University of Colorado
Department of Computer Science

Numerical Computation CSCI 3656

Spring 2016

Problem Set 10 Solutions

Note: these are solutions to the original version of this PS. We're in the process of updating them to match the actual version.

1. *[10 pts]* Problem 2 on page 224

$$y_1 = A_1 = \begin{bmatrix} -4 \\ -2 \\ 4 \end{bmatrix}$$

$$r_{1,1} = ||y_1||_2 = 6$$

$$q_1 = \frac{y_1}{||y_1||_2} = \begin{bmatrix} -2/3 \\ -1/3 \\ 2/3 \end{bmatrix}$$

$$r_{1,2} = q_1^T A_2 = -3$$

$$y_2 = A_2 - q_1 q_1^T A_2 = \begin{bmatrix} -4 \\ 7 \\ -5 \end{bmatrix} - \begin{bmatrix} -2/3 \\ -1/3 \\ 2/3 \end{bmatrix} (-3) = \begin{bmatrix} -6 \\ 6 \\ -3 \end{bmatrix}$$

$$r_{2,2} = ||y_2||_2 = 9$$

$$q_2 = \frac{y_2}{||y_2||_2} = \begin{bmatrix} -2/3 \\ 2/3 \\ -1/3 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$y_3 = A_3 - q_1 q_1^T A_3 - q_2 q_2^T A_3$$

$$y_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} -2/3 \\ -1/3 \\ 2/3 \end{bmatrix} (-1/3) - \begin{bmatrix} -2/3 \\ 2/3 \\ -1/3 \end{bmatrix} (-1/3) = \begin{bmatrix} 5/9 \\ 10/9 \\ 10/9 \end{bmatrix}$$

$$q_3 = \frac{y_3}{||y_3||_2} = \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}$$

$$Q = \begin{bmatrix} -2/3 & -2/3 & 1/3 \\ -1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & 2/3 \end{bmatrix} R = \begin{bmatrix} 6 & -3 \\ 0 & 9 \\ 0 & 0 \end{bmatrix}$$

2. *[10 pts]* Problem 4 on page 224

   We have the same $q_1, q_2, r_{1,1}, r_{1,2}$, and $r_{2,2}$ as the previous problem.

$$A_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$y_3' = A_3 - q_1 q_1^T A_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} -2/3 \\ -1/3 \\ 2/3 \end{bmatrix} (-1/3) = \begin{bmatrix} 7/9 \\ 8/9 \\ 11/9 \end{bmatrix}$$

$$y_3 = y_3' - q_2 q_2^T y_3' = \begin{bmatrix} 7/9 \\ 8/9 \\ 11/9 \end{bmatrix} - \begin{bmatrix} -2/3 \\ 2/3 \\ -1/3 \end{bmatrix} (-1/3) = \begin{bmatrix} 5/9 \\ 10/9 \\ 10/9 \end{bmatrix}$$

$$q_3 = \frac{y_3}{||y_3||_2} = \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}$$

$$Q = \begin{bmatrix} -2/3 & -2/3 & 1/3 \\ -1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & 2/3 \end{bmatrix} R = \begin{bmatrix} 6 & -3 \\ 0 & 9 \\ 0 & 0 \end{bmatrix}$$

3. *[10 pts]* Problem 6 on page 224

$$w_1 = \begin{bmatrix} ||A_1||_2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix}$$

$$v_1 = w_1 - A_1 = \begin{bmatrix} 10 \\ 2 \\ -4 \end{bmatrix}$$

$$H_1 = I - 2\frac{v_1 v_1^T}{v_1^T v_1} = \begin{bmatrix} -2/3 & -1/3 & 2/3 \\ -1/3 & 1/15 & -2/15 \\ -2/3 & -2/15 & 11/15 \end{bmatrix}$$

$$H_1 A = \begin{bmatrix} 6 & -3 \\ 0 & 7.2 \\ 0 & -5.4 \end{bmatrix}$$

$$x = \begin{bmatrix} 7.2 \\ -5.4 \end{bmatrix}$$

$$w_2 = \begin{bmatrix} ||x||_2 \\ 0 \end{bmatrix} = \begin{bmatrix} 9 \\ 0 \end{bmatrix}$$

$$v_2 = w_2 - x = \begin{bmatrix} 1.8 \\ 5.4 \end{bmatrix}$$

$$\hat{H}_2 = I - 2\frac{v_2 v_2^T}{v_2^T v_2} = \begin{bmatrix} 0.8 & -0.6 \\ -0.6 & -0.8 \end{bmatrix}$$

$$H_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.8 & -0.6 \\ 0 & -0.6 & -0.8 \end{bmatrix}$$

$$Q = H_2^{-1} H_1^{-1} = \begin{bmatrix} -2/3 & -2/3 & -1/3 \\ -1/3 & 2/3 & -2/3 \\ 2/3 & -1/3 & -2/3 \end{bmatrix}$$

$$R = H_2 H_1 A = \begin{bmatrix} 6 & -3 \\ 0 & 9 \\ 0 & 0 \end{bmatrix}$$

4. *[10 pts]* Use the two-point forward difference formula to approximate $f'(1)$, where $f(x) = e^x + 0.5$, for $h = 0.1$, $h = 0.01$, and $h = 0.001$. Compare these estimates to the true value. Was this what one should expect, given the progression of values of h? Explain.

   The true value should be $e = 2.71828183$. Using the formula: $f'(x) \approx \frac{f(x+h)-f(x)}{h}$, we have $f'(x) \approx 2.85884195$, 2.73191866, and 2.71964142 for $h = 0.1$, 0.01, and 0.001 (respectively). The limit as h approaches zero defines the derivative; as h decreases, the approximation is closer to the actual derivative.

5. *[10 pts]* Repeat problem 4 using the three-point centered difference formula. Please also compare the results of the forward and centered difference calculations at each h value—to each other and to the true value. Which one is better, for a given h? Will that *always* be true?

   Using the formula: $f'(x) \approx \frac{f(x+h)-f(x-h)}{2h}$, we have $f'(x) \approx 2.72281456$, 2.71832713, and 2.71828228 for $h = 0.1$, 0.01, and 0.001 (respectively). The approximations for the three-point centered differences are closer to the actual derivative than that of the two-point forward differences. The three-point centered difference formula is better, because the error is $O(h^2)$, as opposed to $O(h)$ for the first formula. This is always true for infinite precision numbers, however floating point errors arise if h is too small.