P1.1.1

The first column of M

$$M(:,1) = (A - x_1 I)(A - x_2 I)\cdots[(A - x_r I)e_1], \quad e_1 = \left.\begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right\} n.$$

By computing it in reverse order, we can turn the matrix-matrix multiplication into matrix-vector multiplication.

$\boxed{\text{Alg:}}$

```
e1 = zeros(n,1); e1(1) = 1;
m = (A - x(r)*I) * e1;
for i = 1:(r-1)
    m = (A - x(r-i)*I) * m;
end
```

P1.1.3.

$$\underset{n\times n}{C} = (xy^T)^k = xy^T\underbrace{xy^Txy^Txy^T\cdots}_{k.}xy^T$$

$$= x(y^Tx)\underbrace{(y^Tx)(y^Tx)\cdots(y^Tx)}_{(k-1)}y^T$$

$$= (y^Tx)^{k-1}\cdot xy^T$$

$\boxed{\text{Algorithm:}}$ This involves $O(n^2)$ arithmetic operations.

```
s = 0
for i = 1:n
    s = s + y(i)x(i)
end
```

$s = S^{\wedge}(k-1)$

$C = \text{zeros}(n, n).$

for $i = 1:n$

$\quad C(i,:) = C(i,:) + x(i) \cdot y^T$

end

P1.1.4

According to Table 1.1.2, We have

$$C = C + AB, \ A \in \mathbb{R}^{m\times r}, \ B \in \mathbb{R}^{r\times n}, \ C \in \mathbb{R}^{m\times n}, \quad \text{Flops: } 2mnr.$$

$D = ABC \quad , A \in \mathbb{R}^{m\times n}, \ B \in \mathbb{R}^{n\times p}, \ C \in \mathbb{R}^{p\times q}$

$D = \underbrace{(AB)}_{m\times p}\underbrace{C}_{p\times q}, \quad \text{Flops}((AB)C) = 2mnp + 2mpq$

$D = \underbrace{A}_{m\times n}\underbrace{(BC)}_{n\times q} \quad \text{Flops}(A(BC)) = 2npq + 2mnq$

When $\text{Flops}((AB)C) < \text{Flops}(A(BC))$

$2mnp + 2mpq < 2npq + 2mnq$

$mp(n+q) < nq(p+m)$,

the former procedure is more flop-efficient than the latter.

P1.2.3 column saxpy $n\times n$ $C = C + AB$, A: upper. B: lower

$\boxed{\text{Alg.}}$

for $j = 1:n$

$\quad$ for $k = j:n$

$\quad\quad$ for $i = 1:k$

```
            C(i,j) = C(i,j) + A(i,k)·B(k,j)
        end
    end
end
```

P2.1.1

$A \in \mathbb{R}^{m\times n}$ has rank p, then the column space of A has a basis of p vectors, $v_1, v_2, \cdots, v_p$, they are linearly independent.

Let $X \in \mathbb{R}^{m\times p}$ be $[v_1, v_2, \ldots v_p]$, $\text{rank}(X) = p$.

And each column of A is a linear combination of columns of X,

we have $A = XY^T$, $Y \in \mathbb{R}^{n\times p}$.

$\text{rank}(Y) \le p$ and $\text{rank}(Y) \le n$.

Each row of A is a linear combination of rows of Y^T,

so we have $\text{rank}(A) \le \text{rank}(Y^T) = \text{rank}(Y)$

$$p \le \text{rank}(Y) \le p,$$

therefore, $\text{rank}(Y) = p$.

P2.1.4

$$(A + uv^T)x = b$$

$$x = (A + uv^T)^{-1} b$$

$$Ax = A(A + uv^T)^{-1} b.$$

$$= A(A^{-1} - \frac{1}{\beta} A^{-1} uv^T A^{-1}) b$$

$$= b - \frac{1}{\beta} uv^T A^{-1} b, \quad (\beta = 1 + v^T A^{-1} u)$$

And we have $Ax = b + \alpha u$

$$\therefore \quad b - \frac{1}{\beta} uv^T A^{-1} b = b + \alpha u$$

$$\alpha = -\frac{1}{\beta}v^TA^{-1}b, \quad \text{where } \beta = 1 + v^TA^{-1}u.$$

P2.3.2

Suppose $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{s\times t}$ and B is a submatrix of A.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix} \quad B = A(\alpha,\beta) = \begin{bmatrix} a_{\alpha_1\beta_1} & a_{\alpha_1\beta_2} & \cdots & a_{\alpha_1\beta_t} \\ \vdots & \vdots & \ddots & \vdots \\ a_{\alpha_s\beta_1} & a_{\alpha_s\beta_2} & \cdots & a_{\alpha_s\beta_t} \end{bmatrix}$$

Where $\alpha = [\alpha_1, \alpha_2 \ldots \alpha_s]$

$\beta = [\beta_1, \beta_2 \cdots \beta_t]$ are integer vectors with distinct components that have $1 \le \alpha_i \le m$, $1 \le \beta_i \le n$.

① Keep columns:

$$Q = I_n(:,\beta), \quad Q \in \mathbb{C}^{n\times t} \qquad (1.2.10).$$

② Keep rows:

$$P = I_m(\alpha,:), \quad P \in \mathbb{C}^{s\times m}$$

Then $\underbrace{P}_{s\times m}\underbrace{A}_{m\times n}\underbrace{Q}_{n\times t} = \underbrace{A(\alpha,\beta)}_{s\times t} = B$

Then we have, for any p that satisfies $1 \le p \le \infty$,

$$\|B\|_p = \|P\times A\times Q\|_p \le \|P\|_p \times \|A\|_p \times \|Q\|_p. \qquad (2.3.3)$$

P keeps part of rows of x (p)

$$\|P\|_p = \sup_{x\ne 0}\frac{\|Px\|_p}{\|x\|_p} \le 1$$

The same applies to Q as well.

$$\|Q\|_p = \sup_{x=0} \frac{\|Qx\|_p}{\|x\|_p} \le 1$$

$$\therefore \quad \|B\|_p = \|P \times A \times Q\|_p \le \|P\|_p \times \|A\|_p \times \|Q\|_p \le \|A\|_p .$$