Problem Set 1

Conditional Probabilities

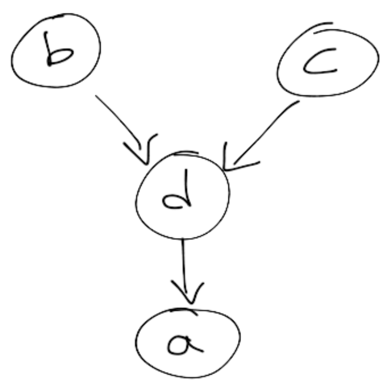
(a) $P(a|b,c) = P(b|a,c)$, then $P(a|c) = P(b|c)$

$$\left.\begin{aligned} P(a,b,c) &= P(a|b,c)\,P(b|c)\,P(c) \\ &= P(b|a,c)\,P(a|c)\,P(c) \end{aligned}\right\} \rightarrow P(a|c) = P(b|c)$$

$\boxed{\text{True}}$

(b) $P(a|b,c) = P(a)$ then $P(b|c) = P(b)$

$a \perp b,c \Rightarrow b \perp c$



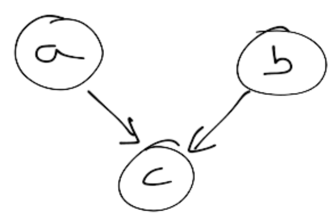
$a \perp (b,c) | d$

$b \not\perp c \,|\, d$

$\boxed{\text{False}}$

(c) If $P(a|b) = P(a)$ then $P(a|b,c) = P(a|c)$

$a \perp b \Rightarrow a \perp b | c$



$a \perp b$

$a \not\perp b | c$

$\boxed{\text{False}}$

Finding the Fake coin

n unbiased coins $\quad n-1$: head and tail
$\qquad\qquad\qquad\qquad 1$: two heads

(a) X: pick one coin $\quad Y$: result of fliping

$$P(x=\text{fake} \mid y=\text{head}) = \frac{P(y=\text{head} \mid x=\text{fake}) \cdot P(x=\text{fake})}{P(y=\text{head})} = \frac{1/n}{\frac{n+1}{2n}} = \frac{2}{n+1}$$

$$P(y=\text{head} \mid x=\text{fake}) = 1 \qquad P(x=\text{fake}) = \frac{1}{n}$$

$$P(y=\text{head}) = \frac{n-1}{n} \cdot \frac{1}{2} + \frac{1}{n} = \frac{n-1+2}{2n} = \frac{n+1}{2n}$$

(b) Y_k : result of k flips

$$P(Y_k=H \mid X=N) = 2^{-k} \qquad P(Y_k=H \mid X=F) = 1$$

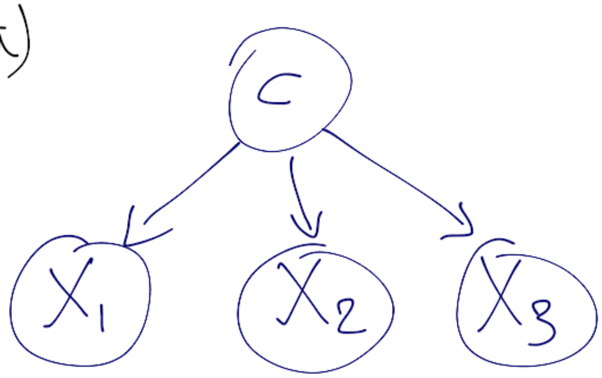
$$P(X=F \mid Y_k=H) = \frac{P(Y_k=H \mid X=F)\, P(X=F)}{P(Y_k=H)} = \frac{P(Y_k=H \mid X=F) P(X=F)}{P(Y_k=H \mid X=F) P(X=F) + P(Y_k=H \mid X=N) P(X=N)} =$$

$$= \frac{1 \cdot 1/n}{1\, \frac{1}{n} + 2^{-k}\, \frac{n-1}{n}} = \frac{1}{1+2^{-k}(n-1)}$$

(c) $P(X=N,\ Y_k=H) = P(Y_k=H \mid X=N)\, P(X=N) = 2^{-k}\, \frac{(n-1)}{n}$

Naive Bayes

(a)



$C=a$	$C=b$	$C=c$
$1/3$	$1/3$	$1/3$

C	$X_i=H$
a	0.2
b	0.6
c	0.8

(b)

$$P(X_1=H, X_2=H, X_3=T \mid C) =$$
$$= P(X_1=H, X_2=T, X_3=H \mid C) =$$
$$= P(X_1=T, X_2=H, X_3=H \mid C) =$$
$$= [P(X_i=H \mid C)]^2 P(X_i=T \mid C)$$

A be the event where two coins are head and one tail.

$$P(A \mid C) = 3[P(X_i=H \mid C)]^2 P(X_i=T \mid C)$$

z : coin drawn

$$P(C=z \mid A) = \frac{P(A \mid C=z) \cdot P(C=z)}{P(A)}$$

$$\underset{z\in\{a,b,c\}}{\operatorname{argmax}} P(C=z \mid A) = \underset{z}{\operatorname{argmax}}\ P(A \mid C=z) =$$
$$= \underset{z}{\operatorname{argmax}} [P(X_i=H \mid C=z)]^2 P(X_i=T \mid C=z) =$$
$$= \operatorname{argmax}\{\underbrace{0.2^2 0.8}_{z=a}, \underbrace{0.6^2 \cdot 0.4}_{z=b}, \underbrace{0.8^2 \cdot 0.2}_{z=c}\} =$$
$$= \operatorname{argmax}\{0.032, 0.144, 0.128\} = b$$