Stochastic Liquidity-Aware Markowitz Model

CS 524 Final Project

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April 2025

1 Introduction

In class, we talked about the classical Markowitz model, where we are managing an investment portfolio and seek to maximize expected return while minimizing variance. Under this classical framework, we are making a one-time decision, based on the current best available estimate of returns and risks.

However, in a more realistic setting, we will rebalancing our portfolio at a regular time interval, and want to reduce transaction amount out of either asset liquidity concern or transaction cost consideration. In this project, we consider a more developed portfolio optimization framework, where we face a multi-period management problem and consider the transaction amount over period while maximizing returns and minimizing risks.

2 Relevant Concepts

1. Liquidity Measure Let L_i denote a liquidity score for asset
2. Transaction Costs with Liquidity Measure

$$Cost_t = \sum_i \gamma_i |w_{i,t} - w_{i,t-1}| f(L_{i,t}) \tag{1}$$

where:

- γ_i is a parameter that scales the cost for asset i.
- $f(L_{i,t})$ is a function of the liquidity measure. Our choice is

$$f(L_{i,t}) = \frac{1}{L_{i,t}},$$

- This term ensures that large trades in illiquid assets (i.e., when $L_{i,t}$ is low) receive a higher penalty.

3 Concrete Formulation

3.1 Intuition

In a dynamic, multi-period setting, our goal is to optimize the cumulative trade-off over all periods by balancing risk, return, and liquidity-based transaction costs. One possible formulation is:

$$\min_{\{\mathbf{w}_t\}_{t=0}^{T}} \quad \sum_{t=0}^{T} \left\{ \lambda\, \mathbf{w}_t^\top \Sigma_t \mathbf{w}_t - (1-\lambda)\, \mathbf{w}_t^\top \mu_t + \sum_{i=1}^{n} \gamma_i \left| w_{i,t} - w_{i,t-1} \right| \, \frac{1}{L_{i,t}} \right\}. \tag{2}$$

Key points in this formulation:

- **Risk-Return Trade-off:** Each period's portfolio is evaluated using the term
$$\lambda\, \mathbf{w}_t^\top \Sigma_t \mathbf{w}_t - (1-\lambda)\, \mathbf{w}_t^\top \mu_t,$$
where λ controls the emphasis on risk versus return.
- **Liquidity-Based Costs:** The additional penalty
$$\sum_{i=1}^{n} \gamma_i \left| w_{i,t} - w_{i,t-1} \right| \, \frac{1}{L_{i,t}}$$
penalizes large or frequent adjustments in portfolio weights, especially for assets with lower liquidity.
- **Dynamic Coupling:** The trading term $|w_{i,t} - w_{i,t-1}|$ links the decisions across periods, thus making the optimization problem multi-stage in nature.

This looks like a promising approach. However, in a real life setting, one might only have the current best available estimate returns and risks, and do not have reliable estimate of these data in future time points. Then this model will "cheat" by choosing the best portfolio weights under the classical Markovitz framework, without changing portfolio weights in the entire future perios, thus incurring no transaction penalty at all.

To circumvent this dilemma and also to model in a more realistic manner, we set our expected return for each asset to be a random variable.

$$r_i \sim \mathcal{N}(\mu_i, \sigma_i^2), \tag{3}$$

With this setting, our actual realized return become a random variable in future time periods (thus the actual portfolio weights before adjusting.)

The wealth invested in asset i evolves as:

$$V_{i,t+1} = V_{i,t}(1 + r_{i,t+1}),$$

so that the portfolio's total wealth at time $t+1$ is

$$V_{t+1} = \sum_{i=1}^{n} V_{i,t+1}.$$

The corresponding portfolio weight for asset i at time $t+1$ is given by:

$$w_{i,t+1} = \frac{V_{i,t+1}}{V_{t+1}}.$$

3.2 Modeling Method

In order to manage the randomness while keeping the optimization problem computationally tractable, we utilizes the scenario tree formulation.

We consider a multi-period portfolio optimization model where the stochastic process of returns is discretized into a finite number of scenarios over the time horizon $t = 0, 1, \ldots, T$. For each time step:

- Let S_t be the set of all nodes (i.e., scenarios) at time t.
- For each node $s \in S_t$, denote by $\mathbf{r}_t^{(s)}$ the realization of asset returns and by p_s the probability of that node.

State Variables

At each node $s \in S_t$, we track the state variables:

- The wealth invested in asset i at time t:

$$V_{i,t}^{(s)}.$$

- The corresponding portfolio weight for asset i:

$$w_{i,t}^{(s)} = \frac{V_{i,t}^{(s)}}{V_t^{(s)}}, \quad \textit{with} \quad V_t^{(s)} = \sum_{i=1}^{n} V_{i,t}^{(s)}.$$

Decision Variables

At each node $s \in S_t$, we introduce decision variables:

$$x_{i,t}^{(s)},$$

which represent the trades (positive for buying, negative for selling) of asset i at time t. These variables enable us to rebalance the portfolio dynamically.

Objective Function

The overall objective is to minimize the expected cost over all scenarios, combining a risk-return trade-off with liquidity-based transaction costs. This can be expressed as:

$$\min_{\{x_{i,t}^{(s)}\}} \sum_{t=0}^{T} \sum_{s \in S_t} p_s \left\{ \lambda \left(\mathbf{w}_t^{(s)} \right)^\top \Sigma \, \mathbf{w}_t^{(s)} - (1-\lambda) \left(\mathbf{w}_t^{(s)} \right)^\top \mu + \sum_{i=1}^{n} \gamma_i \left| x_{i,t}^{(s)} \right| \frac{1}{L_{i,t}^{(s)}} \right\}, \tag{4}$$

where:

- $\lambda \in [0,1]$ is a parameter that balances the trade-off between risk and return.
- Σ is the covariance matrix of asset returns.
- μ is the vector of expected returns.
- γ_i scales the trading cost for asset i.
- $L_{i,t}^{(s)}$ is the liquidity measure for asset i at node s and time t.

Linking Equations

To capture the dynamics of portfolio evolution, the wealth for asset i evolves as follows:

$$V_{i,t+1}^{(s')} = \left(V_{i,t}^{(s)} + x_{i,t}^{(s)} \right) \times \left(1 + r_{i,t+1}^{(s')} \right), \tag{5}$$

where $s' \in S_{t+1}$ is a descendant node of s at time $t+1$, and $r_{i,t+1}^{(s')}$ denotes the return for asset i in node s'.