Circuits
-
Capacitors and Inductors

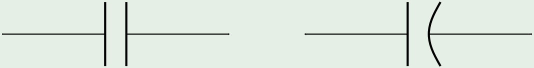
Spring 2022

Storage elements

Time dependence

- Until now, everything was happening instantly (DC mode)
- We will discover **new elements** which have some **dependence on time** time dependent
- These elements can **store and deliver energy**

Capacitor

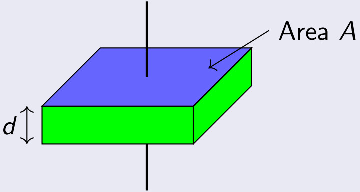


Inductor

Capacitors

Physical concept

Dielectric material between conductive plates



- Suppose we impose the voltage
- Charge builds up on each of the plates
- The charge stored is directly proportional to the voltage

Equation

$$q = C \cdot v$$

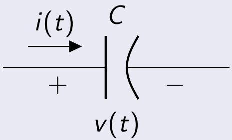
- q: charge in Coulombs (C)
- C: capacitance in Farads (F)
- v: voltage in Volts (V)

$$C = \frac{\epsilon \cdot A}{d}$$

- ϵ: permittivity in $\mathrm{C}^2/\mathrm{N} \cdot \mathrm{m}^2$

Capacitors

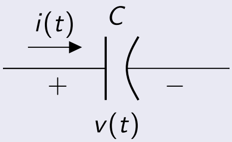
Constitutive equation



- $q(t) = C \cdot v(t)$
- $\frac{d}{dt} q(t) = C \frac{d}{dt} v(t)$
- $\boxed{i(t) = C \frac{dv(t)}{dt}}$

Capacitors

Alternative constitutive equation



- $i(t) = C \cdot \dfrac{dv(t)}{dt}$
- $\displaystyle\int_{t_0}^{t} i(\tau)d\tau = C\int_{t_0}^{t} \frac{dv(\tau)}{d\tau}d\tau = C \cdot [v(t) - v(t_0)]$
- $\boxed{\displaystyle v(t) = v(t_0) + \frac{1}{C}\int_{t_0}^{t} i(\tau)d\tau}$

Very often (but not always!), we will consider that: $v(0) = 0$

Capacitors

Linearity?

- Are capacitors **linear elements**?

For a voltage $v_1(t)$, then the current is: $i_1(t) = C\dfrac{dv_1(t)}{dt}$

For another voltage $v_2(t)$, then the current is: $i_2(t) = C\dfrac{dv_2(t)}{dt}$

Now, we impose a voltage $v(t) = a \cdot v_1(t) + b \cdot v_2(t)$. The current is:

$$i(t) = C\frac{dv(t)}{dt} = C\frac{d(a \cdot v_1(t) + b \cdot v_2(t))}{dt} = aC\frac{dv_1(t)}{dt} + bC\frac{dv_2(t)}{dt}$$

$$i(t) = a \cdot i_1(t) + b \cdot i_2(t)$$

Capacitors are linear

Capacitors

Power absorbed by capacitors

$$p(t) = v(t) \cdot i(t) = v(t) \cdot C\frac{dv(t)}{dt}$$

- If positive: absorbs power (increases the stored energy)
- If negative: delivers power (gives back the stored energy)

Energy stored

The energy stored by a capacitor from time t_0 to time t is:

$$w(t) - w(t_0) = \int_{t_0}^{t} p(\tau)d\tau = \int_{t_0}^{t} v(\tau) \cdot C\frac{dv(\tau)}{d\tau}d\tau$$

$$= C\int_{v(t_0)}^{v(t)} v(\tau)dv(\tau) = \frac{1}{2}C[v(t)^2 - v(t_0)^2]$$

Capacitors

Energy

From previous equation, we can conclude that:

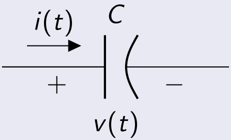
$$w(t) = \frac{1}{2}Cv(t)^2 + cste$$

By convention, the *cste* is chosen to be null. It means that when v is equal to 0, the capacitor does not have any energy stored.

$$\text{Energy stored:} \qquad w(t) = \frac{1}{2}Cv(t)^2$$

Capacitors

Observations



$$i(t) = C \cdot \frac{dv(t)}{dt}$$

- When $v(t)$ is constant, then there is no current ($i(t) = 0$)
- In DC mode, the capacitor is equivalent to an open-circuit

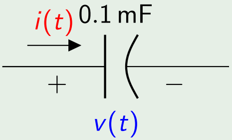
- $v(t)$ **cannot** change too quickly, otherwise it leads to huge currents (physically possible?)
- $v(t)$ is actually continuous (cannot have discontinuities)

- Capacitors do not dissipate energy, they store it then deliver it back

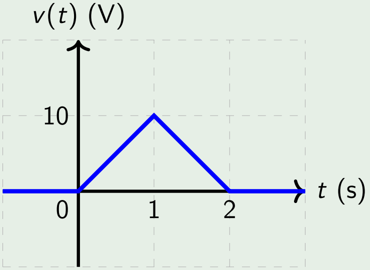
Capacitors

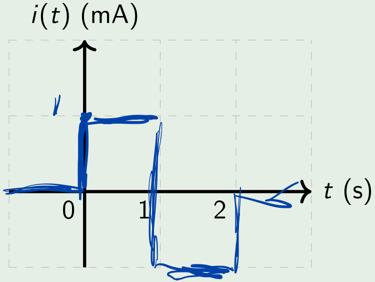
Exercise

Find the expression of $i(t)$. Also plot the corresponding timing diagram.



$$i(t) = C \frac{dv(t)}{dt}$$





Capacitors

Exercise

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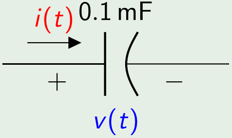
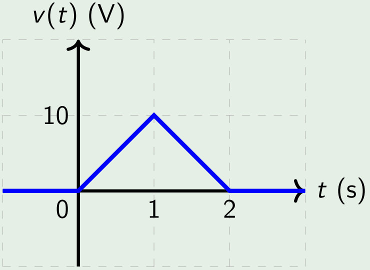
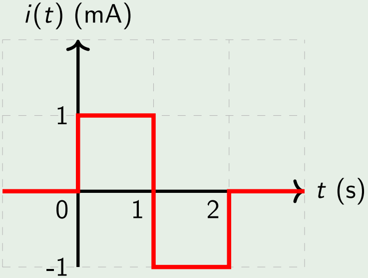






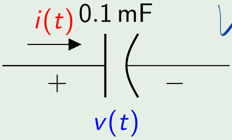
Capacitors

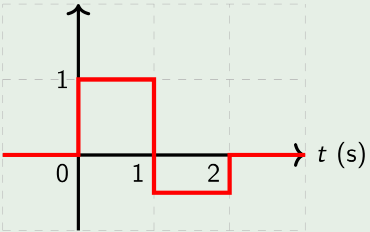
Exercise

Find the expression of $v(t)$. Also plot the corresponding timing diagram. At $t = 0$, the capacitor voltage is $-5\,\mathrm{V}$.

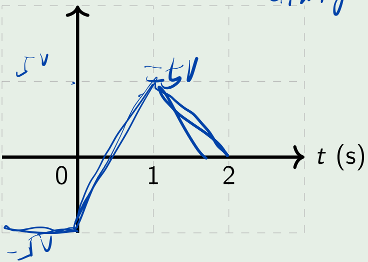
$$V(t) = V(t_0) + \frac{1}{C}\int_{t_0}^{t} i(t)\,dt$$

$$V(1s) = V(0s) + \frac{1}{C}\int_0^t i(t)\,dt$$





$v(t)$ (V) $= -5V + \frac{1}{0.1mF}\int_0^t i(t)\,dt$



Capacitors

Exercise

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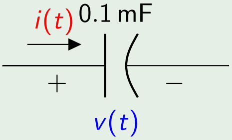
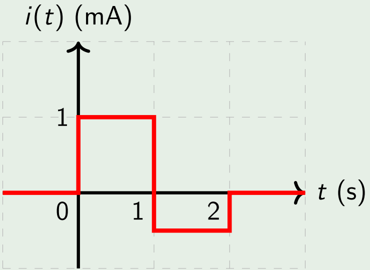
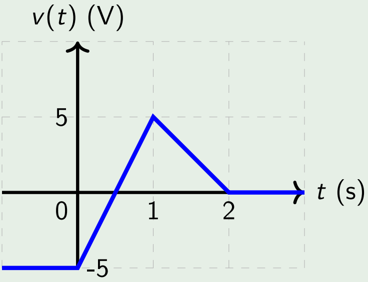
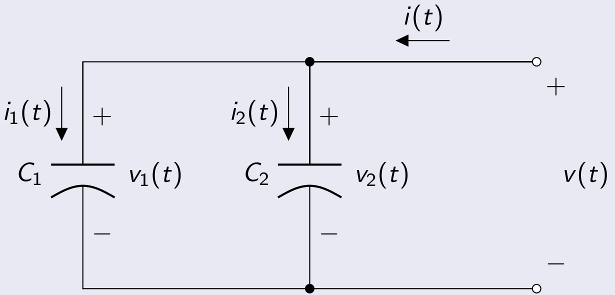






Capacitors

Capacitors in parallel

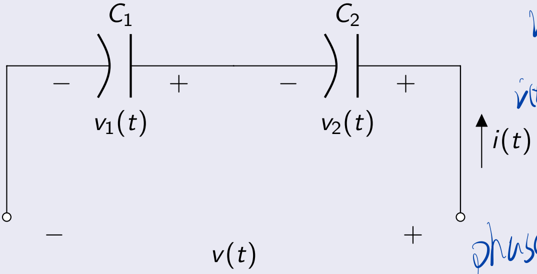


Equivalence

- $v_1(t) = v_2(t) = v(t)$
- $i(t) = i_1(t) + i_2(t) = C_1 \dfrac{dv(t)}{dt} + C_2 \dfrac{dv(t)}{dt} = (C_1 + C_2)\dfrac{dv(t)}{dt}$
- $\boxed{C_{eq} = C_1 + C_2}$

Capacitors

Capacitors in series



$v(t) = \sin \omega t$

$i(t) = C \dfrac{dv(t)}{dt}$

$= C\omega \cos \omega t$

phase difference

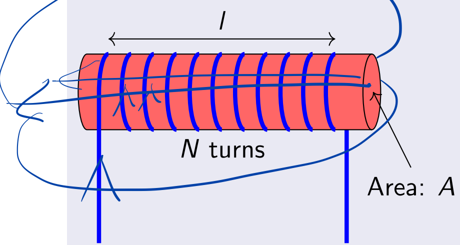
Equivalence

- $v(t) = v_1(t) + v_2(t) = v_1(t_0) + \dfrac{1}{C_1}\displaystyle\int_{t_0}^{t} i(\tau)d\tau + v_2(t_0) + \dfrac{1}{C_2}\displaystyle\int_{t_0}^{t} i(\tau)d\tau$
- $v(t) = v(t_0) + \left(\dfrac{1}{C_1} + \dfrac{1}{C_2}\right)\displaystyle\int_{t_0}^{t} i(\tau)d\tau \implies \boxed{\dfrac{1}{C_{eq}} = \dfrac{1}{C_1} + \dfrac{1}{C_2}}$

Inductors

Physical concept

Coil of conducting wire around some magnetic material



- Suppose we impose the current
- It will create a magnetic flux in the material
- The magnetic flux is directly proportional to the current

Equation

$$N \cdot \phi = L \cdot i$$

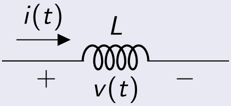
- ϕ: flux in Webers (Wb)
- L: inductance in Henrys (H)
- i: current in Amperes (A)

$$L = \frac{\mu \cdot N^2 \cdot A}{l}$$

- μ: permeability in N/A^2
- N: number of turns

Inductors

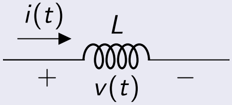
Constitutive equation



- $N \cdot \phi(t) = L \cdot i(t)$
- Faraday's law: $v(t) = N\dfrac{d\phi(t)}{dt}$
- $\boxed{v(t) = L\dfrac{di(t)}{dt}}$

Inductors

Alternative constitutive equation



- $v(t) = L\dfrac{di(t)}{dt}$
- $\displaystyle\int_{t_0}^{t} v(\tau)d\tau = L\int_{t_0}^{t} \frac{di(\tau)}{d\tau}d\tau = L\cdot[i(t) - i(t_0)]$
- $\boxed{\displaystyle i(t) = i(t_0) + \frac{1}{L}\int_{t_0}^{t} v(\tau)d\tau}$

Very often (but not always!), we will consider that: $i(0) = 0$

Inductors

Linearity?

- Are inductors **linear elements**?

For a current $i_1(t)$, then the voltage is: $v_1(t) = L\dfrac{di_1(t)}{dt}$

For another current $i_2(t)$, then the voltage is: $v_2(t) = L\dfrac{di_2(t)}{dt}$

Now, we impose a current $i(t) = a \cdot i_1(t) + b \cdot i_2(t)$. The voltage is:

$$v(t) = L\frac{di(t)}{dt} = L\frac{d(a \cdot i_1(t) + b \cdot i_2(t))}{dt} = aL\frac{di_1(t)}{dt} + bL\frac{di_2(t)}{dt}$$

$$v(t) = a \cdot v_1(t) + b \cdot v_2(t)$$

Inductors are linear

Inductors

Power absorbed by inductors

$$p(t) = v(t) \cdot i(t) = L\frac{di(t)}{dt} \cdot i(t)$$

- If positive: absorbs power (increases the stored energy)
- If negative: delivers power (gives back the stored energy)

Energy stored

The energy stored by an inductor from time t_0 to time t is:

$$w(t) - w(t_0) = \int_{t_0}^{t} p(\tau)d\tau = \int_{t_0}^{t} L\frac{di(\tau)}{d\tau} \cdot i(\tau)d\tau$$

$$= L\int_{i(t_0)}^{i(t)} i(\tau)di(\tau) = \frac{1}{2}L[i(t)^2 - i(t_0)^2]$$

Inductors

Energy

From previous equation, we can conclude that:

$$w(t) = \frac{1}{2}Li(t)^2 + cste$$

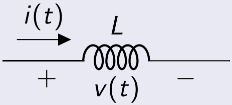
By convention, the *cste* is chosen to be null. It means that when i is equal to 0, the inductor does not have any energy stored.

$$\text{Energy stored:} \qquad w(t) = \frac{1}{2}Li(t)^2$$

Inductors

NYU SHANGHAI 上海纽约大学

Observations



$$v(t) = L \cdot \frac{di(t)}{dt}$$

- When $i(t)$ is constant, then there is no voltage ($v(t) = 0$)
- In DC mode, the inductor is equivalent to a short-circuit
- $i(t)$ **cannot** change too quickly, otherwise it leads to huge voltages (physically possible?)
- $i(t)$ is actually continuous (cannot have discontinuities)

Voltage can be not continuous

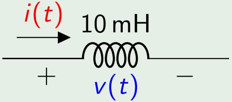
- Inductors do not dissipate energy, they store it then deliver it back

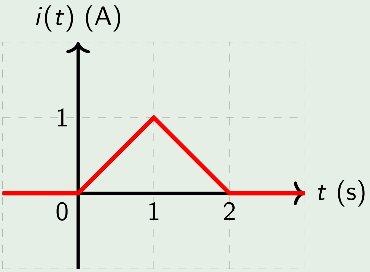
Inductors

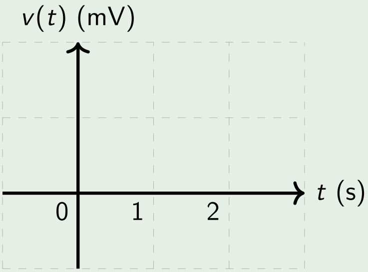
Exercise

Find the expression of $v(t)$. Also plot the corresponding timing diagram.

$V = L \frac{di}{dt}$







Inductors

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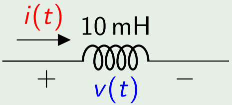
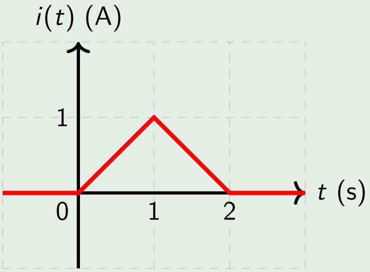
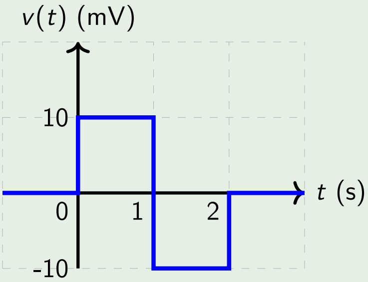
Find the expression of $v(t)$. Also plot the corresponding timing diagram.





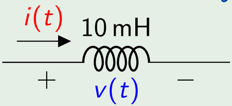


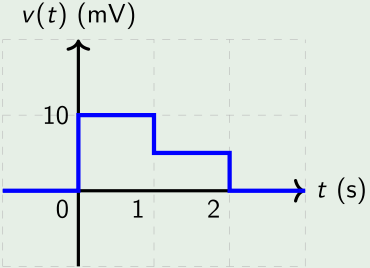
Inductors

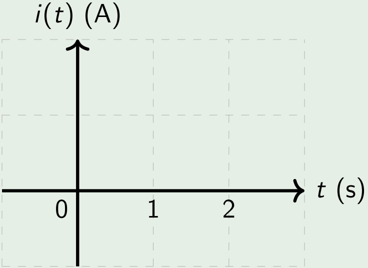
Exercise

Find the expression of $i(t)$. Also plot the corresponding timing diagram. At $t = 0$, the inductor current is $-0.5\,\mathrm{A}$.

the current should always be continuous







Inductors

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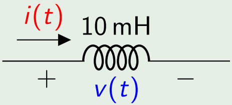
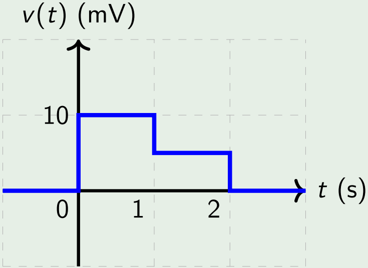
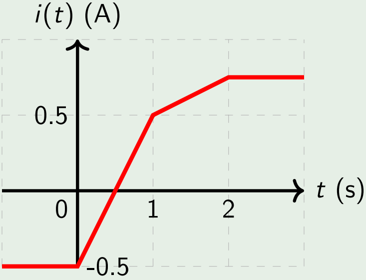
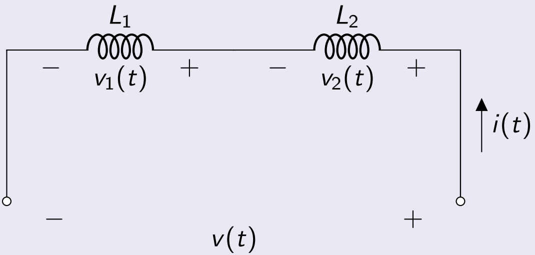






Inductors

Inductors in series

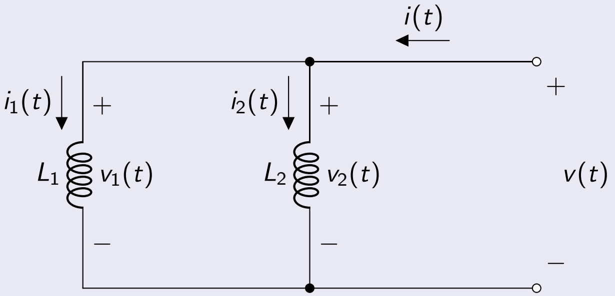


Equivalence

- $v(t) = v_1(t) + v_2(t) = L_1\dfrac{di(t)}{dt} + L_2\dfrac{di(t)}{dt} = (L_1 + L_2)\dfrac{di(t)}{dt}$
- $\boxed{L_{eq} = L_1 + L_2}$

Inductors

Inductors in parallel



Equivalence

- $i(t) = i_1(t) + i_2(t) = i_1(t_0) + \frac{1}{L_1}\int_{t_0}^{t} v(\tau)d\tau + i_2(t_0) + \frac{1}{L_2}\int_{t_0}^{t} v(\tau)d\tau$
- $i(t) = i(t_0) + \left(\frac{1}{L_1} + \frac{1}{L_2}\right)\int_{t_0}^{t} v(\tau)d\tau \implies \boxed{\frac{1}{L_{eq}} = \frac{1}{L_1} + \frac{1}{L_2}}$