Project on Surface Parameterization

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ABSTRACT

In this project, I implement three parametrization algorithms: Euclidean orbifold Tutte embedding [Aigerman and Lipman, 2015], hyperbolic orbifold Tutte embedding [Aigerman and Lipman, 2016], and boundary first flattening [Sawhney and Crane, 2017]. The Euclidean orbifold Tutte embedding uses orbifold structure to generalize the algorithm in [Gortler et al., 2006] to sphere-type meshes. Hyperbolic orbifold Tutte embedding generalize the Euclidean orbifold Tutte embedding in the sense of geometry. But this generalization in non-linear. Besides, the Euclidean orbifold Tutte is also claimed to be conformal, so this algorithm provide a new linear method to compute conformal maps. Boundary first flattening is another linear conformal map solver. But this algorithm is based on differential geometry.

1 Introduction

Parameterization is very important to many graphics applications. In this project, I explored two kinds of parameterization algorithms. The former is what in my **proposal**. The latter is what I did extra for the **bonus points**.

The first one is orbifold Tutte embedding [Aigerman and Lipman, 2015] [Aigerman and Lipman, 2016] . The behaviors of orbifolds is different in different background geometry, and I explored two kinds of geometry: Euclidean geometry and hyperbolic geometry. We will see later that algorithms for these two different settings are very different. One is linear and the other is non-linear. [Aigerman et al., 2017] is about spherical orbfolds. But in the level of implementation, there is no difference with hyperbolic orbifolds, except the distance function and isometries.

The second one is boundary first flattening [Sawhney and Crane, 2017]. This algorithm is in the mindset of differential geometry and conformal geometry. And this algorithm is linear.

This report is organized as follows: First I talk about the tools I used to implement my project. Then I separate the rest of report into two parts to introduce two kinds of algorithm separately.

The whole project and documents are uploaded onto GitHub: `https://github.com/xuan-li/GraphicsProject`

2 Project Overview

2.1 Toolbox

I implemented my whole project on Windows, using Visual Studio Community 2017. And I used some open-source libraries: Eigen [Guennebaud et al., 2010], Libigl [Jacobson et al., 2016], LBFGS++ [Qiu, 2016], and OpenMesh [Botsch et al., 2002]. The needed libraries are compiled into static libraries file (*.lib), and included in the projects.

2.1.1 OpenMesh

OpenMesh [Botsch et al., 2002] is the core data sturcture of my project. This library implements so-called halfedge data structure for polygonal mesh.

Each mesh $M = (V,E,F,H)$ can be represented by four components: V stands for vertex set, E stands for edge set, H stands for halfedge setsand F stands for face set. Each edge is corresponded with two opposite halfedges. The linking relations are shown as follows Fig.1.

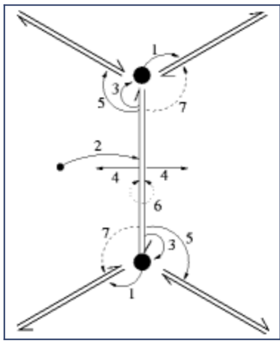
2.1.2 Eigen

Eigen [Guennebaud et al., 2010] is a standard library to handle matrices and their operations in C++. It's a head-only library, so it's very easy to deploy.

I will use this library to handle linear part in my project.

2.1.3 LBFGS++

LBFGS is a first-order optimization algorithm with line search. LBFGS++ [Qiu, 2016] is an implementation for this algorithm in C++. This is a project I found on GitHub, and it is very new.



- Each **vertex** references one outgoing halfedge, i.e. a halfedge that starts at this vertex (1).
- Each **face** references one of the halfedges bounding it (2).
- Each **halfedge** provides a handle to
  - the vertex it points to (3),
  - the face it belongs to (4)
  - the next halfedge inside the face (ordered counter-clockwise) (5),
  - the opposite halfedge (6),
  - (optionally: the previous halfedge in the face (7)).

Figure 1. Halfedge data structure.

I use this algorithm to optimize the energy for hyperbolic orbifold.

2.1.4 Libigl

Libigl [Jacobson et al., 2016] is a simple C++ geometry processing library.

I use this library to implement my GUI. The GUI part of this library is based on Nanogui[1] and OpenGL3.

On the other hand, this library provides lots of functions used in geomettry processing. I also use its quadratic programming algorithm. It is used in the optimization problem for closed loop in boundary first flatten algorithm.

3 Theoretical Backgroud

3.1 Euclidean Orbifold Tutte Embedding

The first part of my project is Euclidean orbifold Tutte embedding [Aigerman and Lipman, 2015]. This algorithm is a generalization of so-called Tutte graph embedding based on the following theorem:

Theorem 3.1 *Let $G = \langle V, E, F\rangle$ be a 3-connected planar graph with boundary vertices $B \subset V$ defining a unique unbounded exterior face f_e . Suppose ∂f_e is embedded in the plane as a (not necessarily strictly) convex planar polygon, and each interior vertex is positioned in the plane as a strictly convex combination of its neighbors, then the straight-line drawing of G with these vertex positions is an embedding. In addition, this embedding has strictly convex interior faces.*

The embedding problem is the following full-rank linear system:

$$
\begin{aligned}
\sum_{v_j \in N(v_i)} w_{ij} x_j &= x_i, i = 1, \ldots, |V - B| \\
\sum_{v_j \in N(v_i)} w_{ij} y_j &= y_i, i = 1, \ldots, |V - B| \\
x_i &= b_i^x, i = |V - B| + 1, \ldots, |V| \\
y_i &= b_i^y, i = |V - B| + 1, \ldots, |V|
\end{aligned}
\tag{1}
$$

where $\sum_j w_{ij} = 1, \forall v_i$, and $w > 0$. The result is guaranteed to be injective.

But this algorithm cannot be generalized onto topological sphere directly. They [Aigerman and Lipman, 2015] use a geometric structure called orbifold, making this method computable on sphere.

An orbifold can be treated as a tiled plane. The tile transformation is isometries. In this paper Euclidean orbifolds are used. We cut the mesh into disk to get a maximal submesh. The maximal submesh can be mapped to a tile of Euclidean plane. So besides Eq.1, we should make the embedding satisfy the following two conditions:

1. The embedding can tail the whole plane seamlessly.
2. The image of every vertex satisfy weighted convex combination on tiled plane.

For example, in Fig.2, not only interior but also boundary vertices of the maximal submesh satisfy convex combination condition on the tiled plane.

[1] https://github.com/wjakob/nanogui

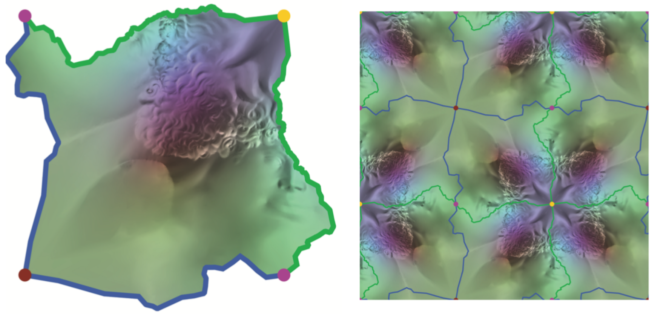


Figure 2. Tiled plane.

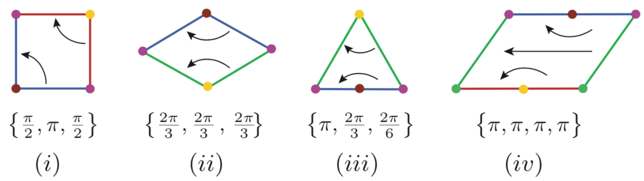


Figure 3. Four kinds of sphere-type Euclidean orbifolds

The system needs to be solved is:

$$\begin{aligned} \sum_{v_j \in N(v_i)} w_{ij}(z_j - z_i) = 0, \quad & v_i \in \bar{V} - \bar{B} \\ \sum_{v_j \in N(v_i)} w_{ij}(z_j - z_i) + \sum_{v_j \in N(v_{i'})} w_{i'j} R_{i'i}(z_j - z_i') = 0, \quad & v_i \in \bar{B} - \bar{C} \\ z_i = z_i^0, \quad & v_i \in \bar{C} \end{aligned} \tag{2}$$

Here we cut the M into a submesh $\bar{M}$, which is a disk. $\bar{C}$ is the cones of the orbifold. $\bar{B}$ is the boundary of the submesh. And $R_{i'i}$ is the rotation part of the isometry from i' to i.

Note that we have only four Euclidean sphere-type orbifold, shown in Fig. 3. And one intresting property is that the map is actually confomal map.

3.2 Hyperbolic Orbifold Tutte Embedding

The second part of my projects is hyperbolic orbifold Tutte embedding [Aigerman and Lipman, 2016]. Hyperbolic orbifold Tutte embedding is the generalization of Euclidean orbifold Tutte embedding. We treat a hyperbolic orbifold as a tiled Poincare disk by a basic tile. But hyperbolic isometries are non linear. Besides we use a non linear generalization of Tutte embedding:

$$\begin{aligned} & \min_{\Phi} \quad E(\Phi) = \frac{1}{2} \sum_{(i,j) \in E} w_{ij} d(z_i, z_j)^2 \\ & s.t \quad z_i = z_i^0, \quad v_i \in B \end{aligned} \tag{3}$$

where d is the distance function in corresponding geometry. This system is equivalent to Eq.1 in Euclidean geometry.

We use this generalization to get hyperbolic orbifolds:

$$\begin{aligned} & \min_{\Phi} E(\Phi), \\ & s.t \quad z_i = m_{i'i}(z_{i'}), \quad v_i \in \bar{B} - \bar{C} \\ & \qquad z_i = z_i^0, \qquad\quad v_i \in \bar{C} \end{aligned} \tag{4}$$

Here d is hyperbolic distance function and $m_{i'i}$ is hyperbolic isometry from i' to i.

The distance has explicit expression whose first order derivative is easily computed. So we can use first-order optimization algorithm to optimaize this problem.

A result from paper is showned in Fig.4

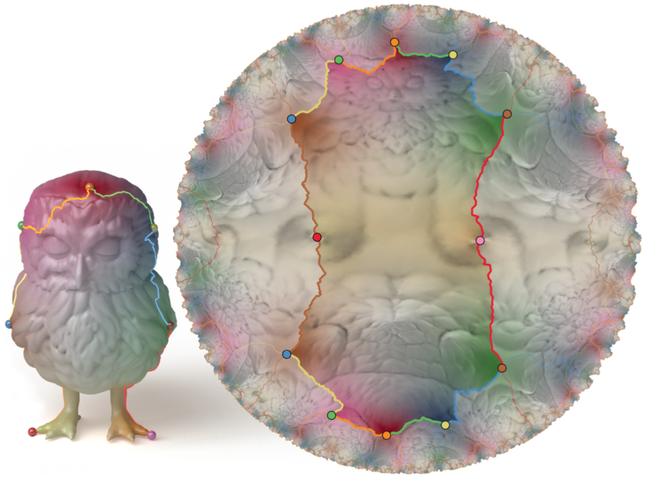


Figure 4. Hyperbolic orbifold Tutte embedding

3.3 Boundary First Flattening

The third part of my project is boundary first flattening [Sawhney and Crane, 2017]. This is a linear method to compute a

4 Implementation Details

4.1 Overview

4.2 Dependencies

5 Experiments

References

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