P5.4.1

$$Q^T x = \begin{bmatrix} \alpha \\ u \end{bmatrix} \qquad x = Q\begin{bmatrix} \alpha \\ u \end{bmatrix}$$

$$Q^T y = \begin{bmatrix} \beta \\ v \end{bmatrix} \qquad y = Q\begin{bmatrix} \beta \\ v \end{bmatrix}$$

$$\begin{aligned} x^T y - \alpha\beta &= \left(Q\begin{bmatrix} \alpha \\ u \end{bmatrix}\right)^T \left(Q\begin{bmatrix} \beta \\ v \end{bmatrix}\right) - \alpha\beta \\ &= [\alpha \ u^T]\, Q^T Q \begin{bmatrix} \beta \\ v \end{bmatrix} - \alpha\beta \\ &= [\alpha \ u^T] \begin{bmatrix} \beta \\ v \end{bmatrix} - \alpha\beta = \alpha\beta + u^T v - \alpha\beta \\ &= u^T v \end{aligned}$$

P5.5.1

$$AXA = \overset{m\times n}{\begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix}} \overset{n\times m}{\begin{bmatrix} T^{-1} & 0 \\ 0 & 0 \end{bmatrix}} \overset{m\times n}{\begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix}}$$

$$= \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix} = A$$

$$\overset{m\times n \ \ n\times m}{AX} = \begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T^{-1} & 0 \\ 0 & 0 \end{bmatrix} = \overset{m\times m}{\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}}$$

$$(AX)^T = \left(\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}\right)^T \quad \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} = AX.$$

For general A, we want to solve $\|Ax_B - b\|_2$ system.

$$x_B = Xb.$$

$$Ax_B = AXb$$

$$A x_B - b = AXb - b$$

$$A x_B - b = (AX - I_m) b.$$

$\because$ A has rank r. so Ax_B has at most rank r.

$$A x_B - b$$

$$= \begin{array}{c} \\ r \\ \\ m \end{array} \begin{pmatrix} \times \\ \times \\ \times \\ \times \\ \hline 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} - \begin{pmatrix} b(1:r) \\ \hline b(r+1:m) \end{pmatrix}.$$

To minimize $\|Ax_B - b\|$, $Ax_B(1:r) = b(1:r)$.

$$\therefore \quad A x_B - b = (AX - I_m) b$$

$$\therefore \quad \begin{array}{c} r \\ \\ \end{array} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ -b(r+1:m) \end{pmatrix} = (AX - I_m) b.$$

$$\therefore \quad AX = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}^{m \times m}$$

We have proved that when X is a pseudoinverse of A.
$AX = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}^{m \times m}$. Therefore, we prove that for general A, $x_B = Xb$ can solve the linear system when X is a pseudoinverse of A.