The Geometry of Celestial Mechanics: Building the Orbits of Earth and Mars

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Earliest observations

The Babylonians were the first people to make significant observations and contributions to astronomy. Most importantly they realized that the phenomena were periodic and therefore could be predicted for later dates.

Observations spanning centuries were recorded in clay tables which led the Babylonians to lay the groundwork for some of the most important concepts in astronomy such as the western astrology and the Zodiac. Additionally they had some idea about the planets, as they observed some dots in night sky would move much more every night compared to the fixed star background. In fact some of the oldest surviving Babylonian texts were observation of what we now call Venus.

For this paper we are particularly interested in the observations they made for Mars. They observed that Mars came back to the same constellation it started with every 780 days. While they did not have any idea that this would be explained by orbits, it was crucial information for later brilliant minds to stand on the shoulders of.

Mars Retrograde and the Geocentric Model

It was actually Aristarchus who first proposed the idea of heliocentric model when he calculated that the mass of the Sun was so much greater than that of the Earth. Unfortunately the rest of Greece did not believe him as they argued if Earth revolved around the sun, then they would see a change in view of the stars at different times of years. Ironically they were correct, but the change in parallax was too small to be observed with the naked eye. So, due to a lack of technology the idea was lost to history and it would be a long time before the heliocentric model would get the investigation it deserved.

One of the biggest pitfalls with geocentric model was the retrograde of planets.

Let all keep silence and hark to Tycho who has devoted thirty-five years to his observations

When watching a planet like Mars move across the night sky every day, observers would periodically see it turn around and move in the backwards direction before turning around again and continuing in its original direction. Astronomers like Ptolemy went great lengths in attempt to explain these pitfalls but they were never completely accurate.

Finally Copernicus concluded that retrograde along with other mysteries could easily be explained by the heliocentric model, as the apparent retrograde motion of Mars would be caused by Earth catching up to Mars, then aligning with Mars, and finally passing Mars in a process taking several days.

Figure 1: Observed Mars Retrograde

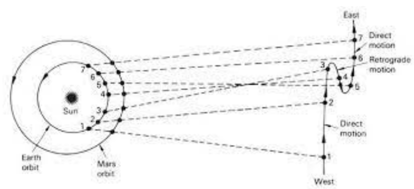


Figure 2: Explained Mars Retrograde

Copernicus did not stop there, now believing that Earth and Mars travel on their own orbits about the Sun he knew that 780 days was not the true period of a martian year. Instead because Earth and Mars had their own angular velocities and because they were orbiting in the same direction the apparent angular velocity was a sum of the true angular velocities of Earth and Mars.

$$\omega_{Earth} + \omega_{Mars} = \omega_{apparent}$$

$$1/365 + 1/T_{mars} = 1/780$$

$$1/365 - 1/780 = 1/T_{mars}$$

$$T_{mars} = 687$$

How far Mars is from sun

While Copernicus is mostly well known for proposing the heliocentric model, he should be more well know for calculated the distances to other planets. He could not actually find their distances in any units, but he was able to find them in ratios of the distance from the Earth to the Sun or astronomical units.

Copernicus believed that the Earth as well as all the other planets took perfectly

For Tycho alone do I wait; he shall explain to me the order and arrangement of the orbits

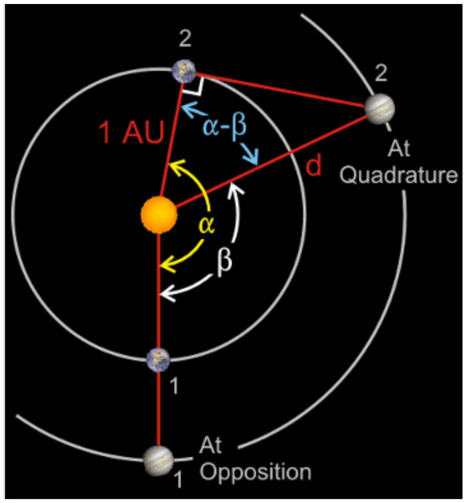


Figure 3: Example with Jupiter for superior planet observations

circular orbits so he started there when finding the distance from the Sun to Mars.

He also had two methods, one for what he called inferior planets which are inner planets relative to Earth, and one for superior planets, those that are outer. Copernicus was good enough at math to able to form triangles to calculate the distance from the Sun to other planets. Of course Mars is a superior planet, which is the far more challenging method of the two. The derivation Copernicus used was as follows:

1. He waited for Mars to be in Opposition. This meant the Sun, then Earth, then Mars were all in a line meaning the elongation of Mars (the angle Sun, Earth, Mars)is 180 degrees.

2. Then he counted the number of days until Mars was seen at Quadrature. This meant that now the angle between looking at the Sun and looking at Mars was 90 degrees.

3. Of course over that period of time, Earth will be in a completely different position on its Orbit. So he had two angles to calculate. Say α for the angle Earth sweeps out in that time T. And β for the angle that the superior planet sweeps out.

$$\alpha = T \cdot \frac{360}{T_{Earth}}$$

$$\beta = T \cdot \frac{360}{T_{Mars}}$$

The work, the work! not to have lived in vain. Into whose hands can I entrust it all?

4. Now a right triangle can be formed, where 90 degrees is at Earth and the hypotenuse is the distance D from the Sun to the outer planet. And the angle at the point of the Sun would be $\alpha - \beta$. D can be calculated as:

$$D = \frac{1}{cos(\alpha - \beta)}$$

Copernicus did this for all the visible Superior planets and with astounding accuracy. With Mars for example Copernicus calculated Mars to have an orbital radius of 1.52 AU, near perfect accuracy.

Understanding Observations

The equinox happens twice per year and can be described as when the Sun is exactly above the equator. This happens because the Suns annual path (the ecliptic) passes through the celestial equator going north. For the Northern hemisphere this happens on March 21 and is referred to as the vernal equinox.

Imagining a circle around the sun with 0 degrees as the vernal equinox, Earth's heliocentric longitude is the angle on the circle based on March 21. So the position of Earth on that circle can either be given by the heliocentric longitude, or the date.

Now imagine a line originating from the Earth which is parallel to the line of the vernal equinox. The geocentric longitude of Mars is the angle of Mars' position to the line. Basically what we did for Sun and Earth, now we do for Earth and Mars. It is this measurement that Tycho Brahe meticulously recorded along with the heliocentric longitude.

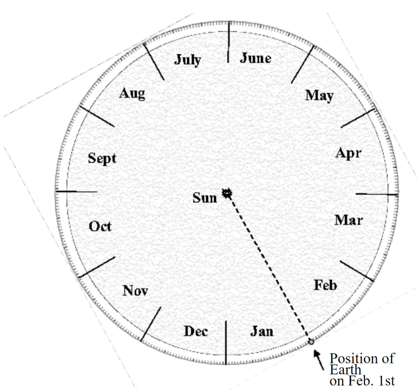


Figure 4: Heliocentric longitude

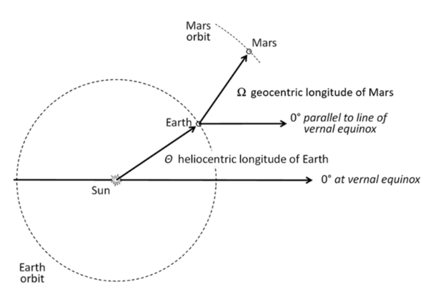


Figure 5: Geocentric longitude

The elongation of Mars is much easier to work with and can be found from the above two measurements. There is actually two elongation angles because

Tycho possesses the best observations, and thus so-to-speak the material for the building of the new edifice

Mars will be measured east or west relative to the line that connects the Sun and Earth.

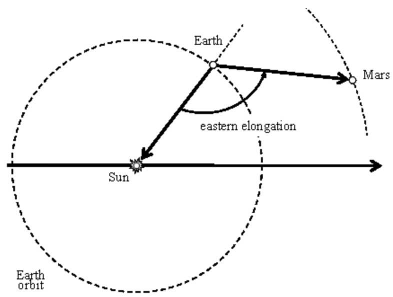


Figure 6: Eastern Elongation

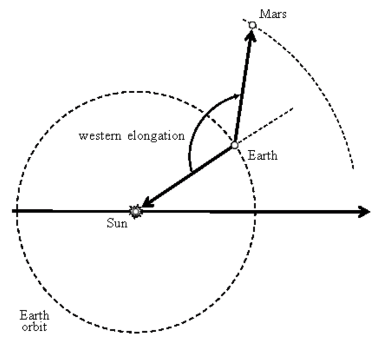


Figure 7: Western Elongation

Constructing Earth's Orbit

Now it comes to Kepler who looked at Tycho Brahe's mountain of data and saw that the data did not fit the orbits as if they were prefect circles as Copernicus predicted. He wanted to find the orbit of Mars but he understood that he should understand the orbit of Earth first, as he was an observer on Earth.

It actually took Kepler hundreds of pages of calculations and many ideas of trial and error before eventually finding orbits were elliptical. He truly loved geometry and proposed wild ideas such as every planet's orbit being described by a growing number of sided polygons.

Figure 8: Kepler's Idea of perfect Solids

Kepler believed in the heliocentric model, and brilliantly used the fact that Mars was in the same place, every 687 days. So in intervals of 687 days Kepler used the data and triangulation to find the location of Earth. This time using the fact that Mars is 1.52 AU as a side length and he had the heliocentric longitude of Earth, and the elongation of Mars he could build triangles with angel side angel, knowing that Earth lies at the intersection of the Earth Mars line, and

I thought to find him standing by the way, waiting to seize the splendor from my hand, the swift the young-eyed runner with the torch

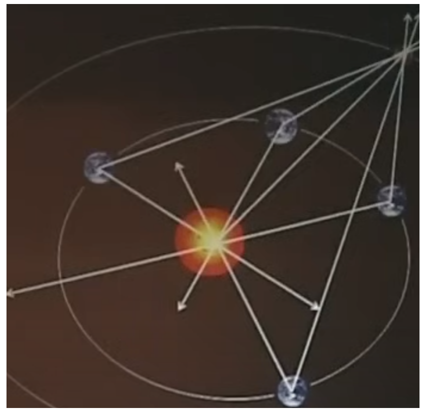


Figure 9: Constructing Earths orbit from 1 position of Mars

the Earth sun line.

In order for this to work, Mars needed to be in the same position that way Kepler could accurately build Earths orbit. This required a tremendous amount of data as only data points separated 687 days apart were relevant. But with barely enough data, Kepler was able to look at his plotted points for long enough to be convinced that path was part a slightly elliptical, but virtually circular, orbit. This was extremely impressive considering that Descartes had not even invented Cartesian Coordinates yet.

Constructing Mars Orbit

Kepler only became more convinced that orbits took ellipses when he found the orbit of Mars. Once he found the orbit of Earth he was able to find the orbit of Mars. Mars was still required to be in the same position, and since he wanted to shape out Mars's orbit he need observation pairs that were 687 days apart.

This allowed for much more data to be used as Kepler no longer needed many dates separated by 687 days, rather just pairs separated by 687 days, a much more abundant data sample.

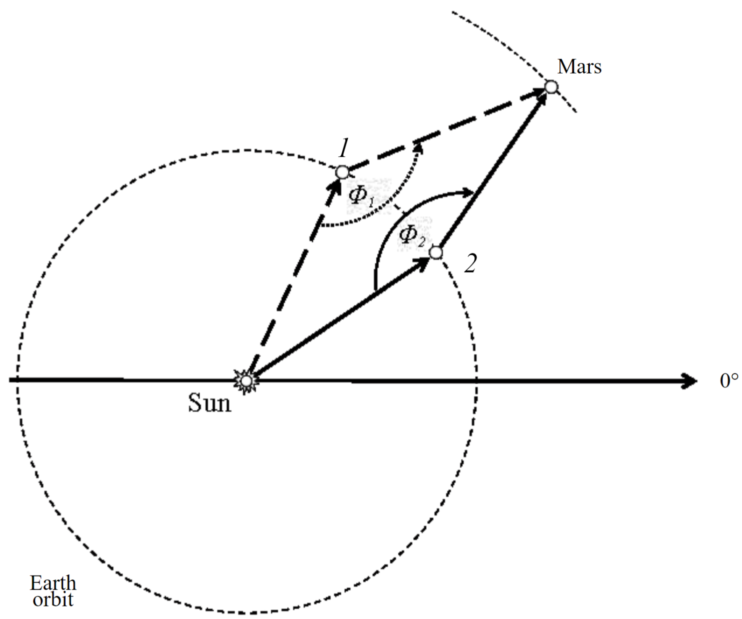
The technique was by using dates in pairs of 687 days you could draw lines in the direction of the elongation of Mars. And where those lines intersect is where Mars lies. This worked because 687 days later Earth would not be in the same position giving a new perspective and therefore new angel to Mars. The angle of heliocentric difference of Earth was required information for the triangle and could be calculated by:

$$360 - ((1 - 687/365.25) * 360$$

Since divine goodness has granted us most diligent observer, Tycho Brahe, from whose observations the error in this calculation eight minutes in Mars is revealed

$$360 - ((1 - 1.8807) * 360$$
$$360 - 317.052$$
$$42.948$$

This also meant that now the distance from the Sun to Mars could be recalculating more accurately knowing Earths orbit.



Elongation angles Φ_1 and Φ_2 are observed.

Figure 10: Basic Location Mars

First it is important to find the distance between Earth 687 days apart. (And we assume Earth has a sufficiently circular orbit)

$$C^2 = A^2 + b^2 - 2ABcos(42.948)$$
$$C^2 = 1 + 1 - 2cos(42.948)$$
$$C^2 = 2(1 - cos(42.948)$$
$$C = 0.73217$$

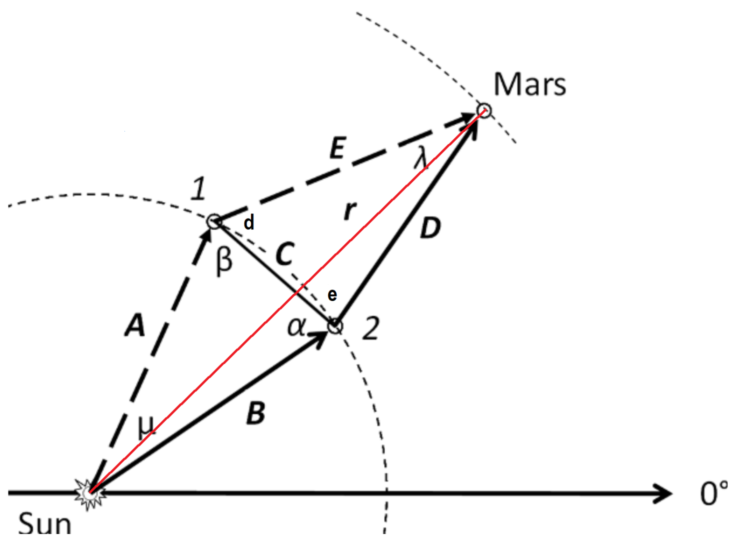


Figure 11: All Angles all sides

Now because we are using a circular orbit, triangle ABC will be an isosceles triangle. Meaning two angel measures will be the same.

$$\alpha = \beta = \frac{180 - 42.948}{2} = 68.526$$

Then angles e and d can be found by subtracting angles α and β from their respective angles of elongation. Then angel λ can found from $180 - e - d$

Using Law of sins: $\frac{C}{sin(\lambda)} = \frac{D}{sin(d)} = \frac{E}{sin(e)}$ Then side D and E can be solved for:

$$E = \frac{sin(e)}{sin(\lambda)} \cdot C$$

$$D = \frac{sin(d)}{sin(\lambda)} \cdot C$$

But e and d can be represented from the original observation angels: $e = \phi_1 - \alpha$ and $d = \phi_2 - \beta$

Therefore sides E and D can be calculated from the original observation angles

$$E = \frac{sin(\phi_1 - \alpha)}{sin(\lambda)} \cdot C$$

It is fitting that we recognize and make use of this good gift of God with a graceful mind

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$$D = \frac{sin(\phi_2 - \beta)}{sin(\lambda)} \cdot C$$

And since λ can be calculated from $180 - e - d$.

$$180 - e - d$$

$$180 - (\phi_1 - \alpha) - (\phi_2 - \beta)$$

Remember α and β are the same angel

$$180 - \phi_1 - \alpha - \phi_2 - \alpha$$

$$180 - ((\phi_1 + \phi_2) - 2\alpha)$$

Now we can express E and D only in terms of our observations

$$E = \frac{sin(\phi_1 - \alpha)}{sin(180 - ((\phi_1 + \phi_2) - 2\alpha))} \cdot C$$

$$D = \frac{sin(\phi_2 - \beta)}{sin(180 - ((\phi_1 + \phi_2) - 2\alpha))} \cdot C$$

In fact, a clever thing to notice is that $sin(180 - ((\phi_1 + \phi_2) - 2\alpha) = sin((\phi_1 + \phi_2) - 2\alpha)$

Our calculations slightly simplify to

$$E = \frac{sin(\phi_1 - \alpha)}{sin((\phi_1 + \phi_2) - 2\alpha)} \cdot C$$

$$D = \frac{sin(\phi_2 - \beta)}{sin((\phi_1 + \phi_2) - 2\alpha)} \cdot C$$

Finally by looking at our big triangle it is possible to find the length r by either using A and E or B and D

$$r^2 = A^2 + E^2 - 2AEcos(\phi_1)$$

$$r = \sqrt{A^2 + E^2 - 2AEcos(\phi_1)}$$
$$r = \sqrt{1 + E^2 - 2Ecos(\phi_1)}$$

I believe it was an act of Divine Providence that I arrived just at the time when Longomontanus was occupied with Mars

$$r = \sqrt{1 + (\frac{sin(\phi_1 - \alpha)}{sin((\phi_1 + \phi_2) - 2\alpha)} \cdot C)^2 - 2(\frac{sin(\phi_1 - \alpha)}{sin((\phi_1 + \phi_2) - 2\alpha)} \cdot C)cos(\phi_1)}$$

Using this equation one could find that the r values varies by pair of measurements showing Kepler that Mars indeed took an elliptical orbit and in fact a very elliptical orbit compared to Earth

My findings

I wrote what initially was supposed to be a small function, that would take the equation for r found above and use it on data to find varying r distances. This proved to be a bigger challenge than I thought, as I spent a great amount of time cleaning, pairing and matching the data. Finally when I got the code running I was able to find 32 different r values. I the found the minimum and maximum in the array to be 1.380589160278675 and 1.6560311335634146 respectively. These values are of course in AU because Kepler could only know how far Mars is relative to Earth's orbit.

When comparing this data to a quick google search I find that Mars' true aphelion and perihelion are 1.666 and 1.3813. It appears an accurate max distance was found but not as accurate min distance. Then the eccentricity can be calculated using the following equations and rearranging.

$$a = \frac{rmin + rmax}{2}$$

$$a = 1.5183101469210447$$

$$rmin = a(1 - e), rmax = a(1 + e)$$

$$\frac{rmin}{a} = 1 - e, \frac{rmax}{a} = 1 + e$$

$$e = 1 - \frac{rmin}{a}, e = \frac{rmax}{a} - 1$$

Now using 1.380589160278675 and 1.6560311335634146 for the rmin and rmax values both calculations get the same eccentricity of 0.09070675508666781. This is a little under the true eccentricity of Mars which according to NASA is 0.0934. I then decided to grow my program to use a python visual library called Turtle to draw the exact process out of finding Mars' orbit. This demonstration is for my presentation and is entirely accurate, using the same data I used to calculate the varying r distances. The program will move to a point given by the heliocentric longitude on the unit circle representing earths orbit. It will then draw an arrow in the direction of Mars, then move the point a martian year later and draw a new line in the direction of Mars. Where these line intersect

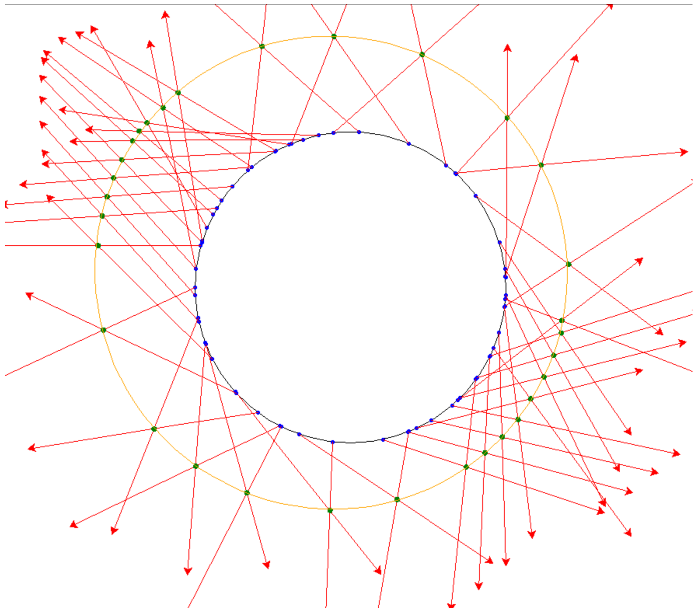

Figure 12: My python program depicted the process

is where Mars is.

I had 64 rows of data meaning 32 data points and it was enough to see an elliptic trend in the data. Then I decided to shape out the ellipse with an algorithm called numerically stable direct least square fitting of ellipses developed by Halir and Flusser. I made this just so I could visualize the complete elliptical orbit. I found that to the naked eye the ellipse looks almost circular but you can slightly tell its not.

I confess that when Tycho died, I quickly took advantage of the absence, or lack of circumspection, of the heirs, by taking the observations under my care, or perhaps usurping them

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Since I was aware that there exist an infinite number of points on the orbit and accordingly an infinite number of distances [from the sun] the idea occurred to me that the sum of these distances is contained in the area of the orbit. For I remembered that in the same manner Archimedes too divided the area of a circle into an infinite number of triangles.