OpenVoronoi notes

Anders Wallin (anders.e.e.wallin@gmail.com)

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Notes on positioning vertices and edge-parametrization for 2D voronoi diagrams with point, line-segment, and circular-arc sites.

These notes should describe the reasoning behind OpenVoronoi, available at `http://github.com/aewallin/openvoronoi`.

1 Sites

Input geometry for the diagram consists of points, (open) line segments, and (open) circular arc segments. These are called sites.

1.1 Point site

Point sites (figure 1) are defined by their xy-coordinates (x_C, y_C). The offset from a point is a circle

$$(x - x_C)^2 + (y - y_C)^2 = t^2 \tag{1}$$

where t is the offset distance ($t \geq 0$).

1.2 Line site

Line sites (figure 2) are defined by

$$ax + by + c = 0 \tag{2}$$

with the normalization $a^2 + b^2 = 1$. The offset of a line is another line

$$ax + by + c + kt = 0 \tag{3}$$

where $k = \{+1, -1\}$ is the offset direction, corresponding to a left or right offset, and t is the offset-distance ($t \geq 0$).

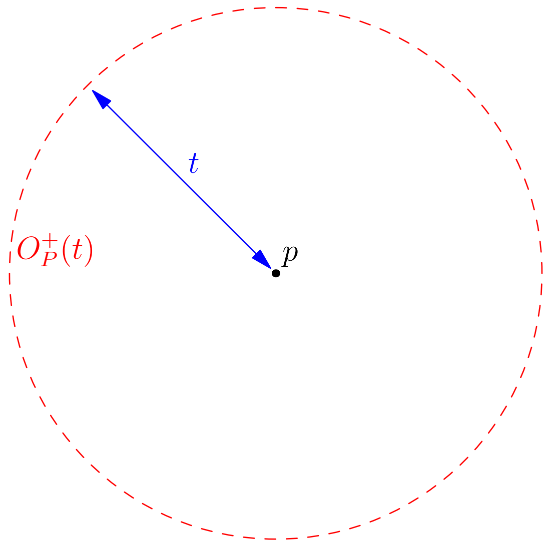

Figure 1: Point site with offset (1) at distance t.

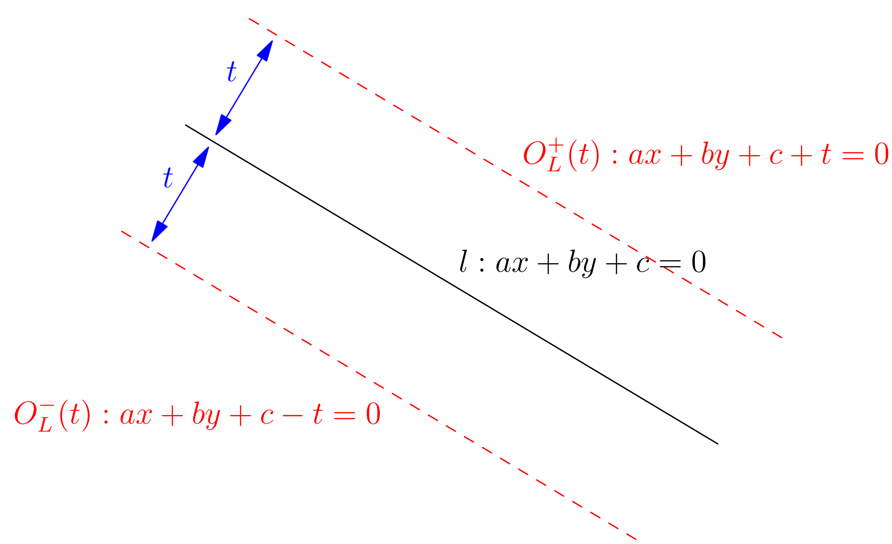

Figure 2: Line site with offsets (3) corresponding to $k = +1$ (top) and $k = -1$ (bottom).

	Point site	Line site	Arc site
q	1	0	1
a	$-2x_C$	a	$-2x_C$
b	$-2y_C$	b	$-2y_C$
c	$x_C^2 + y_C^2$	c	$x_P^2 + y_P^2 - r^2$
d	0	k	$-2kr$

Table 1: General offset equation parameters.

1.3 Arc site

Arc(circle) sites (FIXME: add figure) are defined by

$$(x - x_C)^2 + (y - y_C)^2 = r^2 \tag{4}$$

where (x_C, y_C) is the centerpoint of the circle and r is the radius. The offset is given by

$$(x - x_C)^2 + (y - y_C)^2 = (r + kt)^2 \tag{5}$$

where $k = \{+1, -1\}$ is the offset direction, correspondin to a growing or shirnking radius, and t is the offset-distance ($t \geq 0$).

1.4 General site/offset equation

Equations 1, 3, 5 can be combined into a general equation for the offset of a site

$$q(x^2 + y^2 - t^2) + ax + by + c + dt = 0 \tag{6}$$

where the parameters are given in table 1. Note that Point sites can be thought of as zero radius Arc sites.

2 Solvers

Solvers calculate the position of a new vertex in the diagram. The input for a solver consists of the three sites that are adjacent to the new vertex, and the three offset-directions from the sites. The new vertex should be positioned so that it is equidistant from all three sites.

An alternative approach to finding the position of a vertex in the diagram would be to search for the common intersection point of three edges/bisectors. However many report that this is less numerically stable compared to using the input sites directly.

The solvers are named based on the three input sites.

2.1 Point-Point-Point Solver

(see Sugihara&iri 1994 paper.).

See `solver_ppp.hpp`

2.2 Line-Line-Line Solver

The intersection point between the offsets of three line sites l_1, l_2, and l_3 is given by the solution (x, y, t) to

$$\begin{cases} a_1x + b_1y + c_1 + k_1t = 0 \\ a_2x + b_2y + c_2 + k_2t = 0 \\ a_3x + b_3y + c_3 + k_3t = 0 \end{cases} \tag{7}$$

where each line corresponds to (3). This can be written in matrix-form as $Ax = b$ where

$$A = \begin{bmatrix} a_1 & b_1 & k_1 \\ a_2 & b_2 & k_2 \\ a_3 & b_3 & k_3 \end{bmatrix}, b = \begin{bmatrix} -c_1 \\ -c_2 \\ -c_3 \end{bmatrix}, x = \begin{bmatrix} x \\ y \\ t \end{bmatrix} \tag{8}$$

.

The solutions $x = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^T$ are given by Cramer's rule as

$$x_i = \frac{\det(A_i)}{\det(A)} \tag{9}$$

where A_i is found by replacing column i of A with the column-vector b.

See `solver_lll.hpp`

Todo: when det(A) is zero (or small), an alternative strategy is required!

2.3 Separator Solver

See `solver_sep.hpp`

This solver is used when the input sites consist of a line-site l_1 and a point-site p_2 where p_2 is either end-point of the line-site l_1. This configuration forces the solution to lie on the separator (a line through p_2 and perpendicular to l_1) between the line-site and the point-site. We can then look for a point on the separator which is equidistant to the third input site (a Line site or an Arc site, by design of the higher level algorithm). Points on the separator are located at

$$p_{separator} = p_2 + v_{separator}t \tag{10}$$

where the separator direction is give by

$$\begin{cases} v_{separator} = (-a_1, -b_1) & k_1 = -1 \\ v_{separator} = (a_1, b_1) & k_1 = +1 \end{cases} \tag{11}$$

When the third site is a Line site this leads to

$$\begin{cases} x = x_2 + v_xt \\ y = x_2 + v_yt \\ a_3x + b_3y + c_3 + k_3t = 0 \end{cases} \tag{12}$$

with the solution

$$t = -(a_3 x_2 + b_3 y_2 + c_3)/(v_x a_3 + v_y b_3 + k_3) \tag{13}$$

FIXME: what happens if we get divide by zero? this corresponds to the new line site l3 being parallel to the separator??

3 Edges

Edges (or bisectors) in the diagram are defined by their two adjacent (and equidistant) sites. For offset-generation it is useful to parametrize an edge by the offset-distance (or clearance-disk radius). Edges are split by apex-vertices so that the offset-distance t is either monotonically increasing or decreasing along the edge.

3.1 Point-Point Edge

The edge between two point sites $p_1 = (x_1, y_1)$ and $p_2 = (x_2, y_2)$ is a line with its apex at the mid-point between the sites, and a direction perpendicular to a line connecting the sites (figure 3). A point on the edge is given by

$$e^{\pm}_{PP}(t) = \left(x^{\pm}_{PP}(t), y^{\pm}_{PP}(t)\right) = p_A \pm \hat{v}\sqrt{t^2 - t^2_{min}}, \tag{14}$$

where the apex point is

$$p_A = \tfrac{1}{2}(p_1 + p_2) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right),$$

the minimum offset distance t_{min} is

$$t_{min} = \tfrac{1}{2}\left|p_2 - p_1\right| = \tfrac{1}{2}\sqrt{\left(x_2 - x_1\right)^2 + \left(y_2 - y_1\right)^2}$$

, and the edge is directed along the unit vector

$$\hat{v} = \frac{(p_1 - p_2)^P}{|p_1 - p_2|} = \frac{\left(-\left(y_2 - y_1\right), \left(x_2 - x_1\right)\right)}{\sqrt{\left(x_2 - x_1\right)^2 + \left(y_2 - y_1\right)^2}},$$

where a perpendicular vector to v is denoted by $v^P = (v_x, v_y)^P = (-v_y, v_x)$. To write this more compactly, the following abbreviations are used:

$$\begin{aligned} d &= |p_1 - p_2| \\ d_x &= \tfrac{(x_2 - x_1)}{d} \\ d_y &= \tfrac{(y_2 - y_1)}{d} \end{aligned}$$

Now 14 can be written explicitly in terms of x and y coordinates as

$$\begin{aligned} x^{\pm}_{PP}(t) &= \tfrac{x_1 + x_2}{2} \pm (-d_y)\sqrt{t^2 - \tfrac{d^2}{4}} \\ y^{\pm}_{PP}(t) &= \tfrac{y_1 + y_2}{2} \pm d_x\sqrt{t^2 - \tfrac{d^2}{4}} \end{aligned}$$

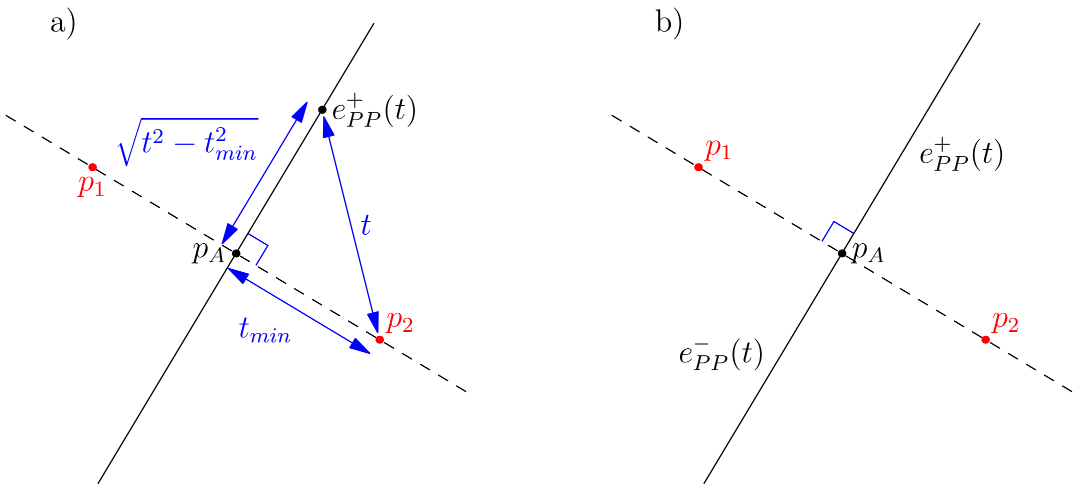

Figure 3: Point-Point edge $e_{PP}(t)$ between point sites $p_1$and p_2.

3.2 Point-Line Edge

The edge between a point site $p_1 = (x_1, y_1)$ and a line site $l_2 = \{(x, y)\,|a_2x + b_2y + c_2 = 0\}$ is a parabola.

3.3 Line-Line Edge

The edge between two line sites $l_1 = \{(x, y)\,|a_1x + b_1y + c_1 = 0\}$ and $l_2 = \{(x, y)\,|a_2x + b_2y + c_2 = 0\}$ is a line. The edge is given by

$$e^{\pm}_{LL}(t) = \left(x^{\pm}_{LL}(t), y^{\pm}_{LL}(t)\right) = p_A \pm \hat{v}t, \tag{15}$$

when the apex p_A is located at the intersection point between l_1 and l_2, and the direction v is is along the average of the tangents to l_1 and l_2. The case where l_1 and l_2 do not intersect is a special case which must be handled separately.

This may also be found by solving:

$$\begin{cases} a_1x + b_1y + c_1 + k_1t = 0 \\ a_2x + b_2y + c_2 + k_2t = 0 \end{cases}$$

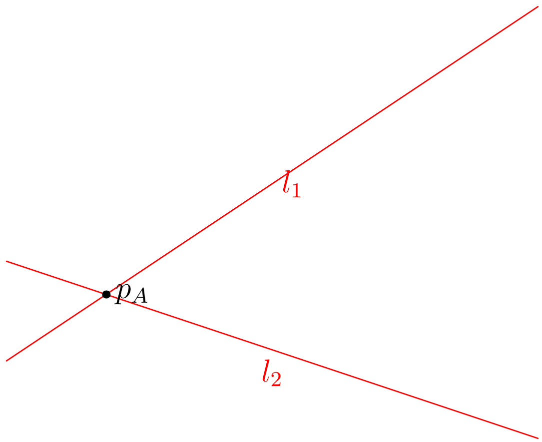

Figure 4: Line-Line edge between line sites l_1 and l_2.

3.4 Point-Arc Edge

3.5 Line-Arc Edge

3.6 Arc-Arc Edge

3.7 A General Edge representation

Held's thesis introduces the following general edge parametrisation, which can be used to represent all edge-types presented above. The parameters are defined in table 2 and table 3.

$$\begin{cases} x^{\pm}_{S_1 S_2}(t) = X_1 - X_2 - X_3 t \pm X_4 \sqrt{(X_5 + X_6 t)^2 - (X_7 + X_8 t)^2} \\ y^{\pm}_{S_1 S_2}(t) = Y_1 - Y_2 - Y_3 t \mp Y_4 \sqrt{(Y_5 + Y_6 t)^2 - (Y_7 + Y_8 t)^2} \end{cases}$$

S_1	S_2	Edge type	X_1 Y_1	X_2 Y_2	X_3 Y_3	X_4 Y_4	X_5 Y_5	X_6 Y_6	X_7 Y_7	X_8 Y_8
p_1	p_2	Line	x_1 y_1	$\alpha_1\alpha_3$ $\alpha_2\alpha_3$	0 0	$-\alpha_2$ $-\alpha_1$	0 0	$+1$ $+1$	α_3 α_3	0 0
p_1	l_2	Parabola	x_1 y_1	$\alpha_1\alpha_3$ $\alpha_2\alpha_3$	$\alpha_1 k_2$ $\alpha_2 k_2$	α_2 α_1	0 0	$+1$ $+1$	α_3 α_3	k_2 k_2
l_1	l_2	Line	α_1 α_2	0 0	$-\alpha_3$ $-\alpha_4$	0 0	0 0	0 0	0 0	0 0
p	arc		x_1 y_1	$\alpha_1\alpha_3$ $\alpha_2\alpha_3$	$\alpha_1\alpha_4$ $\alpha_2\alpha_4$	α_2 α_1	0 0	$+1$ $+1$	α_3 α_3	α_4 α_4
l	arc									
arc	arc									

Table 2: General edge-parameters $X_1...X_8$ and $Y_1...Y_8$

S_1	S_2	α_1	α_2	α_3	α_4	d	Δ
p_1	p_2	$\frac{x_2-x_1}{d}$	$\frac{y_2-y_1}{d}$	$-\frac{d}{2}$		$\lvert p_2-p_1\rvert$	
p_1	l_2	a_2	b_2	$a_2x_1+b_2y_1+c_2$			
l_1	l_2	$\frac{b_1c_2-b_2c_1}{\Delta}$	$\frac{a_2c_1-a_1c_2}{\Delta}$	$-\frac{b_2-b_1}{\Delta}$	$-\frac{a_1-a_2}{\Delta}$		$a_1b_2-b_1a_2$
p	arc	$\frac{x_2-x_1}{d}$	$\frac{y_2-y_1}{d}$	$\frac{r^2-d^2}{2d}$	$\frac{\lambda_2 r_2}{d}$	$\lvert p-c\rvert$	
l	arc						
arc	arc						

Table 3: Abbreviations used in table 2 for edge parameters.