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Bifurcation analysis of the dynamics of a two-link mathematical pendulum in a rotating frame of reference

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Introduction

In this work a mechanical system with two degrees of freedom is considered, which is a two-link mathematical pendulum in a rotating weightless frame of reference moving with a given angular velocity.

Such a system has trivial equilibrium points in the absence of rotation, from which the so-called "skew" equilibria deviate at certain values of the angular velocity. The degrees of stability of these equilibria have been found using the methods described in the work of A. V. Karapetyan [Kar98]. A qualitative analysis of the germs of such solutions in a small neighbourhoods of bifurcation points and the asymptotic behavior of curves at infinity has been carried out.

The square of the angular velocity is considered here as a bifurcation parameter, which allows us to construct a Smale diagram [Kar20; Sma70] for the system.

1 Initial setting.

The task is to describe the dynamics of a two-pendulum system in a rotating suspension using the Lagrange formalism. The system is considered ideal with frictionless constraints. Generalized coordinates describing the position of a pendulum using angles of each link relative to vertical are chosen as the coordinates x and y, and the masses m_1 and m_2 as shown in Figure 1.

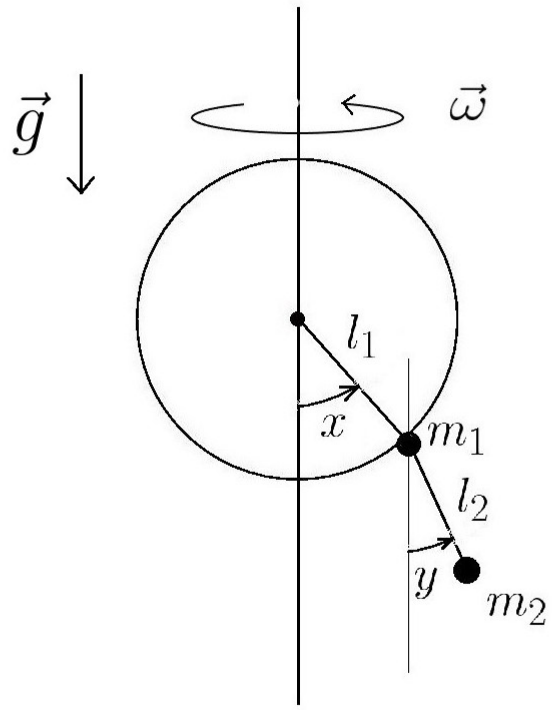

Figure 1: Schematic representation of the system with the main parameters indicated.

Given m_1, m_2, l_1, l_2, $\vec{g}$ и $\vec{\omega}$ the Lagrangian of the system will have the following form:

$$\begin{aligned} L = \frac{1}{2} m_1 \dot{x}^2 l_1^2 + \frac{1}{2} m_2 \left(\dot{x}^2 l_1^2 + 2\, \dot{x}\, \dot{y}\, l_1\, l_2 \cos(x-y) + \dot{y}^2 l_2^2 \right) + \\ + \frac{1}{2}\omega^2 \left(m_1 l_1^2 \sin^2 x + m_2 (l_1 \sin x + l_2 \sin y)^2 \right) + \\ + m_1\, g\, l_1 \cos x + m_2\, g (l_1 \cos x + l_2 \cos y) \end{aligned} \tag{1}$$

The kinetic energy of the pendulum consists of two parts: a quadratic part in the generalized velocities T_2 and an independent part T_0.

$$T_2 = \frac{1}{2}(m_1 + m_2) \dot{x}^2 l_1^2 + m_2\, \dot{x}\, \dot{y}\, l_1\, l_2 \cos(x-y) + \frac{1}{2} m_2\, \dot{y}^2 l_2^2$$

$$T_0 = \frac{1}{2}\omega^2 \left(m_1 l_1^2 \sin^2 x + m_2 (l_1 \sin x + l_2 \sin y)^2 \right)$$

The amended potential function for the system is $V_\omega(x,y)$:

$$V_\omega(x,y) = V(x,y) - T_0(\omega,x,y)$$

Let us redefine $V_\omega(x,y)$ as $V_p(x,y)$, where $p=\omega^2$. In further analysis, we will use the parameter p instead of ω as it is involved only quadratically. The explicit form of this function is as follows:

$$\begin{gathered} V_p(x,y) = -(m_1+m_2)\,g\,l_1\cos x - m_2\,g\,l_2\cos y - \\ -\frac{1}{2}p\left(m_1\,l_1^2\sin^2 x + m_2(l_1\sin x + l_2\sin y)^2\right) \end{gathered} \tag{2}$$

Suppose the mass and length of both pendulums are the same. Introducing a system of units such that $m_1=m_2=1$, $l_1=l_2=1$, $g=1$, expression (2) for $V_p(x,y)$ is transformed as follows:

$$V_p(x,y) = -2\cos x - \cos y - \frac{1}{2}p\left(\sin^2 x + (\sin x + \sin y)^2\right) \tag{3}$$

$$\begin{cases} \varphi = \varphi' \\ p_\varphi = \varepsilon^\alpha\, p'_\varphi \end{cases}$$

$$\begin{cases} \dot\varphi = \dfrac{\partial H}{\partial p_\varphi} \implies \dot\varphi' = \dfrac{1}{\varepsilon^\alpha}\dfrac{\partial H}{\partial p'_\varphi} \\ \dot p_\varphi = -\dfrac{\partial H}{\partial \varphi} \implies \varepsilon^\alpha\,\dot p'_\varphi = -\dfrac{\partial H}{\partial \varphi'} \end{cases} \implies \begin{cases} \dot\varphi' = \dfrac{1}{\varepsilon^\alpha}\dfrac{\partial H}{\partial p'_\varphi} = \dfrac{\partial\left(\frac{1}{\varepsilon^\alpha}H\right)}{\partial p'_\varphi} = \dfrac{\partial \widehat H}{\partial p'_\varphi} \\ \dot p'_\varphi = -\dfrac{1}{\varepsilon^\alpha}\dfrac{\partial H}{\partial \varphi'} = -\dfrac{\partial\left(\frac{1}{\varepsilon^\alpha}H\right)}{\partial \varphi'} = -\dfrac{\partial \widehat H}{\partial \varphi'} \end{cases} \implies$$

$$\implies \widehat H = \frac{1}{\varepsilon^\alpha}\,H\Big|_{\varphi=\varphi',\,p_\varphi=\varepsilon^\alpha p'_\varphi}$$

$$\begin{cases} \psi = \theta \\ p_\psi = \dfrac{p'_\psi}{\varepsilon^\beta} \end{cases}$$

$$\begin{cases} \dot\psi = \dfrac{\partial H}{\partial p_\psi} \implies \dot\theta = \varepsilon^\beta\dfrac{\partial H}{\partial p'_\psi} \\ \dot p_\psi = -\dfrac{\partial H}{\partial \psi} \implies \dfrac{1}{\varepsilon^\beta}\,\dot p'_\psi = -\dfrac{\partial H}{\partial \theta} \end{cases} \implies \begin{cases} \dot\theta = \dfrac{\partial\left(\varepsilon^\beta H\right)}{\partial p'_\psi} = \dfrac{\partial \widehat H}{\partial p'_\psi} \\ \dot p'_\psi = -\dfrac{\partial\left(\varepsilon^\beta H\right)}{\partial \theta} = -\dfrac{\partial \widehat H}{\partial \theta} \end{cases} \implies$$

$$\implies \widehat H = \varepsilon^\beta\,H\Big|_{\psi=\theta,\,p_\psi=\frac{p'_\psi}{\varepsilon^\beta}}$$

$$
(*)\begin{cases}
\varphi = \varphi \\
\psi = \theta \\
p_\varphi = \varepsilon^\alpha\, I \\
p_\psi = \dfrac{p}{\varepsilon^\beta}
\end{cases}
$$

$$
H_* = \frac{\varepsilon^\beta}{\varepsilon^\alpha}\, H\,\Big|_{(*)}
$$

$$
H_* = \varepsilon^{\beta-\alpha}\,\left(\frac{\varepsilon^{2\alpha} I^2}{2\varepsilon} + \frac{p^2/\varepsilon^{2\beta}}{2(1+\varepsilon\sin^2\varphi)} - \varepsilon\sin\varphi\cos\theta\right)
$$

$$
(**)\begin{cases}
\varphi = \varphi \\
\psi = \theta \\
p_\varphi = \varepsilon^\alpha\, I \\
p_\psi = \dfrac{p}{\varepsilon^\beta} \\
t = \varepsilon^\gamma\, \tau
\end{cases}
$$

2 Equilibria.

We will find the extrema of the amended potential $V_p(x, y)$ to find the equilibrium points of the system:

$$\begin{cases} \frac{\partial V_p}{\partial x}(x, y) = 2\sin x - p\,(2\sin x + \sin y)\cos x = 0 \\ \frac{\partial V_p}{\partial y}(x, y) = \sin y - p\,(\sin x + \sin y)\cos y = 0 \end{cases} \tag{4}$$

Equation system (4) has four trivial solutions (x, y mod 2π):

$$(x_1\,, y_1) = (0, 0), \quad (x_2\,, y_2) = (0, \pi), \quad (x_3\,, y_3) = (\pi, 0), \quad (x_4\,, y_4) = (\pi, \pi).$$

2.1 Stability of the equilibria.

To determine the stability of the trivial solutions as a function of p, the eigenvalues of the matrix $\left(\frac{\partial^2 V_p}{\partial q^2}(q)\right)$ will be found with $q_i = (x_i\,, y_i)$ substituted. These eigenvalues are called the Poincaré coefficients:

$$det\left(\frac{\partial^2 V_p}{\partial q^2}(q_i) - \lambda^i E\right) = 0, \quad i = 1, 2, 3, 4 \tag{5}$$

Given that $V_p(x, y)$ depends on the parameter p, coefficients of the Hessian matrix of the amended potential also depend on p, and thus λ^i as functions of the characteristic polynomial coefficients are functions of p.

The matrix $\left(\frac{\partial^2 V_p}{\partial q^2}(q)\right)$ is two-dimensional and its characteristic polynomial is a quadratic polynomial with respect to λ. According to the definition, if equation (5) has no zero roots, then the number of negative eigenvalues is referred to as the degree of instability.

Therefore, the degree of instability for the system considered may have 0, 1 or 2 as its value.

Let us examine the equilibrium $q_1 = (x_1, y_1) = (0, 0)$. The matrix $\left(\frac{\partial^2 V_p}{\partial q^2}(q_1)\right)$ has the form:

$$\begin{pmatrix} 2(1-p) & -p \\ -p & 1-p \end{pmatrix}$$

Matrix degeneration occurs at $p = 2 \pm \sqrt{2}$:

$$det\begin{vmatrix} 2(1-p) & -p \\ -p & 1-p \end{vmatrix} = p^2 - 4p + 2 = \left(p - (2+\sqrt{2})\right)\left(p - (2-\sqrt{2})\right) = 0$$

Its eigenvalues are:

$$\lambda_1^1 = \frac{1}{2}\left(3(1-p) + \sqrt{5p^2 - 2p + 1}\right), \quad \lambda_2^1 = \frac{1}{2}\left(3(1-p) - \sqrt{5p^2 - 2p + 1}\right)$$

It is easy to check that $\lambda_{1,2}^1$ as functions of p have the following signs on the intervals:

$$\begin{array}{llll} (a) & \lambda_1^1(p) > 0, & \lambda_2^1(p) > 0 & \forall p \in [\,0,\, 2-\sqrt{2}\,) \\ (b) & \lambda_1^1(p) < 0, & \lambda_2^1(p) > 0 & \forall p \in (\,2-\sqrt{2},\, 2+\sqrt{2}\,) \\ (c) & \lambda_1^1(p) < 0, & \lambda_2^1(p) < 0 & \forall p \in (\,2+\sqrt{2},\, +\infty\,) \end{array}$$

The stability intervals are as follows: on interval (a) the degree of instability is 0, in case of interval (b) the degree of instability is 1, and for the open ray (c) the degree of instability is equal to 2.

At the degenerate points $p = 2 \pm \sqrt{2}$ corresponding Poincare coefficient becomes zero, and the degree of instability changes as p passes through these points towards increasing. Such values of the bifurcation parameter of the problem are called bifurcation points. At these points, the corresponding solution of equations (4) ceases to be locally unique, and the so-called "skew" equilibrium branches away from the trivial solution. Its degree of instability is equal to the degree of instability of the trivial solution, from which it branches off, depending on the direction. In this particular case, the skew equilibrium exists when $p > 2 - \sqrt{2}$, and therefore carries a zero degree of instability.

For the equilibrium $q_2 = (x_2, y_2) = (0, \pi)$, the Hessian matrix is as follows:

$$\begin{pmatrix} 2(1-p) & p \\ p & -(1+p) \end{pmatrix}$$

Matrix degenerates at $p = \pm\sqrt{2}$:

$$det \begin{vmatrix} 2(1-p) & p \\ p & -(1+p) \end{vmatrix} = p^2 - 2 = 0$$

Since parameter p is the square of the angular velocity, condition $p \geq 0$ applies to it, and the negative root $p = -\sqrt{2}$ is not considered.

Poincaré coefficients (5):

$$\lambda_1^2 = \frac{1}{2}\left(1 - 3p + \sqrt{5p^2 - 6p + 9}\right), \quad \lambda_2^2 = \frac{1}{2}\left(1 - 3p - \sqrt{5p^2 - 6p + 9}\right)$$

Intervals where the sign of $\lambda^2_{1,2}$ is preserved:

$$\begin{aligned} &(d) \quad \lambda^2_1(p) > 0, \quad \lambda^2_2(p) < 0 \quad \forall\, p \in [\,0,\, \sqrt{2}\,) \\ &(e) \quad \lambda^2_1(p) < 0, \quad \lambda^2_2(p) < 0 \quad \forall\, p \in (\,\sqrt{2},\, +\infty\,) \end{aligned}$$

Hence the equilibrium $(x_2, y_2) = (0, \pi)$ has one bifurcation point $p = \sqrt{2}$ which changes the degree of instability from 1 to 2.

Hessian at $q_3 = (x_3, y_3) = (\pi, 0)$:

$$\begin{pmatrix} -2(1+p) & p \\ p & 1-p \end{pmatrix}$$

Its determinant vanishes when $p = \pm\sqrt{2}$:

$$det \begin{vmatrix} -2(1+p) & p \\ p & 1-p \end{vmatrix} = p^2 - 2 = 0$$

The root $p = -\sqrt{2}$ is excluded from consideration due to restrictions on the range of values for p.

Poincaré coefficients $\lambda^3_{1,2}(p)$ and their signs depending on p:

$$\lambda^3_1 = \frac{1}{2}\left(-(1+3p) + \sqrt{5p^2+6p+9}\right), \lambda^3_2 = \frac{1}{2}\left(-(1+3p) - \sqrt{5p^2+6p+9}\right)$$

$$\begin{aligned} &(f) \quad \lambda^4_1(p) > 0, \quad \lambda^4_2(p) < 0 \quad \forall\, p \in [\,0,\, \sqrt{2}\,) \\ &(g) \quad \lambda^4_1(p) < 0, \quad \lambda^4_2(p) < 0 \quad \forall\, p \in (\,\sqrt{2},\, +\infty\,) \end{aligned}$$

Similar to the previous case with $(x_2, y_2) = (0, \pi)$, for the trivial equilibrium $(x_3, y_3) = (\pi, 0)$ there is one bifurcation point $p = \sqrt{2}$ with instability degree changing from 1 to 2.

Equilibrium $q_4 = (x_4, y_4) = (\pi, \pi)$ with the matrix $\left(\frac{\partial^2 V_p}{\partial q^2}(q_4)\right)$:

$$\begin{pmatrix} -(2+p) & -p \\ -p & -(1+p) \end{pmatrix}$$

has a non-degenerate Hessian matrix for all $p \geq 0$:

$$det \begin{vmatrix} -(2+p) & -p \\ -p & -(1+p) \end{vmatrix} = 3p + 2$$

Eigenvalues $\lambda^4_{1,2} < 0$ everywhere at $p \geq 0$:

$$\lambda^4_1 = \frac{1}{2}\left(-(3+2p) + \sqrt{4p^2+1}\right), \quad \lambda^4_2 = \frac{1}{2}\left(-(3+2p) - \sqrt{4p^2+1}\right)$$

Thus, this equilibrium has no bifurcation points and retains instability degree 2 for all non-negative p.

3 Bifurcation analysis of the equilibria.

For this problem, skew equilibria can be found as curves on the closed half-plane $(h, p) \in \mathbb{R}^2; |; h \in \mathbb{R}, ; p \geq 0$, where h is the constant level of the amended potential $V_p(x, y)$, and p is the square of the angular velocity of the weightless frame. In this work, a qualitative analysis of these curves has been conducted: the behaviour of the skew solution germs near the corresponding equilibria and their asymptotics when p approaches infinity have been studied using the small parameter method.

Curves $h = h(p)$ can be considered as parametric curves defined by equation (3) and system (4), where $x, , y$ are parameters to be eliminated.

From (4) have:

$$\begin{cases} \sin y = \dfrac{2(1 - p\cos x)\sin x}{p\cos x} \\ \cos y = \dfrac{2(1 - p\cos x)}{p(2 - p\cos x} \end{cases}$$

Using the Pythagorean identity, one can eliminate y and obtain an equation that relates p and x:

$$4(1 - p\cos x)^2 \left(\cos^2 x + (2 - p\cos x)^2 \sin^2 x\right) = p^2(2 - p\cos x)^2 \cos^2 x \quad (6)$$

Such equation doesn't allow to express variable x explicitly, so the graphs of solution germs were constructed using computer methods via Wolfram Mathematica 12 (Figure 2).

For studying the asymptotic behavior of curves $h(p)$, equilibrium equations (4) are considered with $q = \frac{1}{p}$ substituted.

$$\begin{cases} 2q\sin x = (2\sin x + \sin y)\cos x \\ q\sin y = (\sin x + \sin y)\cos y \end{cases} \quad (7)$$

As $p \to +\infty$, $q \to 0+$ and one can take the limit relative to q:

$$\begin{cases} (2\sin x + \sin y)\cos x = 0 \\ (\sin x + \sin y)\cos y = 0 \end{cases}$$

The system of equations breaks down into a system of logical disjunctions:

$$\begin{cases} \left[\begin{array}{l} 2\sin x + \sin y = 0 \\ \cos x = 0 \end{array} \right. \\ \left[\begin{array}{l} \sin x + \sin y = 0 \\ \cos y = 0 \end{array} \right. \end{cases} \quad (8)$$

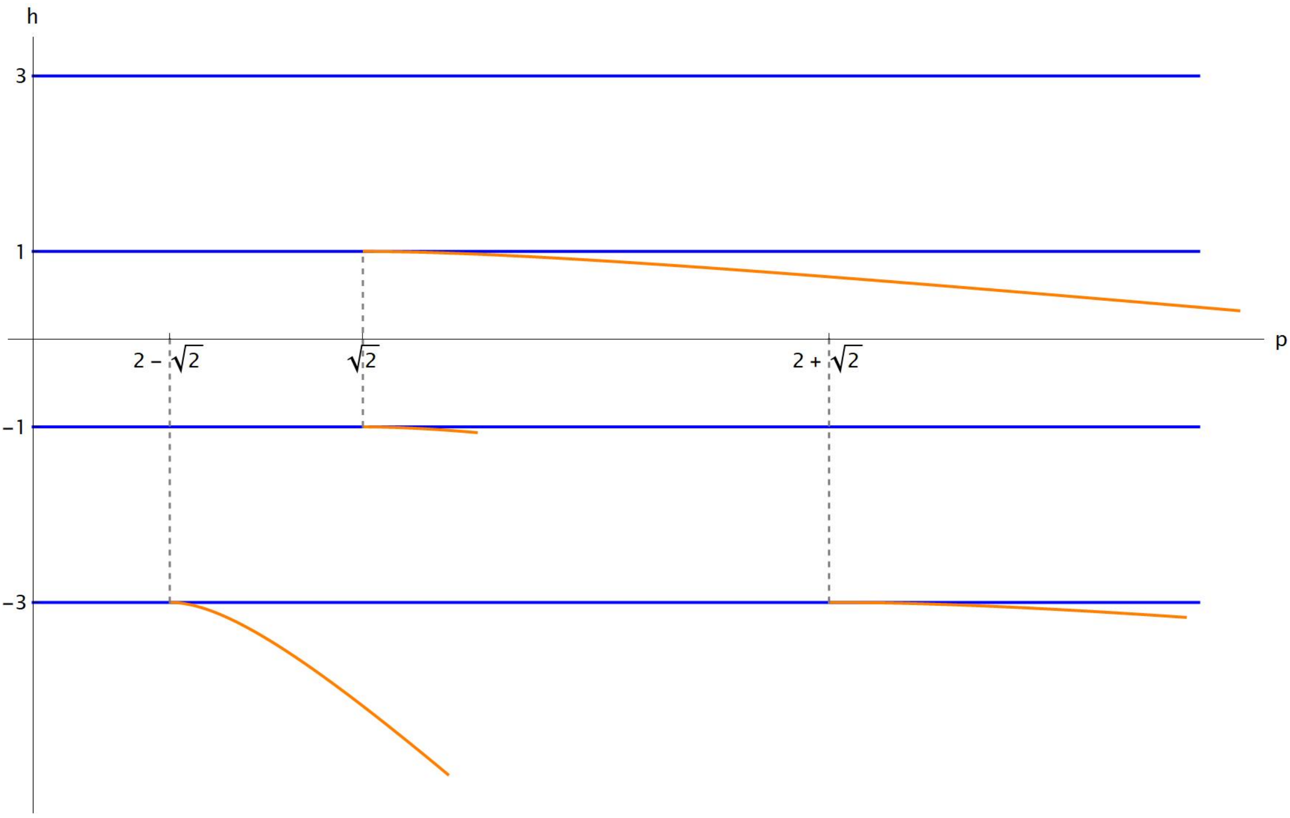

Figure 2: Germs of the skew solutions branching off from the trivial equilibria at the points of bifurcations.

To solve this logical system, it is necessary to determine if the first equations on sums of $\sin x$ and $\sin y$ can hold. To do this we will expand $\sin x$, $\sin y$, $\cos x$, $\cos y$ into Taylor series in a small neighbourhood of the equilibrium up to $O(x^4)$ and, with the help of the small parameter method, we will know the signs of $\sin x$ and $\sin y$ for each skew solution.

Consider case $x = 0$, $y = 0$, $p = 2 - \sqrt{2}$. In its neighbourhood one can set $|x| \ll 1$, $|y| \ll 1$, $p = 2 - \sqrt{2} + \varepsilon$, $|\varepsilon| \ll 1$, ε is as small as x^2. Also we introduce the parameter $z : y = zx, |z| \approx 1$ for convenience.

Then Taylor expansion takes the following form:

$$\sin x = x - \frac{1}{3}x^3 + O(x^4), \quad \cos x = 1 - \frac{1}{2}x^2 + O(x^4)$$
$$\sin y = zx - \frac{1}{3}z^3x^3 + O(x^4), \quad \cos y = 1 - \frac{1}{2}z^2x^2 + O(x^4)$$

Substituting this in (4), we get a system of two equations on x^2, z. Solving it gives $z \approx \sqrt{2} > 0$. As a corollary, we have x and y holding the same sign, and the system (8) transforms into:

$$\begin{cases} \cos x = 0 \\ \cos y = 0 \end{cases} \Leftrightarrow \begin{cases} x = \pm\dfrac{\pi}{2} \\ y = \pm\dfrac{\pi}{2} \end{cases}$$

So the asymptotic estimates on the generalized coordinates are obtained:

$$(x,\, y) \to (\pm\frac{\pi}{2},\, \pm\frac{\pi}{2}), \quad p \to +\infty$$

Taking this into account for the expression (3) with $h = h(x, y, p)$, we get an oblique asymptote:

$$h(p) \to -\frac{5}{2}p, \quad p \to +\infty \tag{9}$$

The degree of instability of the system along this curve is 0 as it branches off from a trivial solution with a Hessian matrix having both eigenvalues positive.

For $x = 0$, $y = 0$, $p = 2 + \sqrt{2}$ similarly have $z \approx -\sqrt{2} < 0$, and therefore different signs for $\sin x$ and $\sin y$. System (8) then has the form:

$$\begin{cases} 2\sin x + \sin y = 0 \\ \sin x + \sin y = 0 \end{cases} \Leftrightarrow \begin{cases} x = \mp\dfrac{\pi}{6} \\ y = \pm\dfrac{\pi}{2} \end{cases}$$

Skew equilibrium that branches off from $(x, y) = (0, 0)$ at $p = 2 + \sqrt{2}$, oblique asymptote as $p \to +\infty$ is expressed by:

$$h(p) \to -\sqrt{3} - \frac{1}{4}p, \quad p \to +\infty \tag{10}$$

The degree of instability of the system on this curve is 1.

The curves that originate from $(x, y) = (0, \pi)$ and $(x, y) = (\pi, 0)$ at $p = \sqrt{2}$ have a common asymptote of $-\frac{1}{2}, p$:. We will demonstrate this.

First, we will apply the same method used to study the two equilibria described above. Consider the case $x = 0$, $y = \pi$, $p = \sqrt{2}$. Taylor series are:

$$y = \pi + \eta, \quad \eta = zx, \quad p = \sqrt{2} + \varepsilon$$
$$\sin y = -\sin\eta = -zx + \frac{1}{3}z^3x^3 + O(x^4),$$
$$\cos y = -\cos\eta = -1 + \frac{1}{2}z^2x^2 + O(x^4)$$

Getting rid of y and solving equations for x^2 and z, one similarly gets $z \approx 2 - \sqrt{2} > 0$ and $\sin x \cdot \sin y < 0$ respectively since $\sin \eta = -\sin y > 0$. Solving (8) gives

$$\begin{cases} x = \pm\dfrac{\pi}{2} \\ y = \mp\dfrac{\pi}{2} \end{cases}$$

Skew solution asymptoptically tends to the straight line $-\frac{1}{2}\,p$:

$$h(p) \to -\frac{1}{2}\,p\,, \quad p \to +\infty \tag{11}$$

Case $x = \pi$, $y = 0$, $p = \sqrt{2}$ has the identical analysis up to the change of variables. Equations for x^2 and z will give $z \approx 1 - \frac{1}{\sqrt{2}} > 0$ which again yields $\sin x \cdot \sin y < 0$ and $h(p) \to -\frac{1}{2}\,p$.

So we described fully qualitative picture of the Smale diagram on the half-plane $(h,\, p)$.

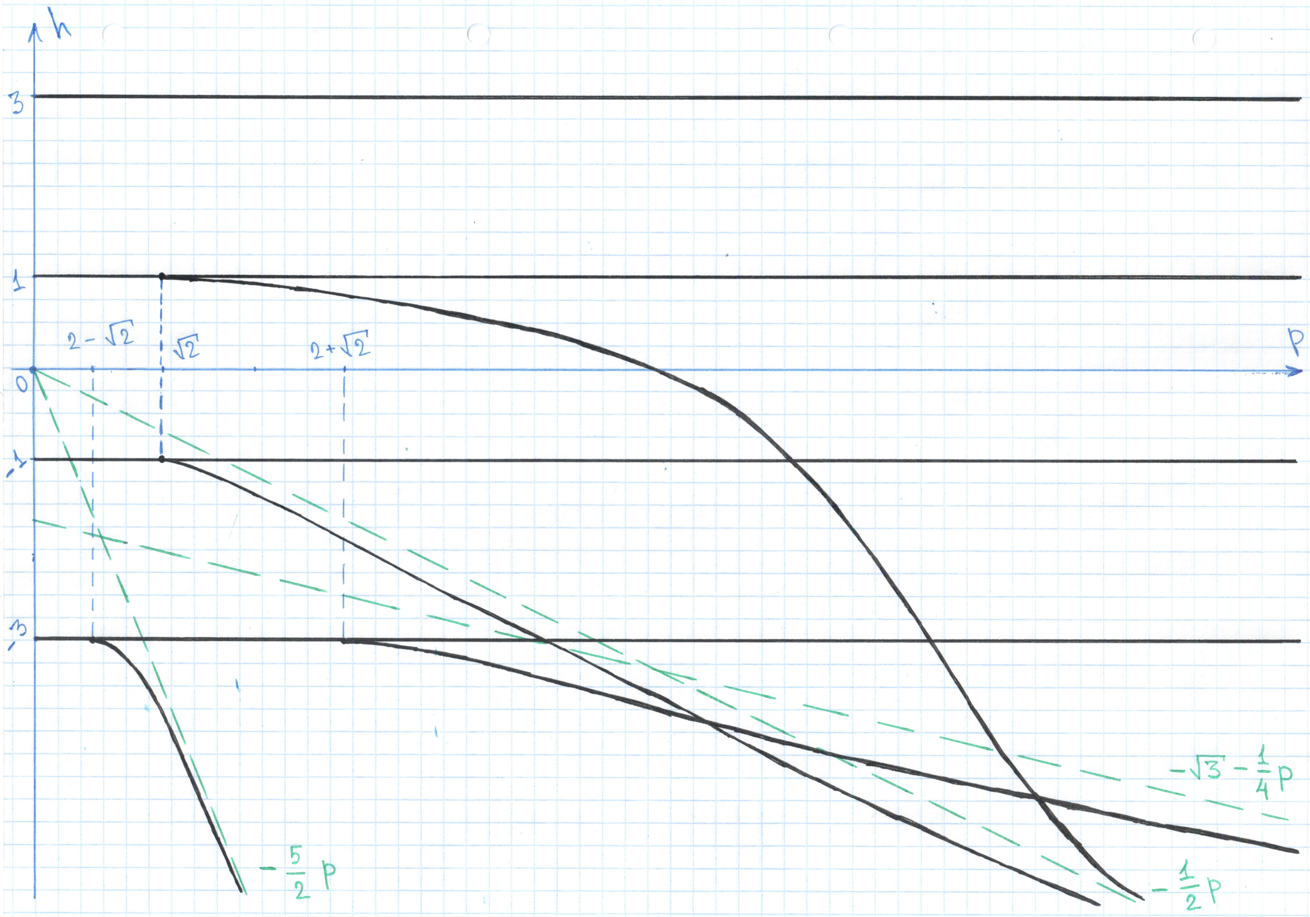

Figure 3: Smale diagram for the two-link mathematical pendulum in a weightless frame rotating with a given square of angular velocity $\omega^2 = p$.

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