p-adic Lie Group

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Abstract

In this paper we seek to introduce the $\boldsymbol{p}$-adic Lie groups. We shall first construct the field of $\boldsymbol{p}$-adic numbers as the completion of the rationals with respect to a non-archimedean metric. We then very briefly develop analysis on the field of $\boldsymbol{p}$-adic numbers and introduce the notion of locally analytic functions. Then, we can finally introduce analytic manifolds. We will then consider the structure of $\boldsymbol{p}$-adic Lie groups, its relation to pro-$\boldsymbol{p}$ groups, and the algebraic theory of $\boldsymbol{p}$-adic Lie groups along with relevant definitions.

To my parents, Zane, and Mr. Nguyen, my high school math teacher.

1 Preliminaries

1.1 p-adic Analysis

Proposition 1.1.1. *Let X be an ultrametric space, then every ball in X is open and closed.*

Proposition 1.1.2. *Let $B, B' \subset X$ be two balls. If $B \cap B' \neq \varnothing$, then $B' \subset B$ or $B \subset B'$.*

Remark 1.1.3 (Total disconnectedness)**.** *A connected ultrametric space contains at most one point.*

Definition 1.1.4 (Spherical complete)**.** *Let (X, d) be an ultrametric space. It is called* ***spherical complete*** *if for every descending chain of balls $B_1 \supseteq B_2 \supseteq \dots$ in X their intersection is non-empty.*

Proposition 1.1.5. *A spherically complete ultrametric space is complete.*

Proposition 1.1.6. *Let X be a complete ultrametric space. If 0 is the only accumulation point of $d(X \times X) \subseteq \mathbb{R}^{\geq 0}$, then X is spherical complete.*

Definition 1.1.7 (Non-archimedean absolute value)**.** *Let K be a field. A* ***non-archimedean absolute value*** *on K is a function*

$$|\cdot| : K \to \mathbb{R}$$

satisfying the following properties:

1. $|a| \geq 0$,
2. $|a| = 0 \iff a = 0$,
3. $|ab| = |a| \cdot |b|$,
4. $|a + b| \leq \max\{|a|, |b|\}$.

A field $(K, |\cdot|)$ is called ***non-archimedean*** *if $|\cdot|$ is non-archimedean and*

1. *$|\cdot|$ is non-trivial*
2. *K is complete with respect to the metric $d(a, b) = |b - a|$.*

Definition 1.1.8 (Field of p-adic numbers)**.** *Let $p \in \mathbb{Z}^+$ be a prime number. We can define*

$$|a|_p := p^{-r} \text{ if } a = p^r \frac{m}{n} \text{ with } r, m, n \in \mathbb{Z} \text{ and } p \nmid mn,$$

which is a non-archimedean absolute value on $\mathbb{Q}$. The completion of $\mathbb{Q}$ under the corresponding metric is called the ***field of p-adic numbers****, which we denote $\mathbb{Q}_p$. It follows from Lemma 1.8 that $\mathbb{Q}_p$ is spherically complete.*

Lemma 1.1.9. *If K is p-adic, then*

$$|n| \geq |n!| \geq |p|^{\frac{n-1}{p-1}}$$

for all $n \in \mathbb{N}$.

Definition 1.1.10. *A (non-archimedean) norm on V is a function $\|\cdot\| : V \to \mathbb{R}$ satisfying:*

1. $\|av\| = |a| \, \|v\|$,

2. $\|v+w\| \leq \max(\|v\|, \|w\|)$,
3. $\|v\| = 0 \implies v = 0$.

for all $v, w \in V, a \in K$.

Definition 1.1.11. *Let V be a normed K-vector space with norm $\|\cdot\|$. Then, V is called a K-Banach space wf V is complete with respect to the metric $d(v,w) = \|w-v\|$.*

Definition 1.1.12. *Let V, W be two normed K-vector spaces. Let $\mathcal{L}(V,W)$ be the set of continuous functions from V to W, and let*

$$\begin{aligned}\|f\| &:= \sup\{\|f(v)\| : \|v\| = 1\} \\ &= \sup\left\{\frac{\|f(v)\|}{\|v\|} : v \neq 0\right\} \\ &= \inf\{C : \|f(v)\| \leq C\|v\| \text{ for all } v \in V\}.\end{aligned}$$

It is clear that $\mathcal{L}(V,W)$ equipped with the operator norm is a normed K-vector space. We also see that W is K-Banach $\implies \mathcal{L}(V,W)$ is K-Banach.

Lemma 1.1.13. *Let K be a fixed non-archimedean field with norm $|\cdot|$ and $(V, \|\cdot\|)$ a K-Banach space. Let $(v_i)_{i\in\mathbb{N}}$ be a sequence in V. Then*

1. *$\sum_{i=1}^{\infty} v_i$ converges $\iff \lim_{i\to\infty} v_i = 0$*
2. *If $\lim_{i\to\infty} v_i$ exists and equals $v \neq 0$, then $\|v_i\| = \|v\|$ for almost every $i \in \mathbb{N}$.*
3. *Rearrangement of convergent series is convergent and converges to the same limit.*

Definition 1.1.14 (Differentiability)**.** *Let V, W be two normed K-vector spaces. Let $U \subset V$ be an open set. Let $f : U \to W$ be a map. We say that f is **differentiable** at $v_0 \in U$ if there is a continuous linear map*

$$D_{v_0} f : V \to W$$

satisfying for any $\epsilon > 0$ there is $U_\epsilon \ni v_0$ open such that

$$\|f(v) - f(v_0) - D_{v_0} f(v - v_0)\| \leq \epsilon \|v - v_0\|$$

for all $v \in U_\epsilon$. Moreover, such a continuous linear map is unique.

Remark 1.1.15. *The usual rules of differentiability applies.*

Definition 1.1.16 (Strict differentiability)**.** *A map $f : V \to W$ is said to be **strictly differentiable** at $v_0 \in U$ if there exists a continuous linear map $D_{v_0} f : V \to W$ such that for all $\epsilon > 0$ there is $U_\epsilon \ni v_0$ with*

$$\|f(v_1) - f(v_2) - D_{v_0} f(v_1 - v_2)\| \leq \epsilon \|v_1 - v_2\|$$

for all $v_1, v_2 \in U_\epsilon$.

Definition 1.1.17 (Power series)**.** *We define $f(X)$, a power series in r variables $X = (X_1, \ldots, X_r)$ with coefficients in V, to be*

$$f(X) = \sum_{\alpha \in \mathbb{N}_0^r} X^\alpha v_\alpha$$

with $v_\alpha \in V$. We let $X^\alpha := X_1^{\alpha_1} \cdots X_r^{\alpha_r}$ and $|\alpha| = \alpha_1 + \ldots + \alpha_r$. We say that $f(X) = \sum_\alpha X^\alpha v_\alpha$ is ϵ-convergent if

$$\lim_{|\alpha| \to \infty} \epsilon^{|\alpha|} \|v_\alpha\| = 0.$$

1.2 Manifolds

Let $(K, |\cdot|)$ be a fixed non-archimedean field. Let M be a Hausdorff topological space.

Definition 1.2.1 (Coordinate chart)**.** *A **coordinate chart** for M is a pair (U, φ) for $U \subseteq M$ open and $\varphi : U \to \varphi(U) \subset K^n$ homeomorphism. We call U a **coordinate domain** or **coordinate neighbourhood**. We say that φ is a **coordinate map**. Let (U, φ), (V, ϕ) be two charts. Then we say that say that they are **compatible** if $\phi \circ \varphi^{-1} : \varphi(U \cap V) \to \phi(U \cap V)$ and $\varphi \circ \phi^{-1} : \phi(U \cap V) \to \varphi(U \cap V)$ are locally analytic.*

It is clear that if two compatible charts overlap, then they are locally euclidean with the same dimension.

Definition 1.2.2 (Atlas)**.** *By an **atlas** for M we mean a set $\mathcal{A} = \{(U, \varphi)\}$ of charts for M such that any two charts in $\mathcal{A}$ are compatible. We say that two atlases are equivalent if their union is also an atlas for M. We say that an atlas $\mathcal{A}$ is **maximal** if for all $\mathcal{B}$ equivalent to $\mathcal{A}$ we have that $\mathcal{B} \subset \mathcal{A}$.*

Charts with empty intersection are automatically compatible. One can check that the definition for equivalence defines an equivalence relation.

Proposition 1.2.3. *If $\mathcal{A}$ is maximal for M, then the coordinate neighbourhoods form a basis for the topology on M.*

Definition 1.2.4 (Dimension of an atlas)**.** *We say that an atlas $\mathcal{A}$ is n-dimensional if all the charts in $\mathcal{A}$ has dimension n.*

Definition 1.2.5 (Analytic manifolds)**.** *A **locally analytic manifold** (over K) is a pair $(M, \mathcal{A})$, where M is a Hausdorff, second-countable topological space equipped with a maximal n-dimensional atlas $\mathcal{A}$. We say that a function $f : M \to E$ is **locally analytic** if $f \circ \varphi^{-1}$ is analytic for all charts (U, φ) for M, where E is a K-Banach space. An analytic map can be defined similarly.*

Definition 1.2.6. *Let X be a topological space. We say that an open covering $\bigcup_{i \in I} U_i$ for X is **locally finite** if for every $x \in X$ there is $U \ni x$ such that $\#\{i \in I : U \cap U_i \neq 0\} < \infty$. We say that X is **paracompact** (respectively strictly paracompact), if any open covering can be refined to a locally finite covering (respectively which consists of pairwise disjoint open subsets).*

We see that any ultrametric space is strictly paracompact.

Proposition 1.2.7. *Let M be a manifold, then the following are equivalent:*

1. *M is paracompact,*

2. M is strictly paracompact,
3. The topology of M can be defined by a metric satisfying the strict triangle inequality.

Definition 1.2.8 (Tangent bundle)**.** *Let M be a manifold, let T_pM denote the tangent space of M at $p \in M$. Then, we define*

$$TM := \coprod_{p \in M} T_pM.$$

One can check routinely that TM is an analytic manifold.
Definition 1.2.9 (Vector field)**.** *Let $U \subset M$ be open. We define a **vector field** ξ to be a locally analytic map $\xi : U \to T(M)$ satisfying $\pi \circ \xi = \mathrm{id}_U$. We denote by*

$$\Gamma(U, T(M))$$

to be the set of all vector fields on U.

1.3 Lie Groups

Definition 1.3.1. *A **Lie Group** over K is a group that is also an analytic manifold with multiplication $\cdot : G \times G \to G$ such that*

$$\begin{aligned} \cdot : G \times G &\longrightarrow G \\ (g, h) &\longmapsto gh \end{aligned}$$

is locally analytic.
Clearly left and right multiplication by an element $g \in G$ are locally analytic isomorphisms of manifolds.
Proposition 1.3.2. *Every continuous homomorphism between p-adic lie groups is analytic. Thus, for a given topological group there is at most one p-adic Lie groups structure.*
One can see that p-adic Lie groups are totally disconnected. This means that the Lie algebra determines the Lie group locally around the identity.
Definition 1.3.3 (Lie group germ)**.** *We define $G_\epsilon := B_\epsilon(0) \subset K^d \cong \mathfrak{g}$. We also define $\{G_\epsilon\}_\epsilon$ to be the **Campbell-Hausdorff Lie group germ** of $\mathfrak{g}$, where $\mathfrak{g}$ is a Lie algebra.*
Theorem 1. *[1]* $\mathrm{Lie}(G_\epsilon) \cong \mathfrak{g}$ *as Lie algebras.*

Proof. Let $c := \left(G_\epsilon, \subseteq, K^d\right)$ be a chart for G_ϵ with the analytic isomorphism

$$\varphi_c : G_\epsilon \times K^d \xrightarrow{\cong} TG_\epsilon,$$

with the corresponding linear isomorphism

$$\begin{aligned} C^{\mathrm{an}}\left(G_\epsilon, K^d\right) &\longrightarrow \Gamma(G_\epsilon, TG_\epsilon) \\ f &\longmapsto \xi_f(g) = [c, f(g)]. \end{aligned}$$

We have the Lie bracket of vector fields

$$[f_1, f_2](g) = D_g f_1 (f_2 (g)) - D_g f_2 (f_1 (g))$$

and

$$T_0 (G_\epsilon) \to \Gamma (G_\epsilon, TG_\epsilon),$$

which induces the Lie bracket on $Lie (G_\epsilon) = T_0 G_\epsilon$. We have the commutative diagram

$$\begin{array}{ccc} K^d & \xrightarrow{D_0 r_g} & K^d \\ \downarrow \cong & & \downarrow \cong \\ T_0 G_\epsilon & \xrightarrow{d(r_g)_0} & T_g (\epsilon) . \end{array}$$

Let $f_v (g) = D_0 r_g (v)$. We get that

$$\xi_{[c,v]} (g) = d (r_g)_0 ([c, v]) = [c, D_0 r_g (v)] = \xi_{f_v (g)}.$$

This gives us another commutative diagram

$$\begin{array}{ccc} K^d & \xrightarrow{r \mapsto f_v} & C^{\mathrm{an}} \left(G_\epsilon, K^d\right) \\ \downarrow \cong & & \downarrow \cong \\ T_0 G_\epsilon & \xrightarrow{t \mapsto \xi_t} & \Gamma (G_\epsilon, TG_\epsilon) . \end{array}$$

Let $[,]''$ be the Lie bracket on K^d, corresponding to the Lie bracket of the Lie algebra $\mathrm{Lie} (G_\epsilon) = T_0 G_\epsilon$. Then, we have that

$$f_{[v,w]''} = [f_v, f_w]$$

for all $v, w \in K^d$. We see that

$$f_v (0) = v$$

and

$$\begin{aligned} [v, w]'' &= [f_v, f_w] (0) \\ &= D_0 f_v (w) - D_0 f_w (v) . \end{aligned}$$

Now, we need to show that $[\ ,\ ]' = [\ ,\ ]''$. To do this we need the following identity

$$r_g (h) = \underline{H} (h, g) ,$$

where

$$\underline{H} (\underline{Y}, \underline{Z}) := \left(H_{(1)} (\underline{Y}, \underline{Z}) , \dots , H_{(d)} (\underline{Y}, \underline{Z})\right)$$

is the tuple of formal power series in $2d$ variables (with abuse of notation). We have that

$$H_{(i)} = Z_i + \sum_{j=1}^{d} P_{(i,j)}\left(\underline{Z}\right) Y_j + \dots.$$

Then

$$d\left(r_g\right)_0 = \left(\frac{\partial H_{(i)}\left(\underline{Y}\right), g}{\partial Y_j}|_{\underline{Y}=0}\right) = \left(P_{i,j}\left(g\right)\right)_{i,j}$$

and

$$f_v\left(g\right) = d\left(r_g\right)_0\left(v\right) = \left(\sum_{j=1}^{d} v_j P_{(1,j)}\left(g\right), \dots, \sum_{j=1}^{d} v_j P_{d,j}\left(g\right)\right), v \in K^d.$$

Then, we see that

$$P_{(i,j)}\left(\underline{Z}\right) = \delta_{ij} + \frac{1}{2}\sum_{k=1}^{d} \gamma_{jk}^{i} Z_k + \dots.$$

Then, we have that

$$\frac{\partial P_{(i,j)(\underline{Z})}}{\partial Z_k}|_{\underline{Z}=0} = \frac{1}{2}\gamma_{jk}^{i},$$

where $\gamma_{ij}^{k} \in K$ is defined by

$$\left[e_i, e_j\right] = \sum_{k=1}^{d} \gamma_{ij}^{k} e_k.$$

Lastly, we have that

$$D_0\left(f_v\right) = \left(\frac{1}{2}\sum_{j=1}^{d} \gamma_{jk}^{i} v_j\right)_{i,k}$$

and

$$D_0\left(f_v\right)\left(w\right) = \left(\frac{1}{2}\sum_{j,k=1}^{d} \gamma_{jk}^{1} v_j w_k, \dots, \frac{1}{2}\sum_{j,k=1}^{d} \gamma_{jk}^{d} v_j w_j\right) = \frac{1}{2}\left[v, w\right]'.$$

Lastly, we have that

$$\left[v, w\right]'' = D_0 f_v\left(w\right) - D_0 f_w\left(v\right) = \frac{1}{2}\left[v, w\right]' - \frac{1}{2}\left[w, v\right]' = \left[v, w\right]'.$$

□

2 The Algebraic Theory of p-adic Lie Groups

2.1 Introduction

Let $\mathcal{O}$ be a complete discrete valuation ring with field of fractions K. Let π be a prime element in $\mathcal{O}$. We see that $\mathcal{O} = \varprojlim_m \mathcal{O}/\pi^m\mathcal{O}$ as a topological ring.

Definition 2.1.1 (Profinite group)**.** *A **profinite group** is a Hausdorff, compact, totally disconnected topological group.*

Definition 2.1.2 (Projective limit)**.** *Let* $(I, \leq)$ *be a directed, partially ordered set. Let* $((A_i)_{i\in I}$ *be groups and* $f_{ij} : A_j \to A_i$ *be homomorphisms for all* $i \leq j$ *such that*

1. $f_{ii} = \mathrm{id}_{A_i}$.
2. $f_{ik} = f_{ij} \circ f_{jk}$ *for all* $i \leq j \leq k$.

We call $\left((A_i)_{i\in I}, (f_{ij})_{i\leq j}\right)$ *an* ***inverse system*** *of groups. We call the homomorphisms* f_{ij} ***transition morphisms****. Now, we can define the* ***inverse limit*** *of* $\left((A_i)_{i\in I}, (f_{ij})_{i\leq j}\right)$ *as follows:*

$$A = \varprojlim_{i\in I} A_i = \left\{(a_1, a_2, \ldots,) \in \prod_{i\in I} A_i : a_i = f_{ij}(a_j), i, j \in I, i \leq j\right\}.$$

Definition 2.1.3. *Let* $\mathcal{N}(G)$ *be the set of open normal subgroups in* G*. Now, we have that*

$$G = \varprojlim_{N\in\mathcal{N}(G)} G/N$$

the projective limit of finite groups G/N.

Definition 2.1.4 (Group ring)**.** *Let* $(G, \cdot)$ *be a multiplicative group, and let* $(R, +, \cdot)$*. We define the group ring* G *over* r*, denoted by* $R[G]$*, to be the set of maps* $f : G \to R$ *such that* f *is zero except for finitely many* $g \in G$*. We define* cf *to be the map* $x \mapsto cf(x)$*,* $c \in R$*. We also define addition of* f, g *to be the map* $x \mapsto f(x) + g(x)$*. We define* fg *to be*

$$x \mapsto \sum_{uv=x} f(u)\, g(v) = \sum_{u\in G} f(u)\, g\left(u^{-1}x\right).$$

One can check that this makes $R[G]$ *into a free module over* R *and a ring. We also have the projective system of rings for* $\mathcal{O}[G/N]$*. Now, define*

$$\Lambda(G) := \mathcal{O}[[G]] := \varprojlim_{N\in\mathcal{N}(G)} \mathcal{O}[G/N]$$

the ***completed group ring*** *or* ***Iwasawa algebra*** *of* G *over* $\mathcal{O}$.

One can see immediately that $\mathcal{O}[G] \to \Lambda(G)$ is a natural inclusion. Thus, we can view $\mathcal{O}[G]$ as a subring of $\Lambda(G)$. Since $\mathcal{O}[G/N]$ is finitely generated and are free $\mathcal{O}$-modules, we see that they are complete topological $\mathcal{O}$-algebra with respect to the π-adic topology. One can see that

$$J_{m,N}(G) := \ker\left(\Lambda(G) \xrightarrow{\mathrm{pr}} (\mathcal{O}/\pi^m\mathcal{O})[G/N]\right)$$

for $m \geq 1$, $N \in \mathcal{N}(G)$ forms a basis of open neighborhoods around zero.

2.2 p-Valued Pro-p-Groups

Definition 2.2.1 (p-valuation)**.** *Let G be a group. We say that $\omega : G \setminus \{1\} \to (0, \infty)$ is a **p-valuation** on G if*

1. $\omega(g) > \frac{1}{p-1}$,
2. $\omega\left(g^{-1}h\right) \geq \min(\omega(g), \omega(h))$,
3. $\omega([g,h]) = \omega\left(ghg^{-1}h^{-1}\right) \geq \omega(g) + \omega(h)$,
4. $\omega(g^p) = \omega(g) + 1$

for all $g, h \in G$. We suppose by convention that $\omega(1) = \infty$.
Remark 2.2.2. *One can see that*

$$\omega(g) = \omega\left(g^{-1}\right)$$

by setting $h = 1$ in 2. We also have that

$$\omega\left(ghg^{-1}\right) = \omega([g,h]\,h) \geq \min(\omega([g,h]), \omega(h)) \geq \min(\omega(g) + \omega(h), \omega(h)) = \omega(h).$$

Thus, we see that $\omega\left(ghg^{-1}\right) = \omega(h)$. One can also show that if $\omega(g) > \omega(h)$, then $\omega(gh) = \omega(h)$. Thus, we see that

$$\omega(gh) = \min(\omega(g), \omega(h)) \text{ for } \omega(g) \neq \omega(h).$$

Now, we define

$$G_\nu := \{g \in G : \omega(g) \geq \nu\}$$

and

$$G_{\nu+} := \{g \in G : \omega(g) > \nu\}$$

for any $\nu \in \mathbb{R}^{\geq 0}$. One can readily check that these are normal subgroups of G. One can show that $G_\nu / G_{\nu+}$ is a central subgroup of $G/G_{\nu+}$.
Proposition 2.2.3. *For all $g, h, k \in G$ we have that*

1. $[gh, k] = g\,[h,k]\,g^{-1}\,[g,k]$,
2. $[g, hk] = [g,h]\,h\,[g,k]\,h^{-1}$,
3. $\left[[g,h], hkh^{-1}\right]\left[[h,k], kgk^{-1}\right]\left[[k,g], ghg^{-1}\right] = 1$.

We will omit the routine check here.
Now, we define

$$\operatorname{gr}_\nu G := G_\nu / G_{\nu+},$$

which is commutative by 3 above. We can also define

$$\operatorname{gr} G := \bigoplus_{\nu > 0} \operatorname{gr}_\nu G.$$

We say that $\xi \in \operatorname{gr} G$ is homogeneous of degree ν if it likes in $\operatorname{gr}_\nu G$. We say that g is a **representative** of ξ if g is such that $\xi = gG_{\nu+}$. We see that $p\xi = 0$ for any homogeneous $\xi \in \operatorname{gr} G$, so $\operatorname{gr} G$ is an F_p-vector space.

Lemma 2.2.4. *The map*

$$\begin{aligned} \operatorname{gr}_\nu G \times \operatorname{gr}_{\nu'} G &\longrightarrow \operatorname{gr}_{\nu+\nu'} G \\ (\xi, \eta) &\longmapsto [\xi, \eta] := [g, h]\, G_{(\nu+\nu')+} \end{aligned}$$

for $\nu, \nu' > 0, g, h$ representatives of ξ, η, is a well-defined bi-additive map. One has that $[\xi, \xi] = 0, [\xi, \eta] = -[\eta, \xi]$.
Thus, we have a graded $\mathbb{F}_p$-bilinear map

$$[\ \ ,\ \ ] : \operatorname{gr} G \times \operatorname{gr} G \to \operatorname{gr} G$$

satisfying the Jacobi Identity and

$$[\xi, \xi] = 0$$

for all $\xi \in \operatorname{gr} G$

Remark 2.2.5. *One can show that the map*

$$\operatorname{gr}_\nu G \to \operatorname{gr}_{\nu+1} G$$

taking

$$gG_{\nu+} \mapsto g^p G_{(\nu+1)+}$$

is well-defined and $\mathbb{F}_p$-linear. Thus, we have a map

$$P : \operatorname{gr} G \to \operatorname{gr} G$$

and we may consider $\operatorname{gr} G$ as a graded module over $\mathbb{F}_p[P]$ in variable P.

Definition 2.2.6 (Rank)**.** *We say that (G, ω) is of* ***finite rank*** *if $\operatorname{gr} G$ is finitely generated as an $\mathbb{F}_p[P]$-module.*

Lemma 2.2.7. *[1] (G, ω) is saturated $\iff$ $P(\operatorname{gr} G) = \oplus_{\nu > \frac{p}{p-1}} \operatorname{gr}_\nu G$.*

Lemma 2.2.8. *[1] Let (G, ω) be of finite rank. Then, we see that the subgroups G^{p^n} forms basis for the open neighborhoods of $e \in G$ and*

$$\operatorname{rank}(G, \omega) = \lim_{n \to \infty} \frac{v\left(\left[G : G^{p^n}\right]\right)}{n}.$$

Theorem 2. *[1] Let G be a p-adic Lie group, then there exists $G' \subseteq G$ compact, open subgroup and ω integral p-valuation on G' defining the topology of G' s.t.*

1. *(G', ω) is saturated;*
2. *$\operatorname{rank} G' = \dim G$.*

Proof. Let d be the dimension of G, let $c = \left(U, \varphi, \mathbb{Q}_p^d\right)$ be a chart around $e \in G$. Normalize φ so that $\varphi(e) = 0$. Since m_G is continuous, we can find an open neighborhood $V \subset U$ of e small enough so that the image of $V \times V$ is contained in U. Then we see that restricting φ to V gives us another chart $\left(V, \varphi|_V, \mathbb{Q}_p^d\right)$ around e. We see that

$$\varphi \circ m_G \circ \left(\varphi^{-1} \times \varphi^{-1}\right) : \varphi(V) \times \varphi(V) \to \varphi(U)$$

is locally analytic and given by $(F_1, \dots, F_d)$, where

$$F_i(X, Y) = \sum_{\alpha,\beta} c_{i,\alpha,\beta} X^\alpha Y^\beta$$

with coefficients in $\mathbb{Q}_p$. Let $n > 0$ be large enough so that

1. $p^n \mathbb{Z}_p^d \subset \varphi(V)$,
2. $\lim_{|\alpha|+|\beta|\to\infty} (v(c_{i,\alpha,\beta} + n(|\alpha| + |\beta|))) = \infty$ for all $1 \le i \le d$,
3. $(F_1(x, y), \dots, F_d(x, y)) = \varphi\left(\varphi^{-1}(x)\varphi^{-1}(y)\right)$ for all $x, y \in p^n \mathbb{Z}_p^d$.

For n large, we also have that

$$v(c_{i,\alpha,\beta}) + n(|\alpha| + |\beta|) \ge n$$

for all α, β. Thus, for n large, we have that $\varphi^{-1}\left(p^n \mathbb{Z}_p^d\right)$ are compact, open, and multiplicatively closed subsets in G. Applying the argument above to the inverse map, we can see that $\varphi^{-1} p^n \mathbb{Z}_p^d$ are subgroups of G for n large. We normalize φ to $\psi := p^{-n}\varphi$ for n large so that the coefficients $c_{i,\alpha,\beta} \in \mathbb{Z}_p$. Thus, $g \mapsto g^p$ and $(g, h) \mapsto [g, h]$ are also locally analytic converging on $\mathbb{Z}_p^d$ with coefficients in $\mathbb{Z}_p$. NOw, set

$$G' := \psi^{-1}\left(p^2 \mathbb{Z}_p^d\right)$$

$$\omega(g) := l + \delta$$

for $g \in \psi^{-1}\left(p^{l+1}\mathbb{Z}_p^d\right) \setminus \psi^{-1}\left(p^{l+2}\mathbb{Z}_p^d\right)$ and

$$\delta = \begin{cases} 1 & p = 2 \\ 0 & p \ne 2. \end{cases}$$

One can check that ω is a p-valuation. Now, we see that for all $l \in \mathbb{N}$ we have that $P : \operatorname{gr}_{l+\delta} G' \to \operatorname{gr}_{l+\delta+1} G'$ is a bijection, which implies that $\operatorname{gr} G'$ is finitely generated by $\operatorname{gr}_{1+\delta} G'$. Thus, we can see by previous lemma that (G', ω) is saturated. Proceeding with the calculation, we see that

$$\begin{aligned} \operatorname{rank} G &= \lim_{n\to\infty} \frac{v\left(\left[G' : G'^{p^n}\right]\right)}{n} \\ &= \lim_{n\to\infty} \frac{v\left(G' : G'_{n+1+\delta}\right)}{n} \\ &= \lim_{n\to\infty} \frac{dn}{n} \\ &= d. \end{aligned}$$

□

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