P2.1.6 $A \in \mathbb{R}^{n\times n}$, symmetric, nonsingular, $u, v \in \mathbb{R}^n$, $\alpha, \beta \in \mathbb{R}$

$$\tilde{A} = A + \alpha(uu^T + vv^T) + \beta(uv^T + vu^T), \quad \tilde{A} \text{ nonsingular}$$

$$= A + \alpha uu^T + \beta uv^T + \alpha vv^T + \beta vu^T$$

$$= A + u(\alpha u^T + \beta v^T) + v(\alpha v^T + \beta u^T)$$

$$= A + \begin{bmatrix} | & | \\ u & v \\ | & | \end{bmatrix}_{n\times 2} \cdot \begin{bmatrix} -(\alpha u^T + \beta v^T)- \\ -(\alpha v^T + \beta u^T)- \end{bmatrix}_{2\times n}$$

Let's denote $U = \begin{bmatrix} | & | \\ u & v \\ | & | \end{bmatrix} \in \mathbb{R}^{n\times 2}$, $V^T = \begin{bmatrix} -(\alpha u^T + \beta v^T)- \\ -(\alpha v^T + \beta u^T)- \end{bmatrix} \in \mathbb{R}^{2\times n}$

Then $\tilde{A} = A + UV^T$

$\operatorname{rank}(UV^T) \leq \min(\operatorname{rank}(U), \operatorname{rank}(V)) \leq 2$

So this is a rank-2 perturbation at most.

According to (2.1.4), $A \in \mathbb{R}^{n\times n}$, $U, V \in \mathbb{R}^{n\times k}$, k is the rank ($k=2$).

$$\tilde{A}^{-1} = (A + UV^T)^{-1} = A^{-1} - A^{-1}U(I + V^T A^{-1} U)^{-1} V^T A^{-1}$$

P2.2.6. For any vector norm on $\mathbb{R}^n$ that $\big| \|x\| - \|y\| \big| \leq \|x - y\|$.

Proof. We know that for any vector $x, y \in \mathbb{R}^n$, satisfy the following triangle inequality.

$$\|\underset{\substack{\uparrow \\ x-y}}{x} + \underset{\substack{\uparrow \\ y}}{y}\| \leq \|\underset{\substack{\uparrow \\ x-y}}{x}\| + \|\underset{\substack{\uparrow \\ y}}{y}\|$$

then $\|x - y + y\| \leq \|x - y\| + \|y\|$

$$\|x\| - \|y\| \leq \|x - y\| \quad ①$$

$$\|x+y\| \le \|x\| + \|y\|$$

$$\begin{array}{cccc} \uparrow & \uparrow & \uparrow & \uparrow \\ y-x & x & y-x & x \end{array}$$

$$\|y-x+x\| \le \|y-x\| + \|x\|$$

$$\|y\| \le \|(-1)(x-y)\| + \|x\| \leftarrow (\|\lambda x\| = |\lambda| \|x\|).$$

$$\|y\| \le \|x-y\| + \|x\|$$

$$-\|x-y\| \le \|x\| - \|y\| \quad ②$$

$\because$ ①② $\Rightarrow -\|x-y\| \le \|x\| - \|y\| \le \|x-y\|$

$$\big|\|x\| - \|y\|\big| \le \|x-y\| \quad \text{holds.}$$

P2.2.7 $\|\cdot\|$: vector norm on $\mathbb{R}^m$, $A \in \mathbb{R}^{m\times n}$, rank$(A) = n$.

Show $\|x\|_A = \|Ax\|$ is a vector norm on $\mathbb{R}^n$.

Proof.

A vector norm on $\mathbb{R}^n$ has the following properties:

① $\|x\|_A \ge 0$, $x \in \mathbb{R}^n$, $\|x\|_A = 0$ iff $x = 0$

$\|\cdot\|$ is a vector norm, so $\|x\|_A = \|Ax\| \ge 0$

We know A is a full column rank matrix, which means it doesn't have free variables, the null space $N(A) = \{$zero vector$\}$.

So if $x = 0$, $\|x\|_A = \|Ax\| = \|0\| = 0$. $\therefore$ ① holds.

② $\|x+y\|_A \le \|x\|_A + \|y\|_A$, $x, y \in \mathbb{R}^n$.

$$\|x+y\|_A = \|A(x+y)\| = \|Ax + Ay\|$$

$\because$ $\|\cdot\|$ is a vector norm, it must satisfy triangular inequality ~~too~~.

$\therefore$ $\|x+y\|_A = \|Ax+Ay\| \le \|Ax\| + \|Ay\| = \|x\|_A + \|y\|_A$.

Therefore, ② holds

③ $\|\alpha x\|_A = |\alpha|\,\|x\|_A \qquad \alpha\in\mathbb{R},\ x\in\mathbb{R}^n.$

$\|\alpha x\|_A = \|A\cdot\alpha x\| = \|\alpha\cdot Ax\| = |\alpha|\,\|Ax\| = |\alpha|\,\|x\|_A$

Therefore, ③ holds.

①②③ hold, $\|x\|_A$ is a vector norm on $\mathbb{R}^n$.

P2.3.6 Verify $\frac{1}{\sqrt{n}}\|A\|_\infty \le \|A\|_2 \le \sqrt{m}\|A\|_\infty$ ①

$\frac{1}{\sqrt{m}}\|A\|_1 \le \|A\|_2 \le \sqrt{n}\|A\|_1$ ② $A\in\mathbb{R}^{m\times n}$

Proof.

① According to (2.5.6), We have

for any $x\in\mathbb{R}^n$, $\|x\|_\infty \le \|x\|_2 \le \sqrt{n}\|x\|_\infty$

$$\|A\|_2 = \sup_{x\neq 0}\frac{\|Ax\|_2}{\|x\|_2} \le \sup_{x\neq 0}\frac{\sqrt{m}\,\|Ax\|_\infty}{\|x\|_2} \le \sqrt{m}\sup_{x\neq 0}\frac{\|Ax\|_\infty}{\|x\|_\infty} = \sqrt{m}\|A\|_\infty$$

(Ax: m×1 vector)

$\therefore \|A\|_2 \le \sqrt{m}\|A\|_\infty$ holds.

$$\|A\|_\infty = \sup_{x\neq 0}\frac{\|Ax\|_\infty}{\|x\|_\infty} \le \sup_{x\neq 0}\frac{\|Ax\|_\infty}{\frac{1}{\sqrt{n}}\|x\|_2} \le \sqrt{n}\sup_{x\neq 0}\frac{\|Ax\|_2}{\|x\|_2} = \sqrt{n}\|A\|_2$$

(x: n×1 vector)

$\therefore \|A\|_\infty \le \sqrt{n}\|A\|_2$, $\frac{1}{\sqrt{n}}\|A\|_\infty \le \|A\|_2$ holds.

Therefore, ① $\frac{1}{\sqrt{n}}\|A\|_\infty \le \|A\|_2 \le \sqrt{m}\|A\|_\infty$ holds.

② According to (2.5.5), We have

for any $x\in\mathbb{R}^n$, $\|x\|_2 \le \|x\|_1 \le \sqrt{n}\|x\|_2$

$$\|A\|_1 = \sup_{x\neq 0}\frac{\|Ax\|_1}{\|x\|_1} \le \sup_{x\neq 0}\frac{\sqrt{m}\,\|Ax\|_2}{\|x\|_1} \le \sqrt{m}\sup_{x\neq 0}\frac{\|Ax\|_2}{\|x\|_2} = \sqrt{m}\|A\|_2$$

$\therefore \|A\|_1 \le \sqrt{m}\|A\|_2$, $\frac{1}{\sqrt{m}}\|A\|_1 \le \|A\|_2$ holds

$$\|A\|_2 = \sup_{x\neq 0}\frac{\|Ax\|_2}{\|x\|_2} \le \sup_{x\neq 0}\frac{\|Ax\|_2}{\frac{1}{\sqrt{n}}\|x\|_1} \le \sqrt{n}\sup_{x\neq 0}\frac{\|Ax\|_1}{\|x\|_1} = \sqrt{n}\|A\|_1$$

$\therefore \|A\|_2 \le \sqrt{n}\,\|A\|_1$ holds

Therefore, ② $\frac{1}{\sqrt{m}}\|A\|_1 \le \|A\|_2 \le \sqrt{n}\,\|A\|_1$ holds.

P2.1.8 $Q \in \mathbb{R}^{n\times n}$ orthogonal, $v \in \mathbb{R}^n$, setting up $n\times n$ A

$a_{ij} = v^T (Q^i)^T (Q^j) v$

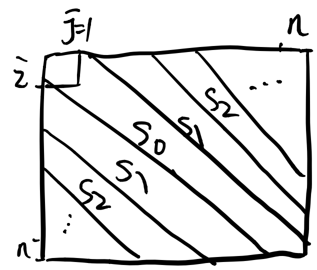
$\because Q^T = Q^{-1}$

$\therefore a_{ij} = v^T (Q^T)^i (Q^j) v = v^T (Q^{-i})(Q^j) v = v^T Q^{j-i} v$

$a_{ji} = v^T Q^{i-j} v = v^T (Q^{j-i})^{-1} v = v^T (Q^{j-i})^T v = \left(v^T (Q^{j-i}) v\right)^T = a_{ij}^T$

$\because a_{ij}, a_{ji}$ are scalars. $\therefore a_{ij} = a_{ji}$. A is a $n\times n$ square matrix.

We only need to compute a_{ij}, where $j-i = 1 \dots n$. and fill the rest.



Alg

```
function [A] = qpower(Q, v)
n = size(Q,1);
A = zeros(n);
i = 1;
for j = 1:n
    S = v' * Q^(j-i) * v;
    a = zeros(n-j+i, 1);
    a(:,1) = S;        % each diag has the same value
    if (j==1)
        A = A + diag(a, j-i);   % 0-diag only adds once
    end
    if (j>1)
```

```
        A = A + diag(a, j-i) + diag(a, i-j); % symmetric
      end
   end
end
```