf{S}_{31}) \tag{358}$$

12.1.3.3 particular linearized + particular linearized

$$\mathcal{A}(\mathbf{S}_{21}) = \mathcal{A}(\mathbf{S}_{31}) \tag{359}$$

$$\mathbf{t} = \mathbf{A}_{31}^T\mathcal{A}(\mathbf{S}_{31}) \tag{360}$$

$$\mathcal{A}(\mathbf{S}_{32}) = \mathbf{t}\mathcal{X}_{12} \tag{361}$$

$$\mathcal{A}(\mathbf{S}_{12}) = \mathbf{R}_{23}^T\mathbf{t} \tag{362}$$

12.1.3.4 unlinearized + homogeneous linearized

$$\mathcal{A}(\mathbf{E}_{31}) = \mathcal{A}(\mathbf{T}_{31})\mathbf{T}_{32}^T \tag{363}$$

$$\mathcal{A}(\mathbf{T}_{32}) = \mathbf{A}_{31}^T\mathcal{A}(\mathbf{T}_{31}) \tag{364}$$

$$\mathcal{A}(\mathbf{E}_{31}) = \mathcal{A}(\mathbf{E}_{31}) + \mathcal{A}(\mathbf{R}_{13})\mathbf{B}_{13}^T \tag{365}$$

$$\mathcal{A}(\mathbf{D}_{31}) = \mathcal{A}(\mathbf{R}_{13})\mathbf{T}_{12}^T \tag{366}$$

$$\mathcal{A}(\mathbf{E}_{31}) = \mathcal{A}(\mathbf{E}_{31}) + \mathcal{A}(\mathbf{S}_{31})\mathbf{C}_{31}^T \tag{367}$$

$$\mathbf{t} = \mathbf{A}_{31}^T\mathcal{A}(\mathbf{S}_{31}) \tag{368}$$

$$\mathcal{A}(\mathbf{S}_{32}) = \mathbf{t}\mathcal{X}_{12} \tag{369}$$

$$\mathcal{A}(\mathbf{R}_{32}) = \mathbf{t}\mathbf{S}_{12}^T \tag{370}$$

$$\mathcal{A}(\mathbf{D}_{31}) = \mathcal{A}(\mathbf{D}_{31}) + \mathcal{A}(\mathbf{E}_{31})\mathbf{P}_{31}^T\mathbf{R}_{21}^T \tag{371}$$

$$\mathcal{A}(\mathbf{R}_{23}) = \mathbf{A}_{31}^T\mathcal{A}(\mathbf{D}_{31}) \tag{372}$$

12.1.3.5 homogeneous linearized + unlinearized

$$\mathcal{A}(\mathbf{E}_{31}) = \mathcal{A}(\mathbf{T}_{31})\mathbf{T}_{32}^T \tag{373}$$

$$\mathcal{A}(\mathbf{R}_{12}) = \mathcal{A}(\mathbf{R}_{13}) \tag{374}$$

$$\mathcal{A}(\mathbf{E}_{31}) = \mathcal{A}(\mathbf{E}_{31}) + \mathcal{A}(\mathbf{R}_{13})\mathbf{B}_{13}^T \tag{375}$$

$$\mathcal{A}(\mathbf{T}_{12}) = \mathbf{D}_{31}^T\mathcal{A}(\mathbf{R}_{13}) \tag{376}$$

$$\mathcal{A}(\mathbf{S}_{21}) = \mathcal{A}(\mathbf{S}_{31}) \tag{377}$$

$$\mathcal{A}(\mathbf{E}_{31}) = \mathcal{A}(\mathbf{E}_{31}) + \mathcal{A}(\mathbf{S}_{31})\mathbf{C}_{31}^T \tag{378}$$

$$\mathbf{t} = \mathbf{A}_{31}^T\mathcal{A}(\mathbf{S}_{31}) \tag{379}$$

$$\mathcal{A}(\mathcal{X}_{12}) = \mathbf{S}_{32}^T\mathbf{t} \tag{380}$$

$$\mathcal{A}(\mathbf{S}_{12}) = \mathbf{R}_{23}^T \mathbf{t} \tag{381}$$

$$\mathbf{t} = \mathcal{A}(\mathbf{E}_{31}) \mathbf{P}_{31}^T \tag{382}$$

$$\mathcal{A}(\mathbf{T}_{21}) = \mathbf{t} \tag{383}$$

$$\mathcal{A}(\mathbf{R}_{21}) = \mathbf{D}_{31}^T \mathbf{t} \tag{384}$$

12.1.3.6 particular linearized + homogeneous linearized

$$\mathcal{A}(\mathbf{E}_{31}) = \mathcal{A}(\mathbf{T}_{31}) \mathbf{T}_{32}^T \tag{385}$$

$$\mathcal{A}(\mathbf{T}_{32}) = \mathbf{A}_{31}^T \mathcal{A}(\mathbf{T}_{31}) \tag{386}$$

$$\mathcal{A}(\mathbf{E}_{31}) = \mathcal{A}(\mathbf{E}_{31}) + \mathcal{A}(\mathbf{R}_{13}) \mathbf{B}_{13}^T \tag{387}$$

$$\mathcal{A}(\mathbf{D}_{31}) = \mathcal{A}(\mathbf{R}_{13}) \mathbf{T}_{12}^T \tag{388}$$

$$\mathcal{A}(\mathbf{S}_{21}) = \mathcal{A}(\mathbf{S}_{31}) \tag{389}$$

$$\mathcal{A}(\mathbf{E}_{31}) = \mathcal{A}(\mathbf{E}_{31}) + \mathcal{A}(\mathbf{S}_{31}) \mathbf{C}_{31}^T \tag{390}$$

$$\mathbf{t} = \mathbf{A}_{31}^T \mathcal{A}(\mathbf{S}_{31}) \tag{391}$$

$$\mathcal{A}(\mathbf{S}_{32}) = \mathbf{t} \mathcal{X}_{12} \tag{392}$$

$$\mathcal{A}(\mathbf{R}_{23}) = \mathbf{t} \mathbf{S}_{12}^T \tag{393}$$

$$\mathcal{A}(\mathbf{S}_{12}) = \mathbf{R}_{23}^T \mathbf{t} \tag{394}$$

$$\mathcal{A}(\mathbf{D}_{31}) = \mathcal{A}(\mathbf{E}_{31}) \mathbf{P}_{31}^T \mathbf{R}_{21}^T \tag{395}$$

$$\mathcal{A}(\mathbf{R}_{23}) = \mathbf{A}_{31}^T \mathcal{A}(\mathbf{D}_{31}) \tag{396}$$

12.1.3.7 homogeneous linearized + particular linearized

$$\mathcal{A}(\mathbf{E}_{31}) = \mathcal{A}(\mathbf{T}_{31})\mathbf{T}_{32}^T \tag{397}$$

$$\mathcal{A}(\mathbf{R}_{12}) = \mathcal{A}(\mathbf{R}_{13}) \tag{398}$$

$$\mathcal{A}(\mathbf{E}_{31}) = \mathcal{A}(\mathbf{E}_{31}) + \mathcal{A}(\mathbf{R}_{13})\mathbf{B}_{13}^T \tag{399}$$

$$\mathcal{A}(\mathbf{T}_{12}) = \mathbf{D}_{31}^T\mathcal{A}(\mathbf{R}_{13}) \tag{400}$$

$$\mathcal{A}(\mathbf{S}_{21}) = \mathcal{A}(\mathbf{S}_{31}) \tag{401}$$

$$\mathcal{A}(\mathbf{E}_{31}) = \mathcal{A}(\mathbf{E}_{31}) + \mathcal{A}(\mathbf{S}_{31})\mathbf{C}_{31}^T \tag{402}$$

$$\mathbf{t} = \mathbf{A}_{31}^T\mathcal{A}(\mathbf{S}_{31}) \tag{403}$$

$$\mathcal{A}(\mathbf{S}_{32}) = \mathbf{t}\mathcal{X}_{12} \tag{404}$$

$$\mathcal{A}(\mathcal{X}_{12}) = \mathbf{S}_{32}^T\mathbf{t} \tag{405}$$

$$\mathcal{A}(\mathbf{S}_{12}) = \mathbf{R}_{23}^T\mathbf{t} \tag{406}$$

$$\mathbf{t} = \mathcal{A}(\mathbf{E}_{31})\mathbf{P}_{31}^T \tag{407}$$

$$\mathcal{A}(\mathbf{T}_{21}) = \mathbf{t} \tag{408}$$

$$\mathcal{A}(\mathbf{R}_{21}) = \mathbf{D}_{31}^T\mathbf{t} \tag{409}$$

12.1.3.8 homogeneous linearized + homogeneous linearized

$$\mathcal{A}(\mathbf{D}_{31}) = \mathcal{A}(\mathbf{T}_{31})\mathbf{T}_{32}^T \tag{410}$$

$$\mathcal{A}(\mathbf{T}_{32}) = \mathbf{A}_{31}^T\mathcal{A}(\mathbf{T}_{31}) \tag{411}$$

$$\mathcal{A}(\mathbf{R}_{12}) = \mathcal{A}(\mathbf{R}_{13}) \tag{412}$$

$$\mathcal{A}(\mathbf{D}_{31}) = \mathcal{A}(\mathbf{D}_{31}) + \mathcal{A}(\mathbf{R}_{13})\mathbf{B}_{13}^T \tag{413}$$

$$\mathbf{t} = \mathbf{A}_{31}^T\mathcal{A}(\mathbf{R}_{13}) \tag{414}$$

$$\mathcal{A}(\mathbf{R}_{23}) = \mathbf{t}\mathbf{T}_{12}^T \tag{415}$$

$$\mathcal{A}(\mathbf{T}_{12}) = \mathbf{R}_{23}^T\mathbf{t} \tag{416}$$

$$\mathcal{A}(\mathbf{S}_{21}) = \mathcal{A}(\mathbf{S}_{31}) \tag{417}$$

$$\mathcal{A}(\mathbf{D}_{31}) = \mathcal{A}(\mathbf{D}_{31}) + \mathcal{A}(\mathbf{S}_{31})\mathbf{C}_{31}^T \tag{418}$$

$$\mathbf{t} = \mathbf{A}_{31}^T\mathcal{A}(\mathbf{S}_{31}) \tag{419}$$

$$\mathcal{A}(\mathbf{S}_{32}) = \mathbf{t}\mathcal{X}_{12} \tag{420}$$

$$\mathcal{A}(\mathcal{X}_{12}) = \mathbf{S}_{32}^T\mathbf{t} \tag{421}$$

$$\mathcal{A}(\mathbf{R}_{23}) = \mathbf{t}\mathbf{S}_{12}^T \tag{422}$$

$$\mathcal{A}(\mathbf{S}_{12}) = \mathbf{R}_{23}^T\mathbf{t} \tag{423}$$

$$\mathbf{t} = \mathcal{A}(\mathbf{D}_{31})\mathbf{P}_{31}^T \tag{424}$$

$$\mathcal{A}(\mathbf{T}_{21}) = \mathbf{t} \tag{425}$$

$$\mathbf{t}_2 = \mathbf{A}_{31}^T\mathbf{t} \tag{426}$$

$$\mathcal{A}(\mathbf{R}_{23}) = \mathbf{t}_2\mathbf{R}_{21}^T \tag{427}$$

$$\mathcal{A}(\mathbf{R}_{21}) = \mathbf{R}_{23}^T\mathbf{t}_2 \tag{428}$$

12.2 Downward: $(\mathbf{R}_{31},\ \mathbf{T}_{13},\ \mathbf{S}_{13})$

12.2.1 Forward

$$\mathbf{P}_{13} = (\mathbf{E} - \mathbf{R}_{21}\mathbf{R}_{23})^{-1} \tag{429}$$

$$\mathbf{A}_{13} = \mathbf{T}_{23}\mathbf{P}_{13} \tag{430}$$

$$\mathbf{B}_{31} = \mathbf{R}_{21}\mathbf{T}_{32} \tag{431}$$

$$\mathbf{C}_{13} = \mathbf{S}_{12} + \mathbf{R}_{21}\mathbf{S}_{32}\mathcal{X}_{12} \tag{432}$$

$$\mathbf{S}_{13} = \mathbf{S}_{23}\mathcal{X}_{12} + \mathbf{A}_{13}\mathbf{C}_{13} \tag{433}$$

$$\mathbf{a}_{13} = \mathbf{A}_{13}(\mathbf{R}_{21}\mathbf{a}_{32} + \mathbf{a}_{12}) + \mathbf{a}_{23} \tag{434}$$

$$\mathbf{Sl}_{13} = \mathbf{A}_{13}\left[\mathbf{R}_{21}(\mathbf{Sl}_{32} + \mathbf{a}_{32}f) + \mathbf{Sl}_{12}\right] + \mathbf{Sl}_{23} + \mathbf{a}_{23}f \tag{435}$$

$$\mathbf{R}_{31} = \mathbf{R}_{32} + \mathbf{A}_{13}\mathbf{B}_{31} \tag{436}$$

$$\mathbf{T}_{13} = \mathbf{A}_{13}\mathbf{T}_{12} \tag{437}$$

12.2.2 Tangent linear

12.2.2.1 unlinearized + particular linearized

$$\mathcal{L}(\mathbf{S}_{13}) = \mathcal{L}(\mathbf{S}_{23})\mathcal{X}_{12} + \mathbf{A}_{13}\mathbf{R}_{21}\mathcal{L}(\mathbf{S}_{32})\mathcal{X}_{12} \tag{438}$$

12.2.2.2 particular linearized + unlinearized

$$\mathcal{L}(\mathbf{S}_{13}) = \mathbf{A}_{13}\mathcal{L}(\mathbf{S}_{12}) \tag{439}$$

12.2.2.3 particular linearized + particular linearized

$$\mathcal{L}(\mathbf{S}_{13}) = \mathcal{L}(\mathbf{S}_{23})\mathcal{X}_{12} + \mathbf{A}_{13}\left[\mathcal{L}(\mathbf{S}_{12}) + \mathbf{R}_{21}\mathcal{L}(\mathbf{S}_{32})\mathcal{X}_{12}\right] \tag{440}$$

12.2.2.4 unlinearized + homogeneous linearized

$$\mathbf{D}_{13} = \mathbf{A}_{13}\mathbf{R}_{21} \tag{441}$$

$$\mathbf{E}_{13} = (\mathcal{L}(\mathbf{T}_{23}) + \mathbf{D}_{13}\mathcal{L}(\mathbf{R}_{23}))\mathbf{P}_{13} \tag{442}$$

$$\mathcal{L}(\mathbf{S}_{13}) = \mathcal{L}(\mathbf{S}_{23})\mathcal{X}_{12} + \mathbf{E}_{13}\mathbf{C}_{13} + \mathbf{A}_{13}\mathbf{R}_{21}\mathcal{L}(\mathbf{S}_{32})\mathcal{X}_{12} \tag{443}$$

$$\mathcal{L}(\mathbf{R}_{31}) = \mathcal{L}(\mathbf{R}_{32}) + \mathbf{E}_{13}\mathbf{B}_{31} + \mathbf{D}_{13}\mathcal{L}(\mathbf{T}_{32}) \tag{444}$$

$$\mathcal{L}(\mathbf{T}_{13}) = \mathbf{E}_{13}\mathbf{T}_{12} \tag{445}$$

12.2.2.5 homogeneous linearized + unlinearized

$$\mathbf{D}_{13} = \mathbf{A}_{13}\mathcal{L}(\mathbf{R}_{21}) \tag{446}$$

$$\mathbf{E}_{13} = \mathbf{D}_{13}\mathbf{R}_{23}\mathbf{P}_{13} \tag{447}$$

$$\mathcal{L}(\mathbf{S}_{13}) = \mathbf{S}_{23}\mathcal{L}(\mathcal{X}_{12}) + \mathbf{E}_{13}\mathbf{C}_{13} + \mathbf{A}_{13}\left[\mathcal{L}(\mathbf{S}_{12}) + \mathcal{L}(\mathbf{R}_{21})\mathbf{S}_{32}\mathcal{X}_{12} + \mathbf{R}_{21}\mathbf{S}_{32}\mathcal{L}(\mathcal{X}_{12})\right] \tag{448}$$

$$\mathcal{L}(\mathbf{R}_{31}) = \mathbf{E}_{13}\mathbf{B}_{31} + \mathbf{D}_{13}\mathbf{T}_{32} \tag{449}$$

$$\mathcal{L}(\mathbf{T}_{13}) = \mathbf{E}_{13}\mathbf{T}_{12} + \mathbf{A}_{13}\mathcal{L}(\mathbf{T}_{12}) \tag{450}$$

12.2.2.6 particular linearized + homogeneous linearized

$$\mathbf{D}_{13} = \mathbf{A}_{13}\mathbf{R}_{21} \tag{451}$$

$$\mathbf{E}_{13} = \left[\mathcal{L}(\mathbf{T}_{23}) + \mathbf{D}_{13}\mathcal{L}(\mathbf{R}_{23})\right]\mathbf{P}_{13} \tag{452}$$

$$\mathcal{L}(\mathbf{S}_{13}) = \mathcal{L}(\mathbf{S}_{23})\mathcal{X}_{12} + \mathbf{E}_{13}\mathbf{C}_{13} + \mathbf{A}_{13}\left[\mathcal{L}(\mathbf{S}_{12}) + \mathbf{R}_{21}\mathcal{L}(\mathbf{S}_{32})\mathcal{X}_{12}\right] \tag{453}$$

$$\mathcal{L}(\mathbf{R}_{31}) = \mathcal{L}(\mathbf{R}_{32}) + \mathbf{E}_{13}\mathbf{B}_{31} + \mathbf{D}_{13}\mathcal{L}(\mathbf{T}_{32}) \tag{454}$$

$$\mathcal{L}(\mathbf{T}_{13}) = \mathbf{E}_{13}\mathbf{T}_{12} \tag{455}$$

12.2.2.7 homogeneous linearized + particular linearized

$$\mathbf{D}_{13} = \mathbf{A}_{13}\mathcal{L}(\mathbf{R}_{21}) \tag{456}$$

$$\mathbf{E}_{13} = \mathbf{D}_{13}\mathbf{R}_{23}\mathbf{P}_{13} \tag{457}$$

$$\mathcal{L}(\mathbf{S}_{13}) = \mathcal{L}(\mathbf{S}_{23})\mathcal{X}_{12} + \mathbf{S}_{23}\mathcal{L}(\mathcal{X}_{12}) + \mathbf{E}_{13}\mathbf{C}_{13} + \mathbf{A}_{13}\left\{\mathcal{L}(\mathbf{S}_{12}) + \mathcal{L}(\mathbf{R}_{21})\mathbf{S}_{32}\mathcal{X}_{12} + \mathbf{R}_{21}\left[\mathcal{L}(\mathbf{S}_{32})\mathcal{X}_{12} + \mathbf{S}_{32}\mathcal{L}(\mathcal{X}_{12})\right]\right\} \tag{458}$$

$$\mathcal{L}(\mathbf{R}_{31}) = \mathbf{E}_{13}\mathbf{B}_{31} + \mathbf{D}_{13}\mathbf{T}_{32} \tag{459}$$

$$\mathcal{L}(\mathbf{T}_{13}) = \mathbf{E}_{13}\mathbf{T}_{12} + \mathbf{A}_{13}\mathcal{L}(\mathbf{T}_{12}) \tag{460}$$

12.2.2.8 homogeneous linearized + homogeneous linearized

$$\mathbf{D}_{13} = \left[\mathcal{L}(\mathbf{T}_{23}) + \mathbf{A}_{13}(\mathcal{L}(\mathbf{R}_{21})\mathbf{R}_{23} + \mathbf{R}_{21}\mathcal{L}(\mathbf{R}_{23}))\right]\mathbf{P}_{13} \tag{461}$$

$$\mathcal{L}(\mathbf{S}_{13}) = \mathcal{L}(\mathbf{S}_{23})\mathcal{X}_{12}+\mathbf{S}_{23}\mathcal{L}(\mathcal{X}_{12})+\mathbf{D}_{13}\mathbf{C}_{13}+\mathbf{A}_{13}\left[\mathcal{L}(\mathbf{S}_{12}) + \mathcal{L}(\mathbf{R}_{21})\mathbf{S}_{32}\mathcal{X}_{12} + \mathbf{R}_{21}(\mathcal{L}(\mathbf{S}_{32})\mathcal{X}_{12} + \mathbf{S}_{32}\mathcal{L}(\mathcal{X}_{12}))\right] \tag{462}$$

$$\mathcal{L}(\mathbf{R}_{31}) = \mathcal{L}(\mathbf{R}_{32}) + \mathbf{D}_{13}\mathbf{B}_{31} + \mathbf{A}_{13}(\mathcal{L}(\mathbf{R}_{21})\mathbf{T}_{32} + \mathbf{R}_{21}\mathcal{L}(\mathbf{T}_{32})) \tag{463}$$

$$\mathcal{L}(\mathbf{T}_{13}) = \mathbf{D}_{13}\mathbf{T}_{12} + \mathbf{A}_{13}\mathcal{L}(\mathbf{T}_{12}) \tag{464}$$

12.2.3 Adjoint of tangent linear

12.2.3.1 unlinearized + particular linearized

$$\mathcal{A}(\mathbf{S}_{23}) = \mathcal{A}(\mathbf{S}_{13})\mathcal{X}_{12} \tag{465}$$

$$\mathcal{A}(\mathbf{S}_{32}) = \mathbf{R}_{21}^T\mathbf{A}_{13}^T\mathcal{A}(\mathbf{S}_{13})\mathcal{X}_{12} \tag{466}$$

12.2.3.2 particular linearized + unlinearized

$$\mathcal{A}(\mathbf{S}_{12}) = \mathbf{A}_{13}^T\mathcal{A}(\mathbf{S}_{13}) \tag{467}$$

12.2.3.3 particular linearized + particular linearized

$$\mathcal{A}(\mathbf{S}_{23}) = \mathcal{A}(\mathbf{S}_{13})\mathcal{X}_{12} \tag{468}$$

$$\mathbf{t} = \mathbf{A}_{13}^T\mathcal{A}(\mathbf{S}_{13}) \tag{469}$$

$$\mathcal{A}(\mathbf{S}_{12}) = \mathbf{t} \tag{470}$$

$$\mathcal{A}(\mathbf{S}_{32}) = \mathbf{R}_{21}^T\mathcal{A}(\mathbf{S}_{13})\mathcal{X}_{12} \tag{471}$$

12.2.3.4 unlinearized + homogeneous linearized

$$\mathcal{A}(\mathbf{E}_{13}) = \mathcal{A}(\mathbf{T}_{13})\mathbf{T}_{12}^T \tag{472}$$

$$\mathcal{A}(\mathbf{R}_{32}) = \mathcal{A}(\mathbf{R}_{31}) \tag{473}$$

$$\mathcal{A}(\mathbf{E}_{13}) = \mathcal{A}(\mathbf{E}_{13}) + \mathcal{A}(\mathbf{R}_{31})\mathbf{B}_{31}^T \tag{474}$$

$$\mathcal{A}(\mathbf{T}_{32}) = \mathbf{D}_{13}^T\mathcal{A}(\mathbf{R}_{31}) \tag{475}$$

$$\mathcal{A}(\mathbf{S}_{23}) = \mathcal{A}(\mathbf{S}_{13})\mathcal{X}_{12} \tag{476}$$

$$\mathcal{A}(\mathbf{E}_{13}) = \mathcal{A}(\mathbf{E}_{13}) + \mathcal{A}(\mathbf{S}_{13})\mathbf{C}_{13}^T \tag{477}$$

$$\mathcal{A}(\mathbf{S}_{32}) = \mathbf{R}_{21}^T\mathbf{A}_{13}^T\mathcal{A}(\mathbf{S}_{13})t_{32} \tag{478}$$

$$\mathbf{t} = \mathcal{A}(\mathbf{E}_{13})\mathbf{P}_{13}^T \tag{479}$$

$$\mathcal{A}(\mathbf{T}_{23}) = \mathbf{t} \tag{480}$$

$$\mathcal{A}(\mathbf{R}_{23}) = \mathbf{D}_{13}^T\mathbf{t} \tag{481}$$

12.2.3.5 homogeneous linearized + unlinearized

$$\mathcal{A}(\mathbf{E}_{13}) = \mathcal{A}(\mathbf{T}_{13})\mathbf{T}_{12}^T \tag{482}$$

$$\mathcal{A}(\mathbf{T}_{12}) = \mathbf{A}_{13}^T\mathcal{A}(\mathbf{T}_{13}) \tag{483}$$

$$\mathcal{A}(\mathbf{E}_{13}) = \mathcal{A}(\mathbf{E}_{13}) + \mathcal{A}(\mathbf{R}_{31})\mathbf{B}_{31}^T \tag{484}$$

$$\mathcal{A}(\mathbf{D}_{13}) = \mathcal{A}(\mathbf{R}_{31})\mathbf{T}_{32}^T \tag{485}$$

$$\mathcal{A}(\mathcal{X}_{12}) = \mathbf{S}_{23}^T\mathcal{A}(\mathbf{S}_{13}) \tag{486}$$

$$\mathcal{A}(\mathbf{E}_{13}) = \mathcal{A}(\mathbf{E}_{13}) + \mathcal{A}(\mathbf{S}_{13})\mathbf{C}_{13}^T \tag{487}$$

$$\mathbf{t} = \mathbf{A}_{13}^T\mathcal{A}(\mathbf{S}_{13}) \tag{488}$$

$$\mathcal{A}(\mathbf{S}_{12}) = \mathbf{t} \tag{489}$$

$$\mathcal{A}(\mathbf{R}_{12}) = \mathcal{A}(\mathbf{S}_{13})\mathbf{S}_{32}^T\mathcal{X}_{12} \tag{490}$$

$$\mathcal{A}(\mathcal{X}_{12}) = \mathcal{A}(\mathcal{X}_{12}) + \mathbf{S}_{32}^T\mathbf{R}_{21}^T\mathcal{A}(\mathbf{S}_{13}) \tag{491}$$

$$\mathcal{A}(\mathbf{D}_{13}) = \mathcal{A}(\mathbf{E}_{13})\mathbf{D}_{13}^T\mathbf{R}_{23}^T \tag{492}$$

$$\mathcal{A}(\mathbf{R}_{21}) = \mathcal{A}(\mathbf{R}_{21}) + \mathbf{A}_{13}^T\mathcal{A}(\mathbf{D}_{13}) \tag{493}$$

12.2.3.6 particular linearized + homogeneous linearized

$$\mathcal{A}(\mathbf{E}_{13}) = \mathcal{A}(\mathbf{T}_{13})\mathbf{T}_{12}^T \tag{494}$$

$$\mathcal{A}(\mathbf{R}_{32}) = \mathcal{A}(\mathbf{R}_{31}) \tag{495}$$

$$\mathcal{A}(\mathbf{E}_{13}) = \mathcal{A}(\mathbf{E}_{13}) + \mathcal{A}(\mathbf{R}_{31})\mathbf{B}_{31}^T \tag{496}$$

$$\mathcal{A}(\mathbf{T}_{32}) = \mathbf{D}_{13}^T\mathcal{A}(\mathbf{R}_{31}) \tag{497}$$

$$\mathcal{A}(\mathbf{S}_{23}) = \mathcal{A}(\mathbf{S}_{13})\mathcal{X}_{12} \tag{498}$$

$$\mathcal{A}(\mathbf{E}_{13}) = \mathcal{A}(\mathbf{E}_{13}) + \mathcal{A}(\mathbf{S}_{13})\mathbf{C}_{13}^T \tag{499}$$

$$\mathbf{t} = \mathbf{A}_{13}^T\mathcal{A}(\mathbf{S}_{13}) \tag{500}$$

$$\mathcal{A}(\mathbf{S}_{12}) = \mathbf{t} \tag{501}$$

$$\mathcal{A}(\mathbf{S}_{32}) = \mathbf{R}_{21}^T\mathcal{A}(\mathbf{S}_{13})\mathcal{X}_{12} \tag{502}$$

$$\mathbf{t} = \mathcal{A}(\mathbf{E}_{13})\mathbf{P}_{13}^T \tag{503}$$

$$\mathcal{A}(\mathbf{T}_{23}) = \mathbf{t} \tag{504}$$

$$\mathcal{A}(\mathbf{R}_{23}) = \mathbf{D}_{13}^T\mathbf{t} \tag{505}$$

12.2.3.7 homogeneous linearized + particular linearized

$$\mathcal{A}(\mathbf{E}_{13}) = \mathcal{A}(\mathbf{T}_{13})\mathbf{T}_{12}^T \tag{506}$$

$$\mathcal{A}(\mathbf{T}_{12}) = \mathbf{A}_{13}^T\mathcal{A}(\mathbf{T}_{13}) \tag{507}$$

$$\mathcal{A}(\mathbf{E}_{13}) = \mathcal{A}(\mathbf{E}_{13}) + \mathcal{A}(\mathbf{R}_{31})\mathbf{B}_{31}^T \tag{508}$$

$$\mathcal{A}(\mathbf{D}_{13}) = \mathcal{A}(\mathbf{R}_{31})\mathbf{T}_{32}^T \tag{509}$$

$$\mathcal{A}(\mathbf{S}_{23}) = \mathcal{A}(\mathbf{S}_{13})\mathcal{X}_{12} \tag{510}$$

$$\mathcal{A}(\mathcal{X}_{12}) = \mathbf{S}_{23}^T\mathcal{A}(\mathbf{S}_{13}) \tag{511}$$

$$\mathcal{A}(\mathbf{E}_{13}) = \mathcal{A}(\mathbf{E}_{13}) + \mathcal{A}(\mathbf{S}_{13})\mathbf{C}_{13}^T \tag{512}$$

$$\mathbf{t} = \mathbf{A}_{13}^T \mathcal{A}(\mathbf{S}_{13}) \tag{513}$$

$$\mathcal{A}(\mathbf{S}_{12}) = \mathbf{t} \tag{514}$$

$$\mathcal{A}(\mathbf{R}_{21}) = \mathbf{t}\mathbf{S}_{32}^T \mathcal{X}_{12} \tag{515}$$

$$\mathbf{t}_2 = \mathbf{R}_{21}^T \mathbf{t} \tag{516}$$

$$\mathcal{A}(\mathbf{S}_{32}) = \mathbf{t}_2 \mathcal{X}_{12} \tag{517}$$

$$\mathcal{A}(\mathcal{X}_{12}) = \mathcal{A}(\mathcal{X}_{12}) + \mathbf{S}_{32}^T \mathbf{t}_2 \tag{518}$$

$$\mathcal{A}(\mathbf{D}_{13}) = \mathcal{A}(\mathbf{E}_{13})\mathbf{P}_{13}^T \mathbf{R}_{23}^T \tag{519}$$

$$\mathcal{A}(\mathbf{R}_{21}) = \mathbf{A}_{31}^T \mathcal{A}(\mathbf{D}_{13}) \tag{520}$$

12.2.3.8 homogeneous linearized + homogeneous linearized

$$\mathcal{A}(\mathbf{D}_{13}) = \mathcal{A}(\mathbf{T}_{13})\mathbf{T}_{12}^T \tag{521}$$

$$\mathcal{A}(\mathbf{T}_{12}) = \mathbf{A}_{13}^T \mathcal{A}(\mathbf{T}_{13}) \tag{522}$$

$$\mathcal{A}(\mathbf{R}_{32}) = \mathcal{A}(\mathbf{R}_{31}) \tag{523}$$

$$\mathcal{A}(\mathbf{D}_{13}) = \mathcal{A}(\mathbf{D}_{13}) + \mathcal{A}(\mathbf{R}_{31})\mathbf{B}_{31}^T \tag{524}$$

$$\mathbf{t} = \mathbf{A}_{13}^T \mathcal{A}(\mathbf{R}_{31}) \tag{525}$$

$$\mathcal{A}(\mathbf{R}_{21}) = \mathbf{t}\mathbf{T}_{32}^T \tag{526}$$

$$\mathcal{A}(\mathbf{T}_{32}) = \mathbf{R}_{21}^T \mathbf{t} \tag{527}$$

$$\mathcal{A}(\mathbf{S}_{23}) = \mathcal{A}(\mathbf{S}_{13})\mathcal{X}_{12} \tag{528}$$

$$\mathcal{A}(\mathcal{X}_{12}) = \mathbf{S}_{23}^T \mathcal{A}(\mathbf{S}_{13}) \tag{529}$$

$$\mathcal{A}(\mathbf{D}_{13}) = \mathcal{A}(\mathbf{D}_{13}) + \mathcal{A}(\mathbf{S}_{13})\mathbf{C}_{13}^T \tag{530}$$

$$\mathbf{t} = \mathbf{A}_{13}^T\mathcal{A}(\mathbf{S}_{13}) \tag{531}$$

$$\mathcal{A}(\mathbf{S}_{12}) = \mathbf{t} \tag{532}$$

$$\mathcal{A}(\mathbf{R}_{21}) = \mathcal{A}(\mathbf{R}_{21}) + \mathbf{t}\mathbf{S}_{32}^T\mathcal{X}_{12} \tag{533}$$

$$\mathbf{t}_2 = \mathbf{R}_{21}^T\mathbf{t} \tag{534}$$

$$\mathcal{A}(\mathbf{S}_{32}) = \mathbf{t}_2\mathcal{X}_{12} \tag{535}$$

$$\mathcal{A}(\mathcal{X}_{12}) = \mathcal{A}(\mathcal{X}_{12}) + \mathbf{S}_{32}^T\mathbf{t}_2 \tag{536}$$

$$\mathbf{t} = \mathcal{A}(\mathbf{D}_{31})\mathbf{P}_{13}^T \tag{537}$$

$$\mathcal{A}(\mathbf{T}_{23}) = \mathbf{t} \tag{538}$$

$$\mathbf{t}_2 = \mathbf{A}_{31}^T\mathbf{t} \tag{539}$$

$$\mathcal{A}(\mathbf{R}_{21}) = \mathcal{A}(\mathbf{R}_{21}) + \mathbf{t}_2\mathbf{R}_{23}^T \tag{540}$$

$$\mathcal{A}(\mathbf{R}_{23}) = \mathbf{R}_{21}^T\mathbf{t}_2 \tag{541}$$

13 Radiance

13.1 Slab radiance

13.1.1 Forward

$$\mathbf{I}_1^+ = \mathbf{R}_{12}^-\mathbf{I}_1^- + \mathbf{T}_{12}^+\mathbf{I}_2^+ + \mathbf{S}_{12}^+ \tag{542}$$

$$\mathbf{I}_2^- = \mathbf{R}_{12}^+\mathbf{I}_2^+ + \mathbf{T}_{12}^-\mathbf{I}_1^- + \mathbf{S}_{12}^- \tag{543}$$

13.1.2 Tangent linear

13.1.2.1 U

$$\mathbf{K}_1^+ = \mathbf{R}_{12}^-\mathbf{K}_1^- + \mathbf{T}_{12}^+\mathbf{K}_2^+ \tag{544}$$

$$\mathbf{K}_2^- = \mathbf{R}_{12}^+\mathbf{K}_2^+ + \mathbf{T}_{12}^-\mathbf{K}_1^- \tag{545}$$

13.1.2.2 S

$$\mathbf{K}_1^+ = \mathbf{R}_{12}^-\mathbf{K}_1^- + \mathbf{T}_{12}^+\mathbf{K}_2^+ + \mathcal{L}(\mathbf{S}_{12}^+) \tag{546}$$

$$\mathbf{K}_2^- = \mathbf{R}_{12}^+\mathbf{K}_2^+ + \mathbf{T}_{12}^-\mathbf{K}_1^- + \mathcal{L}(\mathbf{S}_{12}^-) \tag{547}$$

13.1.2.3 L

$$\mathbf{K}_1^+ = \mathcal{L}(\mathbf{R}_{12}^-)\mathbf{I}_1^- + \mathbf{R}_{12}^-\mathbf{K}_1^- + \mathcal{L}(\mathbf{T}_{12}^+)\mathbf{I}_2^+ + \mathbf{T}_{12}^+\mathbf{K}_2^+ \tag{548}$$

$$\mathbf{K}_2^- = \mathcal{L}(\mathbf{R}_{12}^+)\mathbf{I}_2^+ + \mathbf{R}_{12}^+\mathbf{K}_2^+ + \mathcal{L}(\mathbf{T}_{12}^-)\mathbf{I}_1^- + \mathbf{T}_{12}^-\mathbf{K}_1^- \tag{549}$$

13.1.2.4 B

$$\mathbf{K}_1^+ = \mathcal{L}(\mathbf{R}_{12}^-)\mathbf{I}_1^- + \mathbf{R}_{12}^-\mathbf{K}_1^- + \mathcal{L}(\mathbf{T}_{12}^+)\mathbf{I}_2^+ + \mathbf{T}_{12}^+\mathbf{K}_2^+ + \mathcal{L}(\mathbf{S}_{12}^+) \tag{550}$$

$$\mathbf{K}_2^- = \mathcal{L}(\mathbf{R}_{12}^+)\mathbf{I}_2^+ + \mathbf{R}_{12}^+\mathbf{K}_2^+ + \mathcal{L}(\mathbf{T}_{12}^-)\mathbf{I}_1^- + \mathbf{T}_{12}^-\mathbf{K}_1^- + \mathcal{L}(\mathbf{S}_{12}^-) \tag{551}$$

13.1.3 Adjoint of tangent linear

$$\mathcal{A}(\mathbf{R}_{12}^+) = \mathcal{A}(\mathbf{R}_{12}^+) + \mathcal{A}(\mathbf{I}_2^-)(\mathbf{I}_2^+)^T \tag{552}$$

$$\mathcal{A}(\mathbf{I}_2^+) = \mathcal{A}(\mathbf{I}_2^+) + (\mathbf{R}_{12}^+)^T\mathcal{A}(\mathbf{I}_2^-) \tag{553}$$

$$\mathcal{A}(\mathbf{T}_{12}^-) = \mathcal{A}(\mathbf{T}_{12}^-) + \mathcal{A}(\mathbf{I}_2^-)(\mathbf{I}_1^-)^T \tag{554}$$

$$\mathcal{A}(\mathbf{I}_1^-) = \mathcal{A}(\mathbf{I}_1^-) + (\mathbf{T}_{12}^-)^T\mathcal{A}(\mathbf{I}_2^-) \tag{555}$$

$$\mathcal{A}(\mathbf{S}_{12}^-) = \mathcal{A}(\mathbf{S}_{12}^-) + \mathcal{A}(\mathbf{I}_2^-) \tag{556}$$

$$\mathcal{A}(\mathbf{R}_{12}^-) = \mathcal{A}(\mathbf{R}_{12}^-) + \mathcal{A}(\mathbf{I}_1^+)(\mathbf{I}_1^-)^T \tag{557}$$

$$\mathcal{A}(\mathbf{I}_1^-) = \mathcal{A}(\mathbf{I}_1^-) + (\mathbf{R}_{12}^-)^T\mathcal{A}(\mathbf{I}_1^+) \tag{558}$$

$$\mathcal{A}(\mathbf{T}_{12}^+) = \mathcal{A}(\mathbf{T}_{12}^+) + \mathcal{A}(\mathbf{I}_1^+)(\mathbf{I}_2^+)^T \tag{559}$$

$$\mathcal{A}(\mathbf{I}_2^+) = \mathcal{A}(\mathbf{I}_2^+) + (\mathbf{T}_{12}^+)^T\mathcal{A}(\mathbf{I}_1^+) \tag{560}$$

$$\mathcal{A}(\mathbf{S}_{12}^+) = \mathcal{A}(\mathbf{S}_{12}^+) + \mathcal{A}(\mathbf{I}_1^+) \tag{561}$$

13.2 TOA radiance

13.2.1 Forward

13.2.2 Tangent linear

13.2.3 Adjoint of tangent linear

13.3 BOA radiance

13.3.1 Forward

$$\mathbf{P} = (\mathbf{E} - \mathbf{R}_{12}^{+}\mathbf{R}_{23}^{-})^{-1} \tag{562}$$

$$\mathbf{I}_2^{-} = \mathbf{P}(\mathbf{R}_{12}^{+}\mathbf{I}_3^{+} + \mathbf{T}_{12}^{-}\mathbf{I}_1^{-} + \mathbf{S}_{12}^{-}) \tag{563}$$

$$\mathbf{I}_2^{+} = \mathbf{R}_{23}^{-}\mathbf{I}_2^{-} + \mathbf{I}_3^{+} \tag{564}$$

13.3.2 Tangent linear

13.3.2.1 U_B

$$\mathbf{Q} = \mathbf{R}_{12}^{+}\mathcal{L}(\mathbf{R}_{23}^{-}) \tag{565}$$

$$\mathbf{K}_2^{-} = \mathbf{P}(\mathbf{Q}\mathbf{I}_2^{-} + \mathbf{R}_{12}^{+}\mathbf{K}_3^{+} + \mathbf{T}_{12}^{-}\mathbf{K}_1^{-}) \tag{566}$$

$$\mathbf{K}_2^{+} = \mathcal{L}(\mathbf{R}_{23}^{-})\mathbf{I}_2^{-} + \mathbf{R}_{23}^{-}\mathbf{K}_2^{-} + \mathbf{K}_3^{+} \tag{567}$$

13.3.2.2 L_L

$$\mathbf{Q} = \mathcal{L}(\mathbf{R}_{12}^{+})\mathbf{R}_{23}^{-} + \mathbf{R}_{12}^{+}\mathcal{L}(\mathbf{R}_{23}^{-}) \tag{568}$$

$$\mathbf{K}_2^{-} = \mathbf{P}(\mathbf{Q}\mathbf{I}_2^{-} + \mathcal{L}(\mathbf{R}_{12}^{+})\mathbf{I}_3^{+} + \mathcal{L}(\mathbf{T}_{12}^{-})\mathbf{I}_1^{-} + \mathbf{T}_{12}^{-}\mathbf{K}_1^{-}) \tag{569}$$

$$\mathbf{K}_2^{+} = \mathcal{L}(\mathbf{R}_{23}^{-})\mathbf{I}_2^{-} + \mathbf{R}_{23}^{-}\mathbf{K}_2^{-} \tag{570}$$

13.3.2.3 B_U

$$\mathbf{Q} = \mathcal{L}(\mathbf{U}_{12}^{+})\mathbf{R}_{23}^{-} \tag{571}$$

$$\mathbf{K}_2^{-} = \mathbf{P}(\mathbf{Q}\mathbf{I}_2^{-} + \mathcal{L}(\mathbf{R}_{12}^{+})\mathbf{I}_3^{+} + \mathbf{R}_{12}^{+}\mathbf{K}_3^{+} + \mathcal{L}(\mathbf{T}_{12}^{-})\mathbf{I}_1^{-} + \mathbf{T}_{12}^{-}\mathbf{K}_1^{-} + \mathbf{V}_{12}^{-}) \tag{572}$$

$$\mathbf{K}_2^{+} = \mathbf{R}_{23}^{-}\mathbf{K}_2^{-} + \mathbf{K}_3^{+} \tag{573}$$

13.3.2.4 B_S

$$\mathbf{Q} = \mathcal{L}(\mathbf{R}_{12}^{+})\mathbf{R}_{23}^{-} \tag{574}$$

$$\mathbf{K}_2^{-} = \mathbf{P}(\mathbf{Q}\mathbf{I}_2^{-} + \mathcal{L}(\mathbf{R}_{12}^{+})\mathbf{I}_3^{+} + \mathbf{R}_{12}^{+}\mathbf{K}_3^{+} + \mathcal{L}(\mathbf{T}_{12}^{-})\mathbf{I}_1^{-} + \mathbf{T}_{12}^{-}\mathbf{K}_1^{-} + \mathcal{L}(\mathbf{S}_{12}^{-})) \tag{575}$$

$$\mathbf{K}_2^{+} = \mathbf{R}_{23}^{-}\mathbf{K}_2^{-} + \mathbf{K}_3^{+} \tag{576}$$

13.3.2.5 B_L

$$\mathbf{Q} = \mathcal{L}(\mathbf{R}_{12}^{+})\mathbf{R}_{23}^{-} + \mathbf{R}_{12}^{+}\mathcal{L}(\mathbf{R}_{23}^{-}) \tag{577}$$

$$\mathbf{K}_2^{-} = \mathbf{P}(\mathbf{Q}\mathbf{I}_2^{-} + \mathcal{L}(\mathbf{R}_{12}^{+})\mathbf{I}_3^{+} + \mathcal{L}(\mathbf{T}_{12}^{-})\mathbf{I}_1^{-} + \mathbf{T}_{12}^{-}\mathbf{K}_1^{-} + \mathcal{L}(\mathbf{S}_{12}^{-})) \tag{578}$$

$$\mathbf{K}_2^{+} = \mathcal{L}(\mathbf{R}_{23}^{-})\mathbf{I}_2^{-} + \mathbf{R}_{23}^{-}\mathbf{K}_2^{-} \tag{579}$$

13.3.2.6 B_B

$$\mathbf{Q} = \mathcal{L}(\mathbf{R}_{12}^{+})\mathbf{R}_{23}^{-} + \mathbf{R}_{12}^{+}\mathcal{L}(\mathbf{R}_{23}^{-}) \tag{580}$$

$$\mathbf{K}_2^{-} = \mathbf{P}(\mathbf{Q}\mathbf{I}_2^{-} + \mathcal{L}(\mathbf{R}_{12}^{+})\mathbf{I}_3^{+} + \mathbf{R}_{12}^{+}\mathbf{K}_3^{+} + \mathcal{L}(\mathbf{T}_{12}^{-})\mathbf{I}_1^{-} + \mathbf{T}_{12}^{-}\mathbf{K}_1^{-} + \mathcal{L}(\mathbf{S}_{12}^{-})) \tag{581}$$

$$\mathbf{K}_2^{+} = \mathcal{L}(\mathbf{R}_{23}^{-})\mathbf{I}_2^{-} + \mathbf{R}_{23}^{-}\mathbf{K}_2^{-} + \mathbf{K}_3^{+} \tag{582}$$

13.3.3 Adjoint of tangent linear

13.3.3.1 U_L

13.3.3.2 L_P

13.3.3.3 L_L

$$\mathcal{A}(\mathbf{R}_{23}^{-}) = \mathcal{A}(\mathbf{R}_{23}^{-}) + \mathcal{A}(\mathbf{I}_2^{+})(\mathbf{I}_2^{-})^T \tag{583}$$

$$\mathcal{A}(\mathbf{I}_2^{-}) = \mathcal{A}(\mathbf{I}_2^{-}) + (\mathbf{R}_{23}^{-})^T\mathcal{A}(\mathbf{I}_2^{+}) \tag{584}$$

$$\mathcal{A}(\mathbf{I}_3^{+}) = \mathcal{A}(\mathbf{I}_3^{+}) + \mathcal{A}(\mathbf{I}_2^{+}) \tag{585}$$

$$t = \mathbf{P}\mathcal{A}(\mathbf{I}_2^{-}) \tag{586}$$

$$\mathcal{A}(\mathbf{Q}) = \mathcal{A}(\mathbf{Q}) + t(\mathbf{I}_2^{-})^T \tag{587}$$

$$\mathcal{A}(\mathbf{R}_{12}^{+}) = \mathcal{A}(\mathbf{R}_{12}^{+}) + t(\mathbf{I}_3^{+})^T \tag{588}$$

$$\mathcal{A}(\mathbf{I}_3^{+}) = \mathcal{A}(\mathbf{I}_3^{+}) + (\mathbf{R}_{12}^{+})^T t \tag{589}$$

$$\mathcal{A}(\mathbf{T}_{12}^{-}) = \mathcal{A}(\mathbf{T}_{12}^{-}) + t(\mathbf{I}_{1}^{-})^{T} \tag{590}$$

$$\mathcal{A}(\mathbf{I}_{1}^{-}) = \mathcal{A}(\mathbf{I}_{1}^{-}) + (\mathbf{T}_{12}^{-})^{T} t \tag{591}$$

$$\mathcal{A}(\mathbf{S}_{12}^{-}) = \mathcal{A}(\mathbf{S}_{12}^{-}) + t \tag{592}$$

$$\mathcal{A}(\mathbf{R}_{12}^{+}) = \mathcal{A}(\mathbf{R}_{12}^{+}) + \mathcal{A}(\mathbf{Q})(\mathbf{R}_{23}^{-})^{T} \tag{593}$$

$$\mathcal{A}(\mathbf{R}_{23}^{-}) = \mathcal{A}(\mathbf{R}_{23}^{-}) + (\mathbf{R}_{12}^{+})^{T}\mathcal{A}(\mathbf{Q}) \tag{594}$$

13.4 Internal radiance

13.4.1 Forward

$$\mathbf{P}_{31} = (\mathbf{E} - \mathbf{R}_{23}^{-}\mathbf{R}_{12}^{+})^{-1} \tag{595}$$

$$\mathbf{I}_{2}^{+} = \mathbf{P}_{31}\left[\mathbf{R}_{23}^{-}\mathbf{T}_{12}^{-}\mathbf{I}_{1}^{-} + \mathbf{T}_{23}^{+}\mathbf{I}_{3}^{+} + \mathbf{R}_{23}^{-}\mathbf{S}_{12}^{-} + \mathbf{S}_{23}^{+}\right] \tag{596}$$

$$\mathbf{P}_{13} = (\mathbf{E} - \mathbf{R}_{12}^{+}\mathbf{R}_{23}^{-})^{-1} \tag{597}$$

$$\mathbf{I}_{2}^{-} = \mathbf{P}_{13}\left[\mathbf{R}_{12}^{+}\mathbf{T}_{23}^{+}\mathbf{I}_{3}^{+} + \mathbf{T}_{12}^{-}\mathbf{I}_{1}^{-} + \mathbf{R}_{12}^{+}\mathbf{S}_{23}^{+} + \mathbf{S}_{12}^{-}\right] \tag{598}$$

13.4.2 Tangent linear

$$\begin{aligned}\mathbf{K}_{2}^{+} = \mathbf{P}_{31}\big[&\mathcal{L}(\mathbf{R}_{23}^{-})\mathbf{R}_{12}^{+}\mathbf{I}_{2}^{+} + \mathbf{R}_{23}^{-}\mathcal{L}(\mathbf{R}_{12}^{+})\mathbf{I}_{2}^{+} + \mathcal{L}(\mathbf{R}_{23}^{-})\mathbf{T}_{12}^{-}\mathbf{I}_{1}^{-} + \mathbf{R}_{23}^{-}(\mathcal{L}(\mathbf{T}_{12}^{-})\mathbf{I}_{1}^{-} + \mathbf{T}_{12}^{-}\mathbf{K}_{1}^{-})\\ &+ \mathcal{L}(\mathbf{T}_{23}^{+})\mathbf{I}_{3}^{+} + \mathbf{T}_{23}^{+}\mathbf{K}_{3}^{+} + \mathcal{L}(\mathbf{R}_{23}^{-})\mathbf{S}_{12}^{-} + \mathbf{R}_{23}^{-}\mathcal{L}(\mathbf{S}_{12}^{-}) + \mathcal{L}(\mathbf{S}_{23}^{+})\big]\end{aligned} \tag{599}$$

$$\begin{aligned}\mathbf{K}_{2}^{-} = \mathbf{P}_{13}\big[&\mathcal{L}(\mathbf{R}_{12}^{+})\mathbf{R}_{23}^{-}\mathbf{I}_{2}^{-} + \mathbf{R}_{12}^{+}\mathcal{L}(\mathbf{R}_{23}^{-})\mathbf{I}_{2}^{-} + \mathcal{L}(\mathbf{R}_{12}^{+})\mathbf{T}_{23}^{+}\mathbf{I}_{3}^{+} + \mathbf{R}_{12}^{+}(\mathcal{L}(\mathbf{T}_{23}^{+})\mathbf{I}_{3}^{+} + \mathbf{T}_{23}^{+}\mathbf{K}_{3}^{+})\\ &+ \mathcal{L}(\mathbf{T}_{12}^{-})\mathbf{I}_{1}^{-} + \mathbf{T}_{12}^{-}\mathbf{K}_{1}^{-} + \mathcal{L}(\mathbf{R}_{12}^{+})\mathbf{S}_{23}^{+} + \mathbf{R}_{12}^{+}\mathcal{L}(\mathbf{S}_{23}^{+}) + \mathcal{L}(\mathbf{S}_{12}^{-})\big]\end{aligned} \tag{600}$$

13.4.3 Adjoint of tangent linear

14 Discrete ordinate method

14.1 Layer quantities

14.1.1 Homogeneous solution

14.1.1.1 Forward

14.1.1.2 Tangent linear

14.1.1.3 Adjoint of tangent linear

14.1.2 Particular solution

14.1.2.1 Forward

14.1.2.2 Tangent linear

14.1.2.3 Adjoint of tangent linear

14.2 Boundary value problem

14.2.1 Forward

$$\Lambda_k = \mathrm{diag}(e^{-\nu_{i,k} x_k}) \tag{601}$$

$$U_k^{\pm} = X_k^{\pm} \Lambda_k \tag{602}$$

$$V_k^{\pm} = X_k^{\pm} - X_k^{\mp} R_s \tag{603}$$

$$W_k^{\pm} = V_k^{\pm} \Lambda_k \tag{604}$$

$$\mathbf{A} = \begin{bmatrix}
-X_1^- & -U_1^+ & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
U_1^+ & X_1^- & -X_2^+ & -U_2^- & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
-U_1^- & -X_1^+ & X_2^- & U_2^+ & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & U_2^+ & X_2^- & -X_3^+ & -U_3^- & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & -U_2^- & -X_2^+ & X_3^- & U_3^+ & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & U_3^+ & X_3^- & -X_4^+ & -U_4^- & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -U_3^- & -X_3^+ & X_4^- & U_4^+ & \cdots & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & U_4^+ & X_4^- & -X_K^+ & -U_K^- \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & -U_4^- & -X_4^+ & X_K^- & U_K^+ \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & W_K^+ & V_K^-
\end{bmatrix} \tag{605}$$

$$G_k^+ = F_K^+ - R_s F_K^- \tag{606}$$

$$\mathbf{b} = \begin{bmatrix}
I_0^- - F_1^- \\
F_2^+ - F_1^+ \mathcal{X}_{b,1} \\
F_2^- - F_1^- \mathcal{X}_{b,1} \\
F_3^+ - F_2^+ \mathcal{X}_{b,2} \\
F_3^- - F_2^- \mathcal{X}_{b,2} \\
F_4^+ - F_3^+ \mathcal{X}_{b,3} \\
F_4^- - F_3^- \mathcal{X}_{b,3} \\
\vdots \\
F_K^+ - F_{K-1}^+ \mathcal{X}_{b,K-1} \\
F_K^- - F_{K-1}^- \mathcal{X}_{b,K-1} \\
I_K^+ - G_K^+ \mathcal{X}_{b,K}
\end{bmatrix} \tag{607}$$

$$
\mathbf{x} = \begin{bmatrix} x_1^+ \\ x_1^- \\ x_2^+ \\ x_2^- \\ x_3^+ \\ x_3^- \\ x_4^+ \\ x_4^- \\ \vdots \\ x_K^+ \\ x_K^- \end{bmatrix} \tag{608}
$$

14.2.2 Tangent linear

$$
\mathcal{L}(\Lambda_k) = \mathrm{diag}\left[(-\mathcal{L}(\nu_{i,k})x_k - \nu_{i,k}\mathcal{L}(x_k)\right] e^{-\nu_{i,k}x_k}) \tag{609}
$$

$$
\mathcal{L}(U^\pm) = \mathcal{L}(X_k^\pm)\Lambda_k + X_k^\pm \mathcal{L}(\Lambda_k) \tag{610}
$$

$$
\mathcal{L}(V^\pm) = \mathcal{L}(X_k^\pm) - \mathcal{L}(X_k^\mp)R_s - X_k^\mp \mathcal{L}(R_s) \tag{611}
$$

$$
\mathcal{L}(W^\pm) = \mathcal{L}(V_k^\pm)\Lambda_k + V_k^\pm \mathcal{L}(\Lambda_k) \tag{612}
$$

$$
\mathcal{L}(G^\pm) = \mathcal{L}(F_K^+) - \mathcal{L}(R_s)F_K^- - R_s\mathcal{L}(F_K^-) \tag{613}
$$

$$
\mathcal{L}(\mathbf{b}) = \begin{bmatrix}
\mathcal{L}(I_0^-) - \mathcal{L}(F_1^-) + \mathcal{L}(X_1^-)x_1^+ + \mathcal{L}(U_1^+)x_1^- \\
\mathcal{L}(F_2^+) - \mathcal{L}(F_1^+)\mathcal{X}_{b,1} - F_1^+\mathcal{L}(\mathcal{X}_{b,1}) - \mathcal{L}(U_1^+)x_1^+ - \mathcal{L}(X_1^-)x_1^- + \mathcal{L}(X_2^+)x_2^+ + \mathcal{L}(U_2^-)x_2^- \\
\mathcal{L}(F_2^-) - \mathcal{L}(F_1^-)\mathcal{X}_{b,1} - F_1^-\mathcal{L}(\mathcal{X}_{b,1}) + \mathcal{L}(U_1^-)x_1^+ + \mathcal{L}(X_1^+)x_1^- - \mathcal{L}(X_2^-)x_2^+ - \mathcal{L}(U_2^+)x_2^- \\
\mathcal{L}(F_3^+) - \mathcal{L}(F_2^+)\mathcal{X}_{b,2} - F_2^+\mathcal{L}(\mathcal{X}_{b,2}) - \mathcal{L}(U_2^+)x_2^+ - \mathcal{L}(X_2^-)x_2^- + \mathcal{L}(X_3^+)x_3^+ + \mathcal{L}(U_3^-)x_3^- \\
\mathcal{L}(F_3^-) - \mathcal{L}(F_2^-)\mathcal{X}_{b,2} - F_2^-\mathcal{L}(\mathcal{X}_{b,2}) + \mathcal{L}(U_2^-)x_2^+ + \mathcal{L}(X_2^+)x_2^- - \mathcal{L}(X_3^-)x_3^+ - \mathcal{L}(U_3^+)x_3^- \\
\mathcal{L}(F_4^+) - \mathcal{L}(F_3^+)\mathcal{X}_{b,3} - F_3^+\mathcal{L}(\mathcal{X}_{b,3}) - \mathcal{L}(U_3^+)x_3^+ - \mathcal{L}(X_3^-)x_3^- + \mathcal{L}(X_4^+)x_4^+ + \mathcal{L}(U_4^-)x_4^- \\
\mathcal{L}(F_4^-) - \mathcal{L}(F_3^-)\mathcal{X}_{b,3} - F_3^-\mathcal{L}(\mathcal{X}_{b,3}) + \mathcal{L}(U_3^-)x_3^+ + \mathcal{L}(X_3^+)x_3^- - \mathcal{L}(X_4^-)x_4^+ - \mathcal{L}(U_4^+)x_4^- \\
\vdots \\
\mathcal{L}(F_K^+) - \mathcal{L}(F_{K-1}^+)\mathcal{X}_{b,K-1} - F_{K-1}^+\mathcal{L}(\mathcal{X}_{b,K-1}) - \mathcal{L}(U_{K-1}^+)x_{K-1}^+ - \mathcal{L}(X_{K-1}^-)x_{K-1}^- + \mathcal{L}(X_K^+)x_K^+ + \mathcal{L}(U_K^-)x_K^- \\
\mathcal{L}(F_K^-) - \mathcal{L}(F_{K-1}^-)\mathcal{X}_{b,K-1} - F_{K-1}^-\mathcal{L}(\mathcal{X}_{b,K-1}) + \mathcal{L}(U_{K-1}^-)x_{K-1}^+ + \mathcal{L}(X_{K-1}^+)x_{K-1}^- - \mathcal{L}(X_K^-)x_K^+ - \mathcal{L}(U_K^+)x_K^- \\
\mathcal{L}(I_K^+) - \mathcal{L}(G_K^+)\mathcal{X}_{b,K} - G_K^+\mathcal{L}(\mathcal{X}_{b,K}) - \mathcal{L}(W_K^+)x_K^+ - \mathcal{L}(V_K^-)x_K^-
\end{bmatrix} \tag{614}
$$

14.2.3 Adjoint of tangent linear

14.3 Radiance

14.3.1 At levels

14.3.1.1 Forward

14.3.1.2 Tangent linear

14.3.1.3 Adjoint of tangent linear

14.3.2 At optical depth

14.3.2.1 Forward

14.3.2.2 Tangent linear

14.3.2.3 Adjoint of tangent linear

15 Matrix exponetial method

15.1 Layer quantities

15.1.1 Homogeneous solution

15.1.1.1 Forward

$$\boldsymbol{\alpha} = \mathrm{diag}(e^{-\nu_i x}) \tag{615}$$

$$\mathbf{a} = \mathbf{X}_+ - \mathbf{X}_- \tag{616}$$

$$\mathbf{b} = \mathbf{X}_+ + \mathbf{X}_- \tag{617}$$

$$\mathbf{c} = \frac{1}{2}(\mathbf{a}^{-1} + \mathbf{b}^{-1}) \tag{618}$$

$$\mathbf{d} = \frac{1}{2}(\mathbf{a}^{-1} - \mathbf{b}^{-1}) \tag{619}$$

$$\mathbf{e} = \boldsymbol{\alpha}\mathbf{c} \tag{620}$$

$$\mathbf{f} = \boldsymbol{\alpha}\mathbf{d} \tag{621}$$

$$\mathbf{A}_1 = \begin{bmatrix} \mathbf{e} & \mathbf{f} \\ -\mathbf{d} & -\mathbf{c} \end{bmatrix} \tag{622}$$

$$\mathbf{A}_2 = \begin{bmatrix} -\mathbf{c} & -\mathbf{d} \\ \mathbf{f} & \mathbf{e} \end{bmatrix} \tag{623}$$

15.1.1.2 Tangent linear

$$\mathcal{L}(\boldsymbol{\alpha}) = \mathrm{diag}\left\{[-\mathcal{L}(\nu_i)x - \nu_i\mathcal{L}(x)]\, e^{-\nu_i x}\right\} \tag{624}$$

$$\mathcal{L}(\mathbf{a}) = \mathcal{L}(\mathbf{X}_+) - \mathcal{L}(\mathbf{X}_-) \tag{625}$$

$$\mathcal{L}(\mathbf{b}) = \mathcal{L}(\mathbf{X}_+) + \mathcal{L}(\mathbf{X}_-) \tag{626}$$

$$\mathcal{L}(\mathbf{a}^{-1}) = -\mathbf{a}^{-1}\mathcal{L}(\mathbf{a})\mathbf{a}^{-1} \tag{627}$$

$$\mathcal{L}(\mathbf{b}^{-1}) = -\mathbf{b}^{-1}\mathcal{L}(\mathbf{b})\mathbf{b}^{-1} \tag{628}$$

$$\mathcal{L}(\mathbf{c}) = \frac{1}{2}\left[\mathcal{L}(\mathbf{a}^{-1}) + \mathcal{L}(\mathbf{b}^{-1})\right] \tag{629}$$

$$\mathcal{L}(\mathbf{d}) = \frac{1}{2}\left[\mathcal{L}(\mathbf{a}^{-1}) - \mathcal{L}(\mathbf{b}^{-1})\right] \tag{630}$$

$$\mathcal{L}(\mathbf{e}) = \mathcal{L}(\boldsymbol{\alpha})\mathbf{c} + \boldsymbol{\alpha}\mathcal{L}(\mathbf{c}) \tag{631}$$

$$\mathcal{L}(\mathbf{f}) = \mathcal{L}(\boldsymbol{\alpha})\mathbf{d} + \boldsymbol{\alpha}\mathcal{L}(\mathbf{d}) \tag{632}$$

$$\mathcal{L}(\mathbf{A}_1) = \begin{bmatrix} \mathcal{L}(\mathbf{e}) & \mathcal{L}(\mathbf{f}) \\ -\mathcal{L}(\mathbf{d}) & -\mathcal{L}(\mathbf{c}) \end{bmatrix} \tag{633}$$

$$\mathcal{L}(\mathbf{A}_2) = \begin{bmatrix} -\mathcal{L}(\mathbf{c}) & -\mathcal{L}(\mathbf{d}) \\ \mathcal{L}(\mathbf{f}) & \mathcal{L}(\mathbf{e}) \end{bmatrix} \tag{634}$$

15.1.1.3 Adjoint of tangent linear

15.1.2 Particular solution

15.1.2.1 Forward

$$b_1(\nu_i x) = \frac{e^{-(\tau_k + \nu_i x)} - e^{-\tau_{k+1}}}{\tau_{k+1} - \tau_k - \nu_i x} \tag{635}$$

$$\boldsymbol{\beta_1} = \mathrm{diag}\left[b_1(\nu_i x)\right] \tag{636}$$

$$\begin{aligned} b_2(\nu_i x) &= \frac{e^{-(\tau_{k+1} + \nu_i \Delta\tau)} - e^{-\tau_k}}{\tau_k - \tau_{k+1} - \nu_i \Delta\tau} && (637) \\ &= b_1(-\nu_i x)e^{-\nu_i x} && (638) \end{aligned}$$

$$\boldsymbol{\beta_2} = \mathrm{diag}\left[b_2(\nu_i x)\right] \tag{639}$$

$$\boldsymbol{\Sigma}^{\pm} = \frac{F_0}{4\pi}\mathbf{M}^{-1}\omega\mathbf{P}_0^{\pm} \tag{640}$$

$$\mathbf{o} = \boldsymbol{\beta_1}(\mathbf{c}\boldsymbol{\Sigma}^{+} + \mathbf{d}\boldsymbol{\Sigma}^{-}) \tag{641}$$

$$\mathbf{p} = \boldsymbol{\beta_2}(-\mathbf{d}\boldsymbol{\Sigma}^{+} + -\mathbf{c}\boldsymbol{\Sigma}^{-}) \tag{642}$$

15.1.2.2 Tangent linear

$$\mathcal{L}\left[b_1(\nu_i x)\right] = \frac{\left[-\mathcal{L}(\tau_k) - \mathcal{L}(\nu_i)x - \nu_i\mathcal{L}(x)\right]e^{-(\tau_k+\nu_i x)} + \mathcal{L}(\tau_{k+1})e^{-\tau_{k+1}} - \beta_1\left[\mathcal{L}(\tau_{k+1}) - \mathcal{L}(\tau_k) - \mathcal{L}(\nu_i)x - \nu_i\mathcal{L}(x)\right]}{\tau_{k+1} - \tau_k - \nu_i x} \tag{643}$$

$$\mathcal{L}(\boldsymbol{\beta_1}) = \mathrm{diag}\left\{\mathcal{L}\left[b_1(\nu_i x)\right]\right\} \tag{644}$$

$$\mathcal{L}\left[b_2(\nu_i x)\right] = \frac{\left[-\mathcal{L}(\tau_{k+1}) - \mathcal{L}(\nu_i)x - \nu_i\mathcal{L}(x)\right]e^{-(\tau_{k+1}+\nu_i x)} + \mathcal{L}(\tau_k)e^{-\tau_k} - \beta_2\left[\mathcal{L}(\tau_k) - \mathcal{L}(\tau_{k+1}) - \mathcal{L}(\nu_i)x - \nu_i\mathcal{L}(x)\right]}{\tau_k - \tau_{k+1} - \nu_i x} \tag{645}$$

$$\mathcal{L}(\boldsymbol{\beta_2}) = \mathrm{diag}\left\{\mathcal{L}\left[b_2(\nu_i x)\right]\right\} \tag{646}$$

$$\mathcal{L}(\boldsymbol{\Sigma}^{\pm}) = \frac{F_0}{4\pi}\mathbf{M}^{-1}\left[\mathcal{L}(\omega)\mathbf{P}_0^{\pm} + \omega\mathcal{L}(\mathbf{P}_0^{\pm})\right] \tag{647}$$

$$\mathcal{L}(\mathbf{o}) = \mathcal{L}(\boldsymbol{\beta_1})(\mathbf{cg} + \mathbf{dh}) + \boldsymbol{\beta_1}\left[\mathcal{L}(\mathbf{c})\mathbf{g} + \mathbf{c}\mathcal{L}(\mathbf{g}) + \mathcal{L}(\mathbf{d})\mathbf{h} + \mathbf{d}\mathcal{L}(\mathbf{h})\right] \tag{648}$$

$$\mathcal{L}(\mathbf{p}) = \mathcal{L}(\boldsymbol{\beta_2})(\mathbf{dg} + \mathbf{ch}) + \boldsymbol{\beta_2}\left[\mathcal{L}(\mathbf{d})\mathbf{g} + \mathbf{d}\mathcal{L}(\mathbf{g}) + \mathcal{L}(\mathbf{c})\mathbf{h} + \mathbf{c}\mathcal{L}(\mathbf{h})\right] \tag{649}$$

15.1.2.3 Adjoint of tangent linear

15.2 Boundary value problem

15.2.1 Forward

15.2.2 Tangent linear

15.2.3 Adjoint of tangent linear

15.3 Radiance

15.3.1 At levels

15.3.1.1 Forward

15.3.1.2 Tangent linear

15.3.1.3 Adjoint of tangent linear

15.3.2 At optical depth

15.3.2.1 Forward

$$\mathbf{a} = +(\mathbf{X}^{-1})_{11}\mathbf{I}^{+} + (\mathbf{X}^{-1})_{12}\mathbf{I}^{-} \tag{650}$$

$$\mathbf{b} = -(\mathbf{X}^{-1})_{12}\mathbf{I}^{+} - (\mathbf{X}^{-1})_{11}\mathbf{I}^{-} \tag{651}$$

$$\mathbf{c} = +(\mathbf{X}^{-1})_{11}\mathbf{\Sigma}^{+} + (\mathbf{X}^{-1})_{12}\mathbf{\Sigma}^{-} \tag{652}$$

$$\mathbf{d} = -(\mathbf{X}^{-1})_{12}\mathbf{\Sigma}^{+} - (\mathbf{X}^{-1})_{11}\mathbf{\Sigma}^{-} \tag{653}$$

$$\mathbf{d}_1 = \mathrm{diag}(e^{-\nu_i \upsilon}) \tag{654}$$

$$\mathbf{d}_2 = \mathrm{diag}\left[e^{-\nu_i(x-\upsilon)}\right] \tag{655}$$

$$c_1(y) = \frac{e^{-(\tau_k+y)} - e^{-[(1-\upsilon/x)\tau_k+(\upsilon/x)\tau_{k+1}]}}{\tau_{k+1} - \tau_k - \nu_i x} \tag{656}$$

$$\mathbf{d}_3 = \mathrm{diag}\left[-c_1(\nu_i \upsilon)\right] \tag{657}$$

$$c_2(y) = \frac{e^{-(\tau_{k+1}+y)} - e^{-[(1-\upsilon/x)\tau_k+(\upsilon/x)\tau_{k+1}]}}{\tau_k - \tau_{k+1} - \nu_i x} \tag{658}$$

$$\mathbf{d}_4 = \mathrm{diag}\left\{c_2\left[\nu_i(x-\upsilon)\right]\right\} \tag{659}$$

$$\mathbf{g} = \mathbf{d}_1\mathbf{a} \tag{660}$$

$$\mathbf{h} = \mathbf{d}_2\mathbf{b} \tag{661}$$

$$\mathbf{o} = \mathbf{d}_3\mathbf{c} \tag{662}$$

$$\mathbf{d} = \mathbf{d}_4\mathbf{d} \tag{663}$$

$$\mathbf{I}^{+} = +\mathbf{X}_{+}(\mathbf{g}+\mathbf{o}) + \mathbf{X}_{-}(\mathbf{h}+\mathbf{p}) \tag{664}$$

$$\mathbf{I}^{-} = -\mathbf{X}_{-}(\mathbf{g}+\mathbf{o}) - \mathbf{X}_{+}(\mathbf{h}+\mathbf{p}) \tag{665}$$

15.3.2.2 Tangent linear

$$\mathcal{L}(\mathbf{a}) = +\mathcal{L}(\mathbf{X}^{-1})_{11}\mathbf{I}^{+} + (\mathbf{X}^{-1})_{11}\mathbf{K}^{+} + \mathcal{L}(\mathbf{X}^{-1})_{12}\mathbf{I}^{-} + (\mathbf{X}^{-1})_{12}\mathbf{K}^{-} \tag{666}$$

$$\mathcal{L}(\mathbf{b}) = -\mathcal{L}(\mathbf{X}^{-1})_{12}\mathbf{I}^{+} - (\mathbf{X}^{-1})_{12}\mathbf{K}^{+} - \mathcal{L}(\mathbf{X}^{-1})_{11}\mathbf{I}^{-} - (\mathbf{X}^{-1})_{11}\mathbf{K}^{-} \tag{667}$$

$$\mathcal{L}(\mathbf{c}) = +\mathcal{L}(\mathbf{X}^{-1})_{11}\mathbf{\Sigma}^{+} + (\mathbf{X}^{-1})_{11}\mathcal{L}(\mathbf{\Sigma}^{+}) + \mathcal{L}(\mathbf{X}^{-1})_{12}\mathbf{\Sigma}^{-} + (\mathbf{X}^{-1})_{12}\mathcal{L}(\mathbf{\Sigma}^{-}) \tag{668}$$

$$\mathcal{L}(\mathbf{d}) = -\mathcal{L}(\mathbf{X}^{-1})_{12}\mathbf{\Sigma}^{+} - (\mathbf{X}^{-1})_{12}\mathcal{L}(\mathbf{\Sigma}^{+}) - \mathcal{L}(\mathbf{X}^{-1})_{11}\mathbf{\Sigma}^{-} - (\mathbf{X}^{-1})_{11}\mathcal{L}(\mathbf{\Sigma}^{-}) \tag{669}$$

$$\mathcal{L}(\mathbf{d}_1) = -\mathcal{L}(\nu_i)\upsilon e^{-\nu_i\upsilon} \tag{670}$$

$$\mathcal{L}(\mathbf{d}_2) = -\left[\mathcal{L}(\nu_i)(x-\upsilon) + \nu\mathcal{L}(x)\right]e^{-\nu_i(x-\upsilon)} \tag{671}$$

$$\mathcal{L}(\alpha_1)(y_1,y_2,y_3) = \left[-\mathcal{L}(\tau_k) - y_2\right]e^{-(\tau_k+y_1)} + \left[\frac{\upsilon\mathcal{L}(x)}{(x)^2}\tau_k + (1-\frac{\upsilon}{x})\mathcal{L}(\tau_k) - \frac{\upsilon\mathcal{L}(x)}{(x)^2}\tau_{k+1} + \frac{\upsilon}{x}\mathcal{L}(x_{k+1})\right]e^{-\left[(1-\frac{\upsilon}{x})\tau_k + (\frac{\upsilon}{x})\tau_{k+1}\right]} \tag{672}$$

$$\mathcal{L}(c_1)(y_1,y_2,y_3) = \frac{\mathcal{L}(\alpha_1)(y_1,y_2,y_3) - y_3\left[\mathcal{L}(\tau_{k+1}) - \mathcal{L}(\tau_k) - \mathcal{L}(\nu_i)x - \nu_i\mathcal{L}(x)\right]}{\tau_{k+1} - \tau_k - \nu_i x} \tag{673}$$

$$\mathcal{L}(\mathbf{d}_3) = \operatorname{diag}\left\{-\mathcal{L}(c_1)\left[\nu_i\upsilon, \mathcal{L}(\nu_i)\upsilon, \mathbf{d}_3\right]\right\} \tag{674}$$

$$\mathcal{L}(\alpha_2)(y_1,y_2,y_3) = \left[-\mathcal{L}(\tau_{k+1}) - y_2\right]e^{-(\tau_{k+1}+y_1)} + \left[\frac{\upsilon\mathcal{L}(x)}{(x)^2}\tau_k + (1-\frac{\upsilon}{x})\mathcal{L}(\tau_k) - \frac{\upsilon\mathcal{L}(x)}{(x)^2}\tau_{k+1} + \frac{\upsilon}{x}\mathcal{L}(x_{k+1})\right]e^{-\left[(1-\frac{\upsilon}{x})\tau_k + (\frac{\upsilon}{x})\tau_{k+1}\right]} \tag{675}$$

$$\mathcal{L}(c_2)(y_1,y_2,y_3) = \frac{\mathcal{L}(\alpha_2)(y_1,y_2,y_3) - y_3\left[\mathcal{L}(\tau_k) - \mathcal{L}(\tau_{k+1}) - \mathcal{L}(\nu_i)x - \nu_i\mathcal{L}(x)\right]}{\tau_k - \tau_{k+1} - \nu_i x} \tag{676}$$

$$\mathcal{L}(\mathbf{d}_4) = \operatorname{diag}\left\{\mathcal{L}(c_2)\left[\nu_i(x-\upsilon), \mathcal{L}(\nu_i)(x-\upsilon) + \nu_i\mathcal{L}(x), \mathbf{d}_4\right]\right\} \tag{677}$$

$$\mathcal{L}(\mathbf{g}) = \mathcal{L}(\mathbf{d}_1)\mathbf{a} + \mathbf{d}_1\mathcal{L}(\mathbf{a}) \tag{678}$$

$$\mathcal{L}(\mathbf{h}) = \mathcal{L}(\mathbf{d}_2)\mathbf{b} + \mathbf{d}_2\mathcal{L}(\mathbf{b}) \tag{679}$$

$$\mathcal{L}(\mathbf{o}) = \mathcal{L}(\mathbf{d}_3)\mathbf{c} + \mathbf{d}_3\mathcal{L}(\mathbf{c}) \tag{680}$$

$$\mathcal{L}(\mathbf{p}) = \mathcal{L}(\mathbf{d}_4)\mathbf{d} + \mathbf{d}_3\mathcal{L}(\mathbf{d}) \tag{681}$$

$$\mathbf{K}^{+} = +\mathbf{Y}_{+}(\mathbf{g}+\mathbf{o}) + \mathbf{X}_{+}\left[\mathcal{L}(\mathbf{g}) + \mathcal{L}(\mathbf{o})\right] + \mathbf{Y}_{-}(\mathbf{h}+\mathbf{p}) + \mathbf{X}_{-}\left[\mathcal{L}(\mathbf{h}) + \mathcal{L}(\mathbf{p})\right] \tag{682}$$

$$\mathbf{K}^{-} = -\mathbf{Y}_{-}(\mathbf{g}+\mathbf{o}) - \mathbf{X}_{-}\left[\mathcal{L}(\mathbf{g}) + \mathcal{L}(\mathbf{o})\right] - \mathbf{Y}_{+}(\mathbf{h}+\mathbf{p}) - \mathbf{X}_{+}\left[\mathcal{L}(\mathbf{h}) + \mathcal{L}(\mathbf{p})\right] \tag{683}$$

15.3.2.3 Adjoint of tangent linear

16 Source function integration

16.1 Local source, classical

16.1.1 Upward

16.1.1.1 Forward

$$e_{1,i} = e^{-(x_k - \upsilon_k)/\mu_i} \tag{684}$$

$$I^+_{l,i} = I^+_{l-1,i} e_{1,i} \tag{685}$$

16.1.1.1.1 Solar source

$$\mathbf{F}^+_{\mathrm{u}} = \frac{F_0 \omega}{4\pi} \mathbf{P}^{+-}_{\mathrm{u0}} + (1 + \delta_{0,m}) \frac{\omega}{4} (\mathbf{P}^{++}_{\mathrm{uq}} \mathbf{W} \mathbf{F}^+ + \mathbf{P}^{-+}_{\mathrm{uq}} \mathbf{W} \mathbf{F}^-) \tag{686}$$

$$e_{2,i} = e^{-\upsilon_k/\mu_i} \tag{687}$$

$$E^+_{0,i} = \frac{e_{2,i} - \mathcal{X}_{\mathrm{b},k} e_{1,i}}{1 + \mu_i \lambda_k} \tag{688}$$

$$I^+_{l,i} = I^+_{l,i} + F^+_i E^+_{0,i} \tag{689}$$

16.1.1.1.2 Thermal source

$$\mathbf{A}^+_{\mathrm{u},i} = \frac{\omega}{2} (\mathbf{P}^{++}_{\mathrm{uq}} \mathbf{W} \mathbf{A}^+_i + \mathbf{P}^{-+}_{\mathrm{uq}} \mathbf{W} \mathbf{A}^-_i) \tag{690}$$

$$z_{0,j} = 1 - e_{1,j} \tag{691}$$

$$z_{i,j} = \upsilon^i - x^i e_{1,j} + i \mu_j z_{i-1,j} \tag{692}$$

$$I^+_{l,j} = I^+_{l,j} + \sum_{i=0}^{2} (\mathbf{A}^+_{\mathrm{u},i,j} + (1 - \omega) c_j) z_{i,j} \tag{693}$$

16.1.1.1.3 Homogeneous solution

$$\mathbf{X}^+_{\mathrm{u}} = \mathbf{P}^{++}_{\mathrm{uq}} \mathbf{W} \mathbf{X}^+ + \mathbf{P}^{-+}_{\mathrm{uq}} \mathbf{W} \mathbf{X}^- \tag{694}$$

$$\mathbf{X}^-_{\mathrm{u}} = \mathbf{P}^{++}_{\mathrm{uq}} \mathbf{W} \mathbf{X}^- + \mathbf{P}^{-+}_{\mathrm{uq}} \mathbf{W} \mathbf{X}^+ \tag{695}$$

$$e_{3,i} = e^{-\nu_i \upsilon} \tag{696}$$

$$e_{4,i} = e^{-\nu_i x} \tag{697}$$

$$e_{5,i} = e^{-\nu_i (x-\upsilon)} \tag{698}$$

$$E^{+}_{i,j} = \frac{e_{3,j} - e_{4,j} e_{1,i}}{1 + \mu_i \nu_j} \tag{699}$$

$$E^{-}_{i,j} = \frac{e_{5,j} - e_{1,i}}{1 - \mu_i \nu_j} \tag{700}$$

$$I^{+}_{l,i} = I^{+}_{l,i} + \sum_{j=0}^{N} \omega (b^{+}_{j} \mathbf{X}^{+}_{i,j} E^{+}_{i,j} + b^{-}_{j} \mathbf{X}^{-}_{i,j} E^{-}_{i,j}) \tag{701}$$

16.1.1.2 Tangent linear

16.1.1.2.1 Solar source

16.1.1.2.2 Thermal source

16.1.1.2.3 Homogeneous solution

16.1.1.3 Adjoint of tangent linear

16.1.1.3.1 Solar source

16.1.1.3.2 Thermal source

16.1.1.3.3 Homogeneous solution

16.1.2 Downward

16.1.2.1 Forward

16.1.2.1.1 Solar source

16.1.2.1.2 Thermal source

16.1.2.1.3 Homogeneous solution

16.1.2.2 Tangent linear

16.1.2.2.1 Solar source

16.1.2.2.2 Thermal source

16.1.2.2.3 Homogeneous solution

16.1.2.3 Adjoint of tangent linear

16.1.2.3.1 Solar source

16.1.2.3.2 Thermal source

16.1.2.3.3 Homogeneous solution

16.2 Local source, Green's function

16.2.1 Upward

16.2.1.1 Forward

16.2.1.1.1 Solar source

16.2.1.1.2 Thermal source

16.2.1.1.3 Homogeneous solution

16.2.1.2 Tangent linear

16.2.1.2.1 Solar source

16.2.1.2.2 Thermal source

16.2.1.2.3 Homogeneous solution

16.2.1.3 Adjoint of tangent linear

16.2.1.3.1 Solar source

16.2.1.3.2 Thermal source

16.2.1.3.3 Homogeneous solution

16.2.2 Downward

16.2.2.1 Forward

16.2.2.1.1 Solar source

16.2.2.1.2 Thermal source

16.2.2.1.3 Homogeneous solution

16.2.2.2 Tangent linear

16.2.2.2.1 Solar source

16.2.2.2.2 Thermal source

16.2.2.2.3 Homogeneous solution

16.2.2.3 Adjoint of tangent linear

16.2.2.3.1 Solar source

16.2.2.3.2 Thermal source

16.2.2.3.3 Homogeneous solution

17 Successive orders of scattering

17.0.3 Forward

$$\boldsymbol{\mathcal{T}_k} = \mathrm{diag}(e^{-x_k/\mu_i}) \tag{702}$$

$$\boldsymbol{\mathcal{E}}_k = \mathbf{E} - \boldsymbol{\mathcal{T}}_k \tag{703}$$

$$t_k = \frac{e^{-\tau_{k+1}\lambda_{k+1}} - e^{-\tau_k\lambda_k}}{2} \tag{704}$$

$$\mathbf{I}_1^+(\tau_k) = \boldsymbol{\mathcal{T}}_k \mathbf{I}_1^+(\tau_{k+1}) + \frac{F_0}{4\pi}(\boldsymbol{\mathcal{E}}_k \omega \mathbf{P}_0^+ t)_k \tag{705}$$

$$\mathbf{I}_1^-(\tau_{k+1}) = \boldsymbol{\mathcal{T}}_k \mathbf{I}_1^-(\tau_k) + \frac{F_0}{4\pi}(\boldsymbol{\mathcal{E}}_k \omega \mathbf{P}_0^- t)_k \tag{706}$$

$$\mathcal{L}(t_k) = \frac{[-\mathcal{L}(\tau_{k+1})\lambda_{k+1}]\, e^{-\tau_{k+1}\lambda_{k+1}} - [-\mathcal{L}(\tau_k)\lambda_k]\, e^{-\tau_k\lambda_k}}{2} \tag{707}$$

$$\begin{aligned} \mathbf{K}_1^+(\tau_k) &= \boldsymbol{\mathcal{W}}_k \mathbf{I}_1^+(\tau_{k+1}) + \boldsymbol{\mathcal{T}}_k \mathbf{K}_1^+(\tau_{k+1}) \\ &+ \frac{F_0}{4\pi}\left[\boldsymbol{\mathcal{F}}\omega\mathbf{P}_0^+ t + \boldsymbol{\mathcal{E}}\mathcal{L}(\omega)\mathbf{P}_0^+ t + \boldsymbol{\mathcal{E}}\omega\mathbf{Q}_0^+ t + \boldsymbol{\mathcal{L}}\omega\mathbf{P}_0^+\mathcal{L}(t)\right]_k \end{aligned} \tag{708}$$

$$\begin{aligned} \mathbf{K}_1^-(\tau_{k+1}) &= \boldsymbol{\mathcal{W}}_k \mathbf{I}_1^-(\tau_k) + \boldsymbol{\mathcal{T}}_k \mathbf{K}_1^-(\tau_k) \\ &+ \frac{F_0}{4\pi}\left[\boldsymbol{\mathcal{F}}\omega\mathbf{P}_0^- t + \boldsymbol{\mathcal{E}}\mathcal{L}(\omega)\mathbf{P}_0^- t + \boldsymbol{\mathcal{E}}\omega\mathbf{Q}_0^- t + \boldsymbol{\mathcal{L}}\omega\mathbf{P}_0^-\mathcal{L}(t)\right]_k \end{aligned} \tag{709}$$

$$\boldsymbol{\mathcal{W}_k} = \mathrm{diag}(w_i) \tag{710}$$

$$\mathbf{I}(\tau_{k+0.5}) = \frac{[\mathbf{I}(\tau_k) + \mathbf{I}(\tau_{k+1})]}{2} \tag{711}$$

$$\mathbf{I}_j^+(\tau_k) = \boldsymbol{\mathcal{T}}_k \mathbf{I}_j^+(\tau_{k+1}) + (1+\delta_{0,m})\frac{1}{4}\boldsymbol{\mathcal{E}}_k \omega_k \left[\mathbf{P}_k^{++}\mathbf{I}_{n-1}^+(\tau_{k+0.5}) + \mathbf{P}_k^{-+}\mathbf{I}_{n-1}^-(\tau_{k+0.5})\right]\mathbf{W} \tag{712}$$

$$\mathbf{I}_j^-(\tau_{k+1}) = \boldsymbol{\mathcal{T}}_k \mathbf{I}_j^-(\tau_k) + (1+\delta_{0,m})\frac{1}{4}\boldsymbol{\mathcal{E}}_k \omega_k \left[\mathbf{P}_k^{--}\mathbf{I}_{n-1}^-(\tau_{k+0.5}) + \mathbf{P}_k^{+-}\mathbf{I}_{n-1}^+(\tau_{k+0.5})\right]\mathbf{W} \tag{713}$$

17.0.4 Tangent linear

17.0.5 Adjoint of tangent linear

18 Two orders of scattering

18.1 Forward

18.2 Tangent linear

18.3 Adjoint of tangent linear

19 Two-stream

19.1 Forward

19.2 Tangent linear

19.3 Adjoint of tangent linear

20 Four-stream

20.1 Forward

20.2 Tangent linear

20.3 Adjoint of tangent linear

21 Six-stream

21.1 Forward

21.2 Tangent linear

21.3 Adjoint of tangent linear

22 BRDF Kernels

22.1 Lambertian

22.1.1 Forward

$$f(\theta_i, \theta_r, \phi) = 1 \tag{714}$$

22.1.2 Tangent linear

$$\mathcal{L}\left[f(\theta_i, \theta_r, \phi)\right] = 0 \tag{715}$$

22.1.3 Adjoint of tangent linear

22.2 Roujean

22.2.1 Forward

$$a = \tan\theta_i + \tan\theta_r \tag{716}$$

$$b = \tan^2\theta_i + \tan^2\theta_r \tag{717}$$

$$t = \tan\theta_i \tan\theta_r \tag{718}$$

$$c = 2t \tag{719}$$

$$d = \frac{t}{2\pi} \tag{720}$$

$$t = \begin{cases} -1 & \text{if } \phi < 0 \\ +1 & \text{otherwise} \end{cases} \tag{721}$$

$$f(\theta_i, \theta_r, \phi) = [(\pi - t\phi)\cos\phi + \sin\phi]\, d - \frac{1}{\pi}(a + \sqrt{b - c\cos\phi}) \tag{722}$$

22.2.2 Tangent linear

$$\mathcal{L}\left[f(\theta_i, \theta_r, \phi)\right] = 0 \tag{723}$$

22.2.3 Adjoint of tangent linear

22.3 Li-common

22.3.1 Forward

$$\tan\theta_i' = x\tan\theta_i \tag{724}$$

$$\tan\theta_r' = x\tan\theta_r \tag{725}$$

$$a = \cos\theta_i' \cos\theta_r' \tag{726}$$

$$b = \sin\theta_i' \sin\theta_r' \tag{727}$$

$$c = \tan^2\theta_i' + \tan^2\theta_r' \tag{728}$$

$$d = 2\tan\theta_i' \tan\theta_r' \tag{729}$$

$$e = \tan^2\theta_i' \tan^2\theta_r' \tag{730}$$

$$r = 1/\cos\theta_i' + 1/\sin\theta_i' \tag{731}$$

$$g = y/r \tag{732}$$

$$f(\theta_i, \theta_r, \phi) = \tag{733}$$

22.3.2 Tangent linear

22.3.3 Adjoint of tangent linear

22.4 Li-sparse

22.4.1 Forward

$$f(\theta_i, \theta_r, \phi) = \tag{734}$$

22.4.2 Tangent linear

22.4.3 Adjoint of tangent linear

22.5 Li-dense

22.5.1 Forward

$$f(\theta_i, \theta_r, \phi) = \tag{735}$$

22.5.2 Tangent linear

22.5.3 Adjoint of tangent linear