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QUANTUM MECHANICS I: WAVE FUNCTIONS

VP40.4.1. **IDENTIFY:** This problem involves an electron in a one-dimensional box.

SET UP: $E_n = n^2 \dfrac{\pi^2 h^2}{8mL^2}.$

EXECUTE: **(a)** We want the width L of the box. Solve the energy equation for L with $n = 1$ and use the given energy for E_1. $L = \dfrac{h}{\sqrt{8mE_1}} = 0.549$ nm.

(b) We want the energy. Use $E_n = n^2 E_1$ with the given energy for E_1. $E_2 = 2^2 E_1 = 4(2.00\times10^{-19}\text{ J}) = 8.00\times10^{-19}$ J. $E_3 = 3^2(2.00\times10^{-19}\text{ J}) = 1.80\times10^{-19}$ J.

EVALUATE: Note that the energy levels are not evenly spaced.

VP40.4.2. **IDENTIFY:** This problem involves an electron in a one-dimensional box.

SET UP: $E_n = n^2 \dfrac{\pi^2 h^2}{8mL^2}$. We want the energy difference between levels.

EXECUTE: **(a)** Using the energy equation, the energy difference between levels is

$$\Delta E_{2,1} = \frac{h^2}{8mL^2}\left(2^2 - 1^2\right) = \frac{3h^2}{8mL^2} = \frac{3h^2}{8m\left(5.00\times10^{-15}\text{ m}\right)^2} = 3.94\times10^{-12}\text{ J}.$$

(b) Using the same equation as in part (a) gives

$$\Delta E_{2,1} = \frac{h^2}{8mL^2}\left(3^2 - 2^2\right) = \frac{5h^2}{8mL^2} = 6.57\times10^{-12}\text{ J}.$$

EVALUATE: Note that the energy difference between adjacent levels increases as the levels increase to higher values of n.

VP40.4.3. **IDENTIFY:** This problem involves transitions by an electron in a one-dimensional box.

SET UP: $E_n = n^2 \dfrac{\pi^2 h^2}{8mL^2}$, $E = hc/\lambda$.

EXECUTE: **(a)** We want the energy of the photon. $E = hc/\lambda = hc/(655\text{ nm}) = 3.03 \times 10^{-19}$ J.

(b) We want the length L of the box. The energy of the photon is equal to the energy difference between the $n = 1$ and $n = 2$ levels.

$$E_{\text{ph}} = \frac{h^2}{8mL^2}\left(2^2 - 1^2\right) = \frac{3h^2}{8mL^2}.$$

Solve for L using the photon energy from part (a), giving $L = 0.772$ nm.

(c) We want the wavelength of the photon. The energy of the photon is the energy difference between the $n = 2$ and $n = 3$ levels. Use $E = hc/\lambda$ and $L = 0.772$ nm and solve for λ.

$$E_{\text{ph}} = \frac{h^2}{8mL^2}\left(3^2 - 2^2\right) = \frac{5h^2}{8mL^2} = \frac{hc}{\lambda}. \quad \lambda = \frac{8mL^2c}{5h} = 393 \text{ nm}.$$

EVALUATE: Note that the wavelength for the 3 → 2 transition is less than the wavelength for the 2 → 1 transition. This result is reasonable because the energy difference is greater for the 3 → 2 transition than it is for the 2 → 1 transition.

VP40.4.4. **IDENTIFY:** This problem involves the wave function for a particle in a box.

SET UP: The wave function and Schrödinger equation are

$$\psi(x) = A\cos\left(\frac{x\sqrt{2mE}}{\hbar} + \phi\right), -\frac{\hbar^2}{2m}\frac{d^2\psi}{dx} = E\psi.$$

EXECUTE: **(a)** We want to show that the given wave function satisfied the Schrödinger equation. First take the second derivative of the wave function and then multiply it by $-\hbar^2/2m$.

$$\frac{d^2\psi}{dx^2} = -\frac{A(2mE)}{\hbar^2}\cos\left(\frac{x\sqrt{2mE}}{\hbar} + \phi\right) - \frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = -\frac{\hbar^2}{2m}\left(-\frac{A(2mE)}{\hbar^2}\right)\cos\left(\frac{x\sqrt{2mE}}{\hbar} + \phi\right)$$

$$= EA\cos\left(\frac{x\sqrt{2mE}}{\hbar} + \phi\right) = E\psi.$$

(b) We want ϕ. The wave function must be zero at $x = 0$, which gives $\cos\phi = 0$, so $\phi = \pm\pi/2$.

(c) We want E. At $x = L$ the wave function must be zero, which gives

$$A\cos\left(\frac{L\sqrt{2mE}}{\hbar} \pm \frac{\pi}{2}\right) = 0. \quad \frac{L\sqrt{2mE}}{\hbar} \pm \frac{\pi}{2} = \frac{\pi}{2}.$$

L cannot be zero, so we must have

$$\frac{L\sqrt{2mE_n}}{\hbar} = n\pi. \quad E_n = n^2\frac{\pi^2\hbar^2}{2mL^2}, n = 1, 2, 3, \cdots$$

EVALUATE: The result in part (c) agrees with Eq. (40.31) in the text.

VP40.6.1. **IDENTIFY:** We are dealing with an electron in a finite potential well.

SET UP: For an infinitely deep well the energy levels are

$$E_{n\text{-IDW}} = n^2\frac{\pi^2\hbar^2}{2mL^2}$$

and for a finite well we use Figure 40.15 in the textbook.

EXECUTE: **(a)** We want the ground level energy, so $n = 1$.

$$E_{1\text{-IDW}} = \frac{\pi^2\hbar^2}{2mL^2} = \frac{\pi^2\hbar^2}{2m(0.350 \text{ nm})^2} = 4.92\times10^{-19} \text{ J}.$$

(b) We want the ground level energy. In this case, $U_0 = 6E_{1\text{-IDW}}$. From Figure 40.15b in the text, we see that $E_1 = 0.625E_{1\text{-IDW}}$, which gives

$E_1 = (0.625)(4.92\times10^{-19} \text{ J}) = 3.07\times10^{-19}$ J.

(c) We want the minimum energy to free the electron. The electron already has 3.07×10^{-19} J of energy, and to be free it needs to have a minimum of $U_0 = 6E_{1\text{-IDW}} = 6(4.92\times10^{-19} \text{ J}) = 2.95\times10^{-18}$ J.

So, the additional energy it needs is $U_0 - E_1 = 2.64\times10^{-18}$ J.

EVALUATE: If the well were infinite, the electron would need infinite energy to escape, meaning that it could not escape.

VP40.6.2. **IDENTIFY:** We are dealing with particle in a finite potential well.

SET UP: $U_0 = 6E_{\text{1-IDW}}$, the energy difference between the $n = 2$ and $n = 3$ levels is the energy of the photon. Refer to Figure 40.15 in the textbook.

EXECUTE: **(a)** We want $E_{\text{1-IDW}}$. Figure 40.15b, shows that $E_3 = 5.09E_{\text{1-IDW}}$ and $E_2 = 2.43E_{\text{1-IDW}}$. The energy difference between these levels is $E_3 - E_2 = (5.09 - 2.43)E_{\text{1-IDW}} = 2.50 \times 10^{-19}$ J, which gives $E_{\text{1-IDW}} = 9.40 \times 10^{-20}$ J.

(b) We want U_0. $U_0 = 6E_{\text{1-IDW}} = 6(9.40 \times 10^{-20}\text{ J}) = 5.64 \times 10^{-19}$ J.

(c) We want the width L of the well. Solve $E_{\text{1-IDW}} = n^2h^2/8mL^2$ for L and use the result from part (b) with $n = 1$. This gives $L = 0.800$ nm.

EVALUATE: The ground state energy of this well is $(0.625)E_{\text{1-IDW}} = 5.88 \times 10^{-20}$ J.

VP40.6.3. **IDENTIFY:** We are dealing with an electron in a finite potential well.

SET UP: $U_0 = 6E_{\text{1-IDW}}$, $E = hc/\lambda$. Use Figure 40.15 in the textbook.

EXECUTE: **(a)** We want the initial and final energy levels. In this case, there are only three possible transitions: $3 \to 2$, $2 \to 1$, and $3 \to 1$. The greatest energy difference is from the $3 \to 1$ transition which emits a photon of the shortest wavelength. So the initial state is $n = 3$ and the final state is $n = 1$.

(b) Figure 40.15b in the textbook gives $E_3 = 5.09E_{\text{1-IDW}}$ and $E_1 = 0.625E_{\text{1-IDW}}$. Therefore the energy of the photon is $E_3 - E_1 = (5.09 - 0625)E_{\text{1-IDW}} = 4.465E_{\text{1-IDW}}$. Using $E = hc/\lambda$ with the 355 nm wavelength and solving for $E_{\text{1-IDW}}$ gives $E_{\text{1-IDW}} = hc/[(4.465)(355\text{ nm})]$.

<u>$3 \to 2$ transition</u>: The energy difference is $(5.09 - 2.43)E_{\text{1-IDW}}$, and this is the photon energy.

$$\frac{hc}{\lambda_{3\to2}} = 2.66E_{\text{1-IDW}} = (2.66)\frac{hc}{(4.465)(355\text{ nm})}.\ \lambda_{3\to2} = 596\text{ nm}.$$

<u>$2 \to 1$ transition</u>: The energy difference is $(2.43 - 0.625)E_{\text{1-IDW}}$, and this is the photon energy.

$$\frac{hc}{\lambda_{2\to1}} = 1.805E_{\text{1-IDW}} = (1.805)\frac{hc}{(4.465)(355\text{ nm})}.\ \lambda_{2\to1} = 878\text{ nm}.$$

EVALUATE: As the energy difference between the levels gets larger, the photon wavelength gets shorter, which is physically reasonable.

VP40.6.4. **IDENTIFY:** We are dealing with an electron in a finite potential well.

SET UP: $E_{\text{1-IDW}} = \dfrac{\pi^2\hbar^2}{2mL^2}.$

EXECUTE: **(a)** We want the ground level energy for an infinite well. Using $L = 0.400$ nm and the equation for $E_{\text{1-IDW}}$ gives 2.35 eV.

(b) We want the energy of the bound state if $U_0 = 0.015$ eV. Because $U_0 \ll E_{\text{1-IDW}}$, $E = 0.68U_0$. Therefore $E = (0.68)(0.015\text{ eV}) = 0.010$ eV.

EVALUATE: The energy of the bound state is much less than the ground level for an infinite well of the same length.

VP40.7.1. **IDENTIFY:** This problem is about an electron tunneling through a potential barrier. We want the probability that the electron will tunnel through barriers if different thicknesses if its energy is 3.75 eV and the barrier height is 6.10 eV.

SET UP: The probability T of tunneling us $T = Ge^{-2\kappa L}$, where

$$G = 16\frac{E}{U_0}\left(1 - \frac{E}{U_0}\right) \text{ and } \kappa = \sqrt{\frac{2m(U_0 - E)}{\hbar}}.$$

EXECUTE: **(a)** $L = 0.750$ nm. First calculate G and κ and then use them to find T.

$$G = 16\left(\frac{3.75\text{ eV}}{6.10\text{ eV}}\right)\left(1 - \frac{3.75\text{ eV}}{6.10\text{ eV}}\right) = 3.7893,\quad \kappa = \frac{\sqrt{2m(6.10 - 3.75)\text{ eV}}}{\hbar} = 7.85005\times10^9\text{ m}^{-1}$$

Using these values gives $T = (3.789)e^{-11.763} = 2.92 \times 10^{-5}$.

(b) $L = 0.500$ nm. We get $T = (3.789)e^{-7.85005} = 1.48 \times 10^{-3}$.

EVALUATE: It is about 50 times more likely that the electron will tunnel through the narrower barrier than through the wider one. But probabilities are very low.

VP40.7.2. **IDENTIFY:** This problem is about an electron tunneling through a potential barrier.

SET UP: The probability T of tunneling us $T = Ge^{-2\kappa L}$, where

$$G = 16\frac{E}{U_0}\left(1-\frac{E}{U_0}\right) \text{ and } \kappa = \sqrt{\frac{2m(U_0-E)}{\hbar}}.$$

EXECUTE: **(a)** We want T when $L = 0.800$ nm. First calculate G and κ and then use them to find T.

$$G = 16\left(\frac{3.50 \text{ eV}}{4.00 \text{ eV}}\right)\left(1-\frac{3.50 \text{ eV}}{4.00 \text{ eV}}\right) = 1.7500, \quad \kappa = \frac{\sqrt{2m(4.00-3.50) \text{ eV}}}{\hbar} = 3.62096\times10^9 \text{ m}^{-1}.$$

Using these values gives $T = (1.7500)e^{-5.79354} = 5.33 \times 10^{-3}$.

(b) We want L, so that the probability of tunneling is twice as great as in part (a). Solve $T = Ge^{-2\kappa L}$ for L. G and κ are the same as in part (a). Taking natural logarithms gives $L = -(1/2\kappa)\ln(T/G)$. Using G and κ with T twice what we found in part (a), we get $L = 0.704$ nm.

EVALUATE: Decreasing the width of the barrier from 0.800 nm to 0.704 nm doubled the probability of tunneling.

VP40.7.3. **IDENTIFY:** This problem is about an electron tunneling through a potential barrier. We want to find the probability of tunneling for different energies of the electron with $U_0 = 5.00$ eV and $L = 0.900$ nm.

SET UP: The probability T of tunneling us $T = Ge^{-2\kappa L}$, where

$$G = 16\frac{E}{U_0}\left(1-\frac{E}{U_0}\right) \text{ and } \kappa = \frac{\sqrt{2m(U_0-E)}}{\hbar}.$$

EXECUTE: **(a)** $E = 4.00$ eV. We want T when $L = 0.800$ nm. First calculate G and κ and then use them to find T.

$$G = 16\left(\frac{3.50 \text{ eV}}{4.00 \text{ eV}}\right)\left(1-\frac{3.50 \text{ eV}}{4.00 \text{ eV}}\right) = 1.7500, \quad \kappa = \frac{\sqrt{2m(4.00-3.50) \text{ eV}}}{\hbar} = 3.62096\times10^9 \text{ m}^{-1}$$

Using these values gives $T = (1.7500)e^{-5.79354} = 5.33 \times 10^{-3}$.

(b) $E = 4.30$ eV. $G = 1.9264$, $\kappa = 4.2844 \times 10^9 \text{ m}^{-1}$, $T = (1.9264)e^{-7.71188} = 8.62 \times 10^{-4}$.

(c) $E = 4.60$ eV. $G = 1.1776$, $\kappa = 3.23868 \times 10^9 \text{ m}^{-1}$, $T = (1.1776)e^{-5.82963} = 3.46 \times 10^{-3}$.

EVALUATE: Our results show that as E gets closer to the height of the potential barrier, the probability of tunneling increases, which is physically reasonable.

VP40.7.4. **IDENTIFY:** This problem is about an electron tunneling through a potential barrier. We want to find the width L of the barrier if there is a 1/417 probability that the electron will tunnel through the barrier if the electron's energy is 3.00 eV and the barrier height is 5.00 eV.

SET UP: The probability T of tunneling us $T = Ge^{-2\kappa L}$, where

$$G = 16\frac{E}{U_0}\left(1-\frac{E}{U_0}\right) \text{ and } \kappa = \frac{\sqrt{2m(U_0-E)}}{\hbar}.$$

EXECUTE: Solve $T = Ge^{-2\kappa L}$ for L. Taking natural logarithms gives $L = -(1/2\kappa)\ln(T/G)$.

$$G = 16\left(\frac{3.00 \text{ eV}}{5.00 \text{ eV}}\right)\left(1-\frac{3.00 \text{ eV}}{5.00 \text{ eV}}\right) = 3.8400, \quad \kappa = \frac{\sqrt{2m(5.00-3.00) \text{ eV}}}{\hbar} = 7.24192\times10^9 \text{ m}^{-1}.$$

Using $T = 1/417$ and the G and κ we just calculated gives $L = 0.509$ nm.

EVALUATE: The value of L is comparable to atomic dimensions.

40.1. IDENTIFY: Using the momentum of the free electron, we can calculate k and ω and use these to express its wave function.

SET UP: $\Psi(x,t) = Ae^{ikx}e^{-i\omega t}$, $k = p/\hbar$, and $\omega = \hbar k^2/2m$.

EXECUTE: $k = \dfrac{p}{\hbar} = -\dfrac{4.50\times10^{-24}\ \text{kg}\cdot\text{m/s}}{1.055\times10^{-34}\ \text{J}\cdot\text{s}} = -4.27\times10^{10}\ \text{m}^{-1}$.

$$\omega = \frac{\hbar k^2}{2m} = \frac{(1.055\times10^{-34}\ \text{J}\cdot\text{s})(4.27\times10^{10}\ \text{m}^{-1})^2}{2(9.108\times10^{-31}\ \text{kg})} = 1.05\times10^{17}\ \text{s}^{-1}.$$

$$\Psi(x,t) = Ae^{-i(4.27\times10^{10}\ \text{m}^{-1})x}e^{-i(1.05\times10^{17}\ \text{s}^{-1})t}.$$

EVALUATE: The wave function depends on position and time.

40.2. IDENTIFY: Using the known wave function for the particle, we want to find where its probability function is a maximum.

SET UP: $|\Psi(x,t)|^2 = |A|^2(e^{ikx}e^{-i\omega t} - e^{2ikx}e^{-4i\omega t})(e^{-ikx}e^{+i\omega t} - e^{-2ikx}e^{+4i\omega t})$.

$|\Psi(x,t)|^2 = |A|^2\left[2 - (e^{-i(kx-3\omega t)} + e^{+i(kx-3\omega t)})\right] = 2|A|^2[1-\cos(kx-3\omega t)]$.

EXECUTE: **(a)** For $t=0$, $|\Psi(x,t)|^2 = 2|A|^2(1-\cos(kx))$. $|\Psi(x,t)|^2$ is a maximum when $\cos(kx) = -1$ and this happens when $kx = (2n+1)\pi,\ n = 0, 1, \ldots$. $|\Psi(x,t)|^2$ is a maximum for $x = \dfrac{\pi}{k}, \dfrac{3\pi}{k}$, etc.

(b) $t = \dfrac{2\pi}{\omega}$ and $3\omega t = 6\pi$. $|\Psi(x,t)|^2 = 2|A|^2[1-\cos(kx-6\pi)]$. Maximum for $kx - 6\pi = \pi, 3\pi, \ldots$, which gives maxima when $x = \dfrac{7\pi}{k}, \dfrac{9\pi}{k}$.

(c) From the results for parts (a) and (b), $v_{\text{av}} = \dfrac{7\pi/k - \pi/k}{2\pi/\omega} = \dfrac{3\omega}{k}$. $v_{\text{av}} = \dfrac{\omega_2-\omega_1}{k_2-k_1}$ with $\omega_2 = 4\omega$, $\omega_1 = \omega$, $k_2 = 2k$ and $k_1 = k$ gives $v_{\text{av}} = \dfrac{3\omega}{k}$.

EVALUATE: The expressions in part (c) agree.

40.3. IDENTIFY: Use the wave function from Example 40.1.

SET UP: $|\Psi(x,t)|^2 = 2|A|^2\{1+\cos[(k_2-k_1)x - (\omega_2-\omega_1)t]\}$. $k_2 = 3k_1 = 3k$. $\omega = \dfrac{\hbar k^2}{2m}$, so $\omega_2 = 9\omega_1 = 9\omega$.

$|\Psi(x,t)|^2 = 2|A|^2[1+\cos(2kx - 8\omega t)]$.

EXECUTE: **(a)** At $t = 2\pi/\omega$, $|\Psi(x,t)|^2 = 2|A|^2[1+\cos(2kx-16\pi)]$. $|\Psi(x,t)|^2$ is maximum for $\cos(2kx-16\pi) = 1$. This happens for $2kx - 16\pi = 0, 2\pi, \ldots$. Smallest positive x where $|\Psi(x,t)|^2$ is a maximum is $x = \dfrac{8\pi}{k}$.

(b) From the result of part (a), $v_{\text{av}} = \dfrac{8\pi/k}{2\pi/\omega} = \dfrac{4\omega}{k}$. $v_{\text{av}} = \dfrac{\omega_2-\omega_1}{k_2-k_1} = \dfrac{8\omega}{2k} = \dfrac{4\omega}{k}$.

EVALUATE: The two expressions agree.

40.4. IDENTIFY: Apply the Heisenberg uncertainty principle in the form $\Delta x \Delta p_x \geq \hbar/2$.

SET UP: The uncertainty in the particle position is proportional to the width of $\psi(x)$.

EXECUTE: **(a)** The width of $\psi(x)$ is inversely proportional to $\sqrt{\alpha}$. This can be seen by either plotting the function for different values of α or by finding the full width at half-maximum. The particle's uncertainty in position decreases with increasing α.

(b) Since the uncertainty in position decreases, the uncertainty in momentum must increase.
EVALUATE: As α increases, the function $A(k)$ in Eq. (40.19) must become broader.

40.5. **IDENTIFY** and **SET UP:** $\psi(x) = A\sin kx$. The position probability density is given by $|\psi(x)|^2 = A^2 \sin^2 kx$.

EXECUTE: **(a)** The probability is highest where $\sin kx = 1$ so $kx = 2\pi x/\lambda = n\pi/2,\ n = 1, 3, 5, \ldots$
$x = n\lambda/4,\ n = 1,\ 3,\ 5, \ldots$ so $x = \lambda/4, 3\lambda/4, 5\lambda/4, \ldots$

(b) The probability of finding the particle is zero where $|\psi|^2 = 0$, which occurs where $\sin kx = 0$ and
$kx = 2\pi x/\lambda = n\pi,\ n = 0,\ 1, 2, \ldots$
$x = n\lambda/2,\ n = 0, 1, 2, \ldots$, so $x = 0,\ \lambda/2,\ \lambda,\ 3\lambda/2, \ldots$

EVALUATE: The situation is analogous to a standing wave, with the probability analogous to the square of the amplitude of the standing wave.

40.6. **IDENTIFY:** Determine whether or not $-\dfrac{\hbar^2}{2m}\dfrac{d^2\psi}{dx^2} + U\psi$ is equal to $E\psi$, for some value of E.

SET UP: $-\dfrac{\hbar^2}{2m}\dfrac{d^2\psi_1}{dx^2} + U\psi_1 = E_1\psi_1$ and $-\dfrac{\hbar^2}{2m}\dfrac{d^2\psi_2}{dx^2} + U\psi_2 = E_2\psi_2$.

EXECUTE: $-\dfrac{\hbar^2}{2m}\dfrac{d^2\psi}{dx^2} + U\psi = BE_1\psi_1 + CE_2\psi_2$. If ψ were a solution with energy E, then $BE_1\psi_1 + CE_2\psi_2 = BE\psi_1 + CE\psi_2$ or $B(E_1 - E)\psi_1 = C(E - E_2)\psi_2$. This would mean that ψ_1 is a constant multiple of ψ_2, and ψ_1 and ψ_2 would be wave functions with the same energy. However, $E_1 \neq E_2$, so this is not possible, and ψ cannot be a solution to the equation $-\dfrac{\hbar^2}{2m}\dfrac{d^2\psi}{dx^2} + U\psi = E\psi$.

EVALUATE: ψ is a solution if $E_1 = E_2$.

40.7. **IDENTIFY:** We are dealing with a particle in a box.

SET UP: $E = hc/\lambda,\ \ E_n = n^2\dfrac{\pi^2\hbar^2}{2mL^2}$. We want the wavelength.

EXECUTE: The longest wavelength (lowest energy) photon is from a transition between the $n = 1$ to $n = 2$ states. The energy of this photon is equal to the energy difference between these states.

$$\Delta E_1 = \frac{\pi^2\hbar^2}{2mL^2}\left(2^2 - 1^2\right) = \frac{hc}{\lambda_1}.$$

The next longest wavelength photon is from a transition from the $n = 1$ to the $n = 3$ state.

$$\Delta E_2 = \frac{\pi^2\hbar^2}{2mL^2}\left(3^2 - 1^2\right) = \frac{hc}{\lambda_2}.$$

Dividing these two equations and solving for λ_2 gives

$$\frac{4-1}{9-1} = \frac{\lambda_2}{\lambda_1}.\ \lambda_2 = \frac{3}{8}\lambda_1 = \frac{3}{8}(420\text{ nm}) = 158\text{ nm}.$$

EVALUATE: The next longest photon (starting from the ground state) would be between the $n = 1$ and $n = 4$ states.

40.8. **IDENTIFY:** To describe a real situation, a wave function must be normalizable.

SET UP: $|\psi|^2\, dV$ is the probability that the particle is found in volume dV. Since the particle must be *somewhere*, ψ must have the property that $\int |\psi|^2 dV = 1$ when the integral is taken over all space.

EXECUTE: **(a)** For normalization of the one-dimensional wave function, we have

$$1=\int_{-\infty}^{\infty}|\psi|^2dx=\int_{-\infty}^{0}(Ae^{bx})^2dx+\int_{0}^{\infty}(Ae^{-bx})^2dx=\int_{-\infty}^{0}A^2e^{2bx}dx+\int_{0}^{\infty}A^2e^{-2bx}dx.$$

$$1=A^2\left\{\frac{e^{2bx}}{2b}\bigg|_{-\infty}^{0}+\frac{e^{-2bx}}{-2b}\bigg|_{0}^{\infty}\right\}=\frac{A^2}{b},\ \text{which gives}\ A=\sqrt{b}=\sqrt{2.00\ \text{m}^{-1}}=1.41\ \text{m}^{-1/2}.$$

(b) The graph of the wavefunction versus x is given in Figure 40.8.

(c) (i) $P=\int_{-0.500\ \text{m}}^{+0.500\ \text{m}}|\psi|^2dx=2\int_{0}^{+0.500\ \text{m}}A^2e^{-2bx}dx,$ where we have used the fact that the wave function is an even function of x. Evaluating the integral gives

$$P=\frac{-A^2}{b}(e^{-2b(0.500\ \text{m})}-1)=\frac{-(2.00\ \text{m}^{-1})}{2.00\ \text{m}^{-1}}(e^{-2.00}-1)=0.865.$$

There is a little more than an 86% probability that the particle will be found within 50 cm of the origin.

(ii) $P=\int_{-\infty}^{0}(Ae^{bx})^2dx=\int_{-\infty}^{0}A^2e^{2bx}dx=\frac{A^2}{2b}=\frac{2.00\ \text{m}^{-1}}{2(2.00\ \text{m}^{-1})}=\frac{1}{2}=0.500.$

There is a 50-50 chance that the particle will be found to the left of the origin, which agrees with the fact that the wave function is symmetric about the y-axis.

(iii) $P=\int_{0.500\ \text{m}}^{1.00\ \text{m}}A^2e^{-2bx}dx=\frac{A^2}{-2b}(e^{-2(2.00\ \text{m}^{-1})(1.00\ \text{m})}-e^{-2(2.00\ \text{m}^{-1})(0.500\ \text{m})})=-\frac{1}{2}(e^{-4}-e^{-2})$

$=0.0585.$

EVALUATE: There is little chance of finding the particle in regions where the wave function is small.

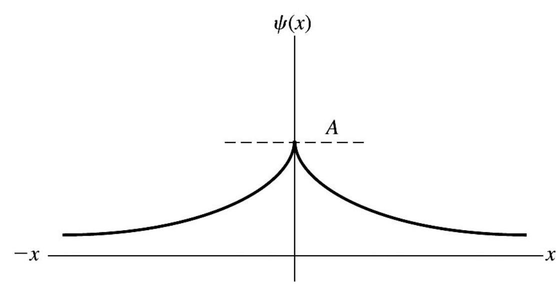

Figure 40.8

40.9. **IDENTIFY** and **SET UP:** The energy levels for a particle in a box are given by $E_n=\frac{n^2h^2}{8mL^2}.$

EXECUTE: **(a)** The lowest level is for $n=1,$ and $E_1=\frac{(1)(6.626\times10^{-34}\ \text{J}\cdot\text{s})^2}{8(0.20\ \text{kg})(1.3\ \text{m})^2}=1.6\times10^{-67}\ \text{J}.$

(b) $E=\frac{1}{2}mv^2,$ so $v=\sqrt{\frac{2E}{m}}=\sqrt{\frac{2(1.2\times10^{-67}\ \text{J})}{0.20\ \text{kg}}}=1.3\times10^{-33}\ \text{m/s}.$ If the ball has this speed the time it would take it to travel from one side of the table to the other is

$$t=\frac{1.3\ \text{m}}{1.3\times10^{-33}\ \text{m/s}}=1.0\times10^{33}\ \text{s}.$$

(c) $E_1=\frac{h^2}{8mL^2},\ E_2=4E_1,$ so $\Delta E=E_2-E_1=3E_1=3(1.6\times10^{-67}\ \text{J})=4.9\times10^{-67}\ \text{J}.$

EVALUATE: **(d)** No, quantum mechanical effects are not important for the game of billiards. The discrete, quantized nature of the energy levels is completely unobservable.

40.10. **IDENTIFY:** Solve the energy-level equation $E_n = \dfrac{n^2h^2}{8mL^2}$ for L.

SET UP: The ground state has $n = 1$.

EXECUTE: $L = \dfrac{h}{\sqrt{8mE_1}} = \dfrac{(6.626\times10^{-34}\ \text{J}\cdot\text{s})}{\sqrt{8(1.673\times10^{-27}\ \text{kg})(5.0\times10^{6}\ \text{eV})(1.602\times10^{-19}\ \text{J/eV})}} = 6.4\times10^{-15}\ \text{m}$

EVALUATE: The value of L we calculated is on the order of the diameter of a nucleus.

40.11. **IDENTIFY:** An electron in the lowest energy state in this box must have the same energy as it would in the ground state of hydrogen.

SET UP: The energy of the n^{th} level of an electron in a box is $E_n = \dfrac{nh^2}{8mL^2}$.

EXECUTE: An electron in the ground state of hydrogen has an energy of $-13.6\ \text{eV}$, so find the width corresponding to an energy of $E_1 = 13.6\ \text{eV}$. Solving for L gives

$$L = \frac{h}{\sqrt{8mE_1}} = \frac{(6.626\times10^{-34}\ \text{J}\cdot\text{s})}{\sqrt{8(9.11\times10^{-31}\ \text{kg})(13.6\ \text{eV})(1.602\times10^{-19}\ \text{J/eV})}} = 1.66\times10^{-10}\ \text{m}.$$

EVALUATE: This width is of the same order of magnitude as the diameter of a Bohr atom with the electron in the K shell.

40.12. **IDENTIFY** and **SET UP:** The energy of a photon is $E = hf = h\dfrac{c}{\lambda}$. The energy levels of a particle in a box are given by $E_n = \dfrac{nh^2}{8mL^2}$.

EXECUTE: **(a)** $E = (6.63\times10^{-34}\ \text{J}\cdot\text{s})\dfrac{(3.00\times10^{8}\ \text{m/s})}{(122\times10^{-9}\ \text{m})} = 1.63\times10^{-18}\ \text{J}$. $\Delta E = \dfrac{h^2}{8mL^2}(n_1^2 - n_2^2)$.

$$L = \sqrt{\frac{h^2(n_1^2 - n_2^2)}{8m\Delta E}} = \sqrt{\frac{(6.63\times10^{-34}\ \text{J}\cdot\text{s})^2(2^2 - 1^2)}{8(9.11\times10^{-31}\ \text{kg})(1.63\times10^{-18}\ \text{J})}} = 3.33\times10^{-10}\ \text{m}.$$

(b) The ground state energy for an electron in a box of the calculated dimensions is

$E = \dfrac{h^2}{8mL^2} = \dfrac{(6.63\times10^{-34}\ \text{J}\cdot\text{s})^2}{8(9.11\times10^{-31}\ \text{kg})(3.33\times10^{-10}\ \text{m})^2} = 5.43\times10^{-19}\ \text{J} = 3.40\ \text{eV}$ (one-third of the original photon energy), which does not correspond to the $-13.6\ \text{eV}$ ground state energy of the hydrogen atom.

EVALUATE: **(c)** Note that the energy levels for a particle in a box are proportional to n^2, whereas the energy levels for the hydrogen atom are proportional to $-\dfrac{1}{n^2}$. A one-dimensional box is not a good model for a hydrogen atom.

40.13. **IDENTIFY** and **SET UP:** The equation $E_n = \dfrac{n^2h^2}{8mL^2}$ gives the energy levels. Use this to obtain an expression for $E_2 - E_1$ and use the value given for this energy difference to solve for L.

EXECUTE: Ground state energy is $E_1 = \dfrac{h^2}{8mL^2}$; first excited state energy is $E_2 = \dfrac{4h^2}{8mL^2}$. The energy separation between these two levels is $\Delta E = E_2 - E_1 = \dfrac{3h^2}{8mL^2}$. This gives

$$L = h\sqrt{\frac{3}{8m\Delta E}} = L = 6.626\times10^{-34}\ \text{J}\cdot\text{s}\sqrt{\frac{3}{8(9.109\times10^{-31}\ \text{kg})(3.0\ \text{eV})(1.602\times10^{-19}\ \text{J/1 eV})}}$$

$= 6.1\times10^{-10}\ \text{m} = 0.61\ \text{nm}$.

EVALUATE: This energy difference is typical for an atom and L is comparable to the size of an atom.

40.14. **IDENTIFY:** The energy of the absorbed photon must be equal to the energy difference between the two states.

SET UP and **EXECUTE:** The second excited state energy is $E_3 = \dfrac{9\pi^2\hbar^2}{2mL^2}$. The ground state energy is $E_1 = \dfrac{\pi^2\hbar^2}{2mL^2}$. $E_1 = 2.00$ eV, so $E_3 = 18.0$ eV. For the transition $\Delta E = \dfrac{4\pi^2\hbar^2}{mL^2}$. $\dfrac{hc}{\lambda} = \Delta E$.

$\lambda = \dfrac{hc}{\Delta E} = \dfrac{(4.136\times10^{-15}\text{ eV}\cdot\text{s})(2.998\times10^{8}\text{ m/s})}{16.0\text{ eV}} = 7.75\times10^{-8}\text{ m} = 77.5\text{ nm}.$

EVALUATE: This wavelength is much shorter than those of visible light.

40.15. **IDENTIFY:** We are dealing with a particle in a box and the de Broglie wavelength.

SET UP: $\lambda = h/p,\ E_n = n^2\dfrac{\pi^2\hbar^2}{2mL^2},\ K = p^2/2m.$

EXECUTE: **(a)** We want the momentum. From the 9 in the numerator of the given energy, we see that $n^2 = 9$, so $n = 3$. The particle's energy is kinetic energy, so $E_3 = K_3 = p_3^2/2m$. Therefore,

$$E_3 = \frac{9\pi^2\hbar^2}{2mL^2} = \frac{p_3^2}{2m}.\ p = \sqrt{\frac{9\pi^2\hbar^2}{L^2}} = \frac{3h}{2L}.$$

(b) We want L/λ. $L/\lambda = L/(h/p) = Lp/h = L(3h/2L)/h = 3/2$.

EVALUATE: The ration in part (b) would be different if the particle were in a different state.

40.16. **IDENTIFY:** Find x where ψ_1 is zero and where it is a maximum.

SET UP: $\psi_1 = \sqrt{\dfrac{2}{L}}\sin\left(\dfrac{\pi x}{L}\right).$

EXECUTE: **(a)** The wave function for $n = 1$ vanishes only at $x = 0$ and $x = L$ in the range $0 \le x \le L$.
(b) In the range for x, the sine term is a maximum only at the middle of the box, $x = L/2$.

EVALUATE: **(c)** The answers to parts (a) and (b) are consistent with the figure.

40.17. **IDENTIFY:** We are dealing with a particle in a box.

SET UP: $E_n = n^2\dfrac{\pi^2\hbar^2}{2mL^2},\ m_A = 9m_B, L_B = 2L_A, E_A = E_B$. We want the lowest possible quantum numbers n_A and n_B of the two states.

EXECUTE: Equate the energies and determine the lowest possible values of n_A and n_B.

$$\frac{n_A^2\pi^2\hbar^2}{2m_A^2L_A^2} = \frac{n_B^2\pi^2\hbar^2}{2m_B^2L_B^2}.\quad \frac{n_A^2}{(9m_B)(L_B/2)^2} = \frac{n_B^2}{m_BL_B^2}.\ 4n_A^2 = 9n_B^2.$$

If $n_B = 2$, then $n_A = 3$, so the lowest possible values of the quantum numbers are $n_A = 3$, $n_B = 2$.

EVALUATE: Other states exist, such as $n_A = 6$, $n_B = 4$, but these are the lowest ones.

40.18. **IDENTIFY:** The energy levels are given by $E_n = \dfrac{n^2h^2}{8mL^2}$. The wavelength λ of the photon absorbed in an atomic transition is related to the transition energy ΔE by $\lambda = \dfrac{hc}{\Delta E}$.

SET UP: For the ground state $n = 1$ and for the third excited state $n = 4$.

EXECUTE: **(a)** The third excited state is $n = 4$, so

$$\Delta E = (4^2 - 1)\frac{h^2}{8mL^2} = \frac{15(6.626\times10^{-34}\text{ J}\cdot\text{s})^2}{8(9.11\times10^{-31}\text{ kg})(0.360\times10^{-9}\text{ m})^2} = 6.973\times10^{-18}\text{ J} = 43.5\text{ eV}.$$

(b) $\lambda = \dfrac{hc}{\Delta E} = \dfrac{(6.63\times10^{-34}\ \text{J}\cdot\text{s})(3.0\times10^{8}\ \text{m/s})}{6.973\times10^{-18}\ \text{J}} = 28.5\ \text{nm}.$

EVALUATE: This photon is an x ray. As the width of the box increases the transition energy for this transition decreases and the wavelength of the photon increases.

40.19. IDENTIFY and **SET UP:** $\lambda = \dfrac{h}{p} = \dfrac{h}{\sqrt{2mE}}$. The energy of the electron in level n is given by the equation $E_n = \dfrac{n^2h^2}{8mL^2}$.

EXECUTE: **(a)** $E_1 = \dfrac{h^2}{8mL^2} \Rightarrow \lambda_1 = \dfrac{h}{\sqrt{2mh^2/8mL^2}} = 2L = 2(3.0\times10^{-10}\ \text{m}) = 6.0\times10^{-10}\ \text{m}$. The wavelength is twice the width of the box. $p_1 = \dfrac{h}{\lambda_1} = \dfrac{(6.63\times10^{-34}\ \text{J}\cdot\text{s})}{6.0\times10^{-10}\ \text{m}} = 1.1\times10^{-24}\ \text{kg}\cdot\text{m/s}.$

(b) $E_2 = \dfrac{4h^2}{8mL^2} \Rightarrow \lambda_2 = L = 3.0\times10^{-10}\ \text{m}$. The wavelength is the same as the width of the box.

$p_2 = \dfrac{h}{\lambda_2} = 2p_1 = 2.2\times10^{-24}\ \text{kg}\cdot\text{m/s}.$

(c) $E_3 = \dfrac{9h^2}{8mL^2} \Rightarrow \lambda_3 = \dfrac{2}{3}L = 2.0\times10^{-10}\ \text{m}$. The wavelength is two-thirds the width of the box.

$p_3 = 3p_1 = 3.3\times10^{-24}\ \text{kg}\cdot\text{m/s}.$

EVALUATE: In each case the wavelength is an integer multiple of $\lambda/2$. In the n^{th} state, $p_n = np_1$.

40.20. IDENTIFY: The energy of the photon is equal to the energy difference ΔE between the energy levels of the electron.

SET UP: The energy levels of an electron in a one-dimensional box are $E_n = \dfrac{n^2h^2}{8mL^2}$. The energy of the absorbed photon is $\Delta E = \dfrac{hc}{\lambda}$.

EXECUTE: **(a)** $\Delta E_{1\to2} = (h^2/8mL^2)(2^2 - 1^2) = 3(h^2/8mL^2) = hc/\lambda_{1\to2}$.

$\Delta E_{2\to3} = (h^2/8mL^2)(3^2 - 2^2) = 5(h^2/8mL^2) = hc/\lambda_{2\to3}$. Take ratios of these two equations, giving

$\dfrac{3}{5} = \dfrac{hc/\lambda_{1\to2}}{hc/\lambda_{2\to3}} = \dfrac{\lambda_{2\to3}}{\lambda_{1\to2}} \;\rightarrow\; \lambda_{2\to3} = (3/5)\lambda_{1\to2} = (3/5)(426\ \text{nm}) = 256\ \text{nm}.$

(b) Follow the same procedure as in part (a), giving

$$\lambda_{1\to3} = (3/8)\lambda_{1\to2} = (3/8)(426\ \text{nm}) = 160\ \text{nm}.$$

(c) From part (a), we know that $\Delta E_{1\to2} = \dfrac{3h^2}{8mL^2} = \dfrac{hc}{\lambda_{1\to2}}$. Solving for L gives

$$L = \sqrt{\frac{3h\lambda_{1\to2}}{8\,mc}} = \sqrt{\frac{3(6.626\times10^{-34}\ \text{J}\cdot\text{s})(426\times10^{-9}\ \text{m})}{8(9.109\times10^{-31}\ \text{kg})(2.998\times10^{8}\ \text{m/s})}} = 6.23\times10^{-10}\ \text{m} = 0.623\ \text{nm}.$$

EVALUATE: The width L of this box is about 6 times the diameter of a hydrogen atom.

40.21. IDENTIFY: We are dealing with a particle in a box.

SET UP: $-\dfrac{\hbar^2}{2m}\dfrac{d^2\psi}{dx} = E\psi.$

EXECUTE: **(a)** $\psi_n = A\cos k_n x$. Take the second derivative and use it to find E_n as follows.

$$-\frac{\hbar^2}{2m}\left(-Ak_n^2 \cos k_n x\right) = EA\cos k_n x.\ \ E_n = \frac{\hbar^2}{2m}k_n^2.$$

The wave function must be zero at $x = \pm L/2$. This gives
$A\cos[k_n(\pm L/2)] = 0.\ k_n(L/2) = \pi/2, 3\pi/2, 5\pi/2, \cdots .k_n = n\pi/L,\ n = 1,3,5,\cdots$

(b) $\psi_n = A\sin k_n x$. Follow the same procedure as in part (a) to obtain $k_n = n\pi/L$, $n = 2, 4, 6, \ldots$

(c) We want the allowed energies.

$\psi_n = A\sin k_n x$: Combine the results for E_n and k_n from part (b) to obtain E_n.

$$E_n = \frac{\hbar^2}{2m}\left(\frac{n\pi}{L}\right)^2 = \frac{\hbar^2\pi^2}{2mL^2}n^2,\ n = 2,4,6,\cdots$$

$\psi_n = A\cos k_n x$: Follow the same procedure using the results from part (a), which gives

$$E_n = \frac{\hbar^2\pi^2}{2mL^2}n^2,\ n = 1,3,5,\cdots$$

EVALUATE: **(d)** The set of energies found here is the *same* as those in Eq. (40.31). This result occurs because the physical system (i.e., the box) does not "know" where we placed the origin of coordinates, so its behavior should be the same in either case.

40.22. **IDENTIFY:** $\lambda = \frac{h}{p}$. p is related to E by $E = \frac{p^2}{2m} + U$.

SET UP: For $x > L$, $U = U_0$. For $0 < x < L$, $U = 0$.

EXECUTE: For $0 < x < L$, $p = \sqrt{2mE} = \sqrt{2m(3U_0)}$ and $\lambda_{\text{in}} = \frac{h}{\sqrt{2m(3U_0)}}$. For $x > L$,

$p = \sqrt{2m(E - U_0)} = \sqrt{2m(2U_0)}$ and $\lambda_{\text{out}} = \frac{h}{\sqrt{2m(E-U_0)}} = \frac{h}{\sqrt{2m(2U_0)}}$. Thus, the ratio of the wavelengths is $\frac{\lambda_{\text{out}}}{\lambda_{\text{in}}} = \frac{\sqrt{2m(3U_0)}}{\sqrt{2m(2U_0)}} = \sqrt{\frac{3}{2}}$.

EVALUATE: For $x > L$ some of the energy is potential and the kinetic energy is less than it is for $0 < x < L$, where $U = 0$. Therefore, outside the box p is less and λ is greater than inside the box.

40.23. **IDENTIFY:** Figure 40.15b in the textbook gives values for the bound state energy of a square well for which $U_0 = 6E_{\text{1-IDW}}$.

SET UP: $E_{\text{1-IDW}} = \frac{\pi^2\hbar^2}{2mL^2}$.

EXECUTE: $E_1 = 0.625E_{\text{1-IDW}} = 0.625\frac{\pi^2\hbar^2}{2mL^2}$; $E_1 = 2.00\text{ eV} = 3.20\times10^{-19}$ J.

$$L = \pi\hbar\left(\frac{0.625}{2(9.109\times10^{-31}\text{ kg})(3.20\times10^{-19}\text{ J})}\right)^{1/2} = 3.43\times10^{-10}\text{ m}.$$

EVALUATE: As L increases the ground state energy decreases.

40.24. **IDENTIFY:** In a finite potential well, the energy levels are lowered compared to the energy levels in an infinite well. The energy of the photon removes the electron from its energy state in the well and any left-over energy is the kinetic energy K of the electron.

SET UP: The energy levels for an infinitely deep well are $E_{n-\text{IDW}} = \frac{n^2h^2}{8mL^2}$, and $n = 1$ is the ground state. The energy of a photon is $E = hc/\lambda$.

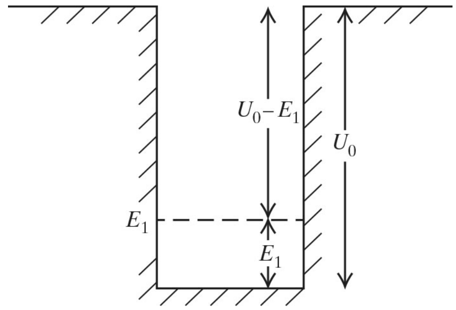

Figure 40.24

EXECUTE: Figure 40.24 illustrates the various energies involved. In this case, $U_0 = 6E_{1\text{-IDW}}$. Figure 40.15b in the textbook shows that the ground state energy E_1 in the finite well is $E_1 = 0.625E_{1\text{-IDW}}$.

The electron already has energy E_1 in the well, so the energy just to remove it from the well is $U_0 - E_1$.

Conservation of energy gives $E_{\text{photon}} = E_{\text{remove el}} + K$, which we can write as

$$\frac{hc}{\lambda} = (U_0 - E_1) + K = 6E_{1-\text{IDW}} - 0.625E_{1-\text{IDW}} + K = 5.375E_{1-\text{IDW}} + K.$$

Solving for K and using $E_{1-\text{IDW}} = \dfrac{h^2}{8mL^2}$ gives

$$K = \frac{hc}{\lambda} - 5.375E_{1-\text{IDW}} = \frac{hc}{\lambda} - \frac{5.735h^2}{8mL^2}.$$

Using $\lambda = 72 \times 10^{-9}$ m and $L = 4.00 \times 10^{-10}$ m, plus the usual values of the constants h, c, and m, we get $K = 2.76 \times 10^{-18}$ J – 2.02×10^{-18} J = 7.4×10^{-19} J, which we can express in electron-volts as K = 17.2 eV – 12.6 eV = 4.6 eV.

EVALUATE: The photon has 17.2 eV and it takes 12.6 eV just to remove the electron from the well, so the remaining 4.6 eV is the kinetic energy of the electron.

40.25. **IDENTIFY:** The energy of the photon is the energy given to the electron.

SET UP: Since $U_0 = 6E_{1\text{-IDW}}$ we can use the result $E_1 = 0.625E_{1\text{-IDW}}$ from Section 40.4. When the electron is outside the well it has potential energy U_0, so the minimum energy that must be given to the electron is $U_0 - E_1 = 5.375E_{1\text{-IDW}}$.

EXECUTE: The maximum wavelength of the photon would be

$$\lambda = \frac{hc}{U_0 - E_1} = \frac{hc}{(5.375)(h^2/8mL^2)} = \frac{8mL^2c}{(5.375)h} = \frac{8(9.11\times10^{-31}\text{ kg})(1.50\times10^{-9}\text{ m})^2(3.00\times10^8\text{ m/s})}{(5.375)(6.63\times10^{-34}\text{ J}\cdot\text{s})}$$
$$= 1.38\times10^{-6}\text{ m}.$$

EVALUATE: This photon is in the infrared. The wavelength of the photon decreases when the width of the well decreases.

40.26. **IDENTIFY:** The longest wavelength corresponds to the smallest energy change.

SET UP: The ground level energy level of the infinite well is $E_{1\text{-IDW}} = \dfrac{h^2}{8mL^2}$, and the energy of the photon must be equal to the energy difference between the two levels.

EXECUTE: The 582-nm photon must correspond to the $n=1$ to $n=2$ transition. Since $U_0 = 6E_{\text{1-IDW}}$, we have $E_2 = 2.43E_{\text{1-IDW}}$ and $E_1 = 0.625E_{\text{1-IDW}}$. The energy of the photon is equal to the energy difference between the two levels, and $E_{\text{1-IDW}} = \dfrac{h^2}{8mL^2}$, which gives

$E_\gamma = E_2 - E_1 \Rightarrow \dfrac{hc}{\lambda} = (2.43-0.625)E_{\text{1-IDW}} = \dfrac{1.805\,h^2}{8mL^2}$. Solving for L gives

$$L = \sqrt{\frac{(1.805)h\lambda}{8mc}} = \sqrt{\frac{(1.805)(6.626\times10^{-34}\ \text{J}\cdot\text{s})(5.82\times10^{-7}\ \text{m})}{8(9.11\times10^{-31}\ \text{kg})(3.00\times10^{8}\ \text{m/s})}} = 5.64\times10^{-10}\ \text{m} = 0.564\ \text{nm}.$$

EVALUATE: This width is slightly more than half that of a Bohr hydrogen atom.

40.27. **IDENTIFY:** Find the transition energy ΔE and set it equal to the energy of the absorbed photon. Use $E = hc/\lambda$, to find the wavelength of the photon.

SET UP: $U_0 = 6E_{\text{1-IDW}}$, as in Figure 40.15 in the textbook, so $E_1 = 0.625E_{\text{1-IDW}}$ and $E_3 = 5.09E_{\text{1-IDW}}$ with $E_{\text{1-IDW}} = \dfrac{\pi^2\hbar^2}{2mL^2}$. In this problem the particle bound in the well is a proton, so $m = 1.673\times10^{-27}$ kg.

EXECUTE: $E_{\text{1-IDW}} = \dfrac{\pi^2\hbar^2}{2mL^2} = \dfrac{\pi^2(1.055\times10^{-34}\ \text{J}\cdot\text{s})^2}{2(1.673\times10^{-27}\ \text{kg})(4.0\times10^{-15}\ \text{m})^2} = 2.052\times10^{-12}$ J. The transition energy is $\Delta E = E_3 - E_1 = (5.09-0.625)E_{\text{1-IDW}} = 4.465E_{\text{1-IDW}}$. $\Delta E = 4.465(2.052\times10^{-12}\ \text{J}) = 9.162\times10^{-12}$ J. The wavelength of the photon that is absorbed is related to the transition energy by $\Delta E = hc/\lambda$, so

$$\lambda = \frac{hc}{\Delta E} = \frac{(6.626\times10^{-34}\ \text{J}\cdot\text{s})(2.998\times10^{8}\ \text{m/s})}{9.162\times10^{-12}\ \text{J}} = 2.2\times10^{-14}\ \text{m} = 22\ \text{fm}.$$

EVALUATE: The wavelength of the photon is comparable to the size of the well.

40.28. **IDENTIFY:** The tunneling probability is $T = Ge^{-2\kappa L}$, with $G = 16\dfrac{E}{U_0}\left(1-\dfrac{E}{U_0}\right)$ and $\kappa = \dfrac{\sqrt{2m(U_0-E)}}{\hbar}$.

So $T = 16\dfrac{E}{U_0}\left(1-\dfrac{E}{U_0}\right)e^{\frac{-2\sqrt{2m(U_0-E)}}{\hbar}L}$.

SET UP: $U_0 = 30.0\times10^6$ eV, $L = 2.0\times10^{-15}$ m, $m = 6.64\times10^{-27}$ kg.

EXECUTE: **(a)** $U_0 - E = 1.0\times10^6$ eV $(E = 29.0\times10^6$ eV$)$, $T = 0.090$.

(b) If $U_0 - E = 10.0\times10^6$ eV $(E = 20.0\times10^6$ eV$)$, $T = 0.014$.

EVALUATE: T is less when $U_0 - E$ is 10.0 MeV than when $U_0 - E$ is 1.0 MeV.

40.29. **IDENTIFY** and **SET UP:** The probability is $T = Ge^{-2\kappa L}$, with $G = 16\dfrac{E}{U_0}\left(1-\dfrac{E}{U_0}\right)$ and $\kappa = \dfrac{\sqrt{2m(U_0-E)}}{\hbar}$. $E = 32$ eV, $U_0 = 41$ eV, $L = 0.25\times10^{-9}$ m. Calculate T.

EXECUTE: **(a)** $G = 16\dfrac{E}{U_0}\left(1-\dfrac{E}{U_0}\right) = 16\dfrac{32}{41}\left(1-\dfrac{32}{41}\right) = 2.741.$

$$\kappa = \frac{\sqrt{2m(U_0-E)}}{\hbar}.$$

$$\kappa = \frac{\sqrt{2(9.109\times10^{-31}\ \text{kg})(41\ \text{eV}-32\ \text{eV})(1.602\times10^{-19}\ \text{J/eV})}}{1.055\times10^{-34}\ \text{J}\cdot\text{s}} = 1.536\times10^{10}\ \text{m}^{-1}.$$

$$T = Ge^{-2\kappa L} = (2.741)e^{-2(1.536\times10^{10}\ \text{m}^{-1})(0.25\times10^{-9}\ \text{m})} = 2.741e^{-7.68} = 0.0013.$$

(b) The only change is the mass *m*, which appears in κ.

$$\kappa = \frac{\sqrt{2m(U_0 - E)}}{\hbar}.$$

$$\kappa = \frac{\sqrt{2(1.673\times10^{-27}\ \text{kg})(41\ \text{eV} - 32\ \text{eV})(1.602\times10^{-19}\ \text{J/eV})}}{1.055\times10^{-34}\ \text{J}\cdot\text{s}} = 6.584\times10^{11}\ \text{m}^{-1}.$$

Then $T = Ge^{-2\kappa L} = (2.741)e^{-2(6.584\times10^{11}\ \text{m}^{-1})(0.25\times10^{-9}\ \text{m})} = 2.741e^{-392.2} = 10^{-143}$.

EVALUATE: The more massive proton has a much smaller probability of tunneling than the electron does.

40.30. **IDENTIFY:** The transmission coefficient is $T = 16\frac{E}{U_0}\left(1 - \frac{E}{U_0}\right)e^{-2\sqrt{2m(U_0-E)}L/\hbar}$.

SET UP: $E = 5.0$ eV, $L = 0.60\times10^{-9}$ m, and $m = 9.11\times10^{-31}$ kg.

EXECUTE: **(a)** $U_0 = 7.0\ \text{eV} \Rightarrow T = 5.5\times10^{-4}$.

(b) $U_0 = 9.0\ \text{eV} \Rightarrow T = 1.8\times10^{-5}$.

(c) $U_0 = 13.0\ \text{eV} \Rightarrow T = 1.1\times10^{-7}$.

EVALUATE: *T* decreases when the height of the barrier increases.

40.31. **IDENTIFY:** The tunneling probability is $T = 16\frac{E}{U_0}\left(1 - \frac{E}{U_0}\right)e^{-2L\sqrt{2m(U_0-E)}/\hbar}$.

SET UP: $\frac{E}{U_0} = \frac{6.0\ \text{eV}}{11.0\ \text{eV}}$ and $E - U_0 = 5\ \text{eV} = 8.0\times10^{-19}$ J.

EXECUTE: **(a)** $L = 0.80\times10^{-9}$ m:

$$T = 16\left(\frac{6.0\ \text{eV}}{11.0\ \text{eV}}\right)\left(1 - \frac{6.0\ \text{ev}}{11.0\ \text{eV}}\right)e^{-2(0.80\times10^{-9}\ \text{m})\sqrt{2(9.11\times10^{-31}\ \text{kg})(8.0\times10^{-19}\ \text{J})}/1.055\times10^{-34}\ \text{J}\cdot\text{s}} = 4.4\times10^{-8}.$$

(b) $L = 0.40\times10^{-9}$ m: $T = 4.2\times10^{-4}$.

EVALUATE: The tunneling probability is less when the barrier is wider.

40.32. **IDENTIFY** and **SET UP:** Use $\lambda = h/p$, where $K = p^2/2m$ and $E = K + U$.

EXECUTE: $\lambda = h/p = h/\sqrt{2mK}$, so $\lambda\sqrt{K}$ is constant. $\lambda_1\sqrt{K_1} = \lambda_2\sqrt{K_2}$; λ_1 and K_1 are for $x > L$ where $K_1 = 2U_0$ and λ_2 and K_2 are for $0 < x < L$ where $K_2 = E - U_0 = U_0$.

$$\frac{\lambda_1}{\lambda_2} = \sqrt{\frac{K_2}{K_1}} = \sqrt{\frac{U_0}{2U_0}} = \frac{1}{\sqrt{2}}.$$

EVALUATE: When the particle is passing over the barrier its kinetic energy is less and its wavelength is larger.

40.33. **IDENTIFY** and **SET UP:** The energy levels are given by $E_n = (n + \tfrac{1}{2})\hbar\omega$, where $\omega = \sqrt{\frac{k'}{m}}$.

EXECUTE: $\omega = \sqrt{\frac{k'}{m}} = \sqrt{\frac{110\ \text{N/m}}{0.250\ \text{kg}}} = 21.0$ rad/s.

The ground state energy is given by $E_n = (n + \tfrac{1}{2})\hbar\omega$, where *n* = 0.

$$E_0 = \tfrac{1}{2}\hbar\omega = \tfrac{1}{2}(1.055\times10^{-34}\ \text{J}\cdot\text{s})(21.0\ \text{rad/s}) = 1.11\times10^{-33}\ \text{J}(1\ \text{eV}/1.602\times10^{-19}\ \text{J}) = 6.93\times10^{-15}\ \text{eV}.$$

$E_n = (n + \tfrac{1}{2})\hbar\omega$, $E_{(n+1)} = (n + 1 + \tfrac{1}{2})\hbar\omega$.

The energy separation between these adjacent levels is

$$\Delta E = E_{n+1} - E_n = \hbar\omega = 2E_0 = 2(1.11\times10^{-33}\ \text{J}) = 2.22\times10^{-33}\ \text{J} = 1.39\times10^{-14}\ \text{eV}.$$

EVALUATE: These energies are extremely small; quantum effects are not important for this oscillator.

40.34. **IDENTIFY:** This problem is about a quantum harmonic oscillator and the uncertainty principle.

SET UP: $K = p^2/2m$, $p_{\text{max}} = mv_{\text{max}}$, $E_n = \left(n+\dfrac{1}{2}\right)\hbar\sqrt{\dfrac{k'}{m}} = \left(n+\dfrac{1}{2}\right)\hbar\,\omega.$

EXECUTE: **(a)** We want p_{max}. When $v = v_{\text{max}}$, $x = 0$, so $K_{\text{max}} = E_n$. Using $K = p^2/2m$ gives

$$\frac{p_{\text{max}}^2}{2m} = \left(n+\frac{1}{2}\right)\hbar\sqrt{\frac{k'}{m}}.\quad p_{\text{max}} = \sqrt{(2n+1)\hbar\sqrt{k'm}}.$$

(b) We want the amplitude A. When $x = A$, $v = 0$, so $\dfrac{1}{2}k'A^2 = E_n$. Use this fact and solve for A.

$$\frac{1}{2}k'A^2 = \left(n+\frac{1}{2}\right)\hbar\sqrt{\frac{k'}{m}}.\ A = \sqrt{\frac{(2n+1)\hbar}{\sqrt{k'm}}}.$$

(c) We want $\Delta x\Delta p_x$. Use the uncertainties given in the problem and the results of (a) and (b).

$$\Delta x\Delta p_x = \frac{A}{\sqrt{2}}\frac{p_{\text{max}}}{\sqrt{2}} = \frac{1}{2}\sqrt{\frac{(2n+1)\hbar}{\sqrt{k'm}}}\sqrt{(2n+1)\hbar\sqrt{k'm}} = \left(n+\frac{1}{2}\right)\hbar.$$

As n increases, the uncertainty product also increases.

EVALUATE: As n increases, so do A and p_{max}. Therefore the uncertainty product should also increase, as we have found.

40.35. **IDENTIFY:** We can model the molecule as a harmonic oscillator. The energy of the photon is equal to the energy difference between the two levels of the oscillator.

SET UP: The energy of a photon is $E_\gamma = hf = hc/\lambda$, and the energy levels of a harmonic oscillator are given by $E_n = (n+\tfrac{1}{2})\hbar\sqrt{\dfrac{k'}{m}} = (n+\tfrac{1}{2})\hbar\omega.$

EXECUTE: **(a)** The photon's energy is $E_\gamma = \dfrac{hc}{\lambda} = \dfrac{(6.63\times10^{-34}\ \text{J}\cdot\text{s})(3.00\times10^8\ \text{m/s})}{5.8\times10^{-6}\ \text{m}} = 0.21\ \text{eV}.$

(b) The transition energy is $\Delta E = E_{n+1} - E_n = \hbar\omega = \hbar\sqrt{\dfrac{k'}{m}}$, which gives $\dfrac{2\pi\hbar c}{\lambda} = \hbar\sqrt{\dfrac{k'}{m}}$. Solving for k', we get $k' = \dfrac{4\pi^2c^2m}{\lambda^2} = \dfrac{4\pi^2(3.00\times10^8\ \text{m/s})^2(5.6\times10^{-26}\ \text{kg})}{(5.8\times10^{-6}\ \text{m})^2} = 5{,}900\ \text{N/m}.$

EVALUATE: This would be a rather strong spring in the physics lab.

40.36. **IDENTIFY:** The energy of the absorbed photon must be equal to the energy difference between the two states.

SET UP and **EXECUTE:** $\Delta E = \dfrac{hc}{\lambda} = \dfrac{(4.136\times10^{-15}\ \text{eV}\cdot\text{s})(2.998\times10^8\ \text{m/s})}{6.35\times10^{-6}\ \text{m}} = 0.1953\ \text{eV} = \hbar\omega.\ \Delta E = \hbar\omega.$

$$E_0 = \frac{\hbar\omega}{2} = \frac{0.1953\ \text{eV}}{2} = 0.0976\ \text{eV}.$$

EVALUATE: The energy of the photon is not equal to the energy of the ground state, but rather it is the energy *difference* between the two states.

40.37. **IDENTIFY:** The photon energy equals the transition energy for the atom.

SET UP: According to the energy level equation $E_n = (n+\tfrac{1}{2})\hbar\omega$, the energy released during the transition between two adjacent levels is twice the ground state energy $E_3 - E_2 = \hbar\omega = 2E_0 = 11.2\ \text{eV}.$

EXECUTE: For a photon of energy E,

$$E = hf \Rightarrow \lambda = \frac{c}{f} = \frac{hc}{E} = \frac{(6.63\times10^{-34}\ \text{J}\cdot\text{s})(3.00\times10^{8}\ \text{m/s})}{(11.2\ \text{eV})(1.60\times10^{-19}\ \text{J/eV})} = 111\ \text{nm}.$$

EVALUATE: This photon is in the ultraviolet.

40.38. IDENTIFY: The energy of the absorbed (or emitted) photon energy is equal to the energy difference between the levels of the oscillator.

SET UP: The energy levels for a harmonic oscillator are $E_n = (n+\tfrac{1}{2})\hbar\omega$, where $\omega = \sqrt{k'/m}$.

EXECUTE: **(a)** The energy difference between *any* two adjacent levels is $\Delta E_{n+1} - \Delta E_n = (n+\tfrac{3}{2})\hbar\omega - (n+\tfrac{1}{2})\hbar\omega = \hbar\omega.$ Therefore transitions between *any* adjacent levels will emit (or absorb) photons of the same energy and hence the same wavelength. So the $2 \rightarrow 3$ transition absorbs a photon of the same wavelength as the $1 \rightarrow 2$ transition, which is $\lambda = 6.50\ \mu\text{m}$.

(b) Since transitions between adjacent levels emit photons of the same energy, transitions between levels for which *n* differs by 2 will emit energy $\hbar\omega + \hbar\omega = 2\hbar\omega$. So the photon absorbed in the $1 \rightarrow 3$ transition will have twice the energy (and therefore half the wavelength) as the photon in the $1 \rightarrow 2$ transition, so its wavelength will be $\tfrac{1}{2}(6.50\ \mu\text{m}) = 3.25\ \mu\text{m}$.

(c) For the $1 \rightarrow 2$ transition, the photon energy is $\hbar\omega$ and $\omega = \sqrt{k'/m}$, so

$\hbar\omega$ = energy of photon = hc/λ. This gives $\omega = \sqrt{k'/m} = \dfrac{hc}{\hbar\lambda} = \dfrac{2\pi c}{\lambda} = \dfrac{2\pi(3.00\times10^{8}\ \text{m/s})}{6.50\times10^{-6}\ \text{m}}$

$= 2.90\times10^{14}\ \text{rad/s}.$

EVALUATE: The frequency of this oscillator would be $f = \omega/2\pi = 4.62\times10^{14}$ Hz, *much* higher than typical classical oscillators.

40.39. IDENTIFY: We model the atomic vibration in the crystal as a harmonic oscillator.

SET UP: The energy levels of a harmonic oscillator are given by $E_n = (n+\tfrac{1}{2})\hbar\sqrt{\dfrac{k'}{m}} = (n+\tfrac{1}{2})\hbar\omega.$

EXECUTE: **(a)** The ground state energy of a simple harmonic oscillator is $E_0 = \tfrac{1}{2}\hbar\omega$

$$= \tfrac{1}{2}\hbar\sqrt{\frac{k'}{m}} = \frac{(1.055\times10^{-34}\ \text{J}\cdot\text{s})}{2}\sqrt{\frac{12.2\ \text{N/m}}{3.82\times10^{-26}\ \text{kg}}} = 9.43\times10^{-22}\ \text{J} = 5.89\times10^{-3}\ \text{eV}.$$

(b) $E_4 - E_3 = \hbar\omega = 2E_0 = 0.0118\ \text{eV}$, so $\lambda = \dfrac{hc}{E} = \dfrac{(6.63\times10^{-34}\ \text{J}\cdot\text{s})(3.00\times10^{8}\ \text{m/s})}{1.88\times10^{-21}\ \text{J}} = 106\ \mu\text{m}.$

(c) $E_{n+1} - E_n = \hbar\omega = 2E_0 = 0.0118\ \text{eV}.$

EVALUATE: These energy differences are much smaller than those due to electron transitions in the hydrogen atom.

40.40. IDENTIFY: Compute the ratio specified in the problem.

SET UP: For $n = 0$, $A = \sqrt{\dfrac{\hbar\omega}{k'}}$. $\omega = \sqrt{\dfrac{k'}{m}}$.

EXECUTE: **(a)** $\dfrac{|\psi(A)|^2}{|\psi(0)|^2} = \exp\left(-\dfrac{\sqrt{mk'}}{\hbar}A^2\right) = \exp\left(-\sqrt{mk'}\dfrac{\omega}{k'}\right) = e^{-1} = 0.368.$ This is consistent with what is shown in Figure 40.27 in the textbook.

(b) $\dfrac{|\psi(2A)^2|}{|\psi(0)|^2} = \exp\left(-\dfrac{\sqrt{mk'}}{\hbar}(2A)^2\right) = \exp\left(-\sqrt{mk'}\,4\dfrac{\omega}{k'}\right) = e^{-4} = 1.83\times10^{-2}.$ This figure cannot be read this precisely, but the qualitative decrease in amplitude with distance is clear.

EVALUATE: The wave function decays exponentially as x increases beyond $x = A$.

40.41. **IDENTIFY:** We know the wave function of a particle in a box.

SET UP and **EXECUTE:** **(a)** $\Psi(x,t)=\frac{1}{\sqrt{2}}\psi_1(x)e^{-iE_1t/\hbar}+\frac{1}{\sqrt{2}}\psi_3(x)e^{-iE_3t/\hbar}$.

$$\Psi^*(x,t)=\frac{1}{\sqrt{2}}\psi_1(x)e^{+iE_1t/\hbar}+\frac{1}{\sqrt{2}}\psi_3(x)e^{+iE_3t/\hbar}.$$

$|\Psi(x,t)|^2=\frac{1}{2}[\psi_1^2+\psi_3^2+\psi_1\psi_3(e^{i(E_3-E_1)t/\hbar}+e^{-i(E_3-E_1)t/\hbar})]=\frac{1}{2}\left[\psi_1^2+\psi_3^2+2\psi_1\psi_3\cos\left(\frac{[E_3-E_1]t}{\hbar}\right)\right]$.

$\psi_1=\sqrt{\frac{2}{L}}\sin\left(\frac{\pi x}{L}\right)$. $\psi_3=\sqrt{\frac{2}{L}}\sin\left(\frac{3\pi x}{L}\right)$. $E_3=\frac{9\pi^2\hbar^2}{2mL^2}$ and $E_1=\frac{\pi^2\hbar^2}{2mL^2}$, so $E_3-E_1=\frac{4\pi^2\hbar^2}{mL^2}$.

$|\Psi(x,t)|^2=\frac{1}{L}\left[\sin^2\left(\frac{\pi x}{L}\right)+\sin^2\left(\frac{3\pi x}{L}\right)+2\sin\left(\frac{\pi x}{L}\right)\sin\left(\frac{3\pi x}{L}\right)\cos\left(\frac{4\pi^2\hbar t}{mL^2}\right)\right]$. At $x=L/2$,

$\sin\left(\frac{\pi x}{L}\right)=\sin\left(\frac{\pi}{2}\right)=1$. $\sin\left(\frac{3\pi x}{L}\right)=\sin\left(\frac{3\pi}{2}\right)=-1$. $|\Psi(x,t)|^2=\frac{2}{L}\left[1-\cos\left(\frac{4\pi^2\hbar t}{mL^2}\right)\right]$.

(b) $\omega_{\text{osc}}=\frac{E_3-E_1}{\hbar}=\frac{4\pi^2\hbar}{mL^2}$.

EVALUATE: Note that $\Delta E=\hbar\omega$.

40.42. **IDENTIFY:** In this problem, we model the hydrogen as an electron in a box.

SET UP: $E=hc/\lambda$.

EXECUTE: **(a)** We want the photon energies. $E_{656}=hc/\lambda=hc/(656\text{ nm})=1.89\text{ eV}$. Using the other wavelenghs in the same way gives $E_{486}=2.55\text{ eV}$, $E_{434}=2.86\text{ eV}$, $E_{410}=3.03\text{ eV}$.

(b) We want the minimum n_i. The transitions must all end on a lower state. The lowest possible ones are n = 1, 2, 3, 4, so the first state above that is n = 5.

(c) We want $\mathcal{E}$. Using $n_\text{i}=5$, the final states are n_f = 4, 3, 2, and 1. We follow the instruction in the problem for the possible transitions and find that the transitions $5\to4$, $5\to3$, $5\to2$, and $5\to1$. The greater the difference between n_i and n_f, the greater the photon energy E. Following the directions in the problem, we have

$$5\to4:\quad \mathcal{E}=\frac{E}{n_\text{i}^2-n_\text{f}^2}=\frac{1.89\text{ eV}}{5^2-4^2}=0.210\text{ eV}$$

$$5\to3:\quad \mathcal{E}=\frac{E}{n_\text{i}^2-n_\text{f}^2}=\frac{2.55\text{ eV}}{5^2-3^2}=0.519\text{ eV}$$

$$5\to2:\quad \mathcal{E}=\frac{E}{n_\text{i}^2-n_\text{f}^2}=\frac{2.86\text{ eV}}{5^2-2^2}=0.136\text{ eV}$$

$$5\to1:\quad \mathcal{E}=\frac{E}{n_\text{i}^2-n_\text{f}^2}=\frac{3.03\text{ eV}}{5^2-1^2}=0.158\text{ eV}$$

Averaging the results gives $\mathcal{E}$ = 0.2 eV.

(d) We want L. $E_n=n^2h^2/8mL^2=n^2\mathcal{E}$ gives $\mathcal{E}=h^2/8mL^2$. Solving for L and using $\mathcal{E}$ = 0.2 eV gives L = 2 nm.

(e) Using $L/2a_0$ with L = 2 nm gives a ratio of 18.

EVALUATE: Our result gives L is about 18 times the diameter of a hydrogen atom, so the particle in a box model is not very good.

40.43. **IDENTIFY:** Let I refer to the region $x<0$ and let II refer to the region $x>0$, so $\psi_I(x)=Ae^{ik_1x}+Be^{-ik_1x}$ and $\psi_{II}(x)=Ce^{ik_2x}$. Set $\psi_I(0)=\psi_{II}(0)$ and $\frac{d\psi_I}{dx}=\frac{d\psi_{II}}{dx}$ at $x=0$.

SET UP: $\frac{d}{dx}(e^{ikx}) = ike^{ikx}$.

EXECUTE: $\psi_I(0) = \psi_{II}(0)$ gives $A + B = C$. $\frac{d\psi_I}{dx} = \frac{d\psi_{II}}{dx}$ at $x = 0$ gives $ik_1A - ik_1B = ik_2C$. Solving this pair of equations for B and C gives $B = \left(\frac{k_1 - k_2}{k_1 + k_2}\right)A$ and $C = \left(\frac{2k_2}{k_1 + k_2}\right)A$.

EVALUATE: The probability of reflection is $R = \frac{B^2}{A^2} = \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2}$. The probability of transmission is $T = \frac{C^2}{A^2} = \frac{4k_1^2}{(k_1 + k_2)^2}$. Note that $R + T = 1$.

40.44. IDENTIFY: The probability of finding the particle between x_1 and x_2 is $\int_{x_1}^{x_2} |\psi|^2\, dx$.

SET UP: For the ground state $\psi_1 = \sqrt{\frac{2}{L}}\sin\frac{\pi x}{L}$. $\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$. $\int \cos\alpha x\, dx = \frac{1}{\alpha}\sin\alpha x$.

EXECUTE: **(a)** $\frac{2}{L}\int_0^{L/4} \sin^2\frac{\pi x}{L}dx = \frac{2}{L}\int_0^{L/4}\frac{1}{2}\left(1 - \cos\frac{2\pi x}{L}\right)dx = \frac{1}{L}\left(x - \frac{L}{2\pi}\sin\frac{2\pi x}{L}\right)_0^{L/4} = \frac{1}{4} - \frac{1}{2\pi}$, which is about 0.0908.

(b) Repeating with limits of $L/4$ and $L/2$ gives $\frac{1}{L}\left(x - \frac{L}{2\pi}\sin\frac{2\pi x}{L}\right)_{L/4}^{L/2} = \frac{1}{4} + \frac{1}{2\pi}$, about 0.409.

(c) The particle is much likely to be nearer the middle of the box than the edge.

EVALUATE: **(d)** The results sum to exactly $\frac{1}{2}$. Since the probability of the particle being anywhere in the box is unity, the probability of the particle being found between $x = L/2$ and $x = L$ is also $\frac{1}{2}$. This means that the particle is as likely to be between $x = 0$ and $L/2$ as it is to be between $x = L/2$ and $x = L$.

(e) These results are consistent with Figure 40.12b in the textbook. This figure shows a greater probability near the center of the box. It also shows symmetry of $|\psi|^2$ about the center of the box.

40.45. IDENTIFY and **SET UP:** The energy levels are given by the equation $E_n = \frac{n^2h^2}{8mL^2}$. Calculate ΔE for the transition and set $\Delta E = hc/\lambda$, the energy of the photon.

EXECUTE: **(a)** Ground level, $n = 1$, $E_1 = \frac{h^2}{8mL^2}$. First excited level, $n = 2$, $E_2 = \frac{4h^2}{8mL^2}$. The transition energy is $\Delta E = E_2 - E_1 = \frac{3h^2}{8mL^2}$. Set the transition energy equal to the energy hc/λ of the emitted photon. This gives $\frac{hc}{\lambda} = \frac{3h^2}{8mL^2}$. $\lambda = \frac{8mcL^2}{3h} = \frac{8(9.109\times10^{-31}\text{ kg})(2.998\times10^{8}\text{ m/s})(4.18\times10^{-9}\text{ m})^2}{3(6.626\times10^{-34}\text{ J}\cdot\text{s})}$.

$\lambda = 1.92\times10^{-5}\text{ m} = 19.2\ \mu\text{m}$.

(b) Second excited level has $n = 3$ and $E_3 = \frac{9h^2}{8mL^2}$. The transition energy is

$\Delta E = E_3 - E_2 = \frac{9h^2}{8mL^2} - \frac{4h^2}{8mL^2} = \frac{5h^2}{8mL^2}$. $\frac{hc}{\lambda} = \frac{5h^2}{8mL^2}$, so $\lambda = \frac{8mcL^2}{5h} = \frac{3}{5}(19.2\ \mu\text{m}) = 11.5\ \mu\text{m}$.

EVALUATE: The energy spacing between adjacent levels increases with n, and this corresponds to a shorter wavelength and more energetic photon in part (b) than in part (a).

40.46. **IDENTIFY:** The probability is $|\psi|^2\,dx,$ with ψ evaluated at the specified value of x.

SET UP: For the ground state, the normalized wave function is $\psi_1 = \sqrt{2/L}\,\sin(\pi x/L).$

EXECUTE: **(a)** $(2/L)\sin^2(\pi/4)dx = dx/L.$

(b) $(2/L)\sin^2(\pi/2)dx = 2dx/L.$

(c) $(2/L)\sin^2(3\pi/4) = dx/L.$

EVALUATE: Our results agree with Figure 40.12b in the textbook. $|\psi|^2$ is largest at the center of the box, at $x = L/2.$ $|\psi|^2$ is symmetric about the center of the box, so is the same at $x = L/4$ as at $x = 3L/4.$

40.47. **IDENTIFY** and **SET UP:** The normalized wave function for the $n = 2$ first excited level is $\psi_2 = \sqrt{\dfrac{2}{L}}\sin\left(\dfrac{2\pi x}{L}\right).$ $P = |\psi(x)|^2\,dx$ is the probability that the particle will be found in the interval x to $x + dx.$

EXECUTE: **(a)** $x = L/4.$

$$\psi(x) = \sqrt{\frac{2}{L}}\sin\left(\left(\frac{2\pi}{L}\right)\left(\frac{L}{4}\right)\right) = \sqrt{\frac{2}{L}}\sin\left(\frac{\pi}{2}\right) = \sqrt{\frac{2}{L}}.$$

$$P = (2/L)dx.$$

(b) $x = L/2.$

$$\psi(x) = \sqrt{\frac{2}{L}}\sin\left(\left(\frac{2\pi}{L}\right)\left(\frac{L}{2}\right)\right) = \sqrt{\frac{2}{L}}\sin(\pi) = 0.$$

$$P = 0.$$

(c) $x = 3L/4.$

$$\psi(x) = \sqrt{\frac{2}{L}}\sin\left(\left(\frac{2\pi}{L}\right)\left(\frac{3L}{4}\right)\right) = \sqrt{\frac{2}{L}}\sin\left(\frac{3\pi}{2}\right) = -\sqrt{\frac{2}{L}}.$$

$$P = (2/L)dx.$$

EVALUATE: Our results are consistent with the $n = 2$ part of Figure 40.12 in the textbook. $|\psi|^2$ is zero at the center of the box and is symmetric about this point.

40.48. **IDENTIFY:** This problem deals with quantum mechanical tunneling.

SET UP and **EXECUTE:** **(a)** We want the average energy per electron. $\Delta U = q\Delta V$ gives $E = eV = e(5\text{ V}) = 5\text{ eV}.$

(b) $U_0 = q\Delta V = e(12\text{ V}) = 12\text{ eV}.$

(c) We want the probability of tunneling, which is $T = Ge^{-2\kappa L},$ where

$$G = 16\frac{E}{U_0}\left(1 - \frac{E}{U_0}\right) \quad \kappa = \frac{\sqrt{2m(U_0 - E)}}{\hbar}.$$

Using E and U_0 from parts (a) and (b), we get $G = 3.89$ and $\kappa = 1.354 \times 10^{10}\text{ m}^{-1}$. Using the equation for T gives a tunneling probability of $T = 6.74 \times 10^{-12}$.

(d) We want the tunneling current. The current density is $J = nev$. On the R side the number of electrons is Tn. $I = JA = TneAv$.

(e) We want the effective speed of tunneled electrons. Using the definition of effective speed gives $v = \sqrt{2eV/m}.$

(f) We want the tunneling current. Using the results from (d) and (e) with the given numbers, we get $I = TneAv = Tnea\sqrt{2eV/m} = Tne\pi(d/2)^2\sqrt{2eV/m} = 95\text{ mA}.$

EVALUATE: In part (c) we see that only a small percent of the electrons tunnel through the barrier.

40.49. **IDENTIFY:** The probability of the particle being between x_1 and x_2 is $\int_{x_1}^{x_2} |\psi|^2\,dx$, where ψ is the normalized wave function for the particle.

(a) SET UP: The normalized wave function for the ground state is $\psi_1 = \sqrt{\frac{2}{L}}\sin\left(\frac{\pi x}{L}\right)$.

EXECUTE: The probability P of the particle being between $x = L/4$ and $x = 3L/4$ is $P = \int_{L/4}^{3L/4} |\psi_1|^2 dx = \frac{2}{L}\int_{L/4}^{3L/4} \sin^2\left(\frac{\pi x}{L}\right)dx$. Let $y = \pi x/L;\, dx = (L/\pi)\,dy$ and the integration limits become $\pi/4$ and $3\pi/4$.

$$P = \frac{2}{L}\left(\frac{L}{\pi}\right)\int_{\pi/4}^{3\pi/4} \sin^2 y\,dy = \frac{2}{\pi}\left[\frac{1}{2}y - \frac{1}{4}\sin 2y\right]_{\pi/4}^{3\pi/4}.$$

$$P = \frac{2}{\pi}\left[\frac{3\pi}{8} - \frac{\pi}{8} - \frac{1}{4}\sin\left(\frac{3\pi}{2}\right) + \frac{1}{4}\sin\left(\frac{\pi}{2}\right)\right].$$

$P = \frac{2}{\pi}\left(\frac{\pi}{4} - \frac{1}{4}(-1) + \frac{1}{4}(1)\right) = \frac{1}{2} + \frac{1}{\pi} = 0.818$. (Note: The integral formula $\int \sin^2 y\,dy = \frac{1}{2}y - \frac{1}{4}\sin 2y$ was used.)

(b) SET UP: The normalized wave function for the first excited state is $\psi_2 = \sqrt{\frac{2}{L}}\sin\left(\frac{2\pi x}{L}\right)$.

EXECUTE: $P = \int_{L/4}^{3L/4} |\psi_2|^2 dx = \frac{2}{L}\int_{L/4}^{3L/4} \sin^2\left(\frac{2\pi x}{L}\right)dx$. Let $y = 2\pi x/L;\, dx = (L/2\pi)\,dy$ and the integration limits become $\pi/2$ and $3\pi/2$.

$$P = \frac{2}{L}\left(\frac{L}{2\pi}\right)\int_{\pi/2}^{3\pi/2} \sin^2 y\,dy = \frac{1}{\pi}\left[\frac{1}{2}y - \frac{1}{4}\sin 2y\right]_{\pi/2}^{3\pi/2} = \frac{1}{\pi}\left(\frac{3\pi}{4} - \frac{\pi}{4}\right) = 0.500.$$

EVALUATE: **(c)** These results are consistent with Figure 40.11b in the textbook. That figure shows that $|\psi|^2$ is more concentrated near the center of the box for the ground state than for the first excited state; this is consistent with the answer to part (a) being larger than the answer to part (b). Also, this figure shows that for the first excited state half the area under $|\psi|^2$ curve lies between $L/4$ and $3L/4$, consistent with our answer to part (b).

40.50. **IDENTIFY:** We start with the penetration distance formula given in the problem.

SET UP: The given formula is $\eta = \frac{\hbar}{\sqrt{2m(U_0 - E)}}$.

EXECUTE: **(a)** Substitute the given numbers into the formula:

$$\eta = \frac{\hbar}{\sqrt{2m(U_0 - E)}} = \frac{1.055\times10^{-34}\ \text{J}\cdot\text{s}}{\sqrt{2(9.11\times10^{-31}\ \text{kg})(20\ \text{eV} - 13\ \text{eV})(1.602\times10^{-19}\ \text{J/eV})}} = 7.4\times10^{-11}\ \text{m}.$$

(b) $\eta = \frac{1.055\times10^{-34}\ \text{J}\cdot\text{s}}{\sqrt{2(1.67\times10^{-27}\ \text{kg})(30\ \text{MeV} - 20\ \text{MeV})(1.602\times10^{-13}\ \text{J/MeV})}} = 1.44\times10^{-15}\ \text{m}.$

EVALUATE: The penetration depth varies widely depending on the mass and energy of the particle.

40.51. **IDENTIFY:** Carry out the calculations that are specified in the problem.

SET UP: For a free particle, $U(x) = 0$, so Schrödinger's equation becomes $\frac{d^2\psi(x)}{dx^2} = -\frac{2m}{\hbar^2}E\psi(x)$.

EXECUTE: **(a)** The graph is given in Figure 40.51.

(b) For $x < 0$: $\psi(x) = e^{+\kappa x}$. $\frac{d\psi(x)}{dx} = \kappa e^{+\kappa x}$. $\frac{d^2\psi(x)}{dx} = \kappa^2 e^{+\kappa x}$. So $\kappa^2 = -\frac{2m}{\hbar^2}E \Rightarrow E = -\frac{\hbar^2\kappa^2}{2m}$.

(c) For $x > 0$: $\psi(x) = e^{-\kappa x}$. $\dfrac{d\psi(x)}{dx} = -\kappa e^{-\kappa x}$. $\dfrac{d^2\psi(x)}{dx} = \kappa^2 e^{-\kappa x}$. So again $\kappa^2 = -\dfrac{2m}{\hbar^2}E \Rightarrow E = \dfrac{-\hbar^2\kappa^2}{2m}$.

Parts (b) and (c) show $\psi(x)$ satisfies the Schrödinger's equation, provided $E = \dfrac{-\hbar^2\kappa^2}{2m}$.

EVALUATE: (d) $\dfrac{d\psi(x)}{dx}$ is discontinuous at $x = 0$. (That is, it is negative for $x > 0$ and positive for $x < 0$.) Therefore, this ψ is not an acceptable wave function; $d\psi/dx$ must be continuous everywhere, except where $U \to \infty$.

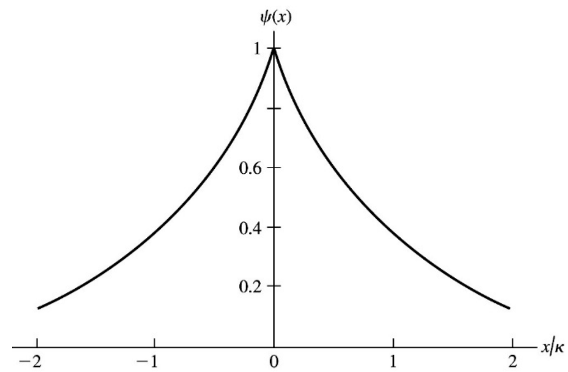

Figure 40.51

40.52. IDENTIFY: $T = Ge^{-2\kappa L}$ with $G = 16\dfrac{E}{U_0}\left(1 - \dfrac{E}{U_0}\right)$ and $\kappa = \dfrac{\sqrt{2m(U_0 - E)}}{\hbar}$, so $L = -\dfrac{1}{2\kappa}\ln\left(\dfrac{T}{G}\right)$.

SET UP: $E = 5.5\text{ eV}, U_0 = 10.0\text{ eV}, m = 9.11\times10^{-31}\text{ kg}$, and $T = 0.0050$.

EXECUTE: $\kappa = \dfrac{\sqrt{2(9.11\times10^{-31}\text{ kg})(4.5\text{ eV})(1.60\times10^{-19}\text{ J/eV})}}{(1.054\times10^{-34}\text{ J}\cdot\text{s})} = 1.09\times10^{10}\text{ m}^{-1}$ and

$G = 16\dfrac{5.5\text{ eV}}{10.0\text{ eV}}\left(1 - \dfrac{5.5\text{ eV}}{10.0\text{ eV}}\right) = 3.96$. Therefore the barrier width L is $L = -\dfrac{1}{2\kappa}\ln\left(\dfrac{T}{G}\right)$

$= -\dfrac{1}{2(1.09\times10^{10}\text{ m}^{-1})}\ln\left(\dfrac{0.0050}{3.96}\right) = 3.1\times10^{-10}\text{ m} = 0.31\text{ nm}$.

EVALUATE: The energies here are comparable to those of electrons in atoms, and the barrier width we calculated is on the order of the diameter of an atom.

40.53. IDENTIFY: This problem is about a quantum mechanical harmonic oscillator.

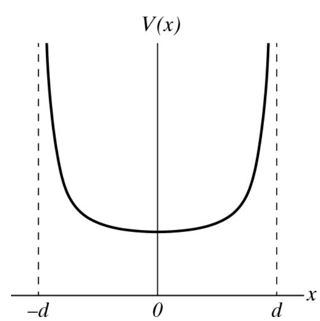

Figure 40.53

SET UP and **EXECUTE:** **(a)** For the sketch, see Figure 40.53.
(b) First combine the terms in the given equation for $V(x)$ to obtain the following:

$$V(x)=\frac{2kq^2}{d}\frac{1}{1-(x/d)^2}, \text{ where } k=1/4\pi\epsilon_0\,.$$

Expand the right-hand fraction using $(1 + z)^n \approx 1 + nz$, where $z \ll 1$, $n = -1$, $z = -(x/d)^2$.

$$V(x)=\frac{2kq^2}{d}\left[1+\frac{x^2}{d^2}\right].$$

Using $q = 6e$ and putting in k gives

$$V(x)=\frac{18e^2}{\pi\epsilon_0 d}+\frac{18e^2}{\pi\epsilon_0 d^3}x^2.$$

(c) We want the spring constant. Using the second term for $V(x)$, we see that

$$V=\frac{1}{2}k'x^2=\frac{18e^2}{\pi\epsilon_0 d^3}x^2$$

$$k=\frac{36e^2}{\pi\epsilon_0 d^3}=265 \text{ N/m}.$$

(d) The classical ground state is $x = 0$, so the energy at this state is

$$V=\frac{18e^2}{\pi\epsilon_0 d}=207 \text{ eV}.$$

(e) We want the energy of the lowest energy photon. Use the quantum energy states.

$$E_0=\frac{1}{2}\hbar\sqrt{\frac{k'}{m}} \text{ and } E_n=\left(n+\frac{1}{2}\right)\hbar\sqrt{\frac{k'}{m}}, \text{ so } E_n=2E_0\left(n+\frac{1}{2}\right).$$

The lowest energy photon is due to a transition from the $n = 1$ state to the $n = 0$ state, so

$$\Delta E=2E_0\left[\left(1+\frac{1}{2}\right)-\left(0+\frac{1}{2}\right)\right]=2E_0=2(0.0379 \text{ eV})=0.0758 \text{ eV}.$$

(f) Use $E=hc/\lambda$ with $E = 0.0758$ eV, giving $\lambda = 16.4$ μm.
EVALUATE: There is obviously a big difference between a quantum oscillator and a classical oscillator.

40.54. **IDENTIFY:** Compare the energy E of the oscillator to the equation $E_n=(n+\frac{1}{2})\hbar\omega$ in order to determine n.
SET UP: At the equilibrium position the potential energy is zero and the kinetic energy equals the total energy.
EXECUTE: **(a)** $E=\frac{1}{2}mv^2=[n+(1/2)]\hbar\omega=[n+(1/2)]hf$, and solving for n,

$$n=\frac{\frac{1}{2}mv^2}{hf}-\frac{1}{2}=\frac{(1/2)(0.020 \text{ kg})(0.480 \text{ m/s})^2}{(6.626\times10^{-34} \text{ J}\cdot\text{s})(1.50 \text{ Hz})}-\frac{1}{2}=2.3\times10^{30}.$$

(b) The difference between energies is $\hbar\omega=hf=(6.63\times10^{-34} \text{ J}\cdot\text{s})(1.50 \text{ Hz})=9.95\times10^{-34} \text{ J}$. This energy is too small to be detected with current technology.
EVALUATE: This oscillator can be described classically; quantum effects play no measurable role.

40.55. **IDENTIFY** and **SET UP:** Calculate the angular frequency ω of the pendulum and apply $E_n=(n+\frac{1}{2})\hbar\omega$ for the energy levels.
EXECUTE: $\omega=\dfrac{2\pi}{T}=\dfrac{2\pi}{0.500 \text{ s}}=4\pi \text{ s}^{-1}.$

The ground-state energy is $E_0 = \frac{1}{2}\hbar\omega = \frac{1}{2}(1.055\times10^{-34}\ \text{J}\cdot\text{s})(4\pi\ \text{s}^{-1}) = 6.63\times10^{-34}\ \text{J}$.

$$E_0 = 6.63\times10^{-34}\ \text{J}(1\ \text{eV}/1.602\times10^{-19}\ \text{J}) = 4.14\times10^{-15}\ \text{eV}.$$
$$E_n = (n+\tfrac{1}{2})\hbar\omega.$$
$$E_{n+1} = (n+1+\tfrac{1}{2})\hbar\omega.$$

The energy difference between the adjacent energy levels is

$$\Delta E = E_{n+1} - E_n = \hbar\omega = 2E_0 = 1.33\times10^{-33}\ \text{J} = 8.30\times10^{-15}\ \text{eV}.$$

EVALUATE: These energies are much too small to detect. Quantum effects are not important for ordinary size objects.

40.56. IDENTIFY: If the given wave function is a solution to the Schrödinger equation, we will get an identity when we substitute that wave function into the Schrödinger equation.

SET UP: The given wave function is $\psi_1(x) = A_1 x e^{-\alpha^2 x^2/2}$ and the Schrödinger equation is

$$-\frac{\hbar}{2m}\frac{d^2\psi(x)}{dx^2} + \frac{k'x^2}{2}\psi(x) = E\,\psi(x).$$

EXECUTE: **(a)** Start by taking the indicated derivatives: $\psi_1(x) = A_1 x e^{-\alpha^2 x^2/2}$.

$$\frac{d\psi_1(x)}{dx} = -\alpha^2 x^2 A_1 e^{-\alpha^2 x^2/2} + A_1 e^{-\alpha^2 x^2/2}.$$
$$\frac{d^2\psi_1(x)}{dx^2} = -A_1\alpha^2 2xe^{-\alpha^2x^2/2} - A_1\alpha^2x^2(-\alpha^2x)e^{-\alpha^2x^2/2} + A_1(-\alpha^2x)e^{-\alpha^2x^2/2}.$$
$$\frac{d^2\psi_1(x)}{dx^2} = \left[-2\alpha^2 + (\alpha^2)^2x^2 - \alpha^2\right]\psi_1(x) = \left[-3\alpha^2 + (\alpha^2)^2x^2\right]\psi_1(x).$$
$$-\frac{\hbar}{2m}\frac{d^2\psi_1(x)}{dx^2} = -\frac{\hbar^2}{2m}\left[-3\alpha^2 + (\alpha^2)^2x^2\right]\psi_1(x).$$

Eq. (40.44) is $-\frac{\hbar}{2m}\frac{d^2\psi(x)}{dx^2} + \frac{k'x^2}{2}\psi(x) = E\,\psi(x)$. Substituting the above result into that equation gives $-\frac{\hbar^2}{2m}\left[-3\alpha^2 + (\alpha^2)^2x^2\right]\psi_1(x) + \frac{k'x^2}{2}\psi_1(x) = E\,\psi_1(x)$. Since $\alpha^2 = \frac{m\omega}{\hbar}$ and $\omega = \sqrt{\frac{k'}{m}}$, the coefficient of x^2 is $-\frac{\hbar^2}{2m}(\alpha^2)^2 + \frac{k'}{2} = -\frac{\hbar^2}{2m}\left(\frac{m\omega}{\hbar}\right)^2 + \frac{m\omega^2}{2} = 0$.

(b) $A_1 = \left(\frac{m\omega}{\hbar}\right)^{3/4}\left(\frac{4}{\pi}\right)^{1/4}$.

(c) The probability density function $|\psi|^2$ is $|\psi_1(x)|^2 = A_1^2 x^2 e^{-\alpha^2x^2}$.

At $x=0$, $|\psi_1|^2 = 0$. $\frac{d|\psi_1(x)|^2}{dx} = A_1^2 2xe^{-\alpha^2x^2} + A_1^2x^2(-\alpha^2 2x)e^{-\alpha^2x^2} = A_1^2 2xe^{-\alpha^2x^2} - A_1^2 2x^3\alpha^2e^{-\alpha^2x^2}$.

At $x=0$, $\frac{d|\psi_1(x)|^2}{dx} = 0$. At $x = \pm\frac{1}{\alpha}$, $\frac{d|\psi_1(x)|^2}{dx} = 0$.

$$\frac{d^2|\psi_1(x)|^2}{dx^2} = A_1^2 2e^{-\alpha^2x^2} + A_1^2 2x(-\alpha^2 2x)e^{-\alpha^2x^2} - A_1^2 2(3x^2)\alpha^2e^{-\alpha^2x^2} - A_1^2 2x^3\alpha^2(-\alpha^2 2x)e^{-\alpha^2x^2}.$$

$\frac{d^2|\psi_1(x)|^2}{dx^2} = A_1^2 2e^{-\alpha^2x^2} - A_1^2 4x^2\alpha^2e^{-\alpha^2x^2} - A_1^2 6x^2\alpha^2e^{-\alpha^2x^2} + A_1^2 8x^4(\alpha^2)^2e^{-\alpha^2x^2}$. At $x=0$,

$\frac{d^2|\psi_1(x)|^2}{dx^2} > 0.$ So at $x = 0,$ the first derivative is zero and the second derivative is positive. Therefore, the probability density function has a minimum at $x = 0.$ At $x = \pm\frac{1}{\alpha},$ $\frac{d^2|\psi_1(x)|^2}{dx^2} < 0.$ So at $x = \pm\frac{1}{\alpha},$ the first derivative is zero and the second derivative is negative. Therefore, the probability density function has maxima at $x = \pm\frac{1}{\alpha},$ corresponding to the classical turning points for $n = 0$ as found in the previous question.

EVALUATE: $\psi_1(x) = A_1 x e^{-\alpha^2 x^2/2}$ is a solution to Eq. (40.44) if $-\frac{\hbar^2}{2m}(-3\alpha^2)\psi_1(x) = E\,\psi_1(x)$ or $E = \frac{3\hbar^2\alpha^2}{2m} = \frac{3\hbar\omega}{2}.$ $E_1 = \frac{3\hbar\omega}{2}$ corresponds to $n = 1$ in Eq. (40.46).

40.57. **IDENTIFY:** For a standing wave in the box, there must be a node at each wall and $n\left(\frac{\lambda}{2}\right) = L.$

SET UP: $p = \frac{h}{\lambda} \neq,$ so $mv = \frac{h}{\lambda}.$

EXECUTE: **(a)** For a standing wave, $n\lambda = 2L,$ and $E_n = \frac{p^2}{2m} = \frac{(h/\lambda)^2}{2m} = \frac{n^2h^2}{8mL^2}.$

(b) With $L = a_0 = 0.5292\times10^{-10}$ m, $E_1 = 2.15\times10^{-17}$ J $= 134$ eV.

EVALUATE: For a hydrogen atom, E_n is proportional to $1/n^2,$ so this is a very poor model for a hydrogen atom. In particular, it gives very inaccurate values for the separations between energy levels.

40.58. **IDENTIFY** and **SET UP:** Follow the steps specified in the problem.

EXECUTE: **(a)** As with the particle in a box, $\psi(x) = A\sin kx,$ where A is a constant and $k^2 = 2mE/\hbar^2.$ Unlike the particle in a box, however, k and hence E do not have simple forms.

(b) For $x > L,$ the wave function must have the form of $\psi(x) = Ce^{\kappa x} + De^{-\kappa x}.$ For the wave function to remain finite as $x \to \infty, C = 0.$ The constant $\kappa^2 = 2m(U_0 - E)/\hbar,$ as in $\psi(x) = Ce^{\kappa x} + De^{-\kappa x}.$

(c) At $x = L, A\sin kL = De^{-\kappa L}$ and $kA\cos kL = -\kappa De^{-\kappa L}.$ Dividing the second of these by the first gives $k\cot kL = -\kappa,$ a transcendental equation that must be solved numerically for different values of the length L and the ratio $E/U_0.$

EVALUATE: When $U_0 \to \infty,$ $\kappa \to \infty$ and $\frac{\cos(kL)}{\sin(kL)} \to \infty.$ The solutions become $k = \frac{n\pi}{L}, n = 1, 2, 3,\ldots,$ the same as for a particle in a box.

40.59. **IDENTIFY** and **SET UP:** The energy levels for an infinite potential well are $E_n = \frac{n^2h^2}{8mL^2} = E_1 n^2.$ The energy of the absorbed photon is equal to the energy difference between the levels. The energy of a photon is $E = hf = hc/\lambda,$ so $\Delta E = hf = E_{n+1} - E_n.$

EXECUTE: **(a)** For the first transition, we have $hf_1 = E_1(n^2 - 1^2),$ and for the second transition we have $hf_2 = E_1[(n+1)^2 - 1^2].$ Taking the ratio of these two equations gives

$$\frac{hf_2}{hf_1} = \frac{16.9}{9.0} = \frac{(n+1)^2 - 1}{n^2 - 1} = \frac{n^2 + 2n + 1 - 1}{n^2 - 1} = \frac{n^2 + 2n}{n^2 - 1}.$$

Rearranging and collecting terms gives the quadratic equation $n^2\left(\frac{16.9}{9.0}-1\right)-2n-\frac{16.9}{9.0}=0.$ Using the quadratic formula and taking the positive root gives $n = 3.0$, so $n = 3$. Therefore the transitions are from the $n = 3$ and $n = 4$ levels to the $n = 1$ level.

(b) Using the $3 \to 1$ transition with $f_1 = 9.0\times10^{14}$ Hz, we have

$hf_1 = (h^2/8mL^2)(3^2-1^2) = h^2/mL^2.$

$$L=\sqrt{\frac{h}{f_1 m}}=\sqrt{\frac{6.626\times10^{-34}\ \text{J}\cdot\text{s}}{(9.0\times10^{14}\ \text{Hz})(9.109\times10^{-31}\ \text{kg})}}=9.0\times10^{-10}\ \text{m}=0.90\ \text{nm}.$$

(c) The longest wavelength is for the smallest energy, and that would be for a transition between $n = 1$ and $n = 2$ levels. Comparing the $1 \to 3$ transition and the $1 \to 2$ transition, we have

$$\frac{hf_{1\to2}}{hf_{1\to3}}=\frac{E_1(2^2-1^2)}{E_1(3^2-1^2)} \quad\to\quad f_{1\to2}=\frac{3}{8}f_{1\to3}=\frac{3}{8}(9.0\times10^{14}\ \text{Hz}).$$

$$\lambda=c/f=\frac{3.00\times10^8\ \text{m/s}}{\frac{3}{8}(9.0\times10^{14}\ \text{Hz})}=890\ \text{nm}.$$

EVALUATE: This wavelength is too long to be visible light. The wavelength of the 9.0×10^{14} Hz photon is 333 nm, which is too short to be visible, as is the 16.9×10^{14} Hz photon. So none of these photons will be visible.

40.60. IDENTIFY and **SET UP:** Provided that $T \ll 1$, the probability T of a particle with energy E and mass m tunneling through a potential barrier of width L and height U_0 is $T = Ge^{-2\kappa L}$, where

$$G=16\frac{E}{U_0}\left(1-\frac{E}{U_0}\right) \text{ and } \kappa=\frac{\sqrt{2m(U_0-E)}}{\hbar}.$$

EXECUTE: **(a)** Using the values of E in the table in the problem, we calculate G and κ. For example, for $E = 4.0$ eV, we have $G=16\frac{E}{U_0}\left(1-\frac{E}{U_0}\right)=16\frac{4.0\ \text{eV}}{8.0\ \text{eV}}\left(1-\frac{4.0\ \text{eV}}{8.0\ \text{eV}}\right)=4.0.$

$$U_0-E=8.0\ \text{eV}-4.0\ \text{eV}=4.0\ \text{eV}=6.4\times10^{-19}\ \text{J}.$$

$$\kappa=\frac{\sqrt{2m(U_0-E)}}{\hbar}=\frac{\sqrt{2(9.11\times10^{-31}\ \text{kg})(6.4\times10^{-19}\ \text{J})}}{1.055\times10^{-34}\ \text{J}\cdot\text{s}}=1.02\times10^{10}\ \text{m}^{-1}.$$

Repeating these calculations for the other values of E, and also calculating $\ln(T/G)$, we get the following:

E(eV)	T	G	κ (m^{-1})	$\ln(T/G)$
4.0	2.40×10^{-6}	4.0	1.02×10^{10}	–14.3
5.0	1.50×10^{-5}	3.75	8.87×10^{9}	–12.4
6.0	1.20×10^{-4}	3.0	7.25×10^{9}	–10.1
7.0	1.30×10^{-3}	1.75	5.12×10^{9}	–7.2
7.6	8.10×10^{-3}	0.76	3.24×10^{9}	–4.5

Figure 40.60 shows the graph of $\ln(T/G)$ versus κ. For $T \ll 1$, we have $T = Ge^{-2\kappa L}$, which we can write as $T/G = e^{-2\kappa L}$. Taking natural logarithms of both sides of the equation gives $\ln(T/G) = -2L\kappa$. From this last equation, we would expect a graph of $\ln(T/G)$ versus κ to be a straight line with slope equal to $-2L$.

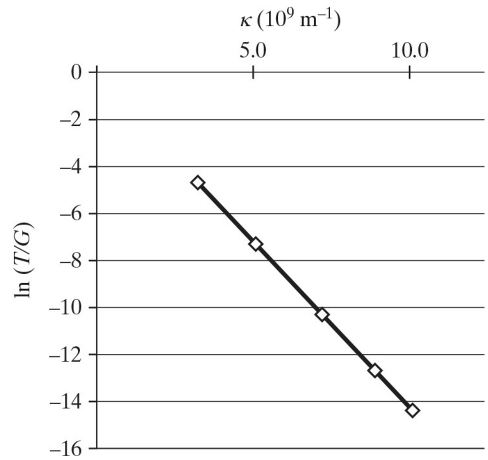

Figure 40.60

(b) The slope of the best-fit straight line for our graph is -1.396×10^{-9} m, so $-2L = -1.396\times10^{-9}$ m, which gives $L = 0.698\times10^{-9}$ m, which rounds to $L = 0.70$ nm.

EVALUATE: This width is about 7 times the width of a hydrogen atom in the Bohr model.

40.61. **IDENTIFY** and **SET UP:** The transmission coefficient T is equal to 1 when the width L of the barrier is $L = \frac{1}{2}\lambda,\ \lambda,\ \frac{3}{2}\lambda,\ 2\lambda,\ \ldots = n\,\lambda/2,$ where $n = 1, 2, 3, \ldots$, and where λ is the de Broglie wavelength of the electron, given by $\lambda = h/p$. The total energy of the electron is $E = U + K$, and $K = p^2/2m$.

EXECUTE: From the condition on λ, we have $\lambda_n = 2L/n$. Therefore $\lambda = h/p = 2L/n$, which gives $p = nh/2L$. The kinetic energy is $K = p^2/2m$, so $K_n = \dfrac{p^2}{2m} = \dfrac{\left(\dfrac{nh}{2L}\right)^2}{2m} = n^2\left(\dfrac{h^2}{8mL^2}\right) = n^2K_1$. The three lowest values of K are for $n = 1$, 2, and 3.

$K_1 = h^2/8mL^2 = (6.626\times10^{-34}\ \text{J}\cdot\text{s})^2/[8(9.11\times10^{-31}\ \text{kg})(1.8\times10^{-10}\ \text{m})^2] = 1.86\times10^{-18}\ \text{J} = 11.6\ \text{eV}.$

The total energy is $E = K + U$, so $E_1 = K_1 + U = 11.6\ \text{eV} + 10\ \text{eV} = 22\ \text{eV}$.

For the $n = 2$ state, we have

$$K_2 = 2^2K_1 = 4(11.6\ \text{eV}) = 46.4\ \text{eV, so } E_2 = 46.4\ \text{eV} + 10\ \text{eV} = 56\ \text{eV}.$$

For the $n = 3$ state, we have

$K_3 = 3^2K_1 = 9(11.6\ \text{eV}) = 104.4\ \text{eV, so } E_3 = 104.4\ \text{eV} + 10\ \text{eV} = 114\ \text{eV},$ which rounds to 110 eV.

EVALUATE: We cannot use Eq. (40.42) because T is not small.

40.62. **IDENTIFY:** This problem deals with the Schrödinger wave equation.

SET UP: $-\dfrac{\hbar^2}{2m}\dfrac{d^2\psi}{dx} = E\psi$

EXECUTE: **(a)** For a free particle, ψ is in the form $\psi = Ae^{\pm in\theta}$. The wave function repeats as θ changes by 2π. If n is an integer, the wave functions must be $\psi_n^+ = A_+e^{+in\theta}$ and $\psi_n^- = A\_e^{-in\theta}$. The – function is for clockwise motion and the + function is for counterclockwise motion.

(b) We normalize the functions to find the coefficients.

$$1 = \int\left|\psi_+\right|^2 dx = \int_0^{2\pi R} A_+^2\, e^{in\theta}e^{-in\theta}\, dx = A_+^2\, 2\pi R.\ A_+ = 1/\sqrt{2\pi R}.$$

The same procedure gives the same value for $A\_$.

(c) We want the energy levels E_n. The wave function must satisfy the time-independent Schrödinger equation. Using $x = R\theta$ for ψ^+ gives the following:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx} = E\psi$$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi_n^+}{dx^2} = -\frac{\hbar^2}{2m}\frac{d^2\psi_n^+}{R^2\,d\theta^2} = -\frac{\hbar^2}{2mR^2}\frac{d^2}{d\theta^2}\left(A_+e^{in\theta}\right) = -\frac{\hbar^2}{2mR^2}A_+(in)^2e^{in\theta} = -\frac{\hbar^2}{2mR^2}(-n^2)A_+e^{in\theta}$$

The right-hand part of the last equation must be $E\psi$, so we find that

$$E_n = \frac{\hbar^2 n^2}{2mR^2}.$$

(d) We want the time-dependent wave function. We use the following:

$$\omega = E_1/\hbar,\ E_n = n^2E_1 = n^2\hbar\omega,\ \ \Psi(x,t) = \psi(x)e^{-iEt/\hbar}.$$

For Ψ_n^+ we have

$$\Psi_n^+(x,t) = \psi_n^+(x)e^{-iE_nt/\hbar} = A_+e^{in\theta}e^{-iE_nt/\hbar} = \frac{1}{\sqrt{2\pi R}}e^{in(\theta-n\omega t)} = \frac{1}{\sqrt{2\pi R}}e^{in(x/R-n\omega t)}.$$

The same procedure leads to

$$\Psi_n^-(x,t) = \frac{1}{\sqrt{2\pi R}}e^{-in(\theta+n\omega t)} = \frac{1}{\sqrt{2\pi R}}e^{-in(x/R+n\omega t)}.$$

(e) We want the probability density.

$$E_n/\hbar = n^2\omega$$

$$\Psi(x,t) = \frac{1}{\sqrt{2}}\left(\psi_1^+(x)e^{-iE_1t/\hbar} + \psi_2^+(x)e^{-iE_2t/\hbar}\right) = \frac{1}{\sqrt{2}}\left(A_1e^{i\theta}e^{-i\omega t} + A_2e^{2i\theta}e^{-4i\omega t}\right)$$

$$\Psi(x,t) = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2\pi R}}\left(e^{i(\theta-\omega t)} + e^{2i(\theta-2\omega t)}\right)$$

To find the probability density, we square the wave function and collect terms:

$$|\Psi(x,t)|^2 = \frac{1}{4\pi R}\left(2 + e^{-i\theta}e^{3i\omega t} + e^{i\theta}e^{-3i\omega t}\right)$$

Now use $e^{\pm i\theta} = \cos\theta \pm i\sin\theta$ to remove the exponentials, giving

$$|\Psi(x,t)|^2 = \frac{1}{2\pi}(1 + \cos\theta\cos 3\omega t + \sin\theta\sin 3\omega t) = \frac{1}{2\pi R}[1 + \cos(\theta - 3\omega t)]$$

Now use the identity suggested in the problem with $\alpha = \theta - 3\omega t$ to obtain

$$|\Psi(x,t)|^2 = \frac{1}{\pi R}\cos^2\left(\frac{\theta - 3\omega t}{2}\right) = \frac{1}{\pi R}\cos^2\left(\frac{x/R - 3\omega t}{2}\right).$$

(f) We want the angular speed of the density peak. The density peak occurs when the probability amplitude is a maximum. This occurs when the cosine factor in the previous result is equal to 1. From this we see that $\theta - 3\omega t = 0$. The angular speed is $\theta/t = 3\omega$.

(g) We want the speed for the electron at the Bohr radius.

$$v = R(\theta/t) = R(3\omega) = 3R\omega.\ \ \omega = E_1/\hbar = \frac{\hbar^2/2mR^2}{\hbar} = \frac{\hbar}{2mR^2}.\ \ v = 3R\omega = 3R\left(\frac{\hbar}{2mR^2}\right) = \frac{3\hbar}{2mR}.$$

Putting in the numbers with $R = a_0 = 5.29 \times 10^{-11}$ m gives $v = 3.28 \times 10^6$ m/s.

EVALUATE: This result is fairly close to the value of 2.2×10^6 m/s that Eq. (39.9) gives.

40.63. **IDENTIFY:** This problem involves commutators and the Schrödinger equation.

SET UP: $[A, B]f = A(Bf) - B(Af)$. The following operators are defined:

$$a_+ = \frac{1}{\sqrt{2m\hbar\omega}}\left(-\hbar\frac{d}{dx} + m\omega x\right) \text{ and } a_- = \frac{1}{\sqrt{2m\hbar\omega}}\left(+\hbar\frac{d}{dx} + m\omega x\right).$$

EXECUTE: (a) We want $[a_-, a_+]$.

$$[a_-, a_+]f = a_-(a_+f) - a_+(a_-f) = \frac{1}{\sqrt{2m\hbar\omega}}\frac{1}{\sqrt{2m\hbar\omega}}\left(\hbar\frac{d}{dx} + m\omega x\right)\left[\left(-\hbar\frac{d}{dx} + m\omega x\right)f\right]$$

$$-\frac{1}{\sqrt{2m\hbar\omega}}\frac{1}{\sqrt{2m\hbar\omega}}\left(-\hbar\frac{d}{dx} + m\omega x\right)\left[\left(\hbar\frac{d}{dx} + m\omega x\right)f\right]$$

Carefully carrying out all the operations gives $[a_-, a_+]f = f$, so $[a_-, a_+] = 1$.

(b) We want $[a_+, a_-]\psi$, where ψ is a wave equation, which means that it is a solution to the Schrödinger equation. Follow the procedure in part (a). The solution is sketched out below.

$$a_+a_-\psi = \frac{1}{\sqrt{2m\hbar\omega}}\left(-\hbar\frac{d}{dx} + m\omega x\right)\left[\frac{1}{\sqrt{2m\hbar\omega}}\left(\hbar\frac{d}{dx} + m\omega x\right)\psi\right].$$

$$a_+a_-\psi = \frac{1}{\sqrt{2m\hbar\omega}}\left(-\hbar\frac{d}{dx} + m\omega x\right)\left(\hbar\frac{d\psi}{dx} + m\omega x\psi\right).$$

$$a_+a_-\psi = \frac{1}{\sqrt{2m\hbar\omega}}\left(-\hbar^2\frac{d^2\psi}{dx^2} - \hbar m\omega\frac{d(x\psi)}{dx} + m\omega x\frac{d\psi}{dx} + m^2\omega^2x^2\psi\right).$$

$$a_+a_-\psi = \frac{1}{\sqrt{2m\hbar\omega}}\left(-\hbar^2\frac{d^2\psi}{dx^2} - \hbar m\omega\psi + m^2\omega^2x^2\psi\right) = \frac{1}{\hbar\omega}\left(-\frac{\hbar^2}{2m}\psi'' + \frac{m\omega^2x^2}{2}\psi\right) - \frac{1}{2}\psi.$$

(c) Use the results of part (b) and the Schrödinger equation given with this problem. Note that the term in parentheses in the answer to (b) is the Schrödinger equation. Solving for it gives

$$\hbar\omega\left(a_+a_-\psi + \frac{1}{2}\psi\right) = -\frac{\hbar^2}{2m}\psi'' + \frac{m\omega^2x^2}{2}\psi$$

Therefore the operator H must be

$$H = \hbar\omega\left(a_+a_- + \frac{1}{2}\right).$$

(d) We want $[H, a_\pm]$. Using $[H, a_\pm]f = H(a_+f) - a_+(Hf)$ gives

$$[H, a_+]f = H(a_+f) - a_+\left[\hbar\omega\left(a_+a_- + \frac{1}{2}\right)\right]f = \hbar\omega\left[\left(\frac{1}{2} + a_+a_-\right)(a_+f) - a_+\frac{f}{2} - a_+(a_+a_-f)\right]$$

$$[H, a_+]f = \hbar\omega(a_+)(a_-a_+ - a_+a_-)f = \hbar\omega a_+f.$$

The same procedure leads to a similar result for $[H, a_\pm]$. Therefore the commutators are

$$[H, a_+] = \hbar\omega a_+ \text{ and } [H, a] = \hbar\omega a_-.$$

(e) We want to relate E_{n+1} to E_n.

For the n state: $H\Psi_n = E_n\Psi_n$

For the $n+1$ state: $H\psi_{n+1} = E_{n+1}\psi_{n+1}$ and $\psi_{n+1} = a_+\psi_n$, so $H\psi_{n+1} = Ha_+\psi_n$

$$[H, a_+]\psi_n = Ha_+\psi_n - a_+H\psi_n \text{ gives } Ha_+\psi_n = [H, a_+]\psi_n + a_+H\psi_n$$

$$H\psi_{n+1} = Ha_+\psi_n = ([H, a_+] + a_+H)\psi_n$$

Using the result from part (d), we can write this result as

$$H\psi_{n+1} = a_+H\psi_n + \hbar\omega a_+\psi_n$$

$$H\psi_n = E_n\psi_n \text{ and } \psi_{n+1} = a_+\psi_n \text{ gives } H\psi_{n+1} = E_n\psi_{n+1} + \hbar\omega\psi_{n+1} = (E_n + \hbar\omega)\psi_{n+1}$$

This result means that $E_{n+1} = E_n + \hbar\omega.$

(f) We want the ground state energy E_0.

Using $H\psi_0 = E_0\psi_0$ gives $\hbar\omega\left(\frac{1}{2}+a_+a_-\right)\psi_0 = E_0\psi_0$. Given that $a_-\psi_0 = 0,$ we have

$\frac{1}{2}\hbar\omega\psi_0 = E_0\psi_0$, which tells us that $E_0 = \frac{1}{2}\hbar\omega.$

(g) We want the energy levels E_n. From (e) we know that the energy of each state is $\hbar\omega$ higher than the previous state, and from (f) we know that the energy of the lowest state is $\hbar\omega/2$. Therefore the energies are

$$E_0 = \hbar\omega\left(\frac{1}{2}\right), E_1 = \hbar\omega\left(1+\frac{1}{2}\right), E_2 = \hbar\omega\left(2+\frac{1}{2}\right), E_3 = \hbar\omega\left(3+\frac{1}{2}\right), \cdots. E_n = \hbar\omega\left(n+\frac{1}{2}\right).$$

EVALUATE: The use of the method developed here depends on selecting appropriate operators $a_\pm$.

40.64. IDENTIFY and **SET UP:** Follow the steps specified in the problem.

EXECUTE: **(a)** $E = K + U(x) = \frac{p^2}{2m} + U(x) \Rightarrow p = \sqrt{2m(E-U(x))}.$ $\lambda = \frac{h}{p} \Rightarrow \lambda(x) = \frac{h}{\sqrt{2m(E-U(x))}}.$

(b) As $U(x)$ gets larger (i.e., $U(x)$ approaches E from below—recall $k \ge 0$), $E-U(x)$ gets smaller, so $\lambda(x)$ gets larger.

(c) When $E = U(x)$, $E-U(x) = 0$, so $\lambda(x) \to \infty$.

(d) $\int_a^b \frac{dx}{\lambda(x)} = \int_a^b \frac{dx}{h/\sqrt{2m(E-U(x))}} = \frac{1}{h}\int_a^b \sqrt{2m(E-U(x))}\,dx = \frac{n}{2} \Rightarrow \int_a^b \sqrt{2m(E-U(x))}\,dx = \frac{hn}{2}.$

(e) $U(x) = 0$ for $0 < x < L$ with classical turning points at $x = 0$ and $x = L$. So,

$\int_a^b \sqrt{2m[E-U(x)]}\,dx = \int_0^L \sqrt{2mE}\,dx = \sqrt{2mE}\int_0^L dx = \sqrt{2mE}L.$ So, from part (d),

$\sqrt{2mE}L = \frac{hn}{2} \Rightarrow E = \frac{1}{2m}\left(\frac{hn}{2L}\right)^2 = \frac{h^2n^2}{8mL^2}.$

EVALUATE: **(f)** Since $U(x) = 0$ in the region between the turning points at $x = 0$ and $x = L$, the result is the *same* as part (e). The height U_0 never enters the calculation. WKB is best used with *smoothly* varying potentials $U(x)$.

40.65. IDENTIFY: Perform the calculations specified in the problem.

SET UP: $U(x) = \frac{1}{2}k'x^2.$

EXECUTE: **(a)** At the turning points $E = \frac{1}{2}k'x_{\text{TP}}^2 \Rightarrow x_{\text{TP}} = \pm\sqrt{\frac{2E}{k'}}.$

(b) $\int_{-\sqrt{2E/k'}}^{+\sqrt{2E/k'}} \sqrt{2m(E-\frac{1}{2}k'x^2)}dx = \frac{nh}{2}.$ To evaluate the integral, we want to get it into a form that matches the standard integral given. $\sqrt{2m\left(E-\frac{1}{2}k'x^2\right)} = \sqrt{2mE - mk'x^2} = \sqrt{mk'}\sqrt{\frac{2mE}{mk'}-x^2} = \sqrt{mk'}\sqrt{\frac{2E}{k'}-x^2}.$

Letting $A^2 = \frac{2E}{k'}$, $a = -\sqrt{\frac{2E}{k'}}$, *and* $b = +\sqrt{\frac{2E}{k'}}$

$$\Rightarrow \sqrt{mk'}\int_a^b \sqrt{A^2-x^2}\,dx = 2\frac{\sqrt{mk'}}{2}\left[x\sqrt{A^2-x^2} + A^2\arcsin\left(\frac{x}{|A|}\right)\right]\Bigg|_0^b$$

$$= \sqrt{mk'}\left[\sqrt{\frac{2E}{k'}}\sqrt{\frac{2E}{k'}-\frac{2E}{k'}} + \frac{2E}{k'}\arcsin\left(\frac{\sqrt{2E/k'}}{\sqrt{2E/k'}}\right)\right] = \sqrt{mk'}\,\frac{2E}{k'}\arcsin(1) = 2E\sqrt{\frac{m}{k'}}(\tfrac{1}{2}).$$

Using WKB, this is equal to $\frac{hn}{2}$, so $E\sqrt{\frac{m}{k'}}\pi = \frac{hn}{2}$. Recall $\omega = \sqrt{\frac{k'}{m}}$, so $E = \frac{h}{2\pi}\omega n = \hbar\omega n.$

EVALUATE: (c) We are missing the zero-point-energy offset of $\frac{\hbar\omega}{2}$ (recall $E = \hbar\omega(n+\frac{1}{2})$). It underestimates the energy. However, our approximation isn't bad at all!

40.66. IDENTIFY and SET UP: Perform the calculations specified in the problem.

EXECUTE: (a) At the turning points $E = A|x_{\text{TP}}| \Rightarrow x_{\text{TP}} = \pm\frac{E}{A}.$

(b) $\int_{-E/A}^{+E/A}\sqrt{2m(E - A|x|)}dx = 2\int_0^{E/A}\sqrt{2m(E - Ax)}\,dx.$ Let $y = 2m(E - Ax)$

$\Rightarrow dy = -2mA\,dx$ when $x = \frac{E}{A}$, $y = 0$, and when $x = 0$, $y = 2mE$. So

$2\int_0^{\frac{E}{A}}\sqrt{2m(E - Ax)}dx = -\frac{1}{mA}\int_{2mE}^{0} y^{1/2}dy = -\frac{2}{3mA}y^{3/2}\Big|_{2mE}^{0} = \frac{2}{3mA}(2mE)^{3/2}.$ Using WKB, this is equal to

$\frac{hn}{2}$. So, $\frac{2}{3mA}(2mE)^{3/2} = \frac{hn}{2} \Rightarrow E = \frac{1}{2m}\left(\frac{3mAh}{4}\right)^{2/3} n^{2/3}.$

EVALUATE: (c) The difference in energy decreases between successive levels. For example:
$1^{2/3} - 0^{2/3} = 1,\ 2^{2/3} - 1^{2/3} = 0.59,\ 3^{3/2} - 2^{3/2} = 0.49,\ldots$

- A sharp ∞ step gave ever-increasing level differences $(\sim n^2)$.
- A parabola $(\sim x^2)$ gave evenly spaced levels $(\sim n)$.
- Now, a linear potential $(\sim x)$ gives ever-decreasing level differences $(\sim n^{2/3})$.

Roughly speaking, if the curvature of the potential (~ second derivative) is bigger than that of a parabola, then the level differences will increase. If the curvature is less than a parabola, the differences will decrease.

40.67. IDENTIFY and SET UP: The energy levels are $E_{m,n} = (m^2 + n^2)(\pi^2\hbar^2)/2ML^2.$

EXECUTE: For a fixed value of m, the spacing between adjacent energy levels is
$\Delta E = [(m^2 + (n+1)^2 - (m^2 + n^2)](\pi^2\hbar^2)/2ML^2 = (2n+1)(\pi^2\hbar^2)/2ML^2.$ As the value of L is decreased, the spacing between adjacent levels increases. In this model, L is the size of the dot, so choice (c) is correct.

EVALUATE: We could also have kept n fixed and varied m by 1.

40.68. IDENTIFY and SET UP: The energy levels are $E_{m,n} = (m^2 + n^2)(\pi^2\hbar^2)/2ML^2.$ $\Delta E = \frac{hc}{\lambda}$ for the photon emitted during a transition through energy difference ΔE.

EXECUTE: Since $E \propto 1/L^2$, it is also the case that $\Delta E \propto 1/L^2$, and we know that $\Delta E = \frac{hc}{\lambda}.$

Therefore
$\lambda \propto L^2$. If $L \to 1.1L$, $\lambda \to (1.1)^2\lambda$. So in this case, $\lambda \to (1.1)^2(550\text{ nm}) = 666\text{ nm} \approx 670\text{ nm}$, which is choice (b).

EVALUATE: Because λ is proportional to the *square* of L instead of just L, a 10% change in L produces around a 20% change in λ.

40.69. **IDENTIFY** and **SET UP:** The energy levels are $E_{m,n} = (m^2 + n^2)(\pi^2\hbar^2)/2ML^2$. $\Delta E = \dfrac{hc}{\lambda}$ for the photon emitted during a transition through energy difference ΔE.

EXECUTE: Since $\Delta E \propto 1/M$ and $\Delta E = \dfrac{hc}{\lambda}$, it follows that $\lambda \propto 1/M$. Since $\lambda_1 > \lambda_2$, it is true that $M_1 > M_2$, which makes choice (a) correct.

EVALUATE: When M is large, the energy difference between states is small, so the energy of an emitted (or absorbed) photon is small, so its wavelength is large.

40.70. **IDENTIFY:** Apply the Heisenberg uncertainty principle, stated in terms of energy and time.
SET UP: $\Delta E \Delta t \ge \hbar/2$.

EXECUTE: Solving for ΔE gives $\Delta E \ge \dfrac{\hbar}{2\Delta t}$. Therefore increasing the lifetime of the excited states results in a smaller energy spread because ΔE is small, meaning that the energies are more well-defined. Therefore choice (a) is correct.

EVALUATE: A long lifetime does not imply that the energy is small; but it does tell us that the *uncertainty* in the energy is small.