University of Colorado
Department of Computer Science

Numerical Computation CSCI 3656

Spring 2016

Problem Set 3 Solutions

1. *[5 pts]* Use your Newton's method code from PS2 to find a root of the function $f(x) = x^2 - x - 2$ starting from the initial guess $x = 1.5$. Note the number of iterations required to get $|f(x)| < 0.0001$.

   This polynomial has roots at $x = -1$ and $x = 2$. From this guess, Newton converges within three iterations to the latter:

```
>> [root,errors]=newton(1.5,0.0001);
>> root = 2.0000
>> errors = 0.3906    0.0144    0.0000
```

2. *[5 pts]* Use your Newton's method program to find a root of the function $f(x) = x^3 - 3x^2 + 4$ starting from the initial guess $x = 1.5$. Note the number of iterations required to get $|f(x)| < 0.0001$.

   This polynomial has a double root at $x = 2$ and a single root at $x = -1$. From this guess, Newton converges to the latter, but (a) more slowly and (b) less accurately. In particular, it takes seven iterations to get to within $f(x) < 0.0001$, and the x value that it finds is off in the third decimal place:

```
>> [root,errors]=newton(1.5,0.0001);
>> root = 1.9968
>> errors = 0.1372    0.0328    0.0080    0.0020    0.0005    0.0001    0.0000
```

3. *[5 pts]* Compare the convergence rates that you determined in the previous two problems for Newton's method on these two polynomials. Are these numbers consistent with what you know about the theoretical convergence rates of this method? Explain.

   Newton should converge quadratically to the single root in problem 1. The error $e_{i+1} \approx M e_i^2$ where $M = |\frac{f''(r)}{2f'(r)}|$ and $e_i = |x_i - r|$. We can check to see if the error decreases quadratically by comparing the error with the square of the previous error – the ratio of which should converge to a constant. For example, $0.0144/0.3609^2 \approx 0.0944$, and $0.00002/0.0144^2 \approx 0.0965$ (note: there are only three iterations to test by, and the error at the third iteration is less than 0.0001, so the number 0.00002 was chosen as an example). The function indeed appears to converge quadratically to the root.

However, Newton should only converge linearly to the double root in problem 2. This is due to the fact that M cannot be used to compare the error at each step, since$f'(r) = 3(r)^2 - 6(r) = 0$, for $r = 2$. If we compare the ratios of successive errors with current error, we see that $\frac{e_{i+1}}{ei} \approx 0.24$ – a constant, which implies linear convergence. Also, the $f(x) < 0.0001$ condition doesn't pin you down close to the root because the function is so flat there, which is why the answer is off. See pp. 53-56 of text.

4. *[5 pts]* If you were to apply the bisection method to a function with a multiple root at $x = 2$, would the errors ($|f(x)|$) converge more slowly than if that function had a *single* root at $x = 2$? Explain.

   The convergence rate of bisection is only a function of the spacing between the initial guesses and the number of iterations. It isn't worse in the case of multiple roots, as many other rootfinders are.

5. *[3 pts]* What would it mean for a rootfinder to be *cubically convergent*?

   The error at the $(n+1)^{th}$ step is proportional to the cube of the error at the n^{th} step.

6. *[3 pts]* What is the next-term rule?

   In general, at any given iteration step, the error can be estimated by the improvement of the next iteration step. In a Taylor series, for example, the sum of the first n terms has error that can be approximated by the size of the $(n+1)^{th}$ term.

7. Dusting off your linear algebra

   - *[3 pts]* Give an example of a function $f(x)$ that is linear.
     Anything of the form $f(x) = ax + b$ where a and b are numbers.
   - *[3 pts]* Give an example of a function $f(x)$ that is nonlinear.
     Anything that has products, powers, or transcendental functions of the variable x.
   - *[3 pts]* If the $n \times n$ matrix A is nonsingular, how many possible solutions are there to the set of n linear equations $A\vec{x} = \vec{b}$?
     One.
   - *[3 pts]* If A^{-1} exists, what is true of $|A|$?
     It is nonzero.

8. *[10 pts]* Solve this system **by hand** using Gaussian elimination. Show your work:

$$\begin{aligned} 3x_1 + x_2 - 2x_3 &= -17 \\ -6x_1 + 2x_2 + 2x_3 &= 4 \\ -x_1 + 3x_2 + 2x_3 &= 1 \end{aligned}$$

$$\left|\begin{array}{ccc|r} 3 & 1 & -2 & -17 \\ -6 & 2 & 2 & 4 \\ -1 & 3 & 2 & 1 \end{array}\right|$$

$$\left|\begin{array}{ccc|r} -6 & 2 & 2 & 4 \\ 3 & 1 & -2 & -17 \\ -1 & 3 & 2 & 1 \end{array}\right|$$

$$\left|\begin{array}{ccc|r} -6 & 2 & 2 & 4 \\ 0 & 2 & -1 & -15 \\ 0 & 2.6667 & 1.6667 & 0.3333 \end{array}\right|$$

$$\left|\begin{array}{ccc|r} -6 & 2 & 2 & 4 \\ 0 & 2 & -1 & -15 \\ 0 & 0 & 3 & 20.3333 \end{array}\right|$$

$x_3 = 20.3333/3 = 6.67778$
$x_2 = (-15 - (-1 * 6.7778))/2 = -4.1111$
$x_1 = (4 - (2 * -4.1111) - (2 * 6.7778))/-6 = 0.2222$

$x_1 = \frac{2}{9} = 0.2222$
$x_2 = \frac{-37}{9} = -4.1111$
$x_3 = \frac{61}{9} = 6.7778$