Circuits
-
Sinusoidal Analysis

NYU SHANGHAI 上海纽约大学

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Context

上海纽约大学 NYU SHANGHAI

DC sources

Up to now, we have only considered loadings from DC sources.

Another classical way is to load circuits with alternating currents (AC). For example, sinusoidal sources are AC sources.

Does it change anything compared to DC?

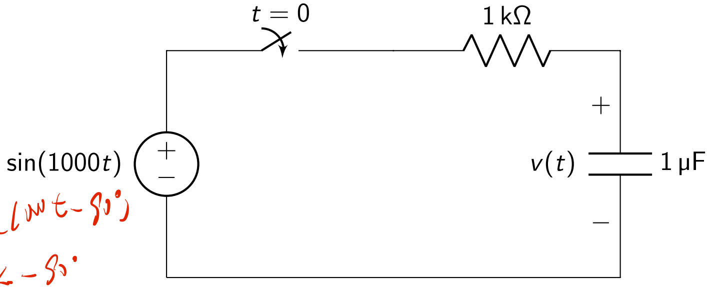
No! Most of the laws/theorems we have seen so far still apply.

But, the repsonse of a circuit can be divided into 2 components:

- a **steady-state response**, the part that remains when $t \to \infty$
- a **transient response**, the part that vanishes when $t \to \infty$

Context

Initial condition: $v(0) = 0$



$\cos(1000t - 90°)$

$V = \angle -90°$

$V = 1k\Omega \cdot I + \frac{1}{j\omega C} I \rightarrow I = \frac{V}{1k\Omega + \frac{1}{j\omega C}} = \frac{1}{1000(1-j)}$

$V_C = \frac{1}{j\omega C} I = \frac{V}{1000(1-j)} (-1000j) = V \frac{-j}{1-j}$

Context

Equation for $t \geq 0$

By applying KVL:

$$v(t) + \tau \frac{dv(t)}{dt} = \sin(1000t)$$

with $\tau = RC = 1\,\text{ms}$

why the phase diff of Vss is 45°?
what's the equivalent DC value of a sin signal?

Solution?

The **steady-state solution** is:

- $v_{ss}(t) = \frac{1}{\sqrt{2}} \sin(1000t - 45°)$

half of the max abs of sin signal

The **transient solution** is:

- $v_{tr}(t) = \frac{1}{2} e^{-t/\tau}$

How Do we find the Vo here?

Context

when timing is infinit. the first part is almost 0

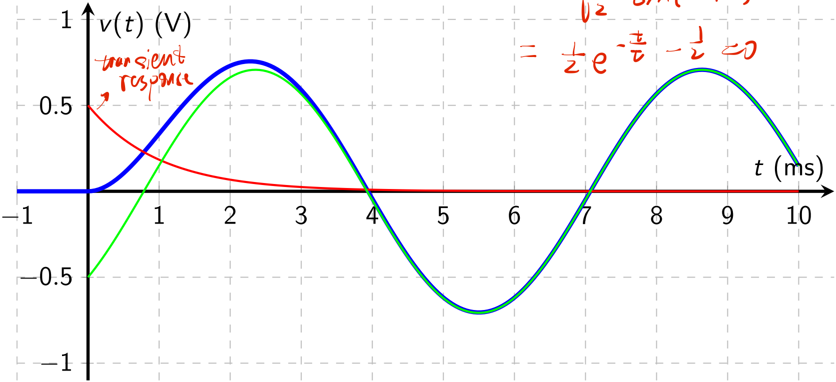
Total solution

$$v(t) = \frac{1}{2}e^{-t/\tau} + \frac{1}{\sqrt{2}}\sin(1000t - 45^\circ)$$

if t is very short, the steady state is

$\frac{1}{\sqrt{2}}\sin(-45^\circ)$

$= \frac{1}{2}e^{-\frac{t}{\tau}} - \frac{1}{2} = 0$



Context

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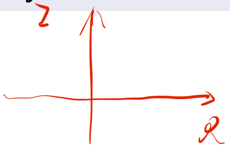
$a + jb = \sqrt{a^2+b^2} \angle \arctan \frac{b}{a}$

How to determine the solution?

In this lecture, we will only focus on **methods** to determine the **steady-state solution**.
The **transient solution** can be retrieved from **DC analysis** and proper initial conditions.

In this course, when dealing with AC circuits, we will **only consider the steady-state solution**.



Complex numbers

Complex representation

Since we are gonna deal with **sinusoidal loadings and responses**, it will be very convenient (mathematically) to represent **every signal** by a **complex number**.

Phasor

Phasor: a complex number that represents the amplitude A and the phase ϕ of a sinusoid

$$A\underline{/\phi}$$

Complex numbers

Equivalence

$$z = a + jb \qquad z = A\underline{/\phi} \qquad z = Ae^{j\phi}$$

$= A(\cos\phi + j\sin\phi)$

where $A = \sqrt{a^2 + b^2}$

$= A\cos\phi + jA\sin\phi$

and $\phi = \text{angle}(a, b) = \arctan\left(\frac{b}{a}\right)$

$a = A\cos\phi \quad b = A\sin\phi$

Euler's identity

$$e^{j\phi} = \cos(\phi) + j\sin(\phi)$$

$e^{-j\phi} = \cos(-\phi) + j\sin(-\phi)$

$= \cos\phi - j\sin\phi$

$$\implies \cos(\phi) = \text{Re}(e^{j\phi}) = \frac{e^{j\phi} + e^{-j\phi}}{2}$$

$$\implies \sin(\phi) = \text{Im}(e^{j\phi}) = \frac{e^{j\phi} - e^{-j\phi}}{2j}$$

Complex numbers

Phasor transform

Suppose we have a sinusoidal voltage:

same ω in the RLC circuit
resonant frequency
oscillating speed

$$v(t) = A\cos(\omega t + \phi)$$

We can represent that voltage by the phasor **V**:

$$\mathbf{V} = A\underline{/\phi}$$

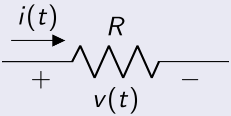
Explanations

$$v(t) = A\cos(\omega t + \phi) = \mathrm{Re}(Ae^{j(\omega t+\phi)}) = \mathrm{Re}(Ae^{j\phi} \cdot e^{j\omega t}) = \mathrm{Re}(\mathbf{V} \cdot e^{j\omega t})$$

As we are gonna see in the next slides, $e^{j\omega t}$ will be a common factor for every signal.
So, in the phasor representation, we only consider the amplitude A and the phase ϕ

Resistor

Resistor and phasors



Suppose we have a current $i(t) = A\cos(\omega t + \phi)$
Then, $v(t) = Ri(t) = RA\cos(\omega t + \phi)$

When representing with phasors, we have:

$$\mathbf{I} = A\underline{/\phi} \qquad \mathbf{V} = RA\underline{/\phi} = R\mathbf{I}$$

Ohm's law

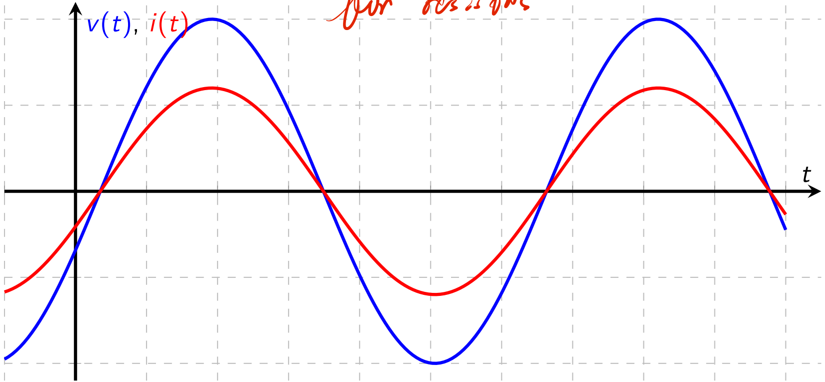
Ohm's law is the same with phasors:

$$\mathbf{V} = R\mathbf{I}$$

Resistor

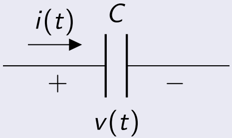
no phase difference between $v(t)$ & $i(t)$ for resistors

$v(t)$ and $i(t)$ are in phase



Capacitor

Capacitor and phasors



Suppose we have a voltage $v(t) = A\cos(\omega t + \phi)$

Then, $i(t) = C\dfrac{dv(t)}{dt} = -CA\omega\sin(\omega t + \phi) = -CA\omega\cos(\omega t + \phi - 90^\circ)$

$= -CA\omega\cos(\omega t + \phi + 270^\circ)$

When representing with phasors, we have:

$$\mathbf{V} = A\angle\phi \qquad \mathbf{I} = -CA\omega e^{j(\phi - 90^\circ)} = -CA\omega e^{j\phi} e^{-j90^\circ}$$

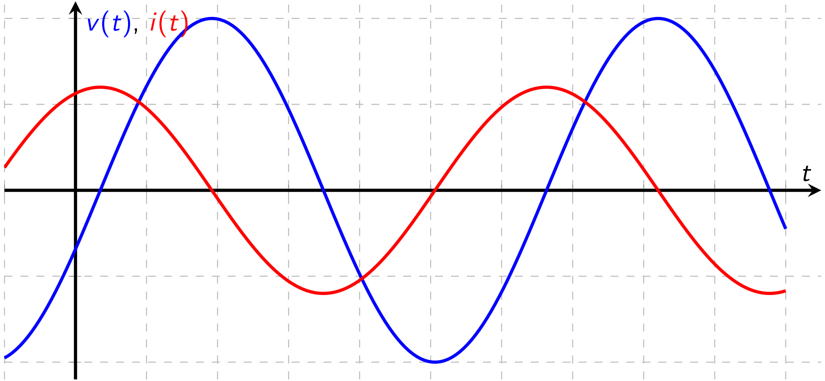
Phasor relationship

Kind of Ohm's law for phasors:

$$\mathbf{I} = jC\omega\mathbf{V} \qquad \text{or} \qquad \mathbf{V} = \frac{1}{jC\omega}\mathbf{I}$$

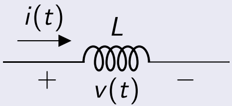
Capacitor

$v(t)$ and $i(t)$ are out of phase
$i(t)$ is leading $v(t)$ by 90°
$v(t)$ is lagging $i(t)$ by 90°



Inductor

Inductor and phasors



Suppose we have a current $i(t) = A\cos(\omega t + \phi)$

Then, $v(t) = L\dfrac{di(t)}{dt} = -LA\omega\sin(\omega t + \phi) = -LA\omega\cos(\omega t + \phi - 90^\circ)$

When representing with phasors, we have:

$$\mathbf{I} = A\underline{/\phi} \qquad \mathbf{V} = -LA\omega e^{j(\phi - 90^\circ)} = -LA\omega e^{j\phi}e^{-j90^\circ}$$

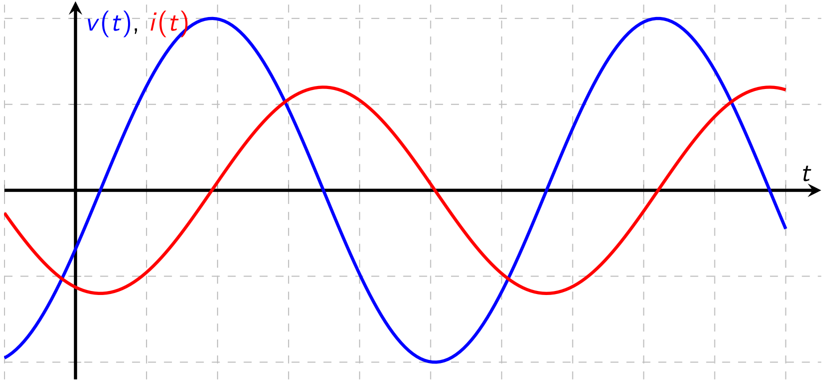
Phasor relationship

Kind of Ohm's law for phasors:

$$\mathbf{V} = jL\omega\mathbf{I} \qquad \text{or} \qquad \mathbf{I} = \frac{1}{jL\omega}\mathbf{V}$$

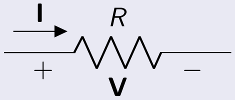
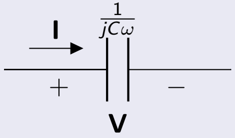
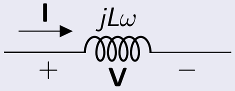
Inductor

$v(t)$ and $i(t)$ are out of phase
$v(t)$ is leading $i(t)$ by 90°
$i(t)$ is lagging $v(t)$ by 90°



Generalization

Generalization of Ohm's law

Element	**Equation**	**Phasor equation**	**Circuit**
Resistor	$v(t) = Ri(t)$	$\mathbf{V} = R\mathbf{I}$	
Capacitor	$i(t) = C\dfrac{dv(t)}{dt}$	$\mathbf{V} = \dfrac{1}{jC\omega}\mathbf{I}$	
Inductor	$v(t) = L\dfrac{di(t)}{dt}$	$\mathbf{V} = jL\omega\mathbf{I}$	

Impedance/Admittance

Impedance

In the previous table, the phasor equation was written in the form:

$$\mathbf{V} = \mathbf{Z}\mathbf{I}$$

$\mathbf{Z}$ is called the **impedance** (unit: ohms - Ω)

Impedance

Sometimes, it can be useful to use another phasor equation:

$$\mathbf{I} = \mathbf{Y}\mathbf{V}$$

$\mathbf{Y}$ is called the **admittance** (unit: siemens - S)

Relationship

Obviously, we have:

$$\mathbf{Z} = \frac{1}{\mathbf{Y}}$$

Impedance/Admittance

Real and imaginary parts

In general, impedances or admittances are complex numbers.

$\mathbf{Z} = R + jX$ and $\mathbf{Y} = G + jB$

- R is called the **resistance** (Ω)
- X is called the **reactance** (Ω)
- G is called the **conductance** (S)
- B is called the **susceptance** (S)

Circuits rules

KVL

Sum of voltages around a closed path is equal to zero.

- $v_1(t) + v_2(t) + \ldots + v_N(t) = 0$
- $A_1 \cos(\omega t + \phi_1) + A_2 \cos(\omega t + \phi_2) + \ldots + A_N \cos(\omega t + \phi_N) = 0$
- $\mathrm{Re}(A_1 e^{j(\omega t + \phi_1)} + A_2 e^{j(\omega t + \phi_2)} + \ldots + A_N e^{j(\omega t + \phi_N)}) = 0$
- $\mathrm{Re}((A_1 e^{j\phi_1} + A_2 e^{j\phi_2} + \ldots + A_N e^{j\phi_N}) e^{j\omega t}) = 0$
- $\mathrm{Re}((\mathbf{V}_1 + \mathbf{V}_2 + \ldots + \mathbf{V}_N) e^{j\omega t}) = 0$

Since $e^{j\omega t} \neq 0$, we have:

$$\mathbf{V}_1 + \mathbf{V}_2 + \ldots + \mathbf{V}_N = 0$$

It means that **KVL still applies to phasors**.

KCL

Similarly, it can be shown that **KCL still applies to phasors**.

Circuit rules

Impedances in series

When several impedances $\mathbf{Z}_i$ are connected in series, it can be replaced by an equivalent impedance $\mathbf{Z}_{eq}$:

$$\mathbf{Z}_{eq} = \sum_{i=1}^{N} \mathbf{Z}_i$$

Impedances in parallel

When several impedances $\mathbf{Z}_i$ (or admittances $\mathbf{Y}_i$) are connected in parallel, it can be replaced by an equivalent impedance $\mathbf{Z}_{eq}$ (admittance $\mathbf{Y}_{eq}$):

$$\mathbf{Z}_{eq} = \frac{1}{\sum_{i=1}^{N} \frac{1}{\mathbf{Z}_i}} \quad \text{or} \quad \mathbf{Y}_{eq} = \sum_{i=1}^{N} \mathbf{Y}_i$$

Circuit rules

Source transformation

Sources can be transformed in phasor representation

Thevenin/Norton equivalences

Thevenin and Norton equivalences can be given in phasor representation

Superposition

Superposition can be applied in the phasor domain

Circuit analysis

Analysis

Since we are only considering the **sinusoidal steady-state response**, the analysis of the circuits will be equivalent to the study of a network of resistors (no differential equation to solve!!!).

The only difference with DC resistor circuits is that we are going to manipulate **phasors** (complex voltages and currents) and **complex impedances**.

Circuit analysis

Different steps

1. Transform independent sources to phasors*
2. Determine the impedance of every passive element
3. Apply the analysis tools learned this semester
4. Find the phasors of interest and transform them back to a time-domain expression

* Be careful is sources have different frequencies!!! Everything detailed earlier only applies for signals with the same frequency.

If different frequencies, you can still apply **superposition**. (Check exercise in recitation)