University of Colorado
Department of Computer Science

Numerical Computation CSCI 3656

Spring 2016

Problem Set 4 Solutions

1. Write a program that:
   - takes as input
     - a 3×3 matrix $\mathbf{A}$
     - a three-element vector $\vec{b}$
   - uses Gaussian elimination — *without pivoting* — to find the solution $\vec{x}$ to the matrix equation $\mathbf{A}\vec{x} = \vec{b}$
   - prints out the augmented matrix at every step

   For this problem, please write the code in python, C, C++, Lisp, Java, etc. That is, do this *from scratch*, without using any linear algebra libraries or calling upon any built-in linear algebra commands from packages like Maple, Matlab, IDL, Mathematica, etc. (You may use those programming environments if you wish, but do not use their built-in linear algebra stuff. One of the points of this problem is to get you thinking about what data structures, loops, etc., you should use for matrix problems.)

   For this assignment, we would like to see both a transcript of your interaction with the program **and a copy of your code.** And do remember that you need to let us know if you worked with other class members on that code.

   Here is some Matlab code that does this.

```
function x = Gauss(A, b)
[n, n] = size(A);
[n, k] = size(b);
x = zeros(n,k);
for i = 1:n-1
    m = -A(i+1:n,i)/A(i,i);
    A(i+1:n,:) = A(i+1:n,:) + m*A(i,:);
    b(i+1:n,:) = b(i+1:n,:) + m*b(i,:);
end;
x(n,:) = b(n,:)/A(n,n);
for i = n-1:-1:1
    x(i,:) = (b(i,:) - A(i,i+1:n)*x(i+1:n,:))/A(i,i);
end
```

(a) *[25 pts]* Use your program to solve the following set of equations:

$$
\begin{aligned}
-11x_1 + 3x_2 + 3x_3 &= 18 \\
-2x_1 - 5x_2 + x_3 &= -3 \\
x_1 - 10x_2 + 8x_3 &= 5
\end{aligned}
$$

Here is the transcript:

```
>> A=[-11,3,3;-2,-5,1;1,-10,8];
>> b=[18;-3;5];
>> Gauss(A,b)
ans =
    -0.6316
    1.3246
    2.3596
```

i.e., $x_1 = -0.6316, x_2 = 1.3246, x_3 = 2.3596$

(b) *[5 pts]* Use your program to solve the following set of equations:

$$
\begin{aligned}
0.16x_1 + 0.85x_2 + 0.34x_3 &= 0.27 \\
-0.25x_1 - 1.5x_2 + 0.50x_3 &= -0.21 \\
1.03x_1 + 5.46x_2 + 1.77x_3 &= 1.75
\end{aligned}
$$

Here is the transcript:

```
>> B=[.16,.85,.34;-.25,-1.5,.5;1.03,5.46,1.77];
>> c=[.27;-.21;1.75];
>> Gauss(B,c)
ans =
    8.0446
   -1.1989
    0.0056
```

i.e., $x_1 = 8.0446, x_2 = -1.1989, x_3 = 0.0056$

(c) *[5 pts]* Use your program to solve the following set of equations:

$$
\begin{aligned}
-2x_1 + x_2 - 4x_3 &= -5 \\
7x_1 - 3x_2 + x_3 &= -2 \\
3x_1 - x_2 - 7x_3 &= 1
\end{aligned}
$$

The matrix is singular – there is no unique solution.

2. *[15 pts]* Compute the condition number of the three matrices in problem 1 using the $||.||_\infty$ norm.

   The condition numbers are: 8.0417 for 1(a), and 4.0898e+03 for 1(b). For 1(c), the condition number should be infinity; however, in Matlab, the condition number calculated using the infinity norm turns out to be very large. Using single precision floating point numbers for the matrix gives 199928432, and double precision gives 1.8400e+17. The wildly different results are due to the limitations of computer representation of numbers — i.e. floating point error. Since A^{-1} is, by definition, very hard to compute accurately if $cond(A)$ is high, high condition numbers are likely to be inaccurate.

   What do those numbers tell you about those matrices? That the first set of equations should have a unique solution, that the second matrix is ill conditioned and thus that the solution will be sensitive to small changes in b, and that the third is singular, so the associated set of equations does not have a unique solution. Does that line up with what happened when you ran your Gaussian elimination code on them? Yes, it did. The code found a solution for the first set of equations and failed on the third set. Here's a demonstration of the second matrix's sensitivity:

```
>> c1=[.2699;-.21;1.75];
>> Gauss(B,c1)
ans =
    8.0846
    -1.2060
    0.0043
```

3. *[15 pts]* Solve the system in problem 1(b) *by hand* using Gaussian elimination *with partial pivoting*. (You're welcome to modify your code from problem 1 to do pivoting in order to check your answers, but it's worth knowing how to do these kinds of problems by hand — e.g., for the hour exams.) Do you get the same answer with and without pivoting? Why or why not?

   See the scans below. Note that these answers do not agree exactly with the solution obtained without pivoting. This is because the roundoff errors are reduced by ensuring that the pivot element has the largest value out of elements in subsequent rows of the same column. This leads to multipliers of magnitude less than one..

4. *[15 pts]* Write down the PA=LU factorization of the system in problem 1(b) and use those factors to solve this system:

$$\begin{aligned}
0.16x_1 + 0.85x_2 + 0.34x_3 &= 0.27 \\
-0.25x_1 - 1.5x_2 + 0.50x_3 &= -0.21 \\
1.03x_1 + 5.46x_2 + 1.77x_3 &= 1.75
\end{aligned}$$

Do this *by hand*, in the most efficient way possible. Please show your work, and be neat enough that we can follow what you're doing. To finish off this problem, please explain why LU factorization is a good idea—i.e., what is the advantage of doing it? In what situation, for example, is that a Good Thing?

See the scan below for the calculations.

$PA = LU$ factorization is advantageous for two primary reasons. First, it averts issues that arise from naive elimination because it uses pivoting to avoid zero pivot elements and swamping. Pivoting allows for zero pivot elements to be swapped out for non-zero elements by row exchanges. Furthermore, pivoting ensures that the multipliers in L be less than one. This avoids a situation in which a large multiplier makes the elements of the row that is being subtracted from become insignificant with respect to computer precision. The second advantage of this method is that $\vec{b}$ is not needed to perform the initial factorization. Since the permutation of $\vec{b}$ is stored in P, the factored matrix can be used to solve the system for any $\vec{b}$.

Problem Set 4
CSCI 3656

Problem #3

$$\begin{array}{c} \mathrm{I} \\ \mathrm{II} \\ \mathrm{III} \end{array} \left[\begin{array}{ccc|c} 0.16 & 0.85 & 0.34 & 0.27 \\ -0.25 & -1.50 & 0.50 & -0.21 \\ 1.03 & 5.46 & 1.77 & 1.75 \end{array}\right]$$

$\downarrow$ Swap I with III

$$\begin{array}{c} \mathrm{I} \\ \mathrm{II} \\ \mathrm{III} \end{array} \left[\begin{array}{ccc|c} 1.03 & 5.46 & 1.77 & 1.75 \\ -0.25 & -1.50 & 0.50 & -0.21 \\ 0.16 & 0.85 & 0.34 & 0.27 \end{array}\right]$$

$\downarrow$ $\mathrm{II} = \mathrm{II} - m_{21}\mathrm{I}$, $m_{21} = -(0.25/1.03)$
$\mathrm{III} = \mathrm{III} - m_{31}\mathrm{I}$, $m_{31} = (0.16/1.03)$

$$\begin{array}{c} \mathrm{I} \\ \mathrm{II} \\ \mathrm{III} \end{array} \left[\begin{array}{ccc|c} 1.03 & 5.46 & 1.77 & 1.75 \\ \boxed{m_{21}} & -0.1748 & 0.9296 & 0.2148 \\ \boxed{m_{31}} & 0.0018 & 0.0650 & -0.0018 \end{array}\right]$$

$\downarrow$ $\mathrm{III} = \mathrm{III} - m_{32}\mathrm{II}$, $m_{32} = -(0.0018/0.1748)$

$$\left[\begin{array}{ccc|c} 1.03 & 5.46 & 1.77 & 1.75 \\ \boxed{m_{21}} & -0.1748 & 0.9296 & 0.2148 \\ \boxed{m_{31}} & \boxed{m_{32}} & 0.0746 & 0.0004 \end{array}\right]$$

$0.0746x_3 = 0.0004$

$\underline{x_3 = 0.0054}$

$-0.1748x_2 + 0.9296x_3 = 0.2148$

$\underline{x_2 = -1.2003}$

$1.03x_1 + 5.46x_2 + 1.77x_3 = 1.75$

$\underline{x_1 = 8.0527}$

Problem #4

$$A = \begin{bmatrix} 0.16 & 0.85 & 0.34 \\ -0.25 & -1.50 & 0.50 \\ 1.03 & 5.46 & 1.77 \end{bmatrix}$$

$$P_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$U = \begin{bmatrix} 1.03 & 5.46 & 1.77 \\ \emptyset & -0.1748 & 0.9296 \\ \emptyset & \emptyset & 0.0746 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & \emptyset & \emptyset \\ m_{21} & 1 & \emptyset \\ m_{31} & m_{32} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -0.2427 & 1 & 0 \\ 0.1553 & -0.0103 & 1 \end{bmatrix}$$

$PA = LU$

(continued →)

Problem # 4 (cont'd)

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad A = \begin{bmatrix} 0.16 & 0.85 & 0.34 \\ -0.25 & -1.50 & 0.50 \\ 1.03 & 5.46 & 1.77 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -0.2427 & 1 & 0 \\ 0.1553 & -0.0103 & 1 \end{bmatrix} \quad U = \begin{bmatrix} 1.03 & 5.46 & 1.77 \\ 0 & -0.1748 & 0.9296 \\ 0 & 0 & 0.0746 \end{bmatrix}$$

$$\vec{b} = \begin{bmatrix} 0.27 \\ -0.21 \\ 1.75 \end{bmatrix}$$

$$L\vec{c} = P\vec{b}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -0.2427 & 1 & 0 \\ 0.1553 & -0.0103 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0.27 \\ -0.21 \\ 1.75 \end{bmatrix}$$

$$c_1 = 1.75$$
$$-0.2427 c_1 + c_2 = -0.21$$
$$c_2 = 0.2147$$
$$0.1553 c_1 - 0.0103 c_2 + c_3 = 0.27$$
$$c_3 = 0.0004$$

$$U\vec{x} = \vec{c}$$

$$\begin{bmatrix} 1.03 & 5.46 & 1.77 \\ 0 & -0.1748 & 0.9296 \\ 0 & 0 & 0.0746 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1.75 \\ 0.2147 \\ 0.0004 \end{bmatrix}$$

$$\underline{x_3 = 0.0054}$$
$$-0.1748 x_2 + 0.9296 x_3 = 0.2147$$
$$\underline{x_2 = -1.1995}$$
$$1.03 x_1 + 5.46 x_2 + 1.77 x_3 = 1.75$$
$$\underline{x_1 = 8.0483}$$