Circuits
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RLC circuits

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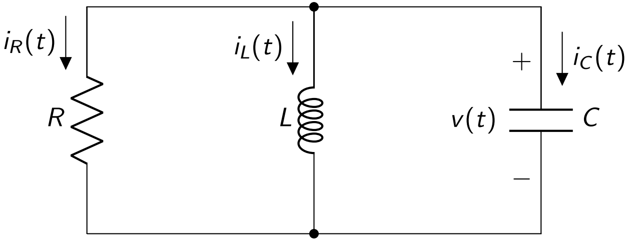
Natural response



Same voltage $v(t)$ for the 3 elements.
Initial conditions:

- $i_L(0^+) = I_0$ (inductor)
- $v(0^+) = V_0$ (capacitor) Steady state of voltage

Natural response

KCL

- $i_R(t) + i_L(t) + i_C(t) = 0$
- $\frac{v(t)}{R} + \underbrace{\boxed{\frac{1}{L}\int_{t_0}^{t} v(t')dt' + i_L(t_0)}}_{i_L(t)} + C\frac{dv(t)}{dt} = 0$

Integro-differential equation

This last equation is not very handy.
Let's differentiate it once with respect to time.

- $C\frac{d^2v(t)}{dt^2} + \frac{1}{R}\frac{dv(t)}{dt} + \frac{1}{L}v(t) = 0$ differentiate time on both sides
- $\frac{d^2v(t)}{dt^2} + \frac{1}{RC}\frac{dv(t)}{dt} + \frac{1}{LC}v(t) = 0$ divide C on both sides

Natural Response

Rewriting the equation

We set the following 2 variables:

- **resonant frequency**: $\omega_0 = \dfrac{1}{\sqrt{LC}}$
- **damping ratio**: $\xi = \dfrac{1}{2R}\sqrt{\dfrac{L}{C}}$

The differential equation is now:

- $\dfrac{d^2 v(t)}{dt^2} + 2\xi\omega_0 \dfrac{dv(t)}{dt} + {\omega_0}^2 v(t) = 0$

Natural Response

Solution

Let's try a similar solution to what we obtained for first-order circuits: $v(t) = A \cdot e^{s \cdot t}$

- $As^2 e^{st} + 2\xi\omega_0 Ase^{st} + {\omega_0}^2 Ae^{st} = 0$

Supposing that $Ae^{st} \neq 0$ (incompatible with initial conditions), we then must respect:

unknown

- $s^2 + 2\xi\omega_0 s + {\omega_0}^2 = 0$ $\quad s = \dfrac{-2\xi\omega_0 \pm \sqrt{4\xi_0^2\omega_0^2 - 4\omega_0^2}}{2}$

Characteristic equation (quadratic)

3 possibilities:

$= -\xi\omega_0 \pm \sqrt{\xi^2\omega_0^2 - \omega_0^2}$

- 2 distinct real roots
- 2 non distinct real roots
- 2 complex roots (conjugate)

Natural Response

Discrimant

$$\Delta = (\xi\omega_0)^2 - {\omega_0}^2 = {\omega_0}^2\left(\xi^2 - 1\right)$$

3 possibilities:

- $\Delta > 0$ (meaning $\xi > 1$): 2 real roots (**overdamped**)
- $\Delta = 0$(meaning $\xi = 1$): 2 non distinct real roots (**critically damped**)
- $\Delta < 0$(meaning $\xi < 1$): 2 complex conjugate roots (**underdamped**)

Role of the resitor R

Supposing that L and C have been fixed (and so is ω_0), we can notice that the damping ratio ξ depends on R
For example, modifying R may have an impact on the discriminant.

In the next slides, we are going to check the 3 possibilities separately

Natural Response (overdamped)

$\Delta > 0$ (meaning $\xi > 1$): 2 real roots

$s = -\xi\omega_0 \pm \sqrt{{\omega_0}^2\left(\xi^2 - 1\right)} = -\omega_0\left(\xi \pm \sqrt{\xi^2 - 1}\right)$

Both roots are **negative**.

- $s_1 = -\omega_0\left(\xi + \sqrt{\xi^2 - 1}\right)$
- $s_2 = -\omega_0\left(\xi - \sqrt{\xi^2 - 1}\right)$

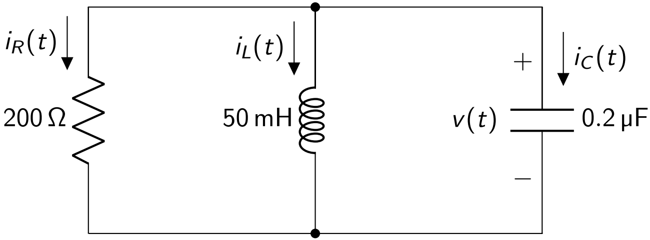
Final solution

We finally have:

$v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$

with two constants A_1 and A_2 to determine from initial conditions

Natural Response (overdamped)



Initial conditions:

- $i_L(0^+) = 30\,\text{mA}$ (inductor)
- $v(0^+) = 12\,\text{V}$ (capacitor)

Natural Response (overdamped)

Evaluate the different values

- $\omega_0 = \dfrac{1}{\sqrt{LC}} = 10\,000\,\mathrm{rad/s}$
- $\xi = \dfrac{1}{2R}\sqrt{\dfrac{L}{C}} = 1.25$ (no unit)
- $\sqrt{\xi^2 - 1} = 0.75$
- $s_1 = -\omega_0\left(\xi + \sqrt{\xi^2 - 1}\right) = -20\,000\,\mathrm{rad/s}$
- $s_2 = -\omega_0\left(\xi - \sqrt{\xi^2 - 1}\right) = -5000\,\mathrm{rad/s}$

Solution

$$v(t) = A_1 e^{-20000t} + A_2 e^{-5000t}$$

Natural Response (overdamped)

Determination of constants

We know that $v(0^+) = 12\,\mathrm{V}$

$$\implies A_1 + A_2 = 12\,\mathrm{V}$$

We also know that $i_L(0^+) = 30\,\mathrm{mA}$

$$i_L(t) = -i_R(t) - i_C(t) = -\frac{v(t)}{R} - C\frac{dv(t)}{dt}$$

$$\frac{dv(t)}{dt} = -20000A_1 e^{-20000t} - 5000A_2 e^{-5000t}$$

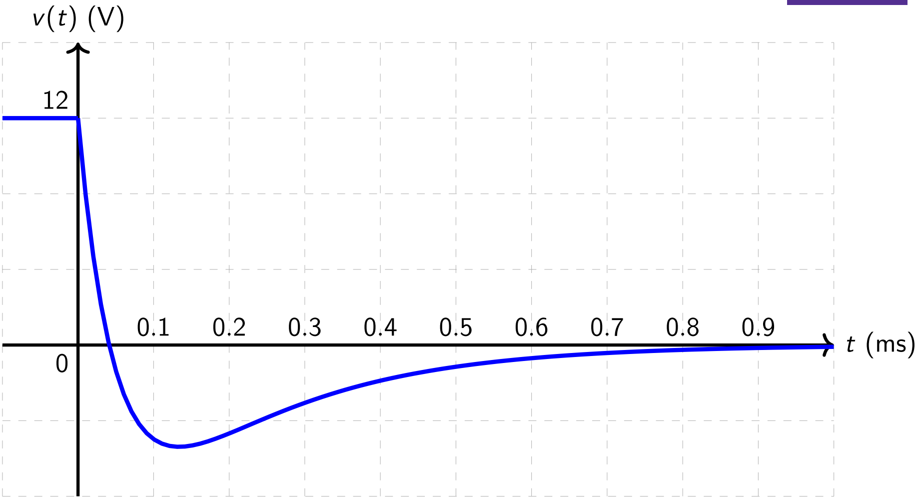
$$i_L(0^+) = -\frac{12\,\mathrm{V}}{200\,\Omega} + 0.2\,\mu\mathrm{F}\,(20000A_1 + 5000A_2) = 30\,\mathrm{mA}$$

$$\implies 20000A_1 + 5000A_2 = 450\,000\,\mathrm{V/s}$$

Final solution

$$v(t) = 26e^{-20000t} - 14e^{-5000t}$$

Natural Response (overdamped)



Natural Response (underdamped)

$\Delta < 0$ (meaning $\xi < 1$): 2 compex conjugate roots

$$s = -\xi\omega_0 \pm j\sqrt{{\omega_0}^2\left(1-\xi^2\right)} = -\omega_0\left(\xi \pm j\sqrt{1-\xi^2}\right)$$

We can set $\omega_d = \omega_0\sqrt{1-\xi^2}$, named the **natural resonant frequency**

- $s_1 = -\xi\omega_0 + j\omega_d$
- $s_2 = -\xi\omega_0 - j\omega_d$

Final solution

We finally have:

$$v(t) = e^{-\xi\omega_0 t}\left(A_1 e^{j\omega_d t} + A_2 e^{-j\omega_d t}\right)$$

with two constants A_1 and A_2 to determine from initial conditions

Natural Response (underdamped)

Complex voltage?

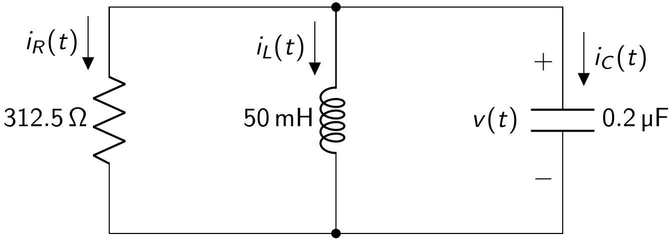
$v(t) = e^{-\xi\omega_0 t}\left(A_1 e^{j\omega_d t} + A_2 e^{-j\omega_d t}\right)$
The voltage $v(t)$ is, of course, **real**.
Actually the 2 constants A_1 and A_2 are complex in order to obtain a real voltage.

Another way to write the voltage equation is:

$$v(t) = e^{-\xi\omega_0 t}\left(B_1 \cos(\omega_d t) + B_2 \sin(\omega_d t)\right)$$

with B_1 and B_2 are 2 real constants

Natural Response (underdamped)



Initial conditions:

- $i_L(0^+) = 30\,\text{mA}$ (inductor)
- $v(0^+) = 12\,\text{V}$ (capacitor)

Natural Response (underdamped)

Evaluate the different values

- $\omega_0 = \dfrac{1}{\sqrt{LC}} = 10\,000\,\mathrm{rad/s}$
- $\xi = \dfrac{1}{2R}\sqrt{\dfrac{L}{C}} = 0.8$ (no unit)
- $\sqrt{1-\xi^2} = 0.6$
- $\xi\omega_0 = 8000\,\mathrm{rad/s}$
- $\omega_d = \omega_0\sqrt{1-\xi^2} = 6000\,\mathrm{rad/s}$

Solution

$$v(t) = e^{-8000t}\left(B_1\cos(6000t) + B_2\sin(6000t)\right)$$

Natural Response (underdamped)

Determination of constants

We know that $v(0^+) = 12\,\mathrm{V}$

$$B_1 = 12\,\mathrm{V}$$

We also know that $i_L(0^+) = 30\,\mathrm{mA}$

$$i_L(t) = -i_R(t) - i_C(t) = -\frac{v(t)}{R} - C\frac{dv(t)}{dt}$$

$$\frac{dv(t)}{dt} = -2000\left(4v(t) - 3e^{-8000t}\left(-B_1\sin(6000t) + B_2\cos(6000t)\right)\right)$$

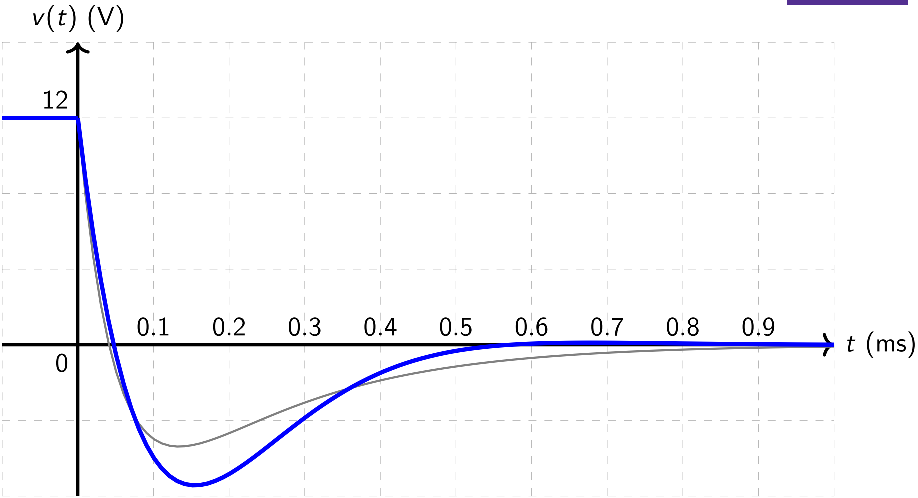
$$i_L(0^+) = -\frac{12\,\mathrm{V}}{312.5\,\Omega} + 0.2\,\mu\mathrm{F}\left(2000(48\,\mathrm{V} - 3B_2)\right) = 30\,\mathrm{mA}$$

$$\implies B_2 = -41\,\mathrm{V}$$

Final solution

$$v(t) = e^{-8000t}\left(12\cos(6000t) - 41\sin(6000t)\right)$$

Natural Response (underdamped)



Natural Response (critically damped)

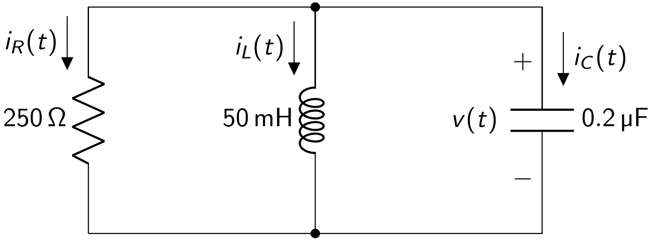
$\Delta = 0$ (meaning $\xi = 1$): 2 real non distinct roots

$s = -\omega_0$
In that particular case only, the solution form we assumed at the beginning is not complete.

Actually the solution will be in the form:

$$v(t) = (A_1 + A_2 \boldsymbol{t})e^{-\omega_0 t}$$

Natural Response (critically damped)



Initial conditions:

- $i_L(0^+) = 30\,\text{mA}$ (inductor)
- $v(0^+) = 12\,\text{V}$ (capacitor)

Natural Response (critically damped)

Evaluate the different values

- $\omega_0 = \dfrac{1}{\sqrt{LC}} = 10\,000\,\mathrm{rad/s}$
- $\xi = \dfrac{1}{2R}\sqrt{\dfrac{L}{C}} = 1$ (no unit)

Solution

$$v(t) = (A_1 + A_2 t)e^{-10000t}$$

Natural Response (critically damped)

Determination of constants

We know that $v(0^+) = 12\,\mathrm{V}$

$$A_1 = 12\,\mathrm{V}$$

We also know that $i_L(0^+) = 30\,\mathrm{mA}$

$$i_L(t) = -i_R(t) - i_C(t) = -\frac{v(t)}{R} - C\frac{dv(t)}{dt}$$

$$\frac{dv(t)}{dt} = -10000v(t) + A_2 e^{-10000t}$$

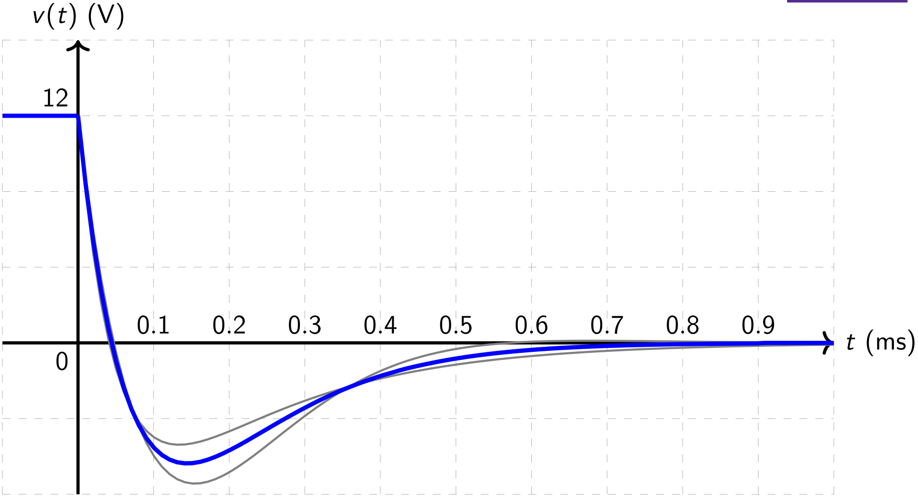
$$i_L(0^+) = -\frac{12\,\mathrm{V}}{250\,\Omega} + 0.2\,\mu\mathrm{F}\left(10000 \cdot 12\,\mathrm{V} - A_2\right) = 30\,\mathrm{mA}$$

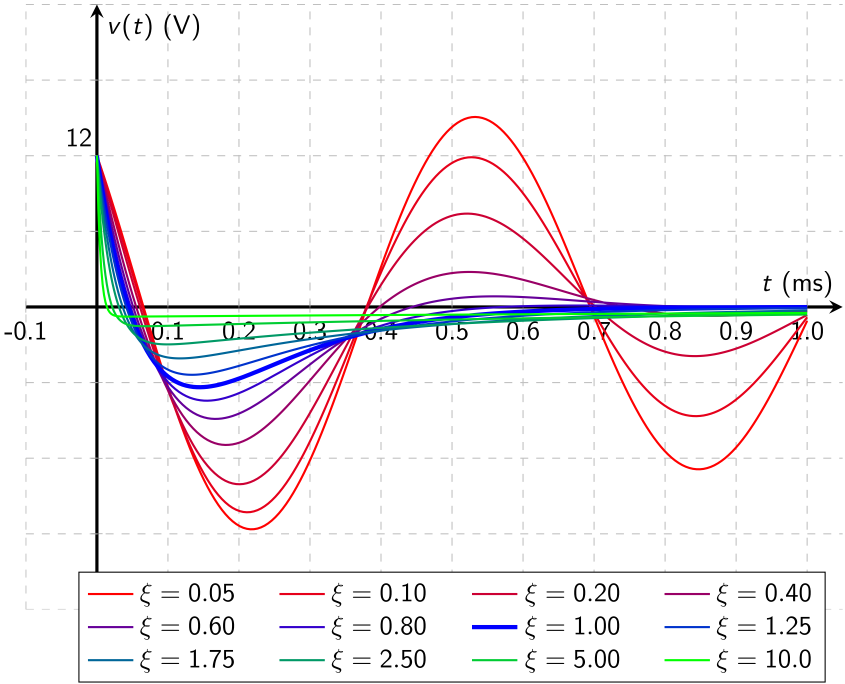
$$\implies A_2 = -270\,000\,\mathrm{V/s}$$

Final solution

$v(t) = 12e^{-10000t} - 270000te^{-10000t}$

Natural Response (critically damped)





$v(t)$ (V)
12
t (ms)
-0.1
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
$\xi = 0.05$
$\xi = 0.10$
$\xi = 0.20$
$\xi = 0.40$
$\xi = 0.60$
$\xi = 0.80$
$\xi = 1.00$
$\xi = 1.25$
$\xi = 1.75$
$\xi = 2.50$
$\xi = 5.00$
$\xi = 10.0$