

Circuits

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RLC circuits

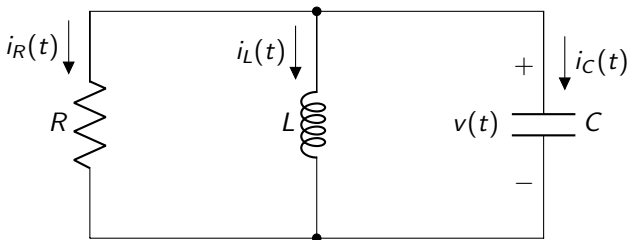
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Natural response

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Same voltage $v(t)$ for the 3 elements.

Initial conditions:

- $i_L(0^+) = I_0$ (inductor)
- $v(0^+) = V_0$ (capacitor)

$$\blacksquare \quad \frac{v(t)}{R} + \frac{1}{L} \int_{t_0}^t v(t') dt' + i_L(t_0) + C \frac{dv(t)}{dt} = 0$$
$$\blacksquare \quad \frac{d^2 v(t)}{dt^2} + \frac{1}{RC} \frac{dv(t)}{dt} + \frac{1}{LC} v(t) = 0$$

Natural Response



Rewriting the equation

We set the following 2 variables:

- **resonant frequency:** $\omega_0 = \frac{1}{\sqrt{LC}}$
- **damping ratio:** $\xi = \frac{1}{2R} \sqrt{\frac{L}{C}}$

The differential equation is now:

- $\frac{d^2v(t)}{dt^2} + 2\xi\omega_0 \frac{dv(t)}{dt} + \omega_0^2 v(t) = 0$

Natural Response



Solution

Let's try a similar solution to what we obtained for first-order circuits:
 $v(t) = A \cdot e^{s \cdot t}$

$$\blacksquare As^2 e^{st} + 2\xi\omega_0 A s e^{st} + \omega_0^2 A e^{st} = 0$$

Supposing that $A e^{st} \neq 0$ (incompatible with initial conditions), we then must respect:

$$\blacksquare s^2 + 2\xi\omega_0 s + \omega_0^2 = 0$$

Characteristic equation (quadratic)

3 possibilities:

- 2 distinct real roots
- 2 non distinct real roots
- 2 complex roots (conjugate)

Natural Response



Discriminant

$$\Delta = (\xi\omega_0)^2 - \omega_0^2 = \omega_0^2 (\xi^2 - 1)$$

3 possibilities:

- $\Delta > 0$ (meaning $\xi > 1$): 2 real roots (**overdamped**)
- $\Delta = 0$ (meaning $\xi = 1$): 2 non distinct real roots (**critically damped**)
- $\Delta < 0$ (meaning $\xi < 1$): 2 complex conjugate roots (**underdamped**)

Role of the resistor R

Supposing that L and C have been fixed (and so is ω_0), we can notice that the damping ratio ξ depends on R

For example, modifying R may have an impact on the discriminant.

In the next slides, we are going to check the 3 possibilities separately

Natural Response (overdamped)



$\Delta > 0$ (meaning $\xi > 1$): 2 real roots

$$s = -\xi\omega_0 \pm \sqrt{\omega_0^2 (\xi^2 - 1)} = -\omega_0 (\xi \pm \sqrt{\xi^2 - 1})$$

Both roots are **negative**.

- $s_1 = -\omega_0 (\xi + \sqrt{\xi^2 - 1})$

- $s_2 = -\omega_0 (\xi - \sqrt{\xi^2 - 1})$

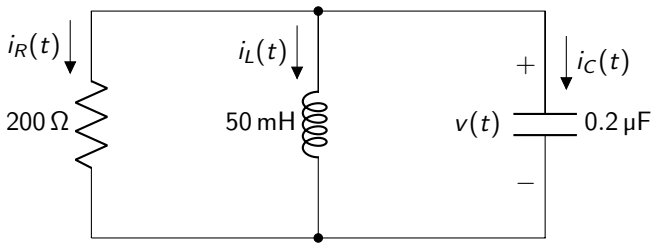
Final solution

We finally have:

$$v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

with two constants A_1 and A_2 to determine from initial conditions

Natural Response (overdamped)



Initial conditions:

- $i_L(0^+) = 30\ \text{mA}$ (inductor)
- $v(0^+) = 12\ \text{V}$ (capacitor)

Natural Response (overdamped)



Evaluate the different values

- $\omega_0 = \frac{1}{\sqrt{LC}} = 10\,000 \text{ rad/s}$
- $\xi = \frac{1}{2R} \sqrt{\frac{L}{C}} = 1.25 \text{ (no unit)}$
- $\sqrt{\xi^2 - 1} = 0.75$
- $s_1 = -\omega_0 \left(\xi + \sqrt{\xi^2 - 1} \right) = -20\,000 \text{ rad/s}$
- $s_2 = -\omega_0 \left(\xi - \sqrt{\xi^2 - 1} \right) = -5\,000 \text{ rad/s}$

Solution

$$v(t) = A_1 e^{-20000t} + A_2 e^{-5000t}$$



Natural Response (overdamped)

Determination of constants

We know that $v(0^+) = 12 \text{ V}$

$$\implies A_1 + A_2 = 12 \text{ V}$$

We also know that $i_L(0^+) = 30 \text{ mA}$

$$i_L(t) = -i_R(t) - i_C(t) = -\frac{v(t)}{R} - C \frac{dv(t)}{dt}$$

$$\frac{dv(t)}{dt} = -20000A_1e^{-20000t} - 5000A_2e^{-5000t}$$

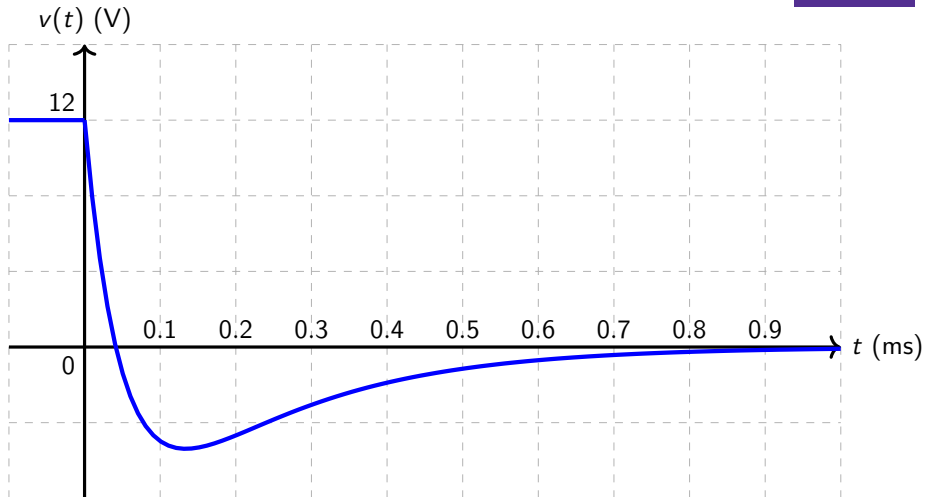
$$i_L(0^+) = -\frac{12 \text{ V}}{200 \Omega} + 0.2 \mu\text{F} (20000A_1 + 5000A_2) = 30 \text{ mA}$$

$$\implies 20000A_1 + 5000A_2 = 450\,000 \text{ V/s}$$

Final solution

$$v(t) = 26e^{-20000t} - 14e^{-5000t}$$

Natural Response (overdamped)



Natural Response (underdamped)



$\Delta < 0$ (meaning $\xi < 1$): 2 complex conjugate roots

$$s = -\xi\omega_0 \pm j\sqrt{\omega_0^2(1 - \xi^2)} = -\omega_0 \left(\xi \pm j\sqrt{1 - \xi^2} \right)$$

We can set $\omega_d = \omega_0\sqrt{1 - \xi^2}$, named the **natural resonant frequency**

- $s_1 = -\xi\omega_0 + j\omega_d$
- $s_2 = -\xi\omega_0 - j\omega_d$

Final solution

We finally have:

$$v(t) = e^{-\xi\omega_0 t} (A_1 e^{j\omega_d t} + A_2 e^{-j\omega_d t})$$

with two constants A_1 and A_2 to determine from initial conditions

Natural Response (underdamped)



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Complex voltage?

$$v(t) = e^{-\xi\omega_0 t} (A_1 e^{j\omega_d t} + A_2 e^{-j\omega_d t})$$

The voltage $v(t)$ is, of course, **real**.

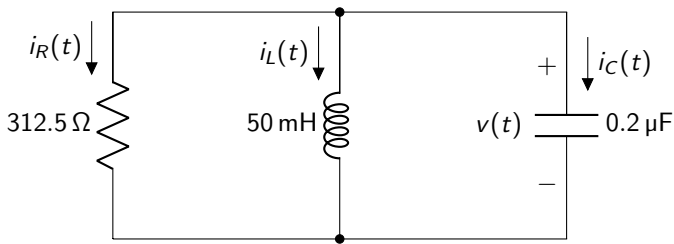
Actually the 2 constants A_1 and A_2 are complex in order to obtain a real voltage.

Another way to write the voltage equation is:

$$v(t) = e^{-\xi\omega_0 t} (B_1 \cos(\omega_d t) + B_2 \sin(\omega_d t))$$

with B_1 and B_2 are 2 real constants

Natural Response (underdamped)



Initial conditions:

- $i_L(0^+) = 30\text{ mA}$ (inductor)
- $v(0^+) = 12\text{ V}$ (capacitor)

Natural Response (underdamped)



Evaluate the different values

- $\omega_0 = \frac{1}{\sqrt{LC}} = 10\,000 \text{ rad/s}$
- $\xi = \frac{1}{2R} \sqrt{\frac{L}{C}} = 0.8 \text{ (no unit)}$
- $\sqrt{1 - \xi^2} = 0.6$
- $\xi\omega_0 = 8000 \text{ rad/s}$
- $\omega_d = \omega_0 \sqrt{1 - \xi^2} = 6000 \text{ rad/s}$

Solution

$$v(t) = e^{-8000t} (B_1 \cos(6000t) + B_2 \sin(6000t))$$

Natural Response (underdamped)



Determination of constants

We know that $v(0^+) = 12 \text{ V}$

$$B_1 = 12 \text{ V}$$

We also know that $i_L(0^+) = 30 \text{ mA}$

$$i_L(t) = -i_R(t) - i_C(t) = -\frac{v(t)}{R} - C \frac{dv(t)}{dt}$$

$$\frac{dv(t)}{dt} = -2000 (4v(t) - 3e^{-8000t} (-B_1 \sin(6000t) + B_2 \cos(6000t)))$$

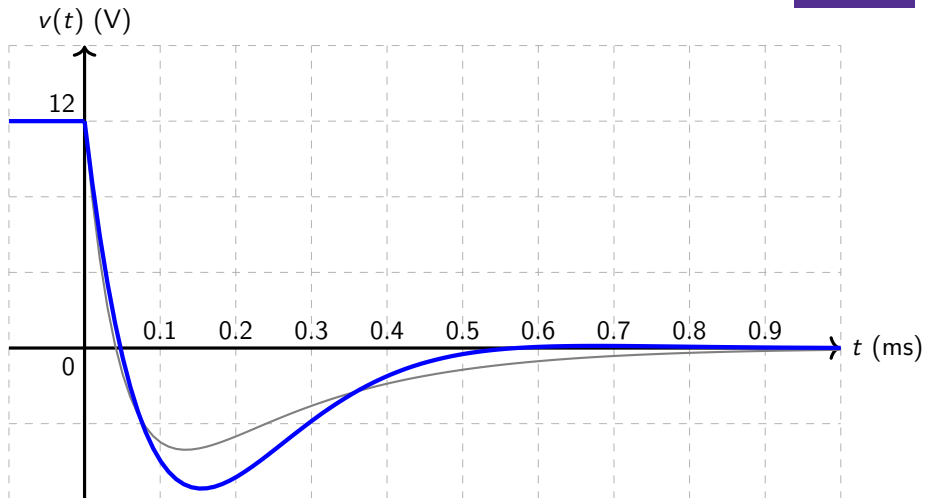
$$i_L(0^+) = -\frac{12 \text{ V}}{312.5 \Omega} + 0.2 \mu\text{F} (2000(48 \text{ V} - 3B_2)) = 30 \text{ mA}$$

$$\implies B_2 = -41 \text{ V}$$

Final solution

$$v(t) = e^{-8000t} (12 \cos(6000t) - 41 \sin(6000t))$$

Natural Response (underdamped)



Natural Response (critically damped)



$\Delta = 0$ (meaning $\xi = 1$): 2 real non distinct roots

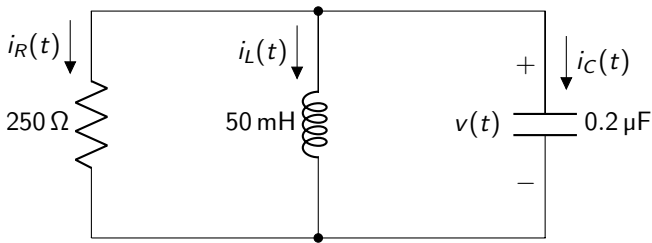
$$s = -\omega_0$$

In that particular case only, the solution form we assumed at the beginning is not complete.

Actually the solution will be in the form:

$$v(t) = (A_1 + A_2 t)e^{-\omega_0 t}$$

Natural Response (critically damped)



Initial conditions:

- $i_L(0^+) = 30\text{ mA}$ (inductor)
- $v(0^+) = 12\text{ V}$ (capacitor)

Natural Response (critically damped)



Evaluate the different values

- $\omega_0 = \frac{1}{\sqrt{LC}} = 10\,000 \text{ rad/s}$
- $\xi = \frac{1}{2R} \sqrt{\frac{L}{C}} = 1 \text{ (no unit)}$

Solution

$$v(t) = (A_1 + A_2 t)e^{-10000t}$$

Natural Response (critically damped)



Determination of constants

We know that $v(0^+) = 12 \text{ V}$

$$A_1 = 12 \text{ V}$$

We also know that $i_L(0^+) = 30 \text{ mA}$

$$i_L(t) = -i_R(t) - i_C(t) = -\frac{v(t)}{R} - C \frac{dv(t)}{dt}$$

$$\frac{dv(t)}{dt} = -10000v(t) + A_2 e^{-10000t}$$

$$i_L(0^+) = -\frac{12 \text{ V}}{250 \Omega} + 0.2 \mu\text{F} (10000 \cdot 12 \text{ V} - A_2) = 30 \text{ mA}$$

$$\implies A_2 = -270\,000 \text{ V/s}$$

Final solution

$$v(t) = 12e^{-10000t} - 270000te^{-10000t}$$

Natural Response (critically damped)

