# ON CARTAN'S THEOREMS A AND B

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Abstract. We present here a report on Cartan's Theorems A and B, central to both the study of several complex variables and sheaf cohomology. We will then use this theorem study the properties of Stein spaces, one of the most studied object in the theory of several complex variables. We will finish with a discussion of its consequences and the relationship between complex analytic geometry and algebraic geometry.

The first two sections introduce the necessary background knowledge needed, both in terms of several complex variables and sheaf theory. We will then present two proofs of Cartan's Theorems A and B, one using more algebraic techniques and the other using methods from PDE. The reference for the first proof will be the papers of Siu ( [6] and [7]), and the reference for the second proof is the book of Hormander [2]. Due to constraint on the length of the paper, most of the proofs will be omitted, and references will be given.

## 1. An Introduction to Several Complex Variables

We start by stating some basic results in the theory of complex variables, most of the proofs can be found in [1], so many of them will be omitted.

1.1. **Preliminaries.** To discuss the theory of several complex variables, we first need some basic definitions. Most of them are very similar to their counterpart in single variable complex analysis.

**Definition 1.1.1.** Let $\mathbb{C}^n$ be the cartesion product of $n$ copies of $\mathbb{C}$, and $z = (z_1, ....., z_n) \in \mathbb{C}^n$, then the absolute value of $z$ is defined as $|z| = \max\{|z_i| \mid 1 \leq i \leq n\}$.

**Definition 1.1.2.** Let $r_i < R_i$, $R_i > 0$, and $z = (z_1, ....., z_n \in \mathbb{C}^n$. Then an open polyannulus centered at $z$, or just polyannulus, is the domain $D(z, r) = \{w = (w_1, ....., w_n) \in \mathbb{C}^n \mid r_i < |w_i - z_i| < R_i, 1 \leq i \leq n\}$.

We will mostly work with polyannulus centered 0. If $r_i = 0$, then we call the domain a polydisk centered at $z$ and $R = (R_1, ....., R_n) \in \mathbb{R}^n$ will be called the polyradius. Polydisks centered at $z$ with polyradius $R$ with be denoted $D(z, R)$. Similarly, an open polydomain is defined as the cartesian product of $n$ domains in $\mathbb{C}$.

**Definition 1.1.3.** A complex-valued function $f$ on $D \subset \mathbb{C}^n$ is a map from $D$ to $\mathbb{C}$, and we will denote the value of $f$ at $z \in D$ by $f(z)$. It is called holomorphic on $D$ if for each $w \in D$, there exists a neighborhood $U$ containing $w$ such that $f$ has power series representation

$$f(z) = \sum_{i_1, ....., i_n = 0}^{\infty} a_{i_1, ....., i_n} (z_1 - w_1)^{i_1} \cdots (z_n - w_n)^{i_n}$$

that converges for all $z \in U$. The set of all holomorphic functions on $D$ is denoted $\mathfrak{O}_D$, and we will add more structures to this set later.

*Remark* 1.1.4. Using elementary analysis, it is clear that the power series converges absolutely and uniformly on sufficiently small polydisks. Moreover, observe that a holomorphic function on $D$ is holomorphic in each variable separately. In fact, the converse is also true.

**Theorem 1.1.5** (Osgood). *Let $f$ be a continuous complex-valued function on an open set $D \subset \mathbb{C}^n$ and holomorphic in each variable, then $f$ is holomorphic on $D$.*

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*Proof.* Repeated application of the single-variable Cauchy integral formula and manipulation of power series. See Theorem I.A.2 of [1] for details. ■

*Remark* 1.1.6. In fact, the hypothesis that $f$ is continuous is not necessary, and this result is known as Hartog's theorem. We will not need this theorem in this report, and the interested reader may refer to Chapter 8 of [8].

We now state the analog of several fundamental results on the theory of single variable complex analysis, none of which we will prove. We will only use some of these, and we stated the other ones for completeness. All of the proofs can be found in Section I.A of [1].

**Theorem 1.1.7** (Multivariable Cauchy integral formula)**.** *Let $f$ be a complex-valued holomorphic function on a open neighborhood of $\overline{D(w,r)}$, then $f$ has the integral representation*

$$f(z) = (\frac{1}{2\pi i})^n \int_{|w_i - \zeta_i| = r_i} \frac{f(\zeta)d\zeta_1 \cdots d\zeta_n}{(\zeta_1 - z_1)\cdots(\zeta_n - z_n)}$$

*Proof.* Repeated application of the single variable Cauchy integral formula. ■

**Definition 1.1.8.** Let $z_i = x_i + iy_i$, then we define the operator

$$\frac{\partial}{\partial \bar{z}_i} = \frac{1}{2}(\frac{\partial}{\partial x_i} + i\frac{\partial}{\partial y_i}).$$

The operator $\frac{\partial}{\partial z_i}$ is defined similarly be replacing the plus with the minus.

**Theorem 1.1.9** (Cauchy-Riemann)**.** *Let $f$ be a complex-valued function on $D \subset \mathbb{C}^n$ that is continuously differentiable in the real sense, then $f$ is holomorphic on $D$ if and only if for all $1 \leq i \leq n$,*

$$\frac{\partial}{\partial \bar{z}_i} f(z) = 0.$$

**Corollary 1.1.10.** *$\mathfrak{O}_D$ is a ring with respect to addition and multiplication of functions. Moreover, any function that is nowhere vanishing has an inverse.*

We shall now define a holomorphic map between two domains in $\mathbb{C}^n$ and $\mathbb{C}^m$.

**Definition 1.1.11.** Let $D \subset \mathbb{C}^n$ and $D' \in \mathbb{C}^m$, then a function $F : D \to D'$ is holomorphic is each component is holomorphic. They are called the holomorphic mappings.

**Proposition 1.1.12.** *The composition of a complex-valued holomorphic function with a holomorphic mapping is holomorphic.*

**Theorem 1.1.13.** *Holomorphic functions defined on a common connected open set $D$ that agrees on a non-empty open subset of $D$ agrees on $D$.*

*Remark* 1.1.14. In addition to these, one can also the state the analog of Maximum Modulus Principle, Schwarz's Lemma, Jensen's Inequality, Implicit and Inverse Function Theorem, and Riemann Removable Singularity Theorem in several complex variables rather easily. The interested reader may refer to Section A,B,C of Chapter 1 of [1] for proofs. Let's now define some basic notions of functional analysis.

**Definition 1.1.15.** Let $E$ be a complex vector space, then a pseudonorm $p$ on $E$ is a map from $E$ to the nonnegative real numbers such that

1 . $p(a+b) \leq p(a) + p(b)$ for all $a, b \in E$.
2 . $p(\lambda a) = |\lambda| p(a)$ for all $a \in E$ and $\lambda \in \mathbb{C}$.

The pseudonorm, which is sometimes called the seminorm, $p$ defines a natural topology on $E$ by defining a basis to be the open neighborhood of radius $\epsilon > 0$. Therefore, $E$ with this topology is called a pseudonormed space, and $p$ is a norm if it satisfies $p(a) = 0$ if and only if $a = 0$.

**Definition 1.1.16.** A Frechet space is a vector space $F$ with a sequence of pseudonorms $\{p_n\}$ such that

1 . $p_n(a) = 0$ for all $n$ if and only if $a = 0$.
2 . If $a_i$ is Cauchy with respect to every pseudo-norm, then there exists an $a$ such that $a_i \to a$ in every pseudo-norm.

*Remark* 1.1.17. It should be noted that the pseudo-norms induce a topology on the Frechet space $F$ with a neighborhood basis of zero given by the sets $N(n, \epsilon) = \{a \in F \mid p_n(a) < \epsilon\}$. This induces the metric

$$d(x, y) = \sum_{n=1}^{\infty} 2^{-n} \frac{p_n(x-y)}{1 + p_n(x) + p_n(y)}.$$

Let $K$ be a compact subset of an open set $D$. Consider the ring of continuous function $C_D$. Let $f \in_D$, define $||f||_K = \sup_{z \in K} |f(z)|$. This gives us a pseudo-norm on $C_D$. If we define a sequence of compact subsets $K_i$ such that $K_i \subset K_{i+1}$ and $\bigcup_{i=1}^{\infty} K_i = D$, then $C_D$ becomes a Frechet space with topology defined by the metric

$$d(f, g) = \sum_{i=1}^{\infty} 2^{-i} \frac{||f-g||_{K_i}}{1 + ||f-g||_{K_i}}.$$

The subring $\mathfrak{O}_D$ inherits this topology from $C_D$ so that it is a topological ring with the topology of uniform convergence with respect to the metric defined above on compact subsets. Moreover, we have the following important result.

**Proposition 1.1.18.** *$\mathfrak{O}_D$ is a closed subring of $C_D$ and is therefore a Frechet space.*

1.2. **The calculus of differential forms.** We now discuss the calculus of exterior differential forms on open domain in $\mathbb{C}^n$, which is a useful tool in the study of several complex variables. Instead of using the intrinsic definitions, we will state everything explicitly.

Let $D \subset \mathbb{C}^n$ an open dimain, and $C_D^{\infty}$ the ring of complex-valued function that is infinitely many times real differentiable with respect to the $2n$ real variables. Let $\mathfrak{E}^1(D)$ be the free module over $C_D^{\infty}$ of rank $2n$ with basis $dz^1, ....., dz^n, d\bar{z}^1, ....., d\bar{z}^n$ so that every element $\phi \in \mathfrak{E}^1(D)$ can be written as

$$\phi = \sum_{i=1}^{n} \phi_i dz^i + \sum_{i=1}^{n} \bar{\phi}_i d\bar{z}^i$$

for $\phi_i, \bar{\phi}_i \in C_D^{\infty}$. Now we introduce formal expressions in the generators $dz^i, d\bar{z}^j$ in the form $dz^1 \wedge \cdots \wedge dz^p \wedge d\bar{z}^1 \wedge \cdots \wedge d\bar{z}^q$, which are fully skew-symmetric. This allows us to define the operation of wedge product on the generators $dz^i, d\bar{z}^j$, which is associative and skew-symmetric.

Define the module $\mathfrak{E}^{p,q}(D)$ to be the free module over the ring $C_D^{\infty}$ generated by $\binom{n}{p}\binom{n}{q}$ elements $dz^{i_1} \wedge \cdots \wedge dz^{i_p} \wedge d\bar{z}^{j_1} \wedge \cdots \wedge d\bar{z}^{j_q}$. The elements in this module are called exterior differential forms of bidegree $(p, q)$. Therefore, any element in $\mathfrak{E}^{p,q}(D)$ can be written as

$$\phi = \sum_{i_1, ....., i_p = 1}^{n} \sum_{j_1, ....., j_q}^{n} \phi_{i_1, ....., i_p, j_1, ....., j_q} dz^{i_1} \wedge \cdots \wedge dz^{i_p} \wedge d\bar{z}^{j_1} \wedge \cdots \wedge d\bar{z}^{j_q},$$

where $\phi_{i_1, ....., i_p, j_1, ....., j_q} \in C_D^{\infty}$ are skew-symmetric in all indices. Moreover $\mathfrak{E}^{0,0}(D) = C_D^{\infty}$.

**Definition 1.2.1.** The algebra of exterior differential forms on the domain $D$ is the direct sum $\mathfrak{E}^*(D) = \sum_{p,q=0}^{n} \mathfrak{E}^{p,q}(D)$, which inherits the wedge product from the wedge product on the generators. The $r$th partial sums, denoted $\mathfrak{E}^r(D)$ are called the modules of differential forms of total degree $r$.

From these definitions, we can conclude that for $\phi \in \mathfrak{E}^r(D)$, $\psi \in \mathfrak{E}^s(D)$, we have $\phi \wedge \psi = (-1)^{rs} \psi \wedge \phi$. We now explicitly define the complex exterior derivations, which are maps $\partial : \mathfrak{E}^{p,q}(D) \to \mathfrak{E}^{p+1,q}(D)$, $\bar{\partial} : \mathfrak{E}^{p,q}(D) \to \mathfrak{E}^{p,q+1}(D)$, and $d : \mathfrak{E}^r(D) \to \mathfrak{E}^{r+1}(D)$, which can be viewed as the analog of the chain maps in the De Rham chain complex.

**Definition 1.2.2.** Let $\phi \in \mathfrak{E}^{p,q}(D)$ be a differential form, then

$$\partial\phi = \sum_{i,j,k} (\partial\phi_{i_1,\ldots,i_p,j_1,\ldots,j_q}/\partial z_k) dz^k \wedge dz^{i_1} \wedge \cdots \wedge dz^{i_p} \wedge d\bar{z}^{j_1} \wedge \cdots \wedge d\bar{z}^{j_q},$$

$$\bar{\partial}\phi = \sum_{i,j,k} (\partial\phi_{i_1,\ldots,i_p,j_1,\ldots,j_q}/\partial \bar{z}_k) d\bar{z}^k \wedge dz^{i_1} \wedge \cdots \wedge dz^{i_p} \wedge d\bar{z}^{j_1} \wedge \cdots \wedge d\bar{z}^{j_q},$$

and $d\phi = \partial\phi + \bar{\partial}\phi$.

**Proposition 1.2.3.** *The three maps are defined above are linear over the constant functions. Moreover for $\phi$ of bidegree $(p, q)$ and $\psi$ of bidegree $(r, s)$, we have*

$$\partial(\phi \wedge \psi) = \partial\phi \wedge \psi + (-1)^{p+q}\phi \wedge \partial\psi \quad \textit{and} \quad \bar{\partial}(\phi \wedge \psi) = \bar{\partial}\phi \wedge \psi + (-1)^{p+q}\phi \wedge \bar{\partial}\psi.$$

*Remark* 1.2.4. Just like in the case of De Rham cohomology, we can easily check that $d^2 = \partial^2 = \bar{\partial}^2 = 0$ and $\partial\bar{\partial} + \bar{\partial}\partial = 0$. Moreover, a $(p, q)$-form $\phi$ is $\bar{\partial}$-closed if $\bar{\partial}\phi = 0$ and is called exact if there exists $(p, q-1)$-form $\psi$ such that $\phi = \bar{\partial}\psi$. It is then clear that every exact form is closed.

Our goal is to eventually state a theorem that tells us when are $\bar{\partial}$-closed forms $\bar{\partial}$-exact. To do so, we first need some lemmas, whose proofs can be found in Section I.D of [1].

**Lemma 1.2.5** (Generalized Cauchy integral formula)**.** *Let $D$ be a region in the complex plane bounded by a rectifiable simple closed curve $\gamma$ and $f$ a function defined in an open neighborhood of $\bar{D}$ that is infinitely many times real differentiable in the real coordinates, then for all $z \in D$, we have*

$$2\pi i f(z) = \int_\gamma f(\zeta)\frac{d\zeta}{\zeta - z} + \iint_D \frac{\partial f}{\partial \bar{\zeta}} \frac{d\zeta \wedge d\bar{\zeta}}{\zeta - z},$$

$$2\pi i f(z) = -\int_\gamma f(\zeta)\frac{d\bar{\zeta}}{\bar{\zeta} - \bar{z}} + \iint_D \frac{\partial f}{\partial \zeta} \frac{d\zeta \wedge d\bar{\zeta}}{\bar{\zeta} - \bar{z}}.$$

*Proof.* Use Stoke's theorem and the rest follows from standard complex analysis. ■

This implies the following lemma asserting the existence of antiderivatives with respect to the operators $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \bar{z}}$.

**Lemma 1.2.6.** *Let $D, \gamma, f$ be the same as above, then there exists $C^\infty$ functions $g, h$ on $D$ such that*

$$\frac{\partial g}{\partial z} f(z) \quad \textit{and} \quad \frac{\partial h}{\partial \bar{z}} = f(z)$$

*Proof.* Use the previous lemma, Stoke's theorem, and standard complex analysis. ■

**Theorem 1.2.7** (Dolbeault)**.** *Let $\bar{D} \subset \mathbb{C}^n$ to be a compact polydisc, and $\omega$ be a $C^\infty$ differential form of bidegree $(p, q)$ in a neighborhood of $\bar{D}$. If $q > 0$ and $\omega$ is $\bar{\partial}$-closed, then there exists a $C^\infty$ form $\eta$ on $D$ of bidegree $(p, q-1)$ such that $\bar{\partial}\eta = \omega$.*

It has the following generalization, which can be obtained by minor modifications of the proof of the previous theorem.

**Corollary 1.2.8.** *Suppose $P$ is a polyannulus in $\mathbb{C}^n$. Let $\omega$ be a $C^\infty$ form on a neighborhood of $\bar{P}$ of bidegree $(0, q)$ with $q > 0$, then there is a $C^\infty$ form $\eta$ of bidegree $(0, q-1)$ on $P$ such that $\bar{\partial}\eta = \omega$.*

*Remark* 1.2.9. By construction, all forms of bidegree $(p, n)$ are closed, and the trivial form is the only form of bidegree $(p, 0)$ for all $p$. Now consider the linear map $\bar{\partial} : \mathfrak{E}^{p,q}(D) \to \mathfrak{E}^{p,q+1}(D)$, which we will denote it by $\bar{\partial}_{p,q}$. The kernel of this map are the closed forms of bidegree $(p, q)$ and the image being the exact forms of bidegree $(p, q+1)$. Thus, the modules $\mathfrak{E}^{p,q}(D)$ with the map $\bar{\partial}_{p,q}$ forms a chain complex, so we can define a cohomology on it.

**Definition 1.2.10.** The Dolbeault cohomology groups of the domain $D$ are the complex vector spaces

$$\mathrm{H}^{p,q}(D) = \ker(\bar{\partial}_{p,q}) / \operatorname{im}(\bar{\partial}_{p,q-1})$$

We will later associate this cohomology with the more well-known sheaf cohomology. Our goal now is to state a theorem regarding approximating holomorphic functions. To start, we introduce a new type of domains.

**Definition 1.2.11.** Suppose $p_j$, $1 \leq j \leq n+r$ are polynomials on $\mathbb{C}^n$ such that $p_i = z_i$ for $1 \leq i \leq n$. Suppose $a_j < b_j$ and $b_j > 0$ for $1 \leq j \leq n+r$. The domain $P = \{z \in \mathbb{C}^n \mid a_j < |p_j(z)| < b_j, 1 \leq j \leq n+r\}$ is called a polynomial polyannulus. Suppose $(k_1, ....., k_{n+r})$ is a permutation of $(1, ....., n+r)$ such that $a_{k_i} \geq 0$ for $1 \leq j \leq m$ and less than 0 for the other $k_i$, then the polynomials $p_{k_i}$, $1 \leq k_i \leq m$ are called the essential defining polynomials for $P$.

We now generalize Corollary 1.2.8 to the case of a polynomial polyannulus.

**Lemma 1.2.12.** *Let $P \subset \mathbb{C}^n$ be a polynomial polyannulus, $\omega$ be a $C^\infty$ closed form of bidegree $(0,q)$ for $q > 0$ on a neighborhood of $\bar{P}$, then there exists a $C^\infty$ form $\eta$ of bidegree $(0,q-1)$ on $P$ such that $\bar{\partial}\eta = \omega$.*

*Proof.* Follows from Corollary 1.2.8 and minor modifications of the proof of Theorem I.F.5 of [1]. ∎

This allows to state the following important result.

**Theorem 1.2.13.** *Let $P$ be a polynomial polyannulus such that $p_k$, $1 \leq k \leq m$ are the essential defining polynomials of $P$. Let $D = \{z \in \mathbb{C}^n \mid p_k(z) \neq 0, 1 \leq k \leq m\}$, then any holomorphic functions on $P$ can be approximated uniformly on compact subsets of $P$ by holomorphic functions on $D$.*

*Proof.* Use the previous lemma and the proof follows by the same argument used in the proof of Theorem I.F.8 of [1] with minor modifications. ∎

**Definition 1.2.14.** A domain $D \subset \mathbb{C}^n$ is polynomially convex if for every compact subset $K$, the set $\hat{K} = \{z \in \mathbb{C}^n \mid |f(z)| \leq ||f||_K \text{ for all polynomials } f\}$ is contained in $D$, and $\hat{K}$ is called the polynomially convex hull.

We will not need the following theorem, but it is a rather interesting and important result in the theory of several complex variables.

**Theorem 1.2.15.** *Let $D$ be a polynomially convex domain, then any holomorphic function on $D$ can be approximated uniformly on compact subsets of $D$ by polynomials.*

*Proof.* See Corollary I.F.9 of [1]. ∎

## 2. An Introduction to Sheaf Theory, Sheaf Cohomology, and Stein Spaces

2.1. **Basic definitions.** In this part, we present three algebraic notions that will be useful in our discussion of sheaf theory and sheaf cohomology. It is common in the study of several complex variables for us to study the behavior of functions in arbitrarily small neighborhood around a fixed point $w$. This naturally leads to the notion of a germ of functions. For our discussion, $w$ is a fixed point and $U, V, W$ are open neighborhoods of $w$. If $f$ is a complex-valued function defined on $U$, we shall temporarily write $f_U$ to emphasize the domain of definition. If $W \subset U$, then we write $f_U|W$ to denote its restriction to $W$. We will assume basic algebra in the next few sections.

**Definition 2.1.1.** Two functions $f_U, g_V$ are equivalent at the point $w$ if there exists a open neighborhood $W \subset U \cap V$ such that $f_U|W = g_V|W$. An equivalence class of such functions will be called a germ of a function at $w$.

**Notation.** We will denote germs of functions by boldface letters $\mathbf{f}$.

It should be noted that any function defined at $w$ belong to some germs, and such germ is called the germ of the function $f$ at $w$. We shall define a ring structure on the set of germs of complex-valued function at $w$. Let $\mathbf{f}$, $\mathbf{g}$ be germs of functions represented by $f_U, g_V$, respectively. Then the sum $f_U + g_V$ and product $f_U g_V$ are complex-valued function on $U \cap V$, and the germs $\mathbf{f} + \mathbf{g}$ and $\mathbf{fg}$ are defined to be the germs of the respective functions. One should check that this construction is independent of the representative functions so that it is well-defined. We can now restrict ourselves to the ring of germs of holomorphic functions at $w$, and we have the following temporary notations.

**Notation.** The ring of germs of holomorphic functions at $w$ is denoted $\mathfrak{O}_w$. If $w = 0$, we will simply write $\mathfrak{O}$. If we want to emphasize the dimension of the underlying complex space $\mathbb{C}^n$, then we write $\mathfrak{O}_{n,w}$. The structure of $\mathfrak{O}_{n,w}$ can be understood in the following ways.

**Theorem 2.1.2.** *The ring $\mathfrak{O}_{n,w}$ is isomorphic to the ring of convergent power series at $w$.*

*Proof.* Follows quickly from the power series definition of a holomorphic function. See Theorem II.B.1 of [1] for details. ■

**Theorem 2.1.3.** *The ring $\mathfrak{O}_n$ is a local, noetherian, UFD(Unique Factorization Domain).*

*Proof.* The fact that it is a domain follows from the identity theorem. The fact it is not local follows from Corollary 1.1.10, which implies that the holomorphic functions that vanished at the origin are exactly the nonunits of $\mathfrak{O}_n$ so forms a unique maximal ideal. The fact that it is noetherian and a UFD follows by induction. See Theorem II.B.7,9 for details. ■

We now briefly discuss the modules over $\mathfrak{O}_n$ in order to introduce free resolutions. We will denote the dimension $k$ free $\mathfrak{O}_n$-module by $\mathfrak{O}_n^k$.

**Definition 2.1.4.** Let $M, N$ by $\mathfrak{O}_n$-modules, $\phi : M \to N$ a module homomorphism. A syzygy for $\phi$ is a module homomorphism $\psi : \mathfrak{O}_n^k \to M$ such that the sequence $\mathfrak{O}_n^k \xrightarrow{\psi} M \xrightarrow{\phi} N$ is exact. A chain of syzygies for an $\mathfrak{O}_n$-module $M$ is an exact sequence of $\mathfrak{O}_n$-modules in the form

$$\cdots \to \mathfrak{O}_n^{k_1} \to \mathfrak{O}_n^{k_0} \xrightarrow{\phi_0} M \to 0.$$

A chain of syzygies is called a free resolution. It is said to break off at the $i$th step if the kernel of $\phi_{i-1}$ is a free module. This allows us to breakdown the free resolution into two chains with at least one of them a finite chain of syzygies or a finite free resolution.

The most important theorem on the theory of free resolutions is the following theorem of Hilbert.

**Theorem 2.1.5** (Hilbert Syzygy Theorem)**.** *Any chain of syzygies of an $\mathfrak{O}_n$-module $M$ breaks off at the $n$th step.*

*Proof.* See Theorem II.C.2 of [1]. ■

We now introduce the notion of a variety, which arise naturally from our study of the vanishing set of holomorphic functions.

**Definition 2.1.6.** Let $U$ be a domain in $\mathbb{C}^n$. A subset $V$ is called a subvariety if for every $z \in U$, there are a neighborhood $U_z$ and holomorphic functions $f_1, ....., f_k$ on $U_z$ such that $V \cap U_z = \{x \in U_z \mid f_1(z) = \cdots = f_k(z) = 0\} = V(f_1, ....., f_k)$.

*Example.* The vanishing set of a collection of holomorphic function on an open set $U$ is a subvariety.

The notion of a variety has a local version.

**Definition 2.1.7.** Let $X, Y$ be subsets of $\mathbb{C}^n$, then they are said to be equivalent at 0 if there is a neighborhood $U$ such that $X \cap U = Y \cap U$. An equivalence class of sets is called the germ of a set, and is denoted $\mathbf{X}$.

Let $\mathbf{f}$ be the germ of a function at 0, then $\mathbf{V}(\mathbf{f})$ is the equivalence class of sets $\{x \in U \mid f(x) = 0\}$, where $f$ is a representative of the germ of function $\mathbf{f}$.

**Definition 2.1.8.** If $\mathbf{f} \in \mathfrak{O}_n$ and $\mathbf{X}$ be the germ of a set, then we say $\mathbf{f}$ vanished on $\mathbf{X}$ if $\mathbf{X} \subset \mathbf{V}(\mathbf{f})$. A germ $\mathbf{X}$ is the germ of a variety if there are elements $\mathbf{f_1}, ....., \mathbf{f_k} \in \mathfrak{O}_n$ such that $\mathbf{V}(\mathbf{f_1}, ....., \mathbf{f_k}) = \mathbf{X}$.

We will refer to the collection of germs of sets at 0 by $\mathfrak{B}_n$.

**Definition 2.1.9.** Let $\mathbf{V} \in \mathfrak{B}_n$, then the ideal of $\mathbf{V}$, denotes $I(\mathbf{V})$ is the set of all $\in \mathfrak{O}_{\mathbf{n}}$ that vanishes on $\mathbf{V}$. Let $\mathfrak{A}$ be a subset of $\mathfrak{O}_n$, then the vanishing set of $\mathfrak{A}$, denotes $V(\mathfrak{A})$ is the intersection of $\mathbf{V}(\mathbf{f})$ for all $\in \mathfrak{O}_{\mathbf{n}}$. Readers familiar with commutative algebra should notice that these are similar to those definitions in the case of polynomial rings.

**Definition 2.1.10.** A germ $\mathbf{V} \in \mathfrak{B}_n$ is irreducible if $\mathbf{V} = \mathbf{V}_1 \cup \mathbf{V}_2$ implies $\mathbf{V} = \mathbf{V}_1$ or $\mathbf{V} = \mathbf{V}_2$, where $\mathbf{V}_i \in \mathfrak{B}_n$.

We have the following theorem relating the irreducibility of $\mathbf{V}$ and the ideal generated by $V$.

**Theorem 2.1.11.** $\mathbf{V}$ *is irreducible if and only if* $I(\mathbf{V})$ *is a prime ideal.*

*Proof.* Follows immediately from the definition. ■

**Theorem 2.1.12.** *Let* $\mathbf{V} \in \mathfrak{B}_n$, *then we can decompose* $\mathbf{V} = \bigcup_{i=1}^{n} \mathbf{V}_i$ *such that* $\mathbf{V}_i$ *are irreducible and no one is contained in another. The* $\mathbf{V}_i$*'s are uniquely determined by* $\mathbf{V}$ *and are called the irreducible branches of* $\mathbf{V}$.

*Proof.* See Theorem II.E.15 of [1]. ■

*Remark* 2.1.13. There is an analog of this for noetherian topological spaces, but we will not state it here. The interested readers may refer to Chapter 1 of [3] for details.

Now we want to introduce some concepts that allow us to define the dimension of a variety.

**Definition 2.1.14.** Let $\mathfrak{P}$ be a prime ideal of $\mathfrak{O}_n$, a coordinate choice $z_1, ....., z_n$ is said to be a regular system of coordinates for the ideal $\mathfrak{P}$ if the following are satisfied:

1 . $\mathfrak{P} \cap \mathfrak{O}_k = (\mathbf{0})$ for some $k$.
2 . $\mathfrak{O}_n/\mathfrak{P}$ is an integral extension over $\mathfrak{O}_k$.
3 . Let $\pi : \mathfrak{O}_n \to \mathfrak{O}_n/\mathfrak{P}$ a projection map and $\mathfrak{F}_k$ be the quotient field of $\mathfrak{O}_k$, then $\mathfrak{F}_n$ is generated over $\mathfrak{F}_k$ by the single element $\pi(z_{k+1})$.

**Definition 2.1.15.** Let $V$ be a subvariety of the domain $U$, then a point $x$ is called a regular point of $V$ if there exists a neighborhood $U_x$ of $x$ such that $V \cap U_x$ is a complex submanifold of $U_x$. A point that is not regular is called a singular point. The set of regular points is denoted by $\mathfrak{R}(V)$.

**Theorem 2.1.16.** *If* $V$ *is an irreducible variety at* 0*, then in a neighborhood of* 0*,* $V$ *contains a connected* $k$*-dimensional submanifold of* $\mathbb{C}^n$*, which is dense and open in* $V$*.*

*Proof.* See Theorem III.A.10 of [1]. ■

**Definition 2.1.17.** The number $k$ in the previous theorem is called the dimension of $V$. It can be checked that it is independent of the choice of coordinates.

**Definition 2.1.18.** Let $\mathbf{V}$ be the germ of a variety at 0 in $\mathbb{C}^n$. Let $V = \bigcup_{i=1}^{k} \mathbf{V}_i$ be its decomposition into irreducible branches, then the dimension of $\mathbf{V}$ is defined as $\dim \mathbf{V} = \max \dim \mathbf{V}_i$. We call $\mathbf{V}$ pure dimensional if all of the branches have the same dimension.

**Theorem 2.1.19.** *Let* $V$ *be a subvariety of a domain* $U$*, then* $\mathfrak{R}(V)$ *is dense in* $U$*. The closures of the components of* $\mathfrak{R}(V)$ *are defined to be the (global) irreducible branches of* $V$*, and* $V$ *is irreducible if and only if* $\mathfrak{R}(V)$ *is connected.*

*Proof.* See the end of Section III.C for details. ■

This gives us all of the necessary background to discuss sheaf theory.

2.2. **Sheaf theory.** We will now define a sheaf, the treatment here follows Hartshorne Section 2.1 [3]. For a more comprehensive treatment of sheaf theory, see the classic paper of Serre [9].

**Definition 2.2.1.** Let $X$ be a topological space, a presheaf $\mathfrak{F}$ of abelian groups on $X$ consists of the data

1 . for every open subset $U$, there is an abelian group $\mathfrak{F}(U)$ and
2 . for every inclusion $V \subset U$ of open subsets, a group homomorphism $\phi_{UV} : \mathfrak{F}(U) \to \mathfrak{F}(V)$,

subject to the conditions $\mathfrak{F}(\emptyset) = 0$, $\phi_{UU}$ is the identity map, and if $W \subset V \subset U$, then $\phi_{UW} = \phi_{VW} \circ \phi_{UV}$.

A presheaf is a way of assigning abelian groups to open subsets of $X$ that satisfies some natural conditions. For those familiar with category theory, a presheaf is simply a contravariant functor from the category of topological space to the category of abelian groups.

*Remark* 2.2.2. As a matter of terminology, if $\mathfrak{F}$ is a presheaf on $X$, we refer to $\mathfrak{F}(U)$ as the sections of the presheaf $\mathfrak{F}$ over the open set $U$, and we will sometimes use the notation $\Gamma(U, \mathfrak{F})$ to denote $\mathfrak{F}(U)$. If $V \subset U$, then the map $\phi_{UV}$ is called the restriction map, and for $s \in \mathfrak{F}(U)$, we sometimes write $s|_V$ instead of $\phi_{UV}(s)$.

**Definition 2.2.3.** A presheaf $\mathfrak{F}$ on a topological space $X$ is called a sheaf if it satisfies the following conditions:

3 . if $U$ is an open set, $\{V_i\}$ an covering of $U$, and $s \in \mathfrak{F}(U)$ such that $s|_{V_i} = 0$, then $s = 0$;
4 . if $U$, $\{V_i\}$ are the same as above and $s_i \in \mathfrak{F}(V_i)$ such that for each $i, j$, $s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}$, then there is an element $s \in \mathfrak{F}(U)$ such that $s|_{V_i} = s_i$ for each $i$. The previous condition implies that such $s$ is unique.

**Definition 2.2.4.** If $\mathfrak{F}$ is a presheaf on $X$, and $P$ a point, we define the stalk $\mathfrak{F}_P$ to be the direct limit of the groups $\mathfrak{F}(U)$ for all open sets containing $P$, via the restriction maps $\phi$.

**Definition 2.2.5.** If $\mathfrak{F}$, $\mathfrak{G}$ are presheaves on $X$, a morphism $\psi : \mathfrak{F} \to \mathfrak{G}$ consists of a morphism of abelian groups $\psi(U) : \mathfrak{F}(U) \to \mathfrak{G}(U)$ for each open set $U$, such that for every subset $V \subset U$, the following diagram commutes.

$$\begin{array}{ccc} \mathfrak{F}(U) & \xrightarrow{\psi(U)} & \mathfrak{G}(U) \\ \Big\downarrow{\scriptstyle \phi_{UV}} & & \Big\downarrow{\scriptstyle \phi'_{UV}} \\ \mathfrak{F}(V) & \xrightarrow{\psi(V)} & \mathfrak{G}(V) \end{array},$$

A morphism between sheaves is defined in exactly the same way.

*Remark* 2.2.6. Every morphism between sheaves on a topological space $X$ induces morphisms on the stalks. Moreover, a morphism between sheaves is an isomorphism(morphism with a two-sided inverse) if and only if the induces map on the stalks are isomorphism of groups. It is important to note that a morphism is not always determined by the map on sections but is always determined by its behavior on stalks. Thus, a sequence of sheaves is exact if and only if it is exact on stalks.

We now define the kernel, cokernel, and image of a morphism of presheaves.

**Definition 2.2.7.** Let $\psi : \mathfrak{F} \to \mathfrak{G}$ a morphism between presheaves. We define the presheaf kernel of $\psi$, presehaf cokernel of $\psi$, and presheaf image of $\psi$ to be the presheaves $U \to \ker(\psi(U))$, $U \to \operatorname{coker} \psi(U)$, and $U \to \operatorname{im} \psi(U)$, respectively.

*Remark* 2.2.8. Note that the presheaf kernel is always a sheaf, which is not true for preshead cokernel and image. Thus, we need the following process.

**Definition 2.2.9.** Let $\mathfrak{F}$ be a presheaf, there is a sheaf $\mathfrak{F}^+$ and a morphism $\theta : \mathfrak{F} \to \mathfrak{F}^+$ such that every morphism $\psi : \mathfrak{F} \to \mathfrak{G}$ factors through $\mathfrak{F}^+$ and $\theta$. Moreover, the pair $(\mathfrak{F}^+, \theta)$ is unique up to unique isomorphism, and $\mathfrak{F}^+$ is called the sheaf associated to the presheaf $\mathfrak{F}$, and this process is called sheafification.

See Hartshorne Proposition II.1.2 [3] for the proof of the existence and uniqueness of such a sheaf.

**Definition 2.2.10.** The subsheaf $\mathfrak{F}'$ of a sheaf $\mathfrak{F}$ is a sheaf such that for every open set $U$, $\mathfrak{F}'(U) \subset \mathfrak{F}(U)$ with the restriction maps induced by those of $\mathfrak{F}$.

**Definition 2.2.11.** Let $\psi : \mathfrak{F} \to \mathfrak{G}$ a morphism between sheaves, then the sheaf of kernel, cokernel, and image are the sheaves associated to the presheaf of kernel, cokernel, and image.

*Remark* 2.2.12. The direct sum, tensor product, and quotient of sheaves can also be defined as the sheaf associated to the direct sum, tensor product, and quotient presheaf, which are natural to define.

Let's now discuss an example of a sheaf that is useful in our study. Let $D$ be a open domain in $\mathbb{C}^n$, and to each open subset, we can associate the ring of holomorphic functions $\mathfrak{O}_U$. If $U \subset V$ are open sets, and $f \in \mathfrak{O}_V$, then the restriction of $f$ to $U$ is the element $\phi_{VU}(f)$. Thus, the collection of rings with the restriction maps define a presheaf of holomorphic functions over $D$. The sheaf associated to this presheaf is the sheaf of germs of holomorphic functions, which is denoted $\mathfrak{O} = \mathfrak{O}(D)$. The stalks at $z$ is the ring $\mathfrak{O}_z$ of germs of holomorphic functions at the point $z$. The set of sections $\Gamma(U, \mathfrak{O})$ over any open set $U$ can be identified with the set $\mathfrak{O}_U$ of holomorphic functions on $U$. We now state an alternate definition of a sheaf that will be used to define a sheaf of modules.

**Definition 2.2.13.** A sheaf of abelian groups over a topological space $X$ is a topological space $\mathfrak{F}$ together with a map $\pi : \mathfrak{F} \to X$ such that

1 . the map $\pi$ is a local homeomorphism;
2 . for each point $z \in X$, $\pi^{-1}(z)$ is an abelian group, and it is called the stalk at $z$;
3 . the group operations are continuous in the topology of $\mathfrak{F}$.

Suppose $\mathfrak{F}, \mathfrak{I}$ are two sheaves of abelian groups over the same base space $X$ with projection maps $\pi : \mathfrak{F} \to X$ and $\tau : \mathfrak{I} \to X$. It is clear that the Cartesian product $\mathfrak{F} \times \mathfrak{I}$ is clearly a sheaf of abelian groups over $X \times X$. We restrict this sheaf to the diagonal, which is a sheaf of abelian groups defined by $\mathfrak{F} \circ \mathfrak{I} = \{(s,t) \in \mathfrak{F} \times \mathfrak{L} \mid \pi(s) = \tau(t)\}$, with the stalk at $z$ being the direct product $\mathfrak{F}_z \times \mathfrak{I}_z$. We are now ready to define a sheaf of modules.

**Definition 2.2.14.** Let $\mathfrak{R}$ be a sheaf of commutative rings and $\mathfrak{F}$ a sheaf of abelian groups over $X$. The sheaf $\mathfrak{F}$ is called a sheaf of $\mathfrak{R}$-modules if there is a homomorphism of sheaves of abelian groups $\mathfrak{R} \circ \mathfrak{F} \to \mathfrak{F}$ such that for every $z \in X$, the induced map on stalks gives $\mathfrak{F}_z$ the structure of an $\mathfrak{R}_z$-modules.

*Remark* 2.2.15. Subsheaf, quotient sheaf, exact sequence, direct sum, and tensor product of sheaves of $\mathfrak{R}$-modules are all defined similarly to the way that they were defined in the case of abelian groups, and we will not repeat them here. Details can be found in Section IV.B of [1].

**Notation.** Let $\mathfrak{R}$ be a sheaf of commutative rings, then $\mathfrak{R}^k$ is the direct sum of $k$ copies of $\mathfrak{R}$ and is therefore a sheaf of $\mathfrak{R}$-modules.

**Definition 2.2.16.** A sheaf $\mathfrak{F}$ of $\mathfrak{R}$-modules on a domain $D$ is called a free sheaf of $\mathfrak{R}$-modules of rank $k$ if it is isomorphic to $\mathfrak{R}^k$.

Recall we have defined previously a chain of syzygies for modules, we now define its analog for sheaf of modules.

**Definition 2.2.17.** A chain of syzygies(or free resolution) for a sheaf of modules $\mathfrak{F}$ over a sheaf of rings $\mathfrak{R}$ is an exact sequence of sheaves of $\mathfrak{R}$-modules in the form

$$\mathfrak{R}^{k_m} \xrightarrow{\lambda_m} \mathfrak{R}^{k_{m-1}} \to \cdots \to \mathfrak{R}^{k_1} \to \mathfrak{R}^{k} \xrightarrow{\lambda_0} \mathfrak{F} \to 0,$$

where the number $m$ is the length of the chain of syzygies.

This naturally leads to the definition of the most important type of sheaf.

**Definition 2.2.18.** Let $\mathfrak{F}$ be a sheaf of $\mathfrak{R}$-modules on a topological space $X$, then $\mathfrak{F}$ is called a coherent sheaf of $\mathfrak{R}$-modules if for every point $x \in X$ and every non-negative integer $k$, there is an open neighborhood $U_x$ of $x$ such that $\mathfrak{F}$ admits a free resolution of length $k$.

It should be noted that coherence is a local condition. We now define a very well-behaved type of sheaf of rings, which fortunately also turns out to be the type that we work with the most in several complex variables.

**Definition 2.2.19.** A sheaf of rings $\mathfrak{R}$ over $X$ is an Oka sheaf of rings if for every open subset $U$, any sheaf morphism $\mu : (\mathfrak{R}|U)^p \to (\mathfrak{R}|U)^q$ has a syzygy in an open neighborhood of any point of $U$, that it, there is a morphism $\psi : (\mathfrak{R}|U)^k \to (\mathfrak{R}|U)^p$ such that $\operatorname{im}\psi = \ker\mu$(the two maps form an exact sequence).

Let's state some nice properties of Oka sheaf of rings, none of which we will prove.

**Proposition 2.2.20.** *Let $\mathfrak{R}$ be Oka sheaf of rings and $\psi : \mathfrak{R}^p \to \mathfrak{R}^q$ a morphism, then the image and kernel of $\psi$ are coherent.*

*Proof.* See Proposition IV.B.8 of [1]. ∎

**Proposition 2.2.21.** *Let $\mathfrak{R}$ be Oka, then the intersection of coherent sheaf of $\mathfrak{R}$-modules is coherent.*

*Proof.* See Proposition IV.B.9 of [1]. ∎

**Proposition 2.2.22.** *Let $\mathfrak{R}$ be Oka, $\mathfrak{F}, \mathfrak{I}, \mathfrak{L}$ be sheaves of $\mathfrak{R}$-modules such that we have an exact sequence $0 \to \mathfrak{F} \to \mathfrak{I} \to \mathfrak{L} \to 0$. If two of the sheaves of modules are coherent, then so is the third one.*

*Proof.* See Proposition IV.B.13 of [1]. ∎

The most important example of a Oka sheaf of rings is the following:

**Theorem 2.2.23** (Oka Syzygy Theorem)**.** *The sheaf $\mathfrak{O}(D)$ of germs of holomorphic functions on an open set $D \subset \mathbb{C}^n$ is Oka.*

*Proof.* See Theorem IV.C.1 of [1]. ∎

**Definition 2.2.24.** An analytic sheaf over a domain $D \subset \mathbb{C}^n$ is a sheaf of $\mathfrak{O}(D)$-modules over $D$, and is a coherent analytic sheaf if it is a coherent sheaf of $\mathfrak{O}(D)$-modules.

*Remark* 2.2.25. We now restrict ourselves to the case of a subvariety. Let $U$ be a open domain and $V$ a subvariety. For each point $x$, we can associate the ideal of the germ of the variety $V$ at $x$, denoted $I(\mathbf{V}_x)$. Let's define $\mathfrak{I}(V)$ the be the union of all $I(\mathbf{V}_x)$ over all $x \in D$. One can easily check that this a sheaf of ideals in the sheaf of rings $\mathfrak{O}(D)$, and it is called the sheaf of ideals of the subvariety $V$. In fact, this sheaf of ideals is a well-behaved one.

**Theorem 2.2.26.** *Let $V$ be an analytic subvariety of a domain $D$, then $\mathfrak{I}(V)$ is a coherent analytic sheaf on $D$.*

*Proof.* See Theorem IV.D.2 of [1]. ∎

This coherence result gives us the following interesting fact.

**Corollary 2.2.27.** *Let $V$ be a subvariety of $D \subset \mathbb{C}^n$, then the set of singular points of $V$ is a subvariety of $V$.*

*Proof.* See Corollary I.D.4 of [1]. ∎

Again let $V$ be an analytic subvariety of domain $D$ in $\mathbb{C}^n$, and $\mathfrak{I}(V)$ the sheaf of ideals defined previously that is contained in the sheaf of rings $\mathfrak{O}(D)$. Consider the quotient sheaf $\mathfrak{O}_V(D) = \mathfrak{O}(D)/\mathfrak{I}(V)$, which gives us a natural exact sequence of analytic sheaves $0 \to \mathfrak{I}(V) \to \mathfrak{O}(D) \to \mathfrak{O}_V(D) \to 0$. By the previous theorem and Proposition 2.2.22, we conclude that the analytic sheaf $\mathfrak{O}_V(D)$ is coherent. Moreover, since $\mathfrak{I}(V)$ is a sheaf of ideal, $\mathfrak{O}_V(D)$ is a sheaf of rings. Now let $x \in D - V$, then the ideal $I(\mathbf{V}_x) = I(\emptyset) = \mathfrak{O}_{n,x}$, which implies that $\mathfrak{O}_V(D)_x = 0$ for all $x \notin V$. This implies that we shall consider the restriction of $\mathfrak{O}_V(D)$ to the subvariety $V$ as a sheaf of rings over $V$. Thus, we have the following definition.

**Definition 2.2.28.** The sheaf of germs of holomorphic functions on the subvariety $V$ is the restriction of $\mathfrak{O}_V(D)$ to $V$, denoted $\mathfrak{O}_V$.

Unsurprisingly, this type of sheaf also has nice behaviors.

**Theorem 2.2.29.** *The sheaf $\mathfrak{O}_V$ defined above is Oka as a sheaf of rings.*

*Proof.* See Theorem IV.D.6 of [1]. ∎

The last thing we need to discuss about before starting cohomology is the analytic space. In order to do so, we first define a ringed space, which is similar to those defined in abstract algebraic geometry.

**Definition 2.2.30.** A ringed space is a pair $(X, \mathfrak{O})$ where $X$ is Hausdorff and $\mathfrak{O}$ a sheaf of subrings of the sheaf of germs of continuous complex-valued function on $X$.

*Example.* 1 . A trivial example: $X$ be any Hausdorff space, and $\mathfrak{O}$ be the sheaf of germs of continuous functions.
2 . Let $X$ be a domain in $\mathbb{C}^n$, and $\mathfrak{O}$ be the sheaf of germs of holomorphic functions on $X$.
3 . Let $V$ be a subvariety of a domain $D$, and let $\mathfrak{O} = \mathfrak{O}_V$.
4 . Let $(X, \mathfrak{O}_X)$ be a ringed space and $U$ an open subset, then $(U, \mathfrak{O}_X|U)$ is also a ringed space.

**Definition 2.2.31.** Let $(X, \mathfrak{O}_X)$, $(Y, \mathfrak{O}_Y)$ be ringed spaces, the a morphism of ringed spaces is a continuous map $f : X \to Y$ such that for every $x \in X$ and $\mathbf{h} \in \mathfrak{O}_{Y,f(x)}$, we have $\mathbf{h}\circ \in \mathfrak{O}_{\mathbf{X},\mathbf{x}}$.

*Remark* 2.2.32. Note $f$ induce a map from $\mathfrak{O}_{Y,f(x)}$ to $\mathfrak{O}_{X,x}$ by sending $\mathbf{h}$ to $\mathbf{h}\circ$, and we will denote this map by $f^*$. If $f$ is one-to-one and $f^*$ is onto, then we call $f$ an injection.

**Definition 2.2.33.** An isomorphism of ringed space is a homeomorphism that is also an injection. If $f^{-1}$ is also an injection, then $(f^{-1})^* = (f^*)^{-1}$.

We now implement more structures on ringed spaces.

**Definition 2.2.34.** A ringed space $(X, \mathfrak{O}_X)$ is a complex analytic manifold if for every point $x$, there is a neighborhood $U$ such that the ringed space $(U, \mathfrak{O}_X|U)$ is isomorphic to a ringed space $(Y, \mathfrak{O}_Y)$, where $Y$ is a domain and $\mathfrak{O}_Y$ is the sheaf of germs of holomorphic functions on $Y$.

Now we can define analytic spaces.

**Definition 2.2.35.** A ringed space $(X, \mathfrak{O}_X)$ is an analytic space if for every point $x$, there is a neighborhood $U$ such that the ringed space $(U, \mathfrak{O}_X|U)$ is isomorphic to a ringed space $(Y, \mathfrak{O}_Y)$, where $Y$ is a subvariety of a domain and $\mathfrak{O}_Y = (\mathfrak{O}_n/\mathfrak{I}(Y)|Y)$, where $\mathfrak{I}(Y)$ was defined previously.

**Definition 2.2.36.** Let $(X, \mathfrak{O}_X)$ an analytic space, then a regular point of $X$ is a point which has a neighborhood $U$ such that $(U, \mathfrak{O}_X|U)$ is a complex manifold. Otherwise, the point is called a singular point.

*Remark* 2.2.37. We can now define dimensions on analytic spaces. Let $(X, \mathfrak{O}_X)$ be an analytic space with a separable topology, so that it has a countable basis of open sets. Thus, $\mathfrak{R}(X)$ is a complex manifold with countably many components, the closure of these are analytic subvarieties, which are the irreducible branches of $X$. See definition in the previous section. The dimension of $X$ is the supremum of the dimension of irreducible branches, and $X$ is said to have pure dimension if all of its branches have the same dimension.

We can now introduce sheaf cohomology.

2.3. **Sheaf cohomology.** We will introduce sheaf cohomology using fine sheaves and fine resolutions. We will be following Gunning and Rossi closely in the next three sections. For a more abstract but elegant approach to sheaf cohomology using right derived functors, see Hartshorne Chapter 3 [3], Grothendieck's EGA Volume 3 [4], or Grothendieck's Tohoku paper [10].

**Definition 2.3.1.** A locally finite open covering of a subset $E$ of $\mathbb{C}^n$ is a collection of open subsets $U_i \subset E$ such that the $U_i$'s cover $E$ and for each point $x \in E$, there is a open neighborhood that intersects finitely many of the $U_i$'s. A refinement of a cover $\{U_i\}$ is an open cover $\{V_j\}$ such that for every $j$, there exists some $i$ such that $V_j \subset U_i$. A paracompact topological space is a a space such that each open cover has a locally finite refinement.

**Notation.** For the next two sections, let $X$ be a paracompact Hausdorff space.

Let $0 \to \mathfrak{I} \to \mathfrak{F} \to \mathfrak{R} \to 0$ be an exact sequence of sheaves of abelian groups over $X$, then for any open subset $U$, this induces an exact sequence $0 \to \Gamma(U, \mathfrak{I}) \to \Gamma(U, \mathfrak{F}) \to \Gamma(U, \mathfrak{R})$. Note here we do not need any conditions on $X$. In categorical language, this means that the functor $\Gamma(U, -)$ is a left exact functor. It is then natural to ask when is this functor right exact. This leads to the following definition:

**Definition 2.3.2.** A sheaf $\mathfrak{F}$ over $X$ is soft if any sections of $\mathfrak{F}$ over a closed subset $E$ can be extended to a section over $X$.

**Theorem 2.3.3.** *Suppose* $0 \to \mathfrak{F} \to \mathfrak{I} \to \mathfrak{R} \to 0$ *is an exact sequence of sheaves over* $X$ *with* $\mathfrak{F}$ *soft, then it induces the exact sequence* $0 \to \Gamma(X, \mathfrak{F}) \to \Gamma(X, \mathfrak{I}) \to \Gamma(X, \mathfrak{R}) \to 0$ *on sections.*

**Corollary 2.3.4.** *If a short exact sequence of sheaves over* $X$ *contains* 2 *soft sheaves, then the third is also soft.*

**Corollary 2.3.5.** *The functor* $\Gamma(X, -)$ *is exact on the category of soft sheaves.*

*Remark* 2.3.6. These three results are Proposition VI.A.2 and Corollary VI.A.3,4 in [1]. One familiar with algebraic geometry should immediately see the similarities between the soft sheaves and the flabby/flasque sheaves in algebraic geometry. See Hartshorne Chapter 3 [3] for details.

Since soft sheaves have such nice properties, it is reasonable to question if there are any nontrivial examples of them. We present the following construction, showing that there is a soft sheaf corresponding to any given sheaf.

*Example.* Let $\mathfrak{F}$ be a sheaf of abelian groups over $X$. Let $U \subset X$ be open, and consider $\mathfrak{F}_U^*$, defined as the set of all maps $f : U \to \mathfrak{F}$ such that $\pi \circ f$ is the identity on $U$. The collection of all groups $\mathfrak{F}_U^*$ forms a presheaf over $X$, and it can be easily verified that the sheaf associated to this presheaf is a soft sheaf of abelian groups.

In addition to extend domains of functions, another very important concept in analysis is partition of unity. Therefore, we want to define a type of sheaf that allows partition of unity. Before that, we first need to define what it means by a partition of unity on a sheaf.

**Definition 2.3.7.** Let $\mathfrak{F}$ be a sheaf of abelian groups over $X$ and $\{U_i\}$ be a locally finite open covering of $X$. A partition of unity of $\mathfrak{F}$ subordinate to the covering $\{U_i\}$ is a collection of sheaf morphisms $\phi_i : \mathfrak{F} \to \mathfrak{F}$ such that $\mathfrak{F}$ is the zero map in an neighborhood of the complement of $U$ and $\sum \phi_i = 1$, the identity map.

**Definition 2.3.8.** A sheaf of abelian groups that admits a partition of unity subordinate to any locally finite open covering is called a fine sheaf.

The following determine the relationship between fine and soft sheaves.

**Theorem 2.3.9.** *Every fine sheaf is soft.*

*Proof.* See Lemma VI.A.6 of [1]. ■

We now give an example of a fine sheaf.

*Example.* Suppose $D$ is an open domain, and $\{U_i\}$ a locally finite open cover of $D$. Then there exists functions $\{p_i\}$ such that $p_i$ are $C^\infty$, non-negative, and equal 0 on an open neighborhood in the complement of $U$. Moreover, $\sum p_i(z) = 1$ for any point $z$. The operation of multiplication by a $C^\infty$ function is clearly a morphism on the sheaf of germs of continuous functions. Thus, multiplication by the functions $p_i$ form a partition of unity subordinate to the open cover $\{U_i\}$. Thu, the sheaf of germs of continuous functions is fine, and is also soft by the previous theorem. Similarly, the sheaf of germs of all $C^\infty$ functions is also fine. For details on the existence of the functions $p_i$, see Appendix A of [1].

**Definition 2.3.10.** A sheaf cohomology theory on $X$ is an assignment of a sequence of abelian groups $\mathrm{H}^q(X, \mathfrak{F})$, $q \geq 0$ to any sheaf of abelian groups $\mathfrak{F}$ over $X$, and the groups $\mathrm{H}^q(X, \mathfrak{F})$ are called the $q$th cohomology groups of $X$ with coefficients in $\mathfrak{F}$. The assignment satisfies the following properties:

1 . $\mathrm{H}^0(X, \mathfrak{F}) = \Gamma(X, \mathfrak{F})$.
2 . If $\mathfrak{F}$ is fine, the for all $q > 0$, $\mathrm{H}^q(X, \mathfrak{F}) = 0$.
3 . For every sheaf morphism $\phi : \mathfrak{F} \to \mathfrak{I}$, there are induced morphisms on the $q$th cohomology groups $\phi_q : \mathrm{H}^q(X, \mathfrak{F}) \to \mathrm{H}^q(X, \mathfrak{I})$ for all $q$.
4 . For every short exact sequence $0 \to \mathfrak{F} \xrightarrow{\phi} \mathfrak{I} \xrightarrow{\psi} \mathfrak{R} \to 0$, there exists boundary maps $\delta : \mathrm{H}^q(X, \mathfrak{R}) \to \mathrm{H}^{q+1}(X, \mathfrak{F})$ such that the following sequence is exact.

$$0 \to \mathrm{H}^0(X, \mathfrak{F}) \xrightarrow{\phi_0} \mathrm{H}^0(X, \mathfrak{I}) \xrightarrow{\psi_0} \mathrm{H}^0(X, \mathfrak{R}) \xrightarrow{\delta} \mathrm{H}^1(X, \mathfrak{F}) \xrightarrow{\phi_1} \cdots$$

This is called the long exact sequence of cohomology.

5 . The map on the zeroth cohomology group induced by a morphism of sheaves coincides with the natural map between the set of sections of a sheaf.
6 . The identity morphism on sheaves induces the identity map on the cohomology groups.
7 . If $\theta = \phi \circ \psi$ as sheaf morphism, then for every $q$, $\theta_q = \phi_q \circ \psi_q$.
8 . For any commutative diagram of sheaves with exact rows

$$\begin{tikzcd} 0 \arrow[r] & \mathfrak{F}' \arrow[r, "\phi'"] \arrow[d, "f"] & \mathfrak{F} \arrow[r, "\phi"] \arrow[d, "g"] & \mathfrak{F}'' \arrow[r] \arrow[d, "h"] & 0 \\ 0 \arrow[r] & \mathfrak{I}' \arrow[r, "\psi'"] & \mathfrak{I} \arrow[r, "\psi"] & \mathfrak{I}'' \arrow[r] & 0 \end{tikzcd},$$

The following induced diagram of cohomology groups is also commutative:

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{H}^0(X, \mathfrak{F}') \arrow[r, "\phi_0'"] \arrow[d, "f_0"] & \mathrm{H}^0(X, \mathfrak{F}) \arrow[r, "\phi_0"] \arrow[d, "g_0"] & \mathrm{H}^0(X, \mathfrak{F}'') \arrow[r, "\delta"] \arrow[d, "h_0"] & \mathrm{H}^1(X, \mathfrak{F}') \cdots \arrow[d, "f_1"] \\ 0 \arrow[r] & \mathrm{H}^0(X, \mathfrak{I}') \arrow[r, "\psi_0'"] & \mathrm{H}^0(X, \mathfrak{I}) \arrow[r, "\psi_0"] & \mathrm{H}^0(X, \mathfrak{I}'') \arrow[r, "\delta"] & \mathrm{H}^1(X, \mathfrak{I}) \cdots \end{tikzcd}.$$

We now discuss methods for calculating the cohomology groups. To do so, we need some definitions.

**Definition 2.3.11.** A cohomological resolution of a sheaf $\mathfrak{F}$ of abelian groups over $X$ is an exact sequence of sheaves in the form

$$0 \to \mathfrak{F} \xrightarrow{\epsilon} \mathfrak{F}_0 \xrightarrow{d_0} \mathfrak{F}_1 \xrightarrow{d_1} \cdots .$$

Such a resolution is called a fine resolution if the sheaves $\mathfrak{F}_i$ are all fine.

*Remark* 2.3.12. For such sequence, there corresponds a sequence of sections

$$0 \to \Gamma(X, \mathfrak{F}) \xrightarrow{\epsilon^*} \Gamma(X, \mathfrak{F}_0) \xrightarrow{d_0^*} \Gamma(X, \mathfrak{F}_1) \xrightarrow{d_1^*} \cdots,$$

which is not necessarily exact as $\mathfrak{F}$ might not be soft. However, we do have $\operatorname{im} d_{i-1}^* \subset \ker d_i^*$, so we have the following useful result.

**Lemma 2.3.13.** *If we have a fine resolution of the sheaf $\mathfrak{F}$ in the form above, then for $\mathrm{H}^0(X, \mathfrak{F}) = \ker d_0^*$ and for all $i \geq 1$, $\mathrm{H}^i(X, \mathfrak{F}) = \ker d_i^* / \operatorname{im} d_{i-1}^*$.*

*Proof.* The proof follows from basic but tedious homological algebra and manipulation with the axioms. See Lemma VI.B.2 for details. ∎

Using this and other result, we have the following theorem.

**Theorem 2.3.14.** *There exists a unique cohomology theory for any paracompact Hausdorff space $X$.*

*Proof.* See Theorem VI.B.4 of [1]. ∎

In De Rham cohomology, one of the most useful computational tools is the Meyer-Vietoris sequence.

**Theorem 2.3.15** (Mayer-Vietoris)**.** *Let $D$ be a domain such that $D = U \cup V$ as a union of open subsets. Then for every coherent analytic sheaf $\mathfrak{F}$, the sequence*

$$0 \to \mathrm{H}^0(D, \mathfrak{F}) \to \mathrm{H}^0(U, \mathfrak{F}) \oplus \mathrm{H}^0(V, \mathfrak{F}) \to \mathrm{H}^0(U \cap V, \mathfrak{F}) \to \mathrm{H}^1(D, \mathfrak{F}) \to \cdots$$

*Proof.* See Lemma 20.8.2(The algebraic counterpart) of [11], the desired theorem follows from simple modifications of this proof. ∎

Although we have already discussed computing cohomologies using fine resolutions, it is not practical in most cases as such fine resolution is rather difficult to construct. In the next section, we introduce two much more practical ways of computing sheaf cohomology groups.

2.4. **Dolbeault and Cech cohomology.** We first restrict ourselves to the cohomology groups of an open domain $D$ with coefficients in the sheaf of germs of holomorphic functions on $D$, which is denoted by $\mathfrak{O}$. Recall on any domain $U$, we have previously defined the group $\mathfrak{E}^{p,q}(U)$ of exterior differential forms of bidegree $(p, q)$ on $U$. The set of all these groups for all open subsets $U \subset D$ forms a presheaf under restriction maps, and associated sheaf will be called the sheaf of germs of exterior differential forms of bidegree $(p, q)$ on $D$, denoted $\mathfrak{E}^{p,q}$. Recall that the groups $\mathfrak{E}^{p,q}(U)$ are free $C_U^\infty$-free module of rank $N = \binom{n}{p}\binom{n}{q}$. Let $\mathfrak{C}^\infty$ denote the sheaf of rings of germs of $C^\infty$ functions, then $\mathfrak{E}^{p,q}$ is a free sheaf of $\mathfrak{C}^\infty$-modules of rank $N$. Since the sheaf of germs of $C^\infty$ functions is fine, so are the sheafs $\mathfrak{E}^{p,q}$.

Recall the differentiation maps $\bar\partial : \mathfrak{E}^{p,q}(U) \to \mathfrak{E}^{p,q+1}(U)$ for open subset $U$. Observe that these operators commute with boundary maps, which induce morphism between sheaves of abelian groups $\mathfrak{E}^{p,q} \to \mathfrak{E}^{p,q+1}$ over $D$. This map will still be denoted by $\bar\partial$ as there is no possible confusion. This gives us the following exact sequence:

**Lemma 2.4.1.** *The following sequence of sheaves of abelian groups is exact over a domain $D$:*

$$0 \to \mathfrak{O} \xrightarrow{\iota} \mathfrak{E}^{0,0} \xrightarrow{\bar\partial} \mathfrak{E}^{0,1} \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} \mathfrak{E}^{0,n} \xrightarrow{\bar\partial} 0,$$

*where the second map is inclusion.*

*Proof.* Direct application of Dolbeault's theorem, Cauchy-Riemann criterion, and manipulation of differential forms. See Lemma VI.C.1 of [1] for details. ∎

*Remark* 2.4.2. Let $\mathfrak{O}^{p,0}$ be the sheaf of holomorphic differential forms of bidegree $(p, 0)$, then we can get a new exact sequence

$$0 \to \mathfrak{O}^{p,0} \xrightarrow{\iota} \mathfrak{E}^{p,0} \xrightarrow{\bar\partial} \mathfrak{E}^{p,1} \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} \mathfrak{E}^{p,n} \xrightarrow{\bar\partial} 0.$$

Note the operator $\partial$ maps each sequence to another sequence with $p$ replaced by $p+1$. Using the exact sequence, we have the following fundamental theorem relating the Dolbeault and sheaf cohomology groups.

**Theorem 2.4.3** (Dolbeault)**.** *Let $D \subset \mathbb{C}^n$ be an open domain, $\mathfrak{O}^{p,0}$ be the same as defined above, $\mathrm{H}^{p,q}(D)$ the Dolbeault cohomology group of bidegree $(p,q)$ for $D$, then $\mathrm{H}^q(D, \mathfrak{O}^{p,0}) \cong \mathrm{H}^{p,q}(D)$.*

*Proof.* The sequence in the remark is a fine resolution of the sheaf $\mathfrak{O}^{p,0}$, so the result follows immediately from Lemma 2.3.13. ■

This theorem of Dolbeault provides us with a method of computing sheaf cohomology with coefficients in the sheaf of germs of holomorphic functions. We now present a method that works in more cases, computing sheaf cohomology using Cech cohomology. In fact, this can even be applied to compute the sheaf cohomology of most schemes with coefficients in a quasi-coherent sheaf. For more details, see Section III.4 of [3]. Many notions defined here will be used in the proof of Cartan's Theorem B.

**Definition 2.4.4.** Let $D$ be a paracompact Hausdorff space, $\mathfrak{U} = \{U\}$ an open covering of $D$. A $p$-simplex $\sigma$ is an ordered $p+1$-tuple of open sets of the covering $\mathfrak{U}$ with nonempty intersection. The nerve of a covering $\mathfrak{U}$, denoted $N(\mathfrak{U})$ is the set of all $p$-simplex. The support of a $p$-simplex $\sigma = (U_0, \ldots.., U_p)$ is the intersection $\bigcap U_i$, and is denoted $|\sigma|$. If $\mathfrak{F}$ is a sheaf of abelian groups over $D$, then a $p$-cochain of $N(\mathfrak{U})$ with coefficients in the sheaf $\mathfrak{F}$ is a function $f$ which associates to each $p$-simplex $\sigma$ a section $f(\sigma) \in \Gamma(|\sigma|, \mathfrak{F})$. Note the set of all $p$-cochains form an abelian group $C^p(N(\mathfrak{U}), \mathfrak{F})$, and there is an coboundary operator $\delta_p : C^p(N(\mathfrak{U}), \mathfrak{F}) \to C^{p+1}(N(\mathfrak{U}), \mathfrak{F})$ defined in the following way: If $f \in C^p(N(\mathfrak{U}), \mathfrak{F})$ and $\sigma = (U_0, \ldots.., U_p)$, then

$$\delta_p f(\sigma) = \sum_{j=0}^{p+1} (-1)^j r_\sigma f(U_0, \ldots.., U_{j-1}, U_{j+1}, \ldots.., U_{p+1}),$$

where $r_\sigma$ denotes the restriction of sections to the open set $|\sigma|$.

The coboundary maps have the following unsuprising property:

**Proposition 2.4.5.** $\delta_{p+1}\delta_p = 0$.

*Proof.* Direct computation, see Lemma VI.D.1 of [1]. ■

**Definition 2.4.6.** The kernel of $\delta_p$ is called the $p$-cocycles, denoted $Z(N(\mathfrak{U}), \mathfrak{F})$ and the image of $\delta_{p-1}$ are called the $p$-coboundaries, denoted $B^p(N(\mathfrak{U}), \mathfrak{F})$. By convention, we have $B^p(N(\mathfrak{U}), \mathfrak{F}) = 0$.

The previous lemma implies that $B^p(N(\mathfrak{U}), \mathfrak{F}) \subset Z^p(N(\mathfrak{U}), \mathfrak{F})$, so we can define a new cohomology.

**Definition 2.4.7.** The $i$th Cech cohomology group is the $i$th cohomology group of the nerve $N(\mathfrak{U})$ with coefficients in the sheaf $\mathfrak{F}$, which is defined to be $\check{\mathrm{H}}^i(N(\mathfrak{U}), \mathfrak{F}) = Z^i(N(\mathfrak{U}), \mathfrak{F})/B^i(N(\mathfrak{U}, \mathfrak{F})$.

We have the following lemma illustrating the similarity between Cech and sheaf cohomology.

**Lemma 2.4.8.** $\check{\mathrm{H}}^0(N(\mathfrak{U}), \mathfrak{F}) = \Gamma(D, \mathfrak{F})$ *and if $\mathfrak{F}$ is fine, then $\check{\mathrm{H}}^i(N(\mathfrak{U}), \mathfrak{F}) = 0$ for all $i > 0$.*

*Proof.* See Lemma VI.D.2,3 of [1]. ■

The following theorem reveals the fundamental relationship between Cech and sheaf cohomology.

**Theorem 2.4.9** (Leray)**.** *Suppose $\mathfrak{F}$ is a sheaf of abelian groups on paracompact Hausdorff space $D$. If $\mathfrak{U}$ is an open covering such that $\mathrm{H}^p(|\sigma|, \mathfrak{F}||\sigma|) = 0$ for all $\sigma \in N(\mathfrak{U})$ and $p \geq 0$, then*

$$\mathrm{H}^p(D, \mathfrak{F}) \cong \check{\mathrm{H}}^p(N(\mathfrak{U}), \mathfrak{F}).$$

*Proof.* See Theorem VI.D.4 of [1]. ■

The previous theorem has the following algebraic geometric analog, due to Serre. I will assume knowledge in algebraic geometry and will not be defining any new terms.

**Theorem 2.4.10** (Serre)**.** *Let $X$ be a noetherian separated scheme, $\mu$ an affine cover of $X$, $\mathfrak{F}$ a quasi-coherent sheaf on $X$, then for all $p \geq 0$,*

$$\mathrm{H}^p(X, \mathfrak{F}) \cong \check{\mathrm{H}}^p(\mathfrak{U}, \mathfrak{F}).$$

*Proof.* See Theorem III.4.5 of [3]. ■

This finishes our discussions of cohomology.

2.5. **Stein spaces.** We now discuss something much more analytic, Stein spaces.

**Notation.** In this section $(X, \mathfrak{O})$ denote an analytic space with a separable topology.

**Definition 2.5.1.** Let $K$ be a compact subset of $X$, and $A$ an algebra of holomorphic functions on $X$, then the $A$-convex hull of $K$ in $X$ is defined as the set $K(A,X) = \{x \in \mid f(x) \leq ||f||_K \text{ for all } f \in A\}$. We say $K$ is $A$-convex if $K = K(A,X)$. In most cases, the algebra $A$ is simple $\mathfrak{O}_X$. We write $\hat{K} = K(\mathfrak{O}_X, X)$ and $\hat{K}$ is called the holomorphically convex hull of $X$.

**Definition 2.5.2.** Let $(X, \mathfrak{O})$ be an analytic space. A tangent vector of $\mathfrak{O}_X$ at $x$ is a map $t : \mathfrak{O}_X \to \mathbb{C}$ such that for all $a, b \in \mathbb{C}$ and $\mathbf{f}_1, \mathbf{f}_2 \in \mathfrak{O}_X$,

1 . $t(a\mathbf{f}_1 + b\mathbf{f}_2) = at(\mathbf{f}_1) + bt(\mathbf{f}_2)$.
2 . $t(\mathbf{f}_1\mathbf{f}_2) = \mathbf{f}_1(x)t(\mathbf{f}_2) + \mathbf{f}_2(x)t(\mathbf{f}_1)$ .

The collection of all such derivations of $\mathfrak{O}_X$ form a vector space over $\mathbb{C}$, denoted $T_{X,x}$, called the tangent space of $X$ at $x$. Its dimension is denoted $\dim t_x X$, and is called the tangential dimension of $X$ at $x$.

**Definition 2.5.3.** Let $(X, \mathfrak{O})$ be an Stein space. $X$ is Stein if all of the following are true

1 . $X$ has a countable topology.
2 . For all compact subset $K$, $\hat{K}$ is also compact. We say $X$ is holomorphically convex.
3 . For $x \in X$, there exists $f_1, \ldots.., f_n \in \mathfrak{O}_X$ such that $\operatorname{rank}_x(f_1, \ldots.., f_n) = \dim t_x X$.
4 . For $x \neq y$, there exists $f \in \mathfrak{O}_X$ that separates the two points(i.e. $f(x) \neq f(y)$).

A Stein manifold is a Stein space that is also a manifold.

Stein spaces have the following simple properties.

**Proposition 2.5.4.** *Suppose $(X, \mathfrak{O}_X)$ is Stein and $x \in X$, then there are holomorphic functions $f_1, \ldots.., f_n$ that gives a local holomorphic embedding of a neighborhood of $x$ into $\mathbb{C}^n$.*

*Proof.* This is basically the third condition. ■

A more important embedding theorem is the following:

**Theorem 2.5.5.** *A map between topological spaces is called proper if inverse images of compact sets are compact. Every Stein manifold of dimension $n$ can be embedded into $\mathbb{C}^{2n+1}$ by a biholomorphic proper map.*

*Proof.* See [12]. ■

*Remark* 2.5.6. For Stein space of positive dimension, it is clear that there exists nonconstant holomorphic function on there. Thus, using the same reasoning as in the proof of holomorphic functions on compact Riemann surfaces are trivial, we deduce that Stein spaces are noncompact. Thus, a necessary condition for Riemann surfaces(connected by convention) to be Stein is that it is not compact. Surprisingly, it is also a sufficient condition.

**Theorem 2.5.7** (Behnke-Stein)**.** *A Riemann surface is Stein if and only if it is noncompact.*

*Proof.* See Theorem IX.B.10 of [1]. ■

Now we have all of the background knowledge needed to prove Cartan's Theorems

## 3. The proof of Cartan's Theorems

3.1. **Theorem B implies Theorem A.** We shall start by stating Cartan's theorems.

**Theorem 3.1.1** (Cartan's Theorem A)**.** *If $\mathfrak{F}$ is a coherent analytic sheaf on a Stein space $X$, then $\Gamma(X, \mathfrak{F})$ generates the stalks $\mathfrak{F}_x$ at every point $x$.*

**Theorem 3.1.2** (Cartan's Theorem B)**.** *If $\mathfrak{F}$ is a coherent analytic sheaf on a Stein space $X$, then $\mathrm{H}^i(X, \mathfrak{F}) = 0$ for all $i > 0$.*

Our goal this section is to prove Theorem A assuming Theorem B. The proof here follows [6].

*Proof of Theorem B implies A.* Let $(X, \mathfrak{O})$ be a Stein space and $\mathfrak{F}$ a coherent analytic sheaf on $X$. Suppose $x \in X$, and consider the sheaf of germs of holomorphic functions vanishing at $x$, denoted $\mathfrak{I}$. For some $n > 0$, we have an epimorphism $\phi : \mathfrak{O}_x^n \to \mathfrak{F}_x$ on the stalks. Let $e_i = (0, \ldots\ldots, 0, 1, 0, \ldots\ldots, 0)$ with a 1 at the $i$th position, $1 \leq i \leq n$. Consider the natural exact sequence of sheaves

$$0 \to \mathfrak{I}\mathfrak{F} \to \mathfrak{F} \xrightarrow{\psi} \mathfrak{F}/\mathfrak{I}\mathfrak{F} \to 0.$$

Here the sheaf $\mathfrak{I}\mathfrak{F}$ is the tensor product of the two sheaves so that away from $x$ the stalk is just the stalk of $\mathfrak{F}$, and on any open set containing $x \in X$ we get sections of $\mathfrak{F}$ vanishing at $x$. Note here we needed that $\mathfrak{I}$ is a subsheaf of the structure sheaf. Observe that stalk of the quotient sheaf $\mathfrak{F}/\mathfrak{I}\mathfrak{F}$ is trivial at every point other than $x$ by construction, which implies that $\psi \circ \phi(e_i) \in (\mathfrak{F}/\mathfrak{I}\mathfrak{F})_x$ define sections $s_i \in \Gamma(X, \mathfrak{F}/\mathfrak{I}\mathfrak{F})$. Taking cohomology of the exact sequence and apply Cartan's Theorem B, we deduce that the map $\Gamma(X, \mathfrak{F}) \to \Gamma(X, \mathfrak{F}/\mathfrak{I}\mathfrak{F})$ induced by $\psi$ is surjective. Thus, there exists $f_i \in \Gamma(X, \mathfrak{F})$ such that $f_i$ is mapped to $s_i$ by the morphism on sections. Suppose $u \in \mathfrak{F}_x$, then for some $v \in \mathfrak{O}_x^n$, we have $\phi(v) = u$. Note $\psi(\phi(e_i) - (f_i)_x) = 0$, so we have $\phi(e_i) - (f_i)_x \in \mathfrak{I}_x\mathfrak{F}_x = \phi(\mathfrak{I}_x\mathfrak{O}_x^n)$. Let $g_i = (g_{i1}, \ldots\ldots, g_{in}) \in \mathfrak{I}_x\mathfrak{O}_x^n$ satisfy $\phi(e_i) - (f_i)_x$. Consider the $n \times n$ matrix $\delta_{ij} - g_{ij}$, whose determinant is a unit in $\mathfrak{O}_x$. Thus, $e_i - g_i$ generates $\mathfrak{O}_x^n$, so there exists $\lambda_i \in \mathfrak{O}_x$ such that $v = \sum_{i=1}^n \lambda_i(e_i - g_i)$. Hence, $u = \phi(v) = \sum_{i=1}^n \lambda_i(f_i)_x$, and we are done. In fact, we have shown that the least number of global sections needed to generate a given stalk is the same as the least number of elements of the stalk to generate the stalk. ■

3.2. **Some analytic lemmas.** The next three sections are devoted to the proof of Cartan's Theorem B.

**Definition 3.2.1.** A $\sigma$-compact topological space is topological space that is the union of countably many compact subspaces.

We now fix some notations that will be used throughout the proof.

**Notation.** Let $\mathfrak{O}_n$ denote the sheaf of germs of holomorphic functions on $\mathbb{C}^n$, sometimes called the structure sheaf of $\mathbb{C}^n$, $B_r^n$ or $B_r$ denote the ball in $\mathbb{C}^n$ with radius $r$ centered at the origin. Let $g = (f_1, \ldots\ldots, f_n)$ be a $n$-tuple of complex-valued function on $K$, then $||f||_J = \sup\{|f_i(z)| \mid z \in K\}$. Lastly, if $\mathfrak{U}$ is an open covering, then $N(\mathfrak{U})$ is the nerve.

**Definition 3.2.2.** Suppose $\mathfrak{F}$ is a coherent analytic sheaf on a $\sigma$-compact complex space $(X, \mathfrak{O}_X)$ and $K$ is a compact subset. Let $\phi : \mathfrak{O}^n \to \mathfrak{F}$ be a surjective morphism of sheaves that the induced map $\bar{\phi}$ on global sections is also surjective. Let $f \in \Gamma(X, \mathfrak{F})$, then $||f||_K^{\phi}$ is defined as $\inf\{||g||_K \mid g \in \Gamma(X, \mathfrak{O}_X^n), \bar{\phi}(g) = f\}$

**Lemma 3.2.3.** *Let $(X, \mathfrak{O}_X)$ be an analytic space, and $\mathfrak{F}$ an analytic subsheaf of $\mathfrak{O}_X^n$, then for any open set $U$, $\Gamma(U, \mathfrak{F})$ is a Frechet subspace of $\Gamma(U, \mathfrak{O}_X^n)$.*

*Proof.* See Proposition VIII.A.2 of [1]. ■

**Lemma 3.2.4.** *The norm defined above define a Frechet space topology on $\Gamma(X, \mathfrak{F})$.*

*Proof.* Let $\mathfrak{R}$ be the kernel sheaf of $\phi$, then the previous lemma implies that $\Gamma(X, \mathfrak{R})$ is a closed subspace of the Frechet space $\Gamma(X, \mathfrak{O}_X^n)$ with the topology of uniform convergence on compact sets. Since $\bar{\phi}$ is surjective, we can conclude the quotient topology induced by $\bar{\phi}$ is a Frechet space topology. ■

*Remark* 3.2.5. The Frechet space topology is canonical as it does not depend on the choice of $\phi$.

**Lemma 3.2.6.** *Let $\phi^1, \ldots.., \phi^k$ be real-valued $C^\infty$ functions on $\mathbb{C}^n$ such that $|\phi^l_{ij}(z)| < \frac{1}{6n^2}$ and $|\phi^l_{i\bar{j}}(z) - \delta_{ij}| < \frac{1}{3n^2}$, where $z \in \mathbb{C}^n$, $1 \le i, j \le n$, $\delta_{ij}$ being the Kronecker delta, $\phi^l_{ij} = \frac{\partial^2 \phi^l}{\partial z_i \partial z_j}$ and $\phi^l_{i\bar{j}}(z) = \frac{\partial^2 \phi^l}{\partial z_i \partial \bar{z}_j}$. If $D = \{z \in \mathbb{C}^n \mid \phi^l(z) < 0, 1 \le l \le k$ is a bounded domain, then $\mathrm{H}^i(D, \mathfrak{O}_n) = 0$ for $i > 0$.*

**Notation.** If $U \subset U'$ such that $\bar{U}$ is compact in $U'$, then we say $U$ is relatively compact in $U'$, denotes $U \subset\subset U'$.

*Proof.* Let $z_0 \in \partial D$, then there exists some $\phi^l$ such that $\phi^l(z_0) = 0$. Define a polynomial $f(z) = \sum_{i=1}^n \frac{\partial^l}{\partial z_i}(z_0)(z_i - z_{i,0})$, where $z_0 = (z_{1,0}, \ldots.., z_{n,0})$. Therefore, there exists some $z^*$, depending on $z$ such that $\phi^l(z) = 2Re(f(z) + \sum_{1 \le i,j \le n} \phi^l_{ij}(z^*)(z_i - z_{i,0})(z_j - z_{j,0}))$. Using the bounds on the partial derivatives of $\phi^l$, we have $\phi^l(z) \ge 2Re(f(z)) + \frac{1}{3}\sum_{i=1}^n |z_i - z_{i,0}|^2$, so $f$ is nowhere zero on $D$. Thus, we have proved that for all $z_0 = (z_{1,0}, \ldots.., z_{n,0}) \in \partial D$, there exists a polynomial $f$ that is zero at $z_0$ and nonzero everywhere on $D$.

This allows us to contruct a sequence of open subsets $U_k$ of $D$ that satisfies the following conditions:

1 . $P_i$ is a union of topological components of a polynomial polyannulua whose essential defining polynomials are nowhere zero on $D$.
2 . $P_i \subset\subset P_{i+1}$.
3 . $\{P_i\}$ covers $D$.

Applying Lemma 1.2.12 and Theorem 1.2.13, we can complete the proof in the same manner as Theorem I.D.5 of [1]. ■

This gives us the following immediate corollary.

**Corollary 3.2.7.** *Suppose $\mathfrak{F}$ is a coherent analytic sheaf on $D$ with a finite chain of syzygies, then $\mathrm{H}^i(D, \mathfrak{F}) = 0$ for $i > 0$.*

## 3.3. **Lemmas on finiteness and vanishing of cohomology.**

**Proposition 3.3.1.** *Suppose $\mathfrak{F}$ is a coherent analytic sheaf on an open neighborhood $O$ of $\bar{B}_r$, then for all $i > 0$, $\mathrm{H}^i(B_r, \mathfrak{F}) = 0$.*

*Proof.* Choose balls $U_i \subset\subset V_i \subset\subset O$, $1 \le i \le m$ such that

1 . $U_i$'s contain $\partial B_r$.
2 . $\mathfrak{F}$ has a finite chain of syzygies on $V_i$.

Let $\psi_i$ be a $C^\infty$ non-negative function on $\mathbb{C}^n$ such that it is identically zero outside of $V_i$ and greater than zero on $U_i$ for all $i$. Let $\phi^i = \sum_{i=1}^n |z_i|^2 - r^2$ and for $1 \le i \le m$, choose positive reals $\lambda_i$ sufficiently small so that $\phi^l = \phi^0 - \sum_{i=1}^l \lambda_i \psi_i$ satisfies the two inequalities in Lemma 3.2.6 for all $z \in \mathbb{C}$ and $1 \le l \le m$. Define $D_l = \{z \in \mathbb{C}^n \mid \phi^l(z) < 0\}$ for $0 \le l \le m$.Thus, by constructions, we have $D_0 = B_r \subset\subset D_m$ and $D_l = D_{l-1} \cup (D_l \cap V_l)$ and $D_{l-1} \cap V_l = D_{l-1} \cap (D_l \cap V_l)$. Apply the previous corollary, we have $\mathrm{H}^i(D_{l-1} \cap V_l, \mathfrak{F}) = 0$ for $i > 0$ and $1 \le l \le m$. Using these, we have the Meyer-Vietoris sequence

$$\cdots \to \mathrm{H}^i(D_l, \mathfrak{F}) \to \mathrm{H}^i(D_{l-1}, \mathfrak{F}) \oplus \mathrm{H}^i(D_l \cap V_l, \mathfrak{F}) \to \mathrm{H}^i(D_{l-1} \cap V_l, \mathfrak{F}) \to \cdots,$$

whose exactness implies the map $\mathrm{H}^i(D_l, \mathfrak{F}) \to \mathrm{H}^i(D_{l-1}, \mathfrak{O})$ is surjective for $i > 0$ and all $l$. Repeating this, we conclude that the restriction map $\mathrm{H}^i(D_m, \mathfrak{F}) \to \mathrm{H}^i(B_r, \mathfrak{F})$ is surjective for $i > 0$. Call this fact $(*)$

Choose two finite collections $\{\tilde{U}^i_j\}_{j=1}^k, i = 1, 2$ of balls in $O$ that satisfies the following conditions:

1 . $\tilde{U}^1_j \subset\subset \tilde{U}^2_j$.

2 . $B_r \subset \bigcup_{j=1}^{k} \tilde{U_j^1}$.
3 . $D_m \subset \bigcup_{j=1}^{k} \tilde{U_j^2}$.
4 . On $\tilde{U_j^2}$, we have a surjective sheaf morphism $\tau_j : \mathfrak{O}_n^{p_j} \to \mathfrak{F}$ that is part of a finite chain of syzygy.

Let $U_j^1 = \tilde{U_j^1} \cap B_r$ and $U_j^2 = \tilde{U_j^2} \cap D_m$, and the cover $\mathfrak{U}_i$ are the collection of open sets $\{U_j^i\}$ for $i = 1, 2$. Now fix $p \geq 0$. By the previous corollary, we have $\mathrm{H}^i(U_{j_0}^i \cap \cdots \cap U_{j_q}^i, \ker \tau_{j_0}) = 0$, which implies that the map $\Gamma(U_{j_0}^i \cap \cdots \cap U_{j_q}^i, \mathfrak{O}_n^{p_{j_0}}) \to \Gamma(U_{j_0}^i \cap \cdots \cap U_{j_q}^i, \mathfrak{F})$ is surjective for $1 \leq j_0, ....., j_q \leq k$ and $i = 1, 2$. By Lemma 3.2.4, the space of global sections $\Gamma(U_{j_0}^i \cap \cdots \cap U_{j_q}^i, \mathfrak{F})$ has a Frechet space topology, which implies that the sets $Z^p(N(\mathfrak{U}_i), \mathfrak{F})$ and $C^{p-1}(N(\mathfrak{U}_i), \mathfrak{F})$ can be given a canonical Frechet space structure. Let $\rho : Z^p(N(\mathfrak{U}_2), \mathfrak{F}) \to Z^p(N(\mathfrak{U}_q), \mathfrak{F})$ be restriction maps and $\delta : C^{p-1}(N(\mathfrak{U}_1), \mathfrak{F}) \to Z^p(N(\mathfrak{U}), \mathfrak{F})$ the coboundary maps. By the previous corollary, $\mathrm{H}^a(U_{j_0}^i \cap \cdots \cap U_{j_q}^i, \mathfrak{F}) = 0$ for $a > 0$ anf all $j_0, ...., j_q$ and $i = 1, 2$, which implies that $\check{\mathrm{H}}^p(N(\mathfrak{U}_1), \mathfrak{F}) \cong \mathrm{H}^p(B_r, \mathfrak{F})$ and a similar result holds for $\mathfrak{U}_2$ and $D_m$.

By $(*)$, the map $\rho \oplus \delta : Z^p(N(\mathfrak{U}_2, \mathfrak{F})) \oplus C^{p-1}(N(\mathfrak{U}_1), \mathfrak{F}) \to Z^p(N(\mathfrak{U}_1), \mathfrak{F})$ is surjective. Using $U_j^1 \subset\subset U_j^2$, the map $\rho \oplus 0$ with the same domain and range defined by $\rho \oplus 0(x \oplus y) = \rho(x)$ is compact. If the map $0 \oplus \delta = \rho \oplus \delta - \rho \oplus 0$ has finite dimensional cokernel, then $\delta$ has finite dimensional cokernel, and we are done. This is an immediate application of the following theorem of functional analysis. ■

**Theorem 3.3.2** (L. Schwartz)**.** *Let $E, F$ be Frechet spaces, $\phi : E \to F$ a surjective transformtion, $\psi : E \to F$ a compact transformation, then $(\phi + \psi)(E)$ is closed and $F/(\phi + \psi)(E)$ is finite dimensional.*

*Proof.* See Appendix B Theorem 12 of [1]. ■

**Theorem 3.3.3.** *With the same assumptions as in the previous theorem,* $\mathrm{H}^i(B_r, \mathfrak{F}) = 0$ *for all* $i > 0$.

This can be proved by working with the support of a sheaf, which is the analog of the support of a function and applying induction on the dimension of the support of $\mathfrak{F}$. Since this requires more sheaf theory, we will not prove it, and the interested reader may refer to Proposition 3 of [7] for a proof.

If we use the same strategy as in the proof of Theorem B implies A, we obtain the following corollary:

**Corollary 3.3.4.** *With the same assumption as the previous theorem, $\mathfrak{F}$ is generated by global sections on $B_r$.*

**Corollary 3.3.5.** *Suppose $\mathfrak{F}$ is a coherent analytic sheaf on Stein space $X$ and $E$ is relatively compact open subset of $X$, then $\mathfrak{F}$ is generated by global sections on $E$.*

*Proof.* Use the fact that some open neighborhood of $\bar{E}$ is biholomorphic to a subvariety of a ball in the complex space and the previous corollary. ■

**Corollary 3.3.6.** *Suppose $D$ is an open subset of Stein space $(X, \mathfrak{O}_X)$ and $\phi : X \to \mathbb{C}^n$ is a holomorphic function such that*

1 . *For some open neighborhood $U$ of $\bar{D}$, $\phi$ maps biholomorphically onto a subvariety of some open set $V$ of $\mathbb{C}^n$.*
2 . *$\phi(D)$ is a subvariety in a ball $B_r$ in $V$. Then $\Gamma(X, \mathfrak{O}_X)$ is dense in $\Gamma(D, \mathfrak{O}_D)$ with the topology of uniform convergence on compact subsets.*

*Proof.* Let $\mathfrak{I}$ be sheaf of ideals of $\phi(D)$ on $V$. By the theorem, $\mathrm{H}^i(B_r, \mathfrak{I}) = 0$, which implies that the canonical map $\Gamma(B_r, \mathfrak{O}_n) \to \Gamma(B_r, \mathfrak{O}_n/\mathfrak{I}) \cong \Gamma(D, \mathfrak{O}_D)$ is surjective. Thus, the map $\alpha : \Gamma(B_r, \mathfrak{O}_n) \to \Gamma(D, \mathfrak{O}_D)$ induced by the restriction of $\phi$ to $D$ is surjective. Let $\beta : \Gamma(\mathbb{C}^n, \mathfrak{O}_n) \to \Gamma(X, \mathfrak{O}_X)$ be the map induced by $\phi$. These two maps along with the restriction maps froma commutative diagram. If $\Gamma(\mathbb{C}^n, \mathfrak{O}_n)$ is dense in $\Gamma(B_r \mathfrak{O}_n)$, then we have the desired result by the diagram. However, this follows directly from Theorem 1.2.15. ■

3.4. **Proof of Theorem B.** We are now ready to prove Cartan's Theorem B.

*Proof of Theorem B.* Let $\mathfrak{F}$ be a coherent analytic sheaf on a Stein space $(X, \mathfrak{O}_X)$. Construct open subsets $X_k$ and holomorphic functions $\phi^l : X \to \mathbb{C}^{n_k}$ such that 1. $X = \bigcup X_k$, 2. $X_k \subset\subset X_{k+1}$, 3. $\phi^k$ maps $X_{k+1}$ biholomorphically onto a subvariety of an open subset of $\mathbb{C}^{n_k}$, and 4. $\phi^k(X_k)$ is a subvariety in a ball of $\mathbb{C}^{n_k}$. By Corollary 3.3.5, we have some surjective sheaf morphism $\psi^k : \mathfrak{O}^{r_k}_{X_k} \to \mathfrak{F}$ for $k \geq 1$. By Theorem 3.3.3, $\mathrm{H}^1(X_k, \ker \psi^{k+s}) = 0$ for $k, s \geq 1$. Thus, the induced map $\psi_{k,s}$ on global sections is surjective. By Lemma 3.2.4, $\Gamma(X_k, \mathfrak{F})$ has a canonical Frechet space structure. The induced map on global sections along with the restriction maps give us the following commutative diagram

$$\begin{tikzcd}
\Gamma(X_{k+s}, \mathfrak{O}^{r_{k+s+1}}) \arrow[r] \arrow[d, "\psi_{k+s,1}"] & \Gamma(X_k, \mathfrak{O}^{r_{k+s+1}}) \arrow[d, "\psi_{k,s+1}"] \\
\Gamma(X_{k+a}, \mathfrak{F}) \arrow[r] & \Gamma(X_k, \mathfrak{F}).
\end{tikzcd}$$

Using the induced maps on global sections and the previous corollary, we have $\Gamma(X_{k+s}, \mathfrak{F})$ is dense in $\Gamma(X_k, \mathfrak{F})$ for all $k, s \geq 1$.

By Theorem 3.3.3, $\mathrm{H}^p(X_k\, mff) = 0$ for $p, k \geq 1$. Let $\mathfrak{U}^k = \{X_m\}_{m=1}^k$ for $k \geq 1$ and $\mathfrak{U} = \{X_m\}_{m=1}^\infty$. Then $\mathrm{H}^p(N(\mathfrak{U}^k), \mathfrak{F}) = 0$ for $p, k \geq 1$ and $\mathrm{H}^p(N(\mathfrak{U}), \mathfrak{F}) \cong \mathrm{H}^p(X, \mathfrak{F})$. Fix $q \geq 1$ and $\sigma \in Z^q(N(\mathfrak{U}), \mathfrak{F})$. Let $\sigma_k$ denote the restriction of $\sigma$ to $N(\mathfrak{U}^k)$, then $\sigma_k = \delta\alpha_k$ for some $\alpha_k \in C^{q-1}(N(\mathfrak{U})^k, \mathfrak{F}$ and $\alpha_k - \alpha_{k-1} \in Z^{q-1}(N(\mathfrak{U}^{k-1}), \mathfrak{F})$. Theorem $B$ then follows from a case-by-case verification($q = 1$ and $q > 1$) that there exists some $\tau \in C^{q-1}(N(\mathfrak{U}), \mathfrak{F})$ such that $\delta\tau = \sigma$. The details can be found at the end of Siu's paper [7]. ■

## 4. A proof using $L^2$ estimates

In this section, we present a different proof of Cartan's theorems using methods from $PDE$. We will closely follow the work of Hormander [2] For completeness, some of the important definitions and results will be restated.

**Definition 4.0.1.** Let $\Omega$ be an open set in $\mathbb{C}^n$, let $F_1, \dots, F_q \in A(\Omega)^p$($p$-tuple), and let

$$R_z(F_1, \dots, F_q) = \{G = (g^1, \dots, g^q) \in A_z^q : \sum_{j=1}^q g^j \gamma_z(F_j) = 0\}, z \in \Omega$$

, which is a submodule of $A_z^q$ called the **module of relations** between $F_1, \dots, F_q$ at $z$, and $\gamma_z(F_j)$ is the germ of $F_j$ at $z$. Let $\mathscr{R}(F_1, \dots, F_q)$ denote the **sheaf of relations**.

**Definition 4.0.2.** Let $\mathscr{F}$ be a sheaf of abelian groups. We say that $\mathscr{F}$ is a **sheaf of $\mathscr{O}$-module** if $\mathscr{F}_x$ is a $\mathscr{O}_x$-module for all $x \in X$ and product of a section in $\mathscr{O}$ and a section in $\mathscr{F}$ gives rise to a section in $\mathscr{F}$. We say that $\mathscr{F}$ is an **analytic sheaf** if $\mathscr{O}$ is the sheaf of germs of analytic functions.

**Theorem 4.0.3.** *Every locally finitely generated subsheaf of $\mathscr{A}^p$ is coherent, where $\mathscr{A}^p$ is the direct sum of $n$ copies of $\mathscr{A}$, the sheaf of germs of analytic functions on $\Omega$.*

*Proof.* This is a restatement of Oka's Theorem 6.4.1. [2]. ■

**Theorem 4.0.4.** *Let $\Omega$ be a Stein manifold, $K \subset \Omega$ compact and $A(\Omega)$-convex, $\mathscr{F}$ coherent analytic sheaf on a neighborhood of $K$. Then*

1. *We can find finitely many sections $f_1, \dots, f_q$ of $\mathscr{F}$ generating $\mathscr{F}$ over a neighborhood of $K$.*
2. *For any section $f$ over a neighborhood of $K$ there are $c_1, \cdots, c_q$ analytic in a neighborhood of $K$ such that $f = \sum_1^q c_j f_j$ in that neighborhood.*

**Definition 4.0.5.** We define, using the previous theorem, seminorms on sections of $\mathscr{F}$ over a neighborhood of a compact $A(\Omega)$-convex subset. Let $f$ be a section of $\mathscr{F}$ over a neighborhood of $K$. Write

$$f = \sum_1^q c_j f_j$$

and set

$$\|f\|_K = \inf_c \sup_{z\in K} \sum_1^q |c_j(z)|,$$

where the infimum is taken over all $c$ such that the first equation holds.

*Remark* 4.0.6. For a different choice of generators we are given an equivalent seminorm.

**Lemma 4.0.7.** *If $f$ is a section of $\mathscr{F}$ over a neighborhood of $K$ and if $\|f\|_K = 0$, then $f_z = 0$ for every $z \in \operatorname{int}(K)$.*

*Proof.* See Lemma 7.2.3. of [2]. ∎

**Lemma 4.0.8.** *Let $K, K' \subset \Omega$ be compact and $A(\Omega)$-convex, $K \subset \operatorname{int}(K')$. Let $g_1, g_2, \ldots$ be a sequence of sections of $\mathscr{F}$ over neighborhood of $K'$ with*

$$\sum_1^\infty \|g_k\|_K' < \infty,$$

*then there is $g$ section of $\mathscr{F}$ over a neighborhood of $K$ with*

$$\left\| g - \sum_i^j g_k \right\|_K \to 0 \text{ as } j \to \infty.$$

*Proof.* See Lemma 7.2.4. of [2]. ∎

**Theorem 4.0.9.** *Let $K_1, K_2, \ldots \subset \Omega$ be compact and $A(\Omega)$-convex with $\cup_p K_p = \Omega, K_1 \subset \operatorname{int}(K_2) \subset \operatorname{int}(K_3) \subset \ldots$. Let $g_p$ be a section of the coherent analytic sheaf $\mathscr{F}$ in $\Omega$ over a nbh of $K_p$ and assume that for each $p$ we have that*

$$\|g_i - g_j\|_{K_p} \to 0 \text{ when } i, j \to \infty.$$

*Proof.* See Theorem 7.2.5. of [2]. ∎

**Theorem 4.0.10.** *Let $K \subset \Omega$ be compact and $A(\Omega)$-convex. Let $f$ be a section of the coherent analytic sheaf $\mathscr{F}$ over a neighborhood of $K$. Then there is a sequence of sections $f_j \in \Gamma(\Omega, \mathscr{F})$ such that*

$$\|f - f_j\|_K \to 0 \text{ when } j \to \infty.$$

*Proof.* See Theorem 7.2.7 of [2]. ∎

*Proof of Cartan theorem A.* Choose $f_1, ..., f_N$ sections of $\mathscr{F}$ over neighborhood of $z \in \Omega$ generating the stalk $\mathscr{F}_z$. $\{z\}$ is $A(\Omega)$-convex, so Theorem 4.0.10 implies that for every $\epsilon$ we can choose $g_1, \ldots, g_N \in \Gamma(\Omega, \mathscr{F})$ such that

$$g_j - f_j = \sum_{k=1}^N c_{jk} f_k \text{ for } j = 1, \ldots, N$$

or

$$\begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_N \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1N} \\ c_{21} & & & \\ \vdots & & & \\ c_{N1} & & & \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_N \end{bmatrix} + \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_N \end{bmatrix}$$

for some $c_{jk}$ analytic in a neighborhood of $z$ with $|c_{jk}| < \epsilon$. For $\epsilon$ small we see that for Id the $N \times N$ identity matrix, $\mathrm{Id} + (c_{ij})$ is invertible with inverse $(b_{jk})$ analytic in a neighborhood of $z$. We have that

$$f_j = \sum_{k=1}^{N} b_{jk} g_k,$$

so $g_1, \dots, g_N$ are the generators for $\mathscr{F}_z$. ■

**Theorem 4.0.11.** *Let $\Omega$ be a Stein manifold. Let $\mathscr{F}$ be a coherent analytic sheaf on $\Omega$. If $f, f_1, \dots, f_q \in \Gamma(\Omega, \mathscr{F})$ and $f_z$ is in the $A_z$-module generated by $(f_1)_z, \dots, (f_q)_z$ for every $z \in \Omega$, then there are $c_1, \dots, c_q \in A(\Omega)$ such that*

$$f = \sum_{j=1}^{q} c_j f_j$$

**Definition 4.0.12.** Let $X$ be a topological space, $\mathscr{F}$ a sheaf of abelian groups on $X$. Let $\mathscr{U} = \{U_i\}_{i \in I}$ be a covering of $X$, and let $\delta$ be as defined in Definition 2.4.4. Then, we write

$$Z^p(\mathscr{U}, \mathscr{F}) := \{c : c \in C^p(\mathscr{U}, \mathscr{F}), \delta c = 0\}$$

and

$$B^p(\mathscr{U}, \mathscr{F}) := \{\delta c : c \in C^{p-1}(\mathscr{U}, \mathscr{F}).$$

Then, we define

$$H^p(\mathscr{U}, \mathscr{F}) := Z^p(\mathscr{U}, \mathscr{F}) / B^p(\mathscr{U}, \mathscr{F}).$$

*Remark* 4.0.13. One can see that

$$H^0(\mathscr{U}, \mathscr{F}) \approx \Gamma(X, \mathscr{F}).$$

**Definition 4.0.14.** Let $\mathscr{F}, \mathscr{G}, \mathscr{H}$ be sheaves of abelian groups over $X$. Let $\varphi, \psi$ be sheaf homomorphisms such that

$$0 \longrightarrow \mathscr{F} \xrightarrow{\varphi} \mathscr{G} \xrightarrow{\psi} \mathscr{H} \longrightarrow 0$$

is exact. Note that this implies that $\varphi$ is injective and $\psi$ is surjective. Now, we see that this induces the exact sequence

$$0 \longrightarrow C^p(\mathscr{U}, \mathscr{F}) \longrightarrow C^p(\mathscr{U}, \mathscr{G}) \longrightarrow C^p(\mathscr{U}, \mathscr{H})$$

with the last map not necessarily surjective. If we replace $C^p(\mathscr{U}, \mathscr{H})$ by the image $C^p_a(\mathscr{U}, \mathscr{H})$, called the ***liftable cochains***, we get the exact sequence

$$0 \longrightarrow C^p(\mathscr{U}, \mathscr{F}) \longrightarrow C^p(\mathscr{U}, \mathscr{G}) \longrightarrow C^p_a(\mathscr{U}, \mathscr{H}) \longrightarrow 0$$

Define $H^p_a(\mathscr{U}, \mathscr{H}) = Z^p_a / B^p_a$ for $Z^p_a$ the liftable $p$-cocycles and $B^p_a$ the boundary of liftable $(p-1)$-cochains.

**Lemma 4.0.15.** *There is a homomorphism $\delta^\star : H^p_a(\mathscr{U}, \mathscr{H}) \to H^{p+1}(\mathscr{U}, \mathscr{F})$ called the **connecting homomorphism**.*

*Proof.* See p179-180 of [2]. ■

**Theorem 4.0.16.** *The sequence*

$$0 \to H^0(X, \mathscr{F}) \to H^0(X, \mathscr{F}) \to H^0(X, \mathscr{G}) \to H^0_a(X, \mathscr{H}) \to H^1(X, \mathscr{F}) \to \dots$$

*is exact.*

*Proof.* See Theorem 7.3.4. of [2]. ■

**Lemma 4.0.17.** *If $X$ is paracompact, then $H^p_a(\mathscr{U},\mathscr{H})$ is isomorphic to $H^p(X,\mathscr{H})$*

**Theorem 4.0.18.** *If $X$ is paracompact and $0\to\mathscr{F}\to\mathscr{G}\to\mathscr{H}\to 0$ is an exact sequence of sheafs of abelian groups on $X$, then the sequence*

$$0\to H^0(X,\mathscr{F})\to H^0(X,\mathscr{F})\to H^0(X,\mathscr{G})\to H^0(X,\mathscr{H})\to H^1(X,\mathscr{F})\to\dots$$

*is exact.*

*Proof.* Follows from Lemma 4.0.17. ■

**Theorem 4.0.19.** *Let $\Omega$ be a Stein manifold, $K\subset\Omega$ relatively compact, and let $\Omega$ be an open neighborhood of $\hat{K}$. Then there is $\varphi\in C^\infty(\Omega)$ such that*

(1) *$\varphi$ is strictly plurisubharmonic.*
(2) *$\varphi<0$ in $K$ but $\varphi>0$ in $\omega^c$.*
(3) *$\{z:z\in\Omega,\varphi(z)<c\}$ is relatively compact in $\Omega$ for every $c\in\mathbb{R}$.*

**Definition 4.0.20.** Let $\Omega\subset\mathbb{C}^n$ be open. If $\varphi$ is a continuous function on $\Omega$, then we have the space of functions in $\Omega$ that are square integrable with respect to the measure $e^{-\varphi}d\lambda$, where $d\lambda$ is the Lebesgue measure. We denote the space above by $L^2(\varphi,\Omega)$. Also, we denote $L^2(\varphi,\mathrm{loc})$ to be functions that are locally square integrable. Let $L^2_{(p,q)}(\Omega,\varphi)$ denote the space of forms of type $(p,q)$ with coefficients in $L^2(\Omega,\varphi)$. Then, $f$ can be written as

$$f=\sum_{|I|=p}{}'\sum_{|J|=q}{}' f_{I,J}dz^I\wedge d\overline{z}^J.$$

Now, set

$$|f|^2=\sum_{I,J}{}'|f_{I,J}|^2,$$

$$\|f\|^2_\varphi=\int|f|^2e^{-\varphi}d\lambda.$$

This makes $L^2(\Omega,\varphi)$ into a Hilbert space. Define $D_{(p,q)}(\Omega)$ to be the set of $C^\infty$) forms of type $(p,q)$ with compact support. For continuous functions $\varphi_1,\varphi_2$ on $\Omega$, we see that $\overline{\partial}$ defines a linear, closed, densely defined operator

$$T:L^2_{(p,q)}(\Omega,\varphi_1)\to L^2_{(p,q+1)}(\Omega,\varphi_2).$$

We say that an element $u\in L^2_{(p,q)}(\Omega,\varphi_1)$ is in $D_T$ if $\overline{\partial}$ is in $L^2_{(p,q+1)}(\Omega,\varphi_2)$ and set $Tu=\overline{\partial}u$.

**Theorem 4.0.21.** *Let $T,S$ be linear, closed, densely defined operator define by the $\overline{\partial}$ such that*

$$T:L^2_{(p,q)}(\Omega,\varphi)\to L^2_{(p,q+1)}(\Omega,\varphi),$$

$$S:L^2_{(p,q+1)}(\Omega,\varphi)\to L^2_{(p,q+2)}(\Omega,\varphi).$$

*Then there exists a function a continuous function $C$ on $\Omega$ such that*

$$\int(\lambda-C)|f|^2e^{-\varphi}dV\le 4(\|T^\star f\|^2_\varphi+\|Sf\|^2_\varphi)$$

*for $f\in D_{(p,q+1)}(\Omega)$, $\varphi\in C^2(\Omega)$ arbitrary, and $\lambda$ lowest eigenvalue of the hermitian symmetric form $\sum\varphi_{jk}t_j\overline{t}_k$.*

For the detailed discussion refer to p115-118 [2].

**Theorem 4.0.22.** *Let $\Omega$ be a complex manifold where there exists a strictly plurisubharmonic function $\varphi$ such that $\{z:z\in\Omega,\varphi(z)<c\}$ is relatively compact in $\Omega$ for every $c\in\mathbb{R}$. Then the equation $\overline{\partial}u=f$ has a solution $u\in L^2_{(p,q)}(\Omega,\mathrm{loc})$ for every $f\in L^2_{(p,q+1)}(\Omega,\mathrm{loc})$ such that $\overline{\partial}f=0$.*

*Proof.* Let $\chi$ be a convex increasing function. We start by replacing $\varphi$ in the previous theorem with with $\chi(\varphi)$. We replace $\lambda$ by $\chi'(\varphi)\lambda$. By previous theorem, we see that

$$\int (\chi'(\varphi)\lambda - C)|f|^2 e^{-\chi(\varphi)} dV \leq 4(\|T^\star f\|^2_{\chi(\varphi)} + \|Sf\|^2_{\chi(\varphi)}), f \in D_{(p,q+1)}(\Omega).$$

By choosing $\chi$ so that $\chi'(\varphi)\lambda - C \geq 4$, we get that

$$\|f\|^2_{\chi(\varphi)} \leq \|T^\star f\|^2_{\chi(\varphi)} + \|Sf\|^2_{\chi(\varphi)}, f \in D_{T^\star} \cap D_S.$$

Applying Lemma 5.2.1. [2], we get that $\overline{\partial} u = f$ has a solution $u \in L^2_{(p,q)}(\Omega, \chi(\varphi))$ for all $f$ satisfying the theorem statement and $u$ can be chosen to satisfy

$$\|u\|_{\chi(\varphi)} \leq \|f\|_{\chi(\varphi)}.$$

■

**Theorem 4.0.23.** *Let $\Omega$ be a complex manifold admitting a strictly plurisubharmonic function $\varphi \in C^\infty(\Omega)$ with $\{z : z \in \Omega, \varphi(z) < c\}$ relatively compact for all $c$. Then $\overline{\partial} u = f$ has a solution $u \in C^\infty_{(p,q)}(\Omega)$ for every $f \in C^\infty_{(p,q+1)}(\Omega)$ with $\overline{\partial} f = 0$.*

**Theorem 4.0.24.** *Let $D$ be an open polydisc and let $f \in C^\infty_{(p,q+1)}(D)$ satisfying $\overline{\partial} f = 0$. If $D' \subset D$ is relatively compact, we can find $u \in C^\infty$ such that $\overline{\partial} u = f$ in $D'$.*

**Theorem 4.0.25.** *Let $\Omega$ be a complex manifold which is $\sigma$-compact and let $\mathscr{U} = \{U_i\}_{i \in I}$ be a covering with $U_i$ Stein manifold for all $i \in I$. Then $H^p(\Omega, \mathscr{A})$ is isomorphic to the*

$$\{f : f \in C^\infty_{(0,p)}(\Omega), \overline{\partial} f = 0\}/\{\overline{\partial} g : g \in C^\infty_{(0,p-1)}(\Omega)\}$$

*for $p > 0$ and $H^p(\Omega, \mathscr{A})$ is isomorphic to $H^p(\mathscr{U}, \mathscr{A})$.*

*Proof.* Let $\mathscr{E}_q$ denote the sheaf of germs of $C^\infty$ forms of type $(0, q)$. By Theorem 4.0.24 we have the exact sequence of sheaf homomorphisms

$$0 \longrightarrow \mathscr{I}_q \xrightarrow{i} \mathscr{E}_q \xrightarrow{\overline{\partial}} \mathscr{I}_{q+1} \longrightarrow 0.$$

for $\mathscr{I}_q$ the sheaf of germs of $\overline{\partial}$ closed forms of type $(0, q)$. From Theorem 4.0.23 we thus obtain

$$0 \longrightarrow C^p(\mathscr{U}, \mathscr{I}_q) \longrightarrow C^p(\mathscr{U}, \mathscr{E}_q) \longrightarrow C^p(\mathscr{U}, \mathscr{I}_{q+1}) \longrightarrow 0,$$

an exact sequence. Then Theorem 4.0.16. gives us

$$0 \longrightarrow \Gamma(\Omega, \mathscr{I}_q) \longrightarrow \Gamma(\Omega, \mathscr{E}_q) \xrightarrow{\overline{\partial}} \Gamma(\Omega, \mathscr{E}_{q+1}) \longrightarrow H^1(\mathscr{U}, \mathscr{I}_q) \longrightarrow \dots$$

By Proposition 7.3.3. [2], we get that $H^p(\Omega, \mathscr{E}_q) = 0$ for all $p > 0$, and

$$H^p(\mathscr{U}, \mathscr{I}_{q+1}) \cong H^{p+1}(\mathscr{U}, \mathscr{I}_q),$$

for $p > 0$ by the exactness of the sequence. We also have that

$$\Gamma(\Omega, \mathscr{I}_q)/\overline{\partial}\Gamma(\Omega, \mathscr{E}_{p-1})$$

by the exactness of the sequence. Then, we see that

$$H^p(\mathscr{U}, \mathscr{A}) = H^p(\mathscr{U}, \mathscr{I}_0) \cong H^{p-1}(\mathscr{U}, \mathscr{I}_1) \cong H^1(\mathscr{U} \mathscr{I}_{p-1}) \cong \Gamma(\Omega, \mathscr{I}_q)/\overline{\partial}\Gamma(\Omega, \mathscr{E}_{p-1})$$

As coverings can be arbitrarily fine, we are done. ■

**Corollary 4.0.26.** *If $\Omega$ is a Stein manifold, then $H^p(\Omega, \mathscr{A}) = 0$ for every $p > 0$. More precisely, We have that $H^p(\mathscr{U}, \mathscr{A}) = 0$ for every covering $\mathscr{U} = \{U_i\}_{i\in I}$ with each $U_i$ Stein manifold.*

*Proof of Cartan theorem B.* Let $\mathscr{U} = \{U_i\}_{i\in I}$ be a covering with $U_i$ a Stein manifold relatively compact in $\Omega$ for each $i \in I$. Let $\Omega'$ be a stein manifold that is relatively compact in $\Omega$ and let $U_i' = U_i \cap \Omega'$. We want to show that $H^p(\mathscr{U}', \mathscr{F}) = 0$ for all $p > 0$ and all coherent analytic sheaf $\mathscr{F}$ over $\Omega$. By Cartan's theorem A, we can choose $f_1, \dots, f_q \in \Gamma(\Omega, \mathscr{F})$ generating $\mathscr{F}$ over $\Omega'$. Then, we have the exact sequence

$$0 \longrightarrow \mathscr{R}(f_1, \dots, f_q) \longrightarrow \mathscr{A}^q \longrightarrow \mathscr{F} \longrightarrow 0.$$

We see that the intersection of any $U_i'$ is against a Stein manifold. Thus, we can use 4.0.11 to get another exact sequence

$$0 \longrightarrow C^p(\mathscr{U}', \mathscr{R}(f_1, \dots, f_q)) \longrightarrow C^p(\mathscr{U}', \mathscr{A}^p) \longrightarrow C^p(\mathscr{U}', \mathscr{F}) \longrightarrow 0,$$

as all-cochains of $C^p(\mathscr{U}', \mathscr{F})$ are liftable. This gives us the exact sequence

$$H^p(\mathscr{U}', \mathscr{A}^q) \longrightarrow H^p(\mathscr{U}', \mathscr{F}) \longrightarrow H^{p+1}(\mathscr{U}', \mathscr{R}) \longrightarrow H^{p+1}(\mathscr{U}', \mathscr{A}^q).$$

By Corollary 4.0.26, we see that $H^p(\mathscr{U}', \mathscr{A}^q) = 0 = H^{p+1}(\mathscr{U}', \mathscr{A}^q)$, so $H^p(\mathscr{U}', \mathscr{F}) \cong H^{p+1}(\mathscr{U}', \mathscr{R})$. Let the covering be chosen so that fore than $N$ sets of $U_i$ have empty intersection, then $H^p(\mathscr{U}', \mathscr{F}) = 0$ for $p > N$ for all $\mathscr{F}$. Thus, we see that $H^p(\mathscr{U}', \mathscr{F}) = 0$ for all $p > 0$. Now, we wish to show the same for $\mathscr{U}$.

(a) Suppose that $p > 1$. Let $\Omega_1, \Omega_2, \dots$ be an increasing sequence Stein manifolds relatively compact in $\Omega$ and their union equals $\Omega$. Let $\mathscr{U}^j = \{\Omega^j \cap U_i\}_{i\in I}$, let $c$ be a $p$-cocycle in $C^p(\mathscr{U}, \mathscr{F})$, and let $c^j$ be the restriction to $C^p(\mathscr{U}^j, \mathscr{F})$. Then, we have a $(p-1)$-cocycle $b^j \in C^{p-1}(\mathscr{U}^j, \mathscr{F})$ with $\delta b^j = c^j$. Then $b^{j+1} - b^j = \delta a$ for some $a \in C^{p-2}(\mathscr{U}^j, \mathscr{F})$, where we consider $b^{j+1}$ to be the restriction. Let $a' \in C^{p-2}(\mathscr{U}, \mathscr{F})$ be such that $a_s' = a_s$ for $s$ such that $U_s \subset \Omega^j$ and $a_s' = 0$ otherwise. We can then subtract the restriction of $\delta a'$ to $\Omega^{j+1}$ to find that $(b^{j+1})_s = (b^j)_s$ if $U_{s_k} \subset \Omega^j$ for every $k$. $U_i$ relatively compact in $\Omega$ implies that $(b^j)_s$ is independent of $j$ for $j$ large for every $s$. Thus, $b^j$ converges to a cochain $b \in C^{p-1}(\mathscr{U}, \mathscr{F}), \delta b = c$.

(b) Suppose that $p = 1$ with $b^j$ chosen as above. Restricting $b^{j+1}$ to $\Omega^j$, we see that the restriction of $b^{j+1}$ differs from $b^j$ by a 0-cocycle, or a section of $\mathscr{F}$ over $\Omega^j$. Let $K^j$ be a compact $A(\Omega)$-convex subset of $\Omega^j$ with $K^j \subset \operatorname{int}(K^{j+1})$ and $\cup K_j = \Omega$. Theorem 4.0.10 shows that there is a section $a \in \Gamma(\Omega, \mathscr{F})$ with

$$\left\|b^{j+1} - b^j - a\right\|_{K^v} < 2^{-j}, v \leq j$$

Relabeling $b^{j+1} - a$ by $b^{j+1}$, we get that

$$\|b^{j+1} - b^j\|_{K_v} < 2^{-j}, v \leq j.$$

By Lemma 4.0.8, we have that that $b^k - b^j$ converges to a section $s^j$ of $\mathscr{F}$ over $\operatorname{int}(K^j)$ as $k \to \infty$ for each $j$. We see that

$$b^j + s^j = b^{j-1} + s^{j-1}$$

on $\operatorname{int}(K^{j-1})$, so there is $b \in C^0(\mathscr{U}, \mathscr{F})$ such that $b = b^j + s^j$ on $\operatorname{int}(K^j)$ for every $j$. Then, by $\delta b^j = c$ in $\Omega^j$ we have that $\delta b = c$, proving that $H^p(\mathscr{U}, \mathscr{F}) = 0$ for every $p > 0$. Since there is an arbitrarily fine covering for $\Omega$ with these properties, we are done. ∎

## 5. Consequences of Cartan's Theorems and Further Discussions

We first assert the converse of Theorem B, which characterize Stein spaces using cohomology. Note that algebraic geometry will be assumed in this section.

**Theorem 5.0.1.** *Let $(X, \mathfrak{O})$ be an analytic space of pure dimension. Suppose $\mathrm{H}^1(X, \mathfrak{F}) = 0$ for all coherent $\mathfrak{O}_X$-subsheaves of $\mathfrak{O}_X$, then $X$ is Stein.*

*Proof.* See Theorem VIII.B.20 of [1]. ■

We now state 5 applications of Cartan's theorems, all of which are very important in the study of several complex variables.

**Notation.** Let $\mathfrak{M}_X$ denote the sheaf of germs of meromorphic functions on $X$, defined in the same way as the sheaf of germs of holomorphic functions.

Suppose $(X, \mathfrak{O})$ is a complex manifold, $\mathfrak{U} = \{U_i\}$ an open covering, and $m_i \in \Gamma(U_i, \mathfrak{M})$.

**Question 5.0.2** (Cousin's First Problem)**.** *Suppose $m_i - m_j \in \Gamma(U_i \cap U_j, \mathfrak{O})$ for all $i, j$, is there an $m \in \Gamma(X, \mathfrak{M})$ such that $m - m_i \in \Gamma(U_i \mathfrak{O})$.*

**Theorem 5.0.3.** *The Cousin's First problem has a solution if $X$ is Stein.*

*Proof.* Consider $0 \to \mathfrak{O} \to \mathfrak{M} \to \mathfrak{M}/\mathfrak{O} \to 0$, take cohomology, and apply Cartan's Theorem B. See Theorem VIII.B.5 of [1] for details. ■

**Question 5.0.4** (Cousin's Second Problem)**.** *Suppose no $m_i$ is identically $0$, suppose $m_i/m_j \in \Gamma(U_i \cap U_j, \mathfrak{O})$ for all $i, j$. Does there exist $m \in \Gamma(X, \mathfrak{O})$ such that $m/m_i$ and $m_i/m$ are in $\Gamma(U_i, \mathfrak{O})$ for all $i$.*

The difficulty of Cousin's Second problem is that we have to work with $\mathfrak{O}^* = \mathfrak{O} \setminus \{0\}$, whose first cohomology group is not necessarily 0 as it is not coherent. However, we can add the following restriction.

**Theorem 5.0.5.** *Cousin's Second problem is solvable if $X$ is Stein and $\mathrm{H}^2(X, \mathbb{Z}) = 0$.*

*Proof.* Theorem VIII.B.6,8, and 13 of [1]. ■

A direct consequence of Cartan's Theorem $B$ is the following:

**Theorem 5.0.6.** *If $X$ is a Stein manifold, then $\mathrm{H}^{p,q}(X) = 0$ for $p \geq 0$, $q \geq 1$.*

*Proof.* Direct application of Dolbeault's theorem. ■

The following is the analog of analytic continuation on Stein manifold:

**Theorem 5.0.7** (Cartan extension theorem)**.** *Holomorphic functions defined on a closed subvariety of a Stein manifold $X$ can be extended to a holomorphic function on $X$,*

The last result we will state is the following:

**Theorem 5.0.8** (Cartan's division theorem)**.** *If $\mathfrak{F}$ is a coherent analytic sheaf on a Stein space $X$ and $f_1, \ldots.., f_k \in \mathfrak{F}(X)$ generates the stalks $\mathfrak{F}_x$ at every $x \in X$, then for every $f \in \mathfrak{F}(X)$ there exists $g_i \in \mathfrak{O}_X$ such that $f = \sum_{i=1}^{k} g_i f_i$*

*Proof.* The proof of both of these results can be found in Section 2.6 of [13]. ■

From Cartan's Theorems B and Theorem 5.0.1, those familiar with algebraic geometry should immediately see the correlation between Stein spaces and affine varieties as both behave the same way cohomologically. One can refer to Section 3.3 of [3] for more information. More importantly, there is a process(functor), denoted by $F$, that implements analytic structures on an affine variety over $\mathbb{C}$ so that it becomes a Stein space. More importantly, such functor preserves all morphisms between varieties. That is, if $X, Y$ are affine varieties over $\mathbb{C}$ and $X', Y'$ are their respective images under $F$, then $\mathrm{Hom}(X, Y) \cong \mathrm{Hom}(X', Y')$. This is the beginning of a series of results relating complex analytic and algebraic geometry, and results of this type are called a GAGA type of result, named after Serre's classic paper Geometrie algebrique et geometrie analytique [5], which is also the best reference on the subject. Many important theorems, such as Chow's theorem and Kodaira Vanishing theorem can be deduced from the results in Serre's paper.

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