

# The Theory behind PageRank<sup>2</sup>

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<sup>2</sup>Advanced topic: not asked at the exam

# The PageRank Algorithm: History and Facts

It was devised at Stanford by Larry Page and Sergei Brin, (the founders of Google) in 1996 as part of a research project on a new search engine.

The paper was published in 1998 [1, 2] and shortly after the authors founded Google. Named after one of the authors L. Page.

It is a *link analysis algorithm*, i.e., only the links between pages (not their content) are considered.

Other link analysis algorithms: community and spam detection, HITS, ...

# Ranking Web Pages

Humans cannot make sense of billions of web pages, which have to be ranked according to their “importance”.

The web pages `www.stanford.edu` and `www.johnsmith.com` have not the same importance. The former one has more than 25K Web pages linking to it, the latter one only 10.

Simple algorithm: Rank the pages according to the # of links to them.

Are all web pages linking to `www.stanford.edu` equally important?  
Cornell University homepage more important than that of J. Smith friend.

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Are all web pages linking to `www.stanford.edu` equally important?  
Cornell University homepage more important than that of J. Smith friend.

⇒ Recursive definition of importance.

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# Computing Importance

Web graph: a directed graph  $G = (V, E)$  where nodes represent web pages, while there is a directed edge between  $u$  and  $v$  if there is a hyperlink between the corresponding web pages.

Importance of  $v$  is proportional to the importance of nodes linking to  $v$ .

It can be modeled by a system of linear equations ...

# System of Linear Equations for PageRank

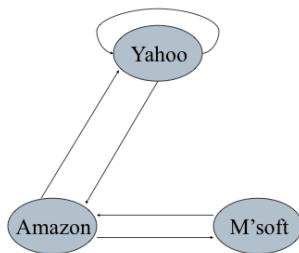
Let  $G = (V, E)$  (web graph) be a directed graph, with  $V = \{v_1, \dots, v_n\}$ . Let  $\delta_{\text{in}}(v)$  be the in-degree of  $v$ , i.e.  $\delta_{\text{in}}(v) = |\{u : (u, v) \in E\}|$ , while let  $\delta_{\text{out}}(v)$  be its out-degree, i.e.  $\delta_{\text{out}}(v) = |\{u : (v, u) \in E\}|$ .

Let  $M_G$  ( $M$  for short) be a  $n \times n$  matrix with entries in  $[0, 1]$  as follows:

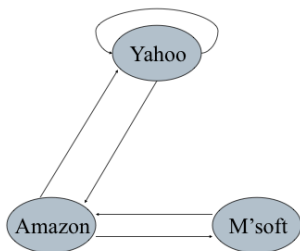
$$M_{ij} = \begin{cases} \frac{1}{\delta_{\text{out}}(v_j)} & \text{if } (v_j, v_i) \in E \\ 0 & \text{if } (v_j, v_i) \notin E \end{cases}, \quad \forall i, j \in [1, n].$$

Any  $\pi$  s.t.  $\pi = M\pi$  and  $\sum_{i=1}^n \pi_i = 1$  meets our requirements.

# Example



# Example

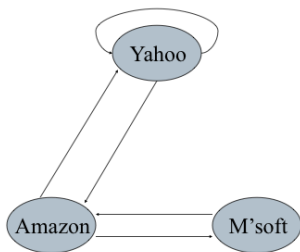


$$M_G = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 1 \\ 0 & \frac{1}{2} & 0 \end{bmatrix}$$

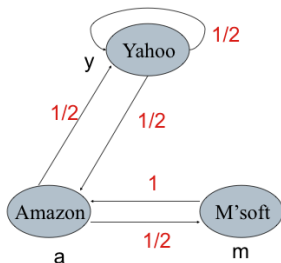
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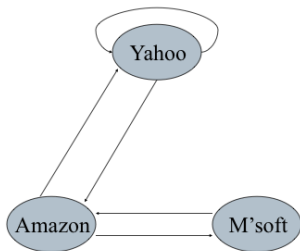


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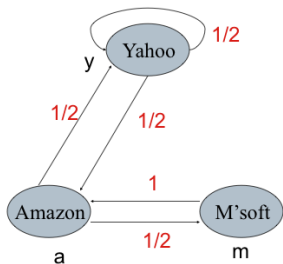


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# Example



$$M_G = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 1 \\ 0 & \frac{1}{2} & 0 \end{bmatrix}$$



$$\begin{aligned} y &= \frac{y}{2} + \frac{a}{2} \\ a &= \frac{y}{2} + m \\ m &= \frac{a}{2} \\ y + a + m &= 1 \end{aligned}$$

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# PageRank

The importance of a web page can be computed by solving the corresponding system of linear equations.

However there are two main issues:

- The solution might not be unique!
- It is expensive to solve large system of linear equations. *Gaussian elimination* requires  $\Omega(n^3)$  operations.

# PageRank: Eigenvector Computation

## Definition 1

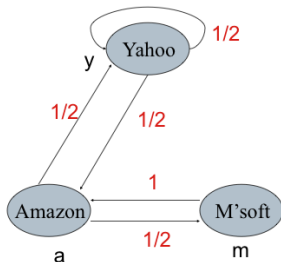
The vector  $x$  is an eigenvector of the matrix  $A$  with eigenvalue  $\lambda$  if

$$Ax = \lambda x.$$

Therefore, one could compute the importance of web pages by computing an eigenvector with eigenvalue 1 of  $M_G$ , which is also very expensive!

# PageRank: Random Surfer

A *random surfer* starts surfing the web from a random page (step 0). At step  $t$ , let  $u$  be the web page currently visited by the random surfer. At step  $t + 1$ , the random surfer visits a web page  $v$  being picked uniformly at random from the  $\delta_{\text{out}}(u)$  neighbors.



$$M_G = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 1 \\ 0 & \frac{1}{2} & 0 \end{bmatrix}$$

$M_{ij}$  is the probability that the random surfer moves from page  $u_j$  to  $u_i$ .

# PageRank = probability of visiting web pages

We would like to compute the importance of a web page  $u$  as the probability that the random surfer visits  $u$  at step  $t > 0$  for some large  $t$ .

Issues:

- Which value of  $t$ ?
- The probability of visiting  $u$  might depend on the starting page.
- Can such a probability be computed efficiently?
- So called *spider traps* and dead-ends might give non-interesting results. A spider trap is a set of web pages  $S$  not containing any link to  $V \setminus S$ . *Dead-ends* are web pages not having any link to any other web page. We assume for simplicity that there are no dead-ends.

## Random Surfer 2.0

The random surfer starts from one page chosen uniformly at random. Then, at step  $t$  it either 1) follows one random link from the current page with prob.  $\beta$  or 2) “jumps” to any of the  $n$  pages uniformly at random.

Common values for  $\beta$  are in  $[0.8, 0.9]$ .

Let  $A$  be the  $n \times n$  matrix where  $A_{ij} = \beta M_{ij} + \frac{1-\beta}{n}$ ,  $i, j \in [1, n]$ .  $A_{ij}$  denotes the probability that the random surfer 2.0 moves from page  $u_j$  to page  $u_i$ .

# PageRank: Interpretation

Given a web graph  $G = (V, E)$  the following facts hold:

- Let  $P(X_t = u)$  be the probability that the random surfer visits page  $u$  at step  $t$ . The PR of page  $u$  is equal to  $\lim_{t \rightarrow \infty} P(X_t = u)$ .
- The PR vector  $\pi$  is the (unique) eigenvector of  $A$  with eigenvalue 1.
- The PageRank vector  $\pi$  satisfies  $\pi = A\pi$  and  $\sum_{i=1}^n \pi_i = 1$ .

To be proved later...



# The PageRank algorithm

**Input:** A directed graph  $G$  with  $n$  nodes (Web pages),  $0 < \beta < 1, \epsilon > 0$ .

**Output:** The PageRank vector  $r$  of the web pages in  $G$ .

- 1: Remove *dead ends* iteratively from  $G$ ;
- 2: Build the stochastic matrix  $M_G$  ( $M$  for short);
- 3: Let  $\pi^{(0)} = [\frac{1}{n}, \dots, \frac{1}{n}]^T$
- 4: **while** (true) **do**
- 5:      $t = t + 1$ ;
- 6:      $\pi^{(t)} = A\pi^{(t-1)}$ ;
- 7:     If  $\|\pi^{(t)} - \pi^{(t-1)}\|_1 < \epsilon$  **break**;
- 8: **end while**
- 9: **return**  $\pi^{(t)}$ .

# Efficiency issues

$A$  is a dense  $n \times n$  matrix.  $n \gg 10^9$  which implies  $A$  contains  $\gg 10^{18}$  non-zero entries. Lemma 2 allows us to deal with  $M_G$  which is sparse.

## Lemma 2

*Let  $x \in \mathbb{R}$  and let  $[x]_n$  be the vector with  $n$  entries equal to  $x$ . Let  $G = (V, E)$  a directed graph, let  $\beta > 0$ , let  $M_G$  and  $A$  be the  $n \times n$  matrices as defined above. For any  $\pi$  with  $\|\pi\|_1 = 1$ , we have:*

$$A\pi = \beta M_G \pi + \left[ \frac{1 - \beta}{n} \right]_n$$

## Proof of Lemma 2

Proof.

For any  $i \in [1, n]$ :

$$\begin{aligned}\sum_{j=1}^n A_{ij} \pi_j &= \sum_{j=1}^n \left( \beta M_{ij} + \frac{1-\beta}{n} \right) \pi_j \\ &= \beta \sum_{j=1}^n M_{ij} \pi_j + \frac{1-\beta}{n} \sum_{j=1}^n \pi_j \\ &= \beta \sum_{j=1}^n M_{ij} \pi_j + \frac{1-\beta}{n}.\end{aligned}$$



# The PageRank algorithm (improved)

**Input:** A directed graph  $G$  with  $n$  nodes (Web pages),  $0 < \beta < 1, \epsilon > 0$ .

**Output:** The PageRank vector  $\pi$  of the web pages in  $G$ .

- 1: Remove *dead ends* iteratively from  $G$ ;
- 2: Build the stochastic matrix  $M_G$  ( $M$  for short);
- 3: Let  $\pi^{(0)} = [\frac{1}{n}, \dots, \frac{1}{n}]^T$
- 4: **while** (true) **do**
- 5:    $t = t + 1$ ;
- 6:    $\pi^{(t)} = \beta M \pi^{(t-1)} + [\frac{1-\beta}{n}]_n$ ;
- 7:   If  $\|\pi^{(t)} - \pi^{(t-1)}\|_1 < \epsilon$  **break**;
- 8: **end while**
- 9: **return**  $\pi^{(t)}$ .

# Events and Probability

Consider a stochastic process (e.g. throw a dice, pick a card from a deck)

- Each possible outcome is a *simple event*.
- The sample space  $\Omega$  is the set of all possible simple events.
- An event is a set of simple events (a subset of the sample space).
- With each simple event  $E$  we associate a real number  $0 \leq P(E) \leq 1$ , which is the probability that event  $E$  happens.

# Probability Space

## Definition 3

A *probability space* has three components:

- A *sample space*  $\Omega$ , which is the set of all possible outcomes of the random process modeled by the probability space;
- A family of sets  $\mathcal{F}$  representing the allowable events, where each set in  $\mathcal{F}$  is a subset of the sample space in  $\Omega$ ;
- a *probability function*  $P : \mathcal{F} \rightarrow \mathbb{R}$ , satisfying the definition below (next slide).

# Probability Function

## Definition 4

A *probability function* is any function  $P : \mathcal{F} \rightarrow \mathbb{R}$  that satisfies the following conditions:

- for any event  $E$ ,  $0 \leq P(E) \leq 1$ ;
- $P(\Omega) = 1$ ;
- for any finite or countably infinite sequence of pairwise mutually disjoint events  $E_1, E_2, E_3, \dots$

$$P\left(\bigcup_{i \geq 1} E_i\right) = \sum_{i \geq 1} P(E_i). \quad (1)$$

# The Union Bound

## Theorem 5

$$P(\cup_{i=1}^n E_i) \leq \sum_{i=1}^n P(E_i). \quad (2)$$

Example: roll a dice:

- let  $E_1 = \text{"result is odd"}$
- let  $E_2 = \text{"result is } \leq 2\text{"}$



# Independent Events

## Definition 6

Two events  $E_1$  and  $E_2$  are *independent* if and only if

$$P(E_1 \cap E_2) = P(E_1) \cdot P(E_2) \quad (3)$$

## Conditional Probability: Example

What is the probability that a random student at Telecom ParisTech was born in Paris?

$E_1$  = the event “born in Paris”.

$E_2$  = the event “student at Telecom ParisTech”.

The conditional probability that a student at Telecom ParisTech was born in Paris is written:

$$P(E_1|E_2).$$

# Conditional Probability: Definition

## Definition 7

The *conditional probability* that event  $E_1$  occurs given that event  $E_2$  occurs is

$$P(E_1|E_2) = \frac{P(E_1 \cap E_2)}{P(E_2)} \quad (4)$$

The conditional probability is only well-defined if  $P(E_2) > 0$ .

By conditioning on  $E_2$  we restrict the sample space to the set  $E_2$ .

# Law of Total Probability

## Theorem 8

Let  $B_1, \dots, B_k$  be a partition of the sample space  $\Omega$ , with  $P(B_i) > 0$ ,  $i = 1, \dots, k$ . Then, for any event  $A \subseteq \Omega$ :

$$P(A) = \sum_{i=1}^k P(A \cap B_i) = \sum_{i=1}^k P(A|B_i)P(B_i). \quad (5)$$

# Random Variable

## Definition 9

A *random variable*  $X$  on a sample space  $\Omega$  is a function on  $\Omega$ ; that is,  $X : \Omega \rightarrow \mathbb{R}$ .

A *discrete random variable* is a random variable that takes on only a finite number of values.

# Examples

In practice, a random variable is some random quantity that we are interested in:

- I roll a die,  $X = \text{result}$ . E.g.  $X = 6$ .
- I pick a card,  $X = 1$  if card is an Ace, 0 otherwise.
- I roll a dice two times.  $X_1 = \text{result of the first experiment}$ ,  $X_2 = \text{result of the second experiment}$ . What is  $P(X_1 + X_2 = 2)$ ?

# Stochastic Processes

## Definition 10

A stochastic process in discrete time  $n \in \mathbb{N}$  is a sequence of random variables  $X_0, X_1, X_2 \dots$  denoted by  $\mathbf{X} = \{X_n\}$ .

We refer to the value  $X_n$  as the *state* of the process at time  $n$ , with  $X_0$  denoting the initial state.

The set of possible values that each random variable can take is denoted by  $S$ . Here, we shall assume that  $S$  is finite and  $S \subseteq \mathbb{N}$ .

# Markov Chains

## Definition 11

A *stochastic process*  $\{X_n\}$  is called a *Markov chain* if for any  $n \geq 0$  and any value  $j_0, j_1, \dots, i, j \in S$ ,

$$P(X_{n+1} = i | X_n = j, X_{n-1} = j_{n-1}, \dots, X_0 = j_0) = P(X_{n+1} = i | X_n = j),$$

which we denote by  $P_{ij}$ .

This can be stated as *the future is independent of the past given the present state*. In other words, the probability of moving to the next state **does not** depend on what happened in the past. Note that  $P_{ij} \neq P_{ji}$ .



# One-step Transition Matrix

$P_{ij}$  denotes the probability that the chain, whenever in state  $j$ , moves next into state  $i$ .

The square matrix  $\mathbf{P} = (P_{ij})$ ,  $i, j \in S$ , is called the *one-step transition matrix*. Note that for each  $j \in S$  we have:

$$\sum_{i \in S} P_{ij} = 1. \quad (6)$$

# n-step Transition Matrix

The *n*-step transition matrix  $\mathbf{P}^{(n)}$ ,  $n \geq 1$ , where

$$P_{ij}^n = P(X_n = i | X_0 = j) = P(X_{m+n} = i | X_m = j), \quad \forall m \quad (7)$$

denotes the probability that *n* steps later the Markov chain will be in state *i* given that at step *m* is in state *j*.

## Theorem 12

$$\mathbf{P}^{(n)} = \mathbf{P}^n = \mathbf{P} \times \mathbf{P} \times \cdots \times \mathbf{P}, n \geq 1.$$

# Stationary Distribution

## Definition 13

A probability distribution  $\pi$  over the states of the Markov chain ( $\sum_{j \in S} \pi_j = 1$ ) is called a *stationary distribution*<sup>a</sup> if

$$\pi = P\pi. \quad (8)$$

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<sup>a</sup>in literature the transpose of  $P$  is often used, in that case  $\pi = \pi P$ .

# Irreducible Markov Chains

## Definition 14

A Markov chain is called *irreducible*<sup>a</sup> iff for any  $i, j \in S$ , there is  $n \geq 1$  s.t.

$$P_{ij}^n > 0. \quad (9)$$

---

<sup>a</sup>definition slightly different when  $S$  is not finite.

That is, we can move from any state  $i$  to any state  $j$ , in one or more steps. If a Markov chain is irreducible then there must be  $n$  such that  $P_{ii}^n > 0$ .

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## Theorem 15

*If a Markov chain is irreducible, there is a unique stationary distribution.*

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<sup>36</sup>Advanced topic: not asked at the exam

# Aperiodic Markov Chains

A state  $i$  has period  $k$  if any return to  $i$  occurs at step  $k \cdot l$ , for some  $l > 0$ . Formally,

$$k = \gcd\{n : P(X_n = i | x_0 = i) > 0\} \quad (10)$$

where  $\gcd$  denotes the *greatest common divisor*. If  $k = 1$  then state  $i$  is said to be *aperiodic*.

## Definition 16

A Markov chain is called *aperiodic* if every state is aperiodic.

# Main Theorem

## Theorem 17

*If a Markov chain is irreducible and aperiodic<sup>a</sup>, then the Markov chain converges to its (unique) stationary distribution, that is,*

$$\pi_j = \lim_{n \rightarrow \infty} P(X_n = j) = \lim_{n \rightarrow \infty} P(X_n = j | X_0 = i), \quad \forall i, j \in S. \quad (11)$$

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<sup>a</sup>in this case the Markov chain is called *ergodic*

Note: Equation (11) holds for any initial state  $i$  of the Markov chain.

# Markov chains and the Random Surfer

Consider the Markov chain (MC) of the random surfer.

- What is  $P(X_n = j | X_0 = i)$ ?



# Markov chains and the Random Surfer

Consider the Markov chain (MC) of the random surfer.

- What is  $P(X_n = j | X_0 = i)$ ? The probability that at step  $n$  the random surfer visits page  $j$  given that at step 0 he/she visited page  $i$ .
- If there are no random jumps, is the MC irreducible/aperiodic ?

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<sup>40</sup>Advanced topic: not asked at the exam

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- If there are no random jumps, is the MC irreducible/aperiodic? Not necessarily (because of spider traps/dead ends).
- What if we add random jumps? It is both irreducible and aperiodic, which implies that the PageRank vector converges to the unique stationary distribution  $\pi$  of MC (see next slide).

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# The Random Surfer and its Stationary Distribution

**Fact:** The stationary distribution  $\pi$  of the MC is the PageRank vector!

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Sketch: At each step  $n$  of the PageRank algorithm we compute  $\pi^{(n)} = P\pi^{(n-1)} = P^n\pi^{(0)} = P^{(n)}\pi^{(0)}$ . From the law of total probability (Theorem 8) and from the fact that  $\pi^{(n)} = P^{(n)}\pi^{(0)}$  it follows that  $\pi_j^{(n)} = P(X_n = j)$ ,  $\forall j$ , which converges to  $\pi$  (Theorem 17).

Therefore, the PageRank algorithm computes for every page  $j$  the probability that the random surfer visits page  $j$ , i.e.  $\lim_{n \rightarrow \infty} P(X_n = j)$ .

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