# 7

# Random measures

## 7.1 Fundamentals

### 7.1.1 Introduction

This chapter presents a brief discussion of random measures on $\mathbb{R}^d$. Random measures are an elegant tool of stochastic geometry often leading to a deeper understanding. In particular curvature measures and marked measures are very useful.

Random measures are here considered as special random set functions taking finite values on bounded sets. Attention is confined mainly to the *nonnegative* case; however, there is a general theory for signed random measures, which closely follows the theory for the nonnegative case. Signed random measures are of importance for the study of curvatures (see Section 7.3) as well as for providing a unifying theoretical framework.

This chapter only provides an introduction to the theory of random measures, being a short sketch of the theory in so far as it is relevant to stochastic geometry. A more detailed account can be found in Mecke (1972), Kallenberg (1976b, 1986, 2002) and Schneider and Weil (2008).

Random measures are mostly auxiliary entities in stochastic geometry, giving alternative, often elegant descriptions of random structures. Random closed sets, fibre processes, and surface processes all have various random measures associated with them. This allows the use of notation and methods from measure and integration theory, for example Palm distributions or moment measures.

The following example illustrates the concept of a random measure. Consider a random collection $\Xi$ of fibres in space. For a Borel set $B$ let $\Phi(B)$ be the total length of the fibre pieces within $B$. For each realisation of $\Xi$ the function $\Phi$ has the properties of a measure: value zero for $B = \emptyset$ and $\sigma$-additivity. So $\Phi$ is genuinely a *random* measure.

Further (partly more abstract, but important) examples of random measures are given in Section 7.3.

*Stochastic Geometry and its Applications*, Third Edition.
Sung Nok Chiu, Dietrich Stoyan, Wilfrid S. Kendall and Joseph Mecke.

### 7.1.2 Definitions and facts

The random measures considered here are all measures on Euclidean spaces. It is straightforward to extend the discussion to more general spaces.

Let $\mathbb{M}$ be the set of all locally finite measures $\varphi$ on $[\mathbb{R}^d, \mathcal{B}^d]$. Let $\mathcal{M}$ denote the smallest $\sigma$-algebra of subsets of $\mathbb{M}$ making all mappings $\varphi \mapsto \varphi(B)$ measurable, where $B$ runs through the bounded Borel sets.

A *random measure* $\Phi$ on $\mathbb{R}^d$ is a random element of $[\mathbb{M}, \mathcal{M}]$. So $\Phi$ is a measurable mapping of a probability space $[\Omega, \mathcal{A}, \mathbf{P}]$ into the measurable space $[\mathbb{M}, \mathcal{M}]$. For each given fixed $B$, $\Phi(B)$ is a real-valued random variable. The *distribution* of $\Phi$ is the probability measure $P$ given by

$$P(Y) = \mathbf{P}(\Phi \in Y) \qquad \text{for } Y \in \mathcal{M}. \tag{7.1}$$

Here $Y$ is a configuration set, for example $Y = \{\varphi : \varphi\big(B(o, r)\big) = 0\}$, the set of all measures giving the ball $B(o, r)$ the measure zero. For a measure $\varphi \in \mathbb{M}$, let $\varphi_x = T_{-x}\varphi$ be the *translated measure* defined by

$$\varphi_x(B) = (T_{-x}\varphi)(B) = \varphi(B - x) \qquad \text{for all Borel sets } B.$$

This notion makes it possible to define stationarity: a random measure $\Phi$ and its distribution $P$ are said to be *stationary* if $\Phi_x$ has the same distribution as $\Phi$ for all $x \in \mathbb{R}^d$. This can be rewritten as

$$P(Y_x) = P(Y) \qquad \text{for all } x \in \mathbb{R}^d \text{ and all } Y \in \mathcal{M}, \tag{7.2}$$

with the notation

$$Y_x = \{\varphi \in \mathbb{M} : \varphi_{-x} \in Y\}. \tag{7.3}$$

The definition of *isotropy* is analogous to that of stationarity, instead of translation- now rotation-invariance is considered.

The standard first-order characteristic of a random measure is the (deterministic) measure $\Lambda$ on $[\mathbb{R}^d, \mathcal{B}^d]$, which is defined by

$$\Lambda(B) = \mathbf{E}\big(\Phi(B)\big) \qquad \text{for all Borel sets } B. \tag{7.4}$$

It is called the *intensity measure* of $\Phi$. This measure is not necessarily locally finite, but in the following this will be assumed. In addition, the null measure will be excluded in the discussion so that $\Lambda(B) \neq 0$ for some $B$. It can be shown that in the stationary case $\Lambda$ is a multiple of Lebesgue measure $\nu_d$ and there is some $\lambda$ in $(0, \infty)$ such that

$$\Lambda(B) = \lambda \nu_d(B) \qquad \text{for all Borel sets } B. \tag{7.5}$$

The constant $\lambda$ is called the *intensity* of $\Phi$. Its interpretation is similar to that in the case of point processes: $\lambda$ is the mean measure per volume unit or the mean measure of a set of volume one.

The notion of ergodicity is analogous to that in the case of point processes in Section 4.1.6, and there is an ergodic theorem analogous to (4.14); see Kallenberg (2002, Corollary 10.19).

There are important *operations* that transform given random measures into others, similar to the operations on point processes given in Section 5.1. By way of example, the operations of superposition and compounding are described here.

Suppose that $\Phi_1$ and $\Phi_2$ are two random measures defined on the same probability space. Then $\Phi$, given by

$$\Phi(B) = \Phi_1(B) + \Phi_2(B) \qquad \text{for all Borel sets } B, \tag{7.6}$$

is again a random measure, the *superposition* of $\Phi_1$ and $\Phi_2$.

If $\Phi$ is a random measure and $\{Z(x)\}$ a nonnegative random field independent of $\Psi$, then the integral

$$Z \circ \Phi(B) = \int_B Z(x)\Phi(\mathrm{d}x) \qquad \text{for all Borel sets } B \tag{7.7}$$

allows construction of a further random measure, the so-called *Z-compound* of $\Phi$ (or the measure *derived* from $\Phi$ by $Z$). Some properties of such derived measures are studied by Karr (1978).

The *support* of a locally finite measure $\varphi$ is the set

$$\operatorname{supp}(\varphi) = \{x \in \mathbb{R}^d : x \in G \text{ implies } \varphi(G) > 0 \text{ for an open } G \subset \mathbb{R}^d\}. \tag{7.8}$$

The set $\operatorname{supp}(\varphi)$ is closed. In the case of a random measure $\Phi$ the support $\operatorname{supp}(\Phi)$ is a random closed set, whose distribution is uniquely determined by the distribution of $\Phi$ (Molchanov, 2005, p. 116).

### 7.1.3 Palm distributions

Just as in the case of point processes, for random measures it is possible to define Palm distributions. These are still conditional distributions, which can be interpreted analogously to Palm distributions of point processes. Their definition is based on Campbell measures, as in the point process case.

Consider a random measure $\Phi$ with distribution $P$. Its *Campbell measure* $\mathscr{C}$ is given by

$$\mathscr{C}(B \times Y) = \mathbf{E}\big(\Phi(B)\mathbf{1}_Y(\Phi)\big) = \int_{\mathbb{M}} \varphi(B)\mathbf{1}_Y(\varphi)P(\mathrm{d}\varphi) \tag{7.9}$$

for Borel $B$ and for $Y \in \mathcal{M}$.

In analogy with the case of point processes, because the Campbell measure $\mathscr{C}$ is absolutely continuous with respect to the intensity measure $\Lambda$, and the latter is a locally finite measure, there is a family $\{P_x\}$ of distributions on $(\mathbb{M}, \mathcal{M})$ such that

$$\mathscr{C}(B \times Y) = \int_B P_x(Y)\Lambda(\mathrm{d}x) \qquad \text{for all Borel } B \text{ and } Y \in \mathcal{M}. \tag{7.10}$$

The $P_x$ are called *Palm distributions*. If $\Psi$ is stationary then

$$P_x(Y_x) = P_o(Y) \qquad \text{for } Y \in \mathcal{M} \text{ and for } Y_x \text{ defined as in (7.3).}$$

So in the stationary case it is sufficient to consider $P_o$, the Palm distribution with respect to the origin. It satisfies

$$\lambda P_o(Y) = \int_{\mathbb{M}} \int_{\mathbb{R}^d} g(x)\mathbf{1}_Y(\varphi_{-x})\varphi(\mathrm{d}x)P(\mathrm{d}\varphi) = \mathbf{E}\left(\int_{\mathbb{R}^d} g(x)\mathbf{1}_Y(\Phi_{-x})\Phi(\mathrm{d}x)\right) \quad (7.11)$$

for every $g : \mathbb{R}^d \to [0, \infty)$ with $\int_{\mathbb{R}^d} g(x)\,\mathrm{d}x = 1$.

Mecke (1967) proved the following properties of the Palm distribution $P_o$ of a stationary random measure $\Phi$.

**Theorem 7.1.** *(a) For all nonnegative measurable functions $f$ on $\mathbb{R}^d \times \mathbb{M}$,*

$$\lambda \int_{\mathbb{M}} \int_{\mathbb{R}^d} f(x, \varphi)\,\mathrm{d}x\, P_o(\mathrm{d}\varphi) = \int_{\mathbb{M}} \int_{\mathbb{R}^d} f(x, \varphi_{-x})\varphi(\mathrm{d}x)P(\mathrm{d}\varphi). \quad (7.12)$$

*(b) For all nonnegative measurable functions $u$ on $\mathbb{M}$ such that $u$ is zero if evaluated at the null measure,*

$$\int_{\mathbb{M}} u(\varphi)P(\mathrm{d}\varphi) = \lambda \int_{\mathbb{M}} \int_{\mathbb{R}^d} h(x, \varphi_x)u(\varphi_x)\,\mathrm{d}x\, P_o(\mathrm{d}\varphi), \quad (7.13)$$

*where $h$ is an arbitrary nonnegative measurable function on $\mathbb{R}^d \times \mathbb{M}$ with*

$$\int_{\mathbb{R}^d} h(x, \varphi)\varphi(\mathrm{d}x) = 1 \qquad \textit{for all } \varphi \textit{ except the null measure}.$$

*(c) For all $f$ as in (a)*

$$\int_{\mathbb{M}} \int_{\mathbb{R}^d} f(-x, \varphi_{-x})\varphi(\mathrm{d}x)P_o(\mathrm{d}\varphi) = \int_{\mathbb{M}} \int_{\mathbb{R}^d} f(x, \varphi)\varphi(\mathrm{d}x)P_o(\mathrm{d}\varphi). \quad (7.14)$$

**Remarks** (1) Formula (7.12) corresponds to Formula (4.64). There is only one distribution $P_o$ satisfying (7.12) for any given stationary $P$.

(2) Formula (7.13) allows the determination of $P$ for given $P_o$, so it is the basis for the so-called *inversion formulae*.

(3) U. Zähle (1984d) showed that, just as in the point process case (see pp. 48 and 130), a local interpretation of the Palm distributions for random measures corresponding to stationary random surface and fibre processes is possible.

This book makes much use of the term 'typical', for example 'typical fibre point', 'typical cell' or 'typical line'. Underlying this language there is always a Palm distribution, which, however, is often not explicitly defined. The idea is explained in detail in Chapter 4 in the point process context. Below, in Section 7.3.4, the Palm distribution of the volume measure is explained, where the typical point is, so to say, 'any' point in the random set considered. In the case of fibre and surface processes, the Palm distribution is nowhere explicitly defined in the book. Nevertheless, the authors hope that the reader can follow: heuristically, the typical fibre (surface) point could be thought of as the result of random uniform sampling on the fibres (surfaces) of the fibre (surface) process.

The case of 'typical cell' or 'typical line' is however different. The underlying Palm distribution is that of a point process in some representation space, and 'typical cell' or

'typical line' then corresponds to the typical point of this point process. Note that it is not customary to speak about 'typical fibre' (with some effort this also could be defined — using some representation space), but only about 'typical fibre point', in the sense of random uniform point sampling on fibres.

### 7.1.4 Marked random measures

Marked (or weighted) random measures are to ordinary random measures as marked point processes are to ordinary point processes. Marks give additional information about the random structure investigated. A simple example is a fibre process, where at each fibre point an additional value is given. This value may characterise the direction of the fibre in that point, or, when the fibres help to model a system of tubes, the tube diameter at that point.

Let $\mathbb{W}$ be the space of possible *marks*, with corresponding $\sigma$-algebra $\mathcal{W}$. (In the example with the fibres, $\mathbb{W}$ is simply $\mathbb{R}$. Also note that there is a change of notation here; $(\mathbb{M}, \mathcal{M})$ in earlier chapters denotes the measurable space of marks but in this chapter denotes that of locally finite measures, and hence $(\mathbb{W}, \mathcal{W})$, originated from the word 'weight', is used for the marks here.) Let $\mathbb{M}_{\mathbb{W}}$ be the set of all measures $\psi$ on $[\mathbb{R}^d \times \mathbb{W}, \mathcal{B}^d \otimes \mathcal{W}]$ with $\psi(B \times \mathbb{W}) < \infty$ for all bounded Borel sets $B$. Let $\mathcal{M}_{\mathcal{W}}$ be the smallest $\sigma$-algebra on $\mathbb{M}_{\mathbb{W}}$ making all mappings $\psi \mapsto \psi(B \times U)$ measurable for $B$ a bounded Borel set and $U$ in $\mathcal{W}$. A marked random measure on $\mathbb{R}^d \times \mathbb{W}$ is then a random variable $\Psi$ on $[\mathbb{M}_{\mathbb{W}}, \mathcal{M}_{\mathcal{W}}]$.

Examples of marked random measures occur in Section 8.6, in the context of fibre and surface processes. The random measures that arise are the length or surface measures; and the marks are the directions of tangents or normals, curvatures or other quantities. Random marked sets (see p. 211) are closely related to marked random measures.

Translations of marked random measures are defined to be shifts which leave the marks unchanged:

$$T_x A = A - x = \{(a, b) : (a - x, b) \in A\} \qquad \text{for } A \in \mathcal{B}^d \otimes \mathcal{W}.$$

If $\psi \in \mathbb{M}_{\mathbb{W}}$ then $\psi_x = T_{-x}\psi$ is the translated marked measure:

$$\psi_x(A) = T_{-x}\psi(A) = \psi(A - x).$$

Using this notion of translation, stationary marked random measures can be defined.

It is possible to define for marked random measures an intensity $\lambda$, which is simply the intensity of the corresponding unmarked random measure $\Phi$ where $\Phi(B) = \Psi(B \times \mathbb{W})$. Also moment measures (see Stoyan and Ohser, 1984) and Palm distributions can be defined for marked random measures. As with the mark distribution for a marked point process, so one can define a mark distribution $M$ for a marked random measure $\Psi$ by

$$M(U) = \frac{1}{\lambda}\mathbf{E}\big(\Psi([0, 1]^d \times U)\big) \qquad \text{for } U \in \mathcal{W}, \tag{7.15}$$

and there is a Campbell theorem which is analogous to Formula (4.29):

$$\mathbf{E}\left(\int_{\mathbb{R}^d \times \mathbb{W}} f(x, w)\Psi\big(\mathrm{d}(x, w)\big)\right) = \lambda \int_{\mathbb{R}^d}\int_{\mathbb{W}} f(x, w)M(\mathrm{d}w)\,\mathrm{d}x. \tag{7.16}$$

## 7.2 Moment measures and related characteristics

### 7.2.1 The Laplace functional

In the theory of random measures the Laplace functional plays a rôle similar to that of the Laplace transform in the theory of random variables. Each random measure $\Phi$ on $\mathbb{R}^d$ is connected with a functional $L_\Phi$ defined on the set $\mathbf{U}$ of all bounded nonnegative measurable functions $u$ of bounded support. The Laplace functional is given by

$$L_\Phi(u) = \mathbf{E}\left(\exp\Big(-\int_{\mathbb{R}^d} u(x)\Phi(\mathrm{d}x)\Big)\right) \qquad \text{for } u \in \mathbf{U}. \tag{7.17}$$

Note that $0 < L_\Phi(u) \le 1$. The distribution of a random measure is uniquely determined by its Laplace functional; see Mecke (1972). If $\Phi$ is a random measure and $B_1, \ldots, B_n$ are bounded Borel sets, then the Laplace transform of the random vector $\big(\Phi(B_1), \ldots, \Phi(B_n)\big)$ can be determined from knowledge of $L_\Phi$:

$$\mathbf{E}\big(\exp(-s_1\Phi(B_1) - \cdots - s_n\Phi(B_n))\big) = L_\Phi(s_1\mathbf{1}_{B_1} + \cdots + s_n\mathbf{1}_{B_n}) \tag{7.18}$$

for $s_1, \ldots, s_n \ge 0$. If $\Phi_1$ and $\Phi_2$ are independent then the Laplace functional $L_\Phi$ of their superposition $\Phi = \Phi_1 + \Phi_2$ is given by

$$L_\Phi = L_{\Phi_1} \cdot L_{\Phi_2}. \tag{7.19}$$

### 7.2.2 Moment measures

Consider as before a random measure $\Phi$ on $\mathbb{R}^d$ with distribution $P$. The $n^{th}$ *moment measure* $\mu^{(n)}$ of $\Phi$ is a measure on $\mathcal{B}^{nd}$ given by

$$\begin{aligned}\int_{\mathbb{R}^{nd}} f(x_1, \ldots, x_n)\mu^{(n)}\big(\mathrm{d}(x_1, \ldots, x_n)\big) \\ &= \mathbf{E}\left(\int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} f(x_1, \ldots, x_n)\Phi(\mathrm{d}x_1)\cdots\Phi(\mathrm{d}x_n)\right) \\ &= \int_{\mathbb{M}} \left(\int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} f(x_1, \ldots, x_n)\varphi(\mathrm{d}x_1)\cdots\varphi(\mathrm{d}x_n)\right) P(\mathrm{d}\varphi)\end{aligned} \tag{7.20}$$

for all nonnegative measurable functions $f(x_1, \ldots, x_n)$ on $\mathbb{R}^{nd}$. In particular

$$\begin{aligned}\mu^{(n)}(B_1 \times \cdots \times B_n) &= \mathbf{E}\big(\Phi(B_1)\cdot\ldots\cdot\Phi(B_n)\big) \\ &= \int_{\mathbb{M}} \big(\varphi(B_1)\cdot\ldots\cdot\varphi(B_n)\big) P(\mathrm{d}\varphi)\end{aligned}$$

and $\mu^{(n)}(B^n) = \mathbf{E}\big(\Phi(B)^n\big)$. Thus the first moment measure coincides with the intensity measure $\Lambda$ introduced by (7.4).

As in the case of point processes, densities with respect to the Lebesgue measure may exist. They are also called *product densities*. In the motion-invariant case the second-

order product density is denoted by $\varrho^{(2)}(r)$. Normalisation yields the pair correlation function $g(r)$,

$$g(r) = \frac{\varrho^{(2)}(r)}{\lambda^2}. \tag{7.21}$$

The rôle of the moment measures of random measures is similar to that of moments of random variables. They can be obtained from the Laplace functional by differentiation:

$$\mu^{(n)}(B_1 \times \cdots \times B_n) = (-1)^n \lim_{s_1,\ldots,s_n \downarrow 0} \frac{\partial}{\partial s_1} \cdots \frac{\partial}{\partial s_n} L_\Phi(s_1 \mathbf{1}_{B_1} + \cdots + s_n \mathbf{1}_{B_n}) \tag{7.22}$$

for bounded Borel sets $B_1, \ldots, B_n$.

Conversely, under certain conditions the Laplace functional can be determined by the moment measures. Suppose all moment measures $\mu^{(n)}$ corresponding to a distribution $P$ are locally finite, and the right-hand side of (7.23) converges. Then for $u$ in $\mathbf{U}$ with $0 \le u(x) \le 1$ for all $x$ in $\mathbb{R}^d$,

$$L_\Phi(u) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \int_{\mathbb{R}^{nd}} u(x_1) \cdot \ldots \cdot u(x_n) \mu^{(n)}\big(\mathrm{d}(x_1 \ldots x_n)\big). \tag{7.23}$$

Zessin (1983) gave conditions to ensure uniqueness of a random measure when the moment measures are given. All this is in close analogy to the *moment problem* for random variables.

Again as in the case of point processes (Section 4.5), it is possible to *reduce* the moment measures. In the stationary and isotropic case, for a diffuse measure a $K$-function can be used which is defined by

$$\lambda K(r) = \int \varphi\big(B(o,r)\big) P_o(\mathrm{d}\varphi) \qquad \text{for } r \ge 0. \tag{7.24}$$

Note that for point processes, which are atomic, in an analogous equation the reduced Palm distribution $P_o^!$ appears.

The variance of $\Phi(B)$ is given in the stationary and isotropic case by the following formula (compare with Formula (4.106))

$$\mathbf{var}\big(\Phi(B)\big) = \lambda^2 \int \overline{\gamma}_B(r)\, \mathrm{d}K(r) - \big(\lambda \nu_d(B)\big)^2, \tag{7.25}$$

where $\overline{\gamma}_B(r)$ is the isotropised set covariance of $B$.

Cross-correlation measures and reduced cross-correlation measures can be defined for any two random measures that are based on the same probability space. They can be used to study correlations between two different random structures such as for example a point process and a fibre process. Stoyan and Ohser (1982, 1984) studied examples of this, for instance when the point process describes tree locations and the fibre process the courses of small brooks in a forest. The case of correlation between a random set and a point process is considered in Foxall and Baddeley (2002).

## 7.3 Examples of random measures

### 7.3.1 Random measures constructed from point processes

Each point process on $\mathbb{R}^d$ can be interpreted as a random integer-valued measure, since $\Phi$, given by

$$\Phi(B) = \text{number of points in } B, \quad \text{for all Borel sets } B, \tag{7.26}$$

is a random measure. It is clearly nonnegative and $\sigma$-additive. For such a random measure, with probability one the realisations are so-called *counting measures*: integer-valued locally finite measures on $[\mathbb{R}^d, \mathcal{B}^d]$. Consequently each point process corresponds to a random counting measure. If the notion of a point process is generalised to allow for multiple points then the correspondence is one-to-one.

The Laplace functional of the random measure $\Phi$ is connected with the generating functional $G$ of the point process $\Phi$ as follows:

$$L_\Phi(u) = G(\mathrm{e}^{-u}) \qquad \text{for } u \in \mathbf{U}. \tag{7.27}$$

For a marked point process with nonnegative marks, the mark sum measure $S_m$ as introduced in Formula (4.33) is a further random measure, associated to the point process.

### 7.3.2 Random measures constructed from random fields

Let $\{Z(x)\}$ be a nonnegative random field on $\mathbb{R}^d$. Then

$$\Phi_Z(B) = \int_B Z(x)\,\mathrm{d}x \qquad \text{for all Borel sets } B \tag{7.28}$$

defines a set function, with the tacit assumption that $Z(x)$ is integrable over all Borel sets $B$. In fact $\Phi_Z$ is a random measure on $[\mathbb{R}^d, \mathcal{B}^d]$, which is almost surely diffuse. Under continuity assumptions, each realisation of $\Phi_Z$ delivers almost surely the corresponding realisation of the random field $\{Z(x)\}$ as the density of $\Phi_Z$ with respect to Lebesgue measure $\nu_d$.

### 7.3.3 Completely random measures

Let $\Phi$ be the random counting measure corresponding to a Poisson process on $\mathbb{R}^d$. Then (by definition) for every family of pairwise disjoint bounded Borel sets, $B_1, \ldots, B_n$, the random variables $\Phi(B_1), \ldots, \Phi(B_n)$ are independent; see Section 2.3.1. It is natural to ask: which other random measures have also this independence property. Such random measures are called *completely random*.

It is clear that the mark sum measure of an independently marked Poisson process is completely random, because sums of independent random variables are involved. More generally, suppose that $H$ is a Borel measure on $\mathbb{R}^d \times (0, \infty)$, with

$$\int_{B\times(0,\infty)} (1 - \mathrm{e}^{-a}) H\big(\mathrm{d}(x, a)\big) < \infty$$

for all bounded Borel sets $B$. Let $\Phi_H$ be the random counting measure corresponding to a Poisson process on $\mathbb{R}^d \times (0, \infty)$ with intensity measure $H$. (If $H$ has an atomic part then $\Phi_H$

can have multiple points.) The measure $\Phi_H$ given by

$$\Phi_H(B) = \int_{B\times(0,\infty)} a\,\Phi_H\big(\mathrm{d}(x,a)\big) \qquad \text{for all Borel sets } B \tag{7.29}$$

is also completely random. The general structure of completely random measures is described in Kingman (1993).

The Laplace functional of $\Phi_H$ is

$$L_{\Phi_H}(u) = \exp\left(-\int_{\mathbb{R}^d\times(0,\infty)} \big(1-\exp(-au(x))\big)H\big(\mathrm{d}(x,a)\big)\right) \qquad \text{for } u \in \mathbf{U}. \tag{7.30}$$

Its intensity measure $\Lambda_{\Phi_H}$ is given by

$$\Lambda_{\Phi_H}(B) = \int_{B\times(0,\infty)} aH\big(\mathrm{d}(x,a)\big) \qquad \text{for all bounded Borel sets } B. \tag{7.31}$$

The higher moment measures are given by

$$\mu^{(2)}(B_1\times B_2) = m_1(B_1)\cdot m_1(B_2) + m_2(B_1\cap B_2)$$

$$\begin{aligned}\mu^{(3)}(B_1\times B_2\times B_3) = {} & m_1(B_1)\cdot m_1(B_2)\cdot m_1(B_3) + m_1(B_1)\cdot m_2(B_2\cap B_3)\\ & + m_1(B_2)\cdot m_2(B_3\cap B_1) + m_1(B_3)\cdot m_2(B_1\cap B_2)\\ & + m_3(B_1\cap B_2\cap B_3) \qquad \text{for } B_1,\ B_2,\ B_3 \text{ Borel sets,}\end{aligned}$$

etc. Here

$$m_n(B) = \int_{B\times(0,\infty)} a^n H\big(d(x,a)\big) \qquad \text{for Borel } B.$$

The $n^{\text{th}}$ moment measure is locally finite if and only if the measure $m_n$ is locally finite.

### 7.3.4 Random measures generated by random closed sets: Curvature measures

In geometrical studies an important rôle is played by measures, in particular curvature measures. These measures generalise and replace the functionals and characteristics introduced in Chapter 1. A simple example is the following.

*Volume measure $V_\Xi$*

Given a random closed set $\Xi$, its volume measure or coverage measure is defined by

$$V_\Xi(B) = \nu_d(B\cap\Xi) \qquad \text{for all Borel sets } B; \tag{7.32}$$

it is always well-defined.

By definition $V_\Xi$ ignores lower-dimensional parts of $\Xi$, for example fibres or surface pieces if in $\mathbb{R}^3$. If $\Xi$ is regular closed (the closure of an open set) then $\operatorname{supp}(V_\Xi) = \Xi$. Therefore, $V_\Xi$ is well-fitted to situations where $\Xi$ is regular closed, as it is the case for usual models of porous media. Ayala *et al.* (1991) discussed the problem of recovering the distribution of $\Xi$ from that of the measure $V_\Xi$. In the following it is assumed that $\Xi$ is almost surely regular closed.

If $\Xi$ is stationary and of volume fraction $p$, then $V_\Xi$ is likewise stationary and its intensity is $p$.

It is mathematically interesting to consider the Palm distribution of $V_\Xi$. This may help to understand the idea of Palm distributions for random measures and sheds light to the nature of contact distribution functions.

It holds that

$$P_o(Y) = \mathbf{P}(V_\Xi \in Y | o \in \Xi) \qquad \text{for any configuration set } Y \text{ in } \mathcal{M}. \tag{7.33}$$

*Proof of (7.33) (Ballani, 2011).* Note that $\lambda = p = \mathbf{P}(o \in \Xi)$. By definition (7.11) it holds for any nonnegative $g(x)$ with $\int_{\mathbb{R}^d} g(x)\mathrm{d}x = 1$,

$$\begin{aligned}
\lambda P_o(Y) &= \mathbf{E}\left(\int_{\mathbb{R}^d} g(x)\mathbf{1}_Y(\Phi_{-x})\Phi(\mathrm{d}x)\right) \\
&= \mathbf{E}\left(\int_{\mathbb{R}^d} g(x)\mathbf{1}_Y(V_{\Xi-x})\mathbf{1}_\Xi(x)\mathrm{d}x\right) \\
&= \mathbf{E}\left(\int_{\mathbb{R}^d} g(x)\mathbf{1}_Y(V_{\Xi-x})\mathbf{1}_{\Xi-x}(o)\mathrm{d}x\right) \\
&= \int_{\mathbb{R}^d} \mathbf{E}\big(\mathbf{1}_Y(V_{\Xi-x})\mathbf{1}_{\Xi-x}(o)\big)g(x)\mathrm{d}x \\
&= \int_{\mathbb{R}^d} \mathbf{E}\big(\mathbf{1}_Y(V_\Xi)\mathbf{1}_\Xi(o)\big)g(x)\mathrm{d}x \\
&= \mathbf{E}\big(\mathbf{1}_Y(V_\Xi)\mathbf{1}_\Xi(o)\big) = \mathbf{P}(V_\Xi \in Y, o \in \Xi),
\end{aligned}$$

and (7.33) follows. □

The statement given in (7.33) can be re-expressed in terms of configuration sets in the trace $\mathcal{F}_{\mathrm{rc}}$ of the hitting $\sigma$-algebra on the set of regular closed sets. Let $Y_{\mathrm{set}}$ be the counterpart of $Y \in \mathcal{M}$ in $\mathcal{F}_{\mathrm{rc}}$. Then

$$P_o(Y) = \mathbf{P}(\Xi \in Y_{\mathrm{set}} | o \in \Xi). \tag{7.34}$$

In order to discuss the spherical contact distribution, the case $Y = \{\varphi : \varphi\big(B(o,r)\big) = b_d r^d\}$ is considered, to which $Y_{\mathrm{set}} = \{X : B(o,r) \subset X\}$ belongs, where $\varphi$ denotes a measure and $X$ a regular closed set. That is, $Y_{\mathrm{set}}$ is the set of all regular closed sets that contain the ball $B(o,r)$.

Equation (7.34) yields

$$P_o(Y) = \mathbf{P}\big(B(o,r) \subset \Xi | o \in \Xi\big). \tag{7.35}$$

The term on the right-hand side can be considered as a function of $r$, and

$$F(r) = 1 - \mathbf{P}\big(B(o,r) \subset \Xi | o \in \Xi\big)$$

is a distribution function. Comparison with (6.59) shows that $F(r)$ is the 'inner' spherical contact distribution function of $\Xi$. When the $\Xi$ considered here is the closure of the complement of a given random closed set, then $F(r)$ is the spherical contact distribution function of that set.

This shows that contact distributions of random sets with positive $p$ have the nature of Palm distributions, as also do the nearest-neighbour distance distribution functions $D(r)$ for point processes, as explained in Section 4.4.4.

The second-order characteristics of the volume measure $V_\Xi$ in the case $p > 0$ are closely related to the set-theoretic second-order characteristics of $\Xi$. In particular, $C(r)$ is the second-order product density of $V_\Xi$ and $C(r)/p^2$ its pair correlation function.

Measure-theoretic ideas play some rôle in statistics of porous media. A powerful method to characterise statistically the variability of random sets is the study of *local porosity*; see Hilfer (1991, 2000). The idea is simple and described as follows.

Take a cube $C_L$ of side-length $L$ and consider the 'volume fraction' of $\Xi$ in $C_L$, that is, the random variable $p_L = V_\Xi(C_L)/\nu_d(C_L)$. Here $\Xi$ is assumed to be motion-invariant; otherwise the location and orientation of $C_L$ also plays a rôle. If $\Xi$ serves as a model for the empty part of a porous medium, then $p_L$ is just the local porosity of $\Xi$ in $C_L$. The random variable is rather difficult to analyse theoretically, even for Boolean models, but it can be easily measured and analysed statistically. Clearly, it is

$$\mathbf{E}(p_L) = p, \tag{7.36}$$

and the variance can be calculated if the covariance of $\Xi$ is known, as $\sigma_v^2$ in (6.83). For large $L$, the local volume fraction $p_L$ is approximately constant, but for smaller $L$ its distribution is rather complicated. For example, $\mathbf{P}(p_L = 0) > 0$ is possible. Hilfer (1991, 2000) showed how the distribution of $p_L$ can be used in the characterisation of porous media.

Hilfer (2000) also demonstrated the use of so-called *local percolation probabilities*. He considered the probability that in the cube $C_L$ there is a path lying in $\Xi$ and connecting a given side of the cube with its opposite side.

*Surface area measure $S_\Xi$*

Given a random closed set $\Xi$ in $\mathbb{R}^d$, its surface area measure is defined by

$$S_\Xi(B) = h_{d-1}(B \cap \partial\Xi) \qquad \text{for all Borel sets } B, \tag{7.37}$$

where $h_{d-1}$ is the $(d-1)$-dimensional Hausdorff measure and $h_{d-1}(B \cap \partial\Xi)$ (not $h_{d-1}(B \cap \Xi)$!) is the surface area of all pieces of the boundary $\partial\Xi$ in $B$. It is well-defined if $\partial\Xi$ has certain smoothness properties, as in the case that $\Xi$ is a regular closed random $\mathbb{S}$-set.

If $\Xi$ is stationary then so is $S_\Xi$. Likewise $S_\Xi$ inherits isotropy from $\Xi$. The intensity of $S_\Xi$, denoted by $S_V^{(d)}$, is called the *specific surface area* of $\Xi$. In the linear, planar and spatial cases special symbols adopted from stereology are used:

$$S_V = S_V^{(3)},$$
$$L_A = S_V^{(2)},$$
$$P_L = S_V^{(1)}.$$

In the linear or one-dimensional case $\Xi$ is a union of non-overlapping closed intervals, and $P_L$ is the intensity of the point process of all interval endpoints.

Pair correlation functions and reduced second moment functions of $S_\Xi$ for some models are considered in Sections 8.3.1 and 8.5.2.

*Curvature measures*

To each random $\mathbb{S}$-set $\Xi$ in $\mathbb{R}^d$ there correspond related random measures $\Phi_{\Xi,k}$, where $k$ runs through $0, 1, \ldots, d$. Two of them have already been described:

$$V_\Xi = \Phi_{\Xi,d},$$

$$S_\Xi = 2\Phi_{\Xi,d-1}.$$

For $k = 0, 1, \ldots, d-1$ the measures $\Phi_{\Xi,k}$ are concentrated on $\partial\Xi$. (Note that the connection with $S_\Xi$ only holds if $\Xi$ is a regular closed set. The comment on (7.41) below indicates what will happen if $\Xi$ possesses lower-dimensional parts.)

The following exposition follows Schneider and Weil (1992); see also Schneider (1993) (but note that there are notational differences between these books and the present text). First curvature measures for deterministic convex bodies are explained, then there is an account of their extension to deterministic polyconvex sets, and finally there is a description of random curvature measures corresponding to random $\mathbb{S}$-sets.

Convex sets

Let $K$ be a convex body of $\mathbb{R}^d$ and suppose that $x \in \mathbb{R}^d$. Then $p(x, K)$ denotes the (uniquely determined) nearest point to $x$ in $K$. For any $B \subset \mathbb{R}^d$ the *local parallel set* $M_r(K, B)$ of $K$ at distance $r$ is defined by

$$M_r(K, B) = \{x \in \mathbb{R}^d : \|x - p(x, K)\| \le r,\ p(x, K) \in B\} \qquad \text{for } r \ge 0.$$

If $B$ is a Borel set, then so is $M_r(K, B)$. Finite Borel measures are defined for each positive $r$ by

$$\mu_r(K, B) = \nu_d\big(M_r(K, B)\big) \qquad \text{for all Borel sets } B. \tag{7.38}$$

These measures $\mu_r(K, \cdot)$ satisfy the *local Steiner formula*

$$\mu_r(K, B) = \sum_{k=0}^{d} b_{d-k}\Phi_k(K, B)r^{d-k} \qquad \text{for } r \ge 0 \text{ and } B \text{ a Borel set;} \tag{7.39}$$

compare this with Formula (1.39). The measures $\Phi_k(K, \cdot)$ appearing here are called *curvature measures* of $K$. For $k = d$ and $k = d-1$,

$$\Phi_d(K, B) = V_K(B) = \nu_d(B \cap K) \tag{7.40}$$

and

$$2\Phi_{d-1}(K, B) = S_K(B) = h_{d-1}(B \cap \partial K) \tag{7.41}$$

for all Borel sets $B$. The formula for the surface measure is true if $K$ is $d$-dimensional. If $K$ is $(d-1)$-dimensional, that is, a subset of a $(d-1)$-dimensional flat, then

$$\Phi_{d-1}(K, B) = h_{d-1}(\partial K \cap B),$$

as with $S(K)$ on p. 12.

Under smoothness conditions the values $\Phi_k(K, B)$ for $k < d$ can be obtained as integrals with respect to curvatures over $B \cap \partial K$. This explains the name 'curvature measure'; see the explanation of $\Phi_0$ for a planar polyconvex set below.

For $B = \mathbb{R}^d$ the measures yield the intrinsic volumes,

$$\Phi_k(K, \mathbb{R}^d) = V_k(K) \qquad \text{for } k = 0, 1, \ldots, d.$$

Polyconvex sets

As the intrinsic volumes and Minkowski functionals, the curvature measures can be extended to sets in the convex ring $\mathcal{R}$, with the additivity property

$$\Phi_k(A_1 \cup A_2, B) + \Phi_k(A_1 \cap A_2, B) = \Phi_k(A_1, B) + \Phi_k(A_2, B)$$

for $A_1$ and $A_2$ in $\mathcal{R}$. This *additive extension* can be achieved by means of the theory of extensions of valuations; see Schneider (1980), Schneider and Weil (1992) and Schneider (1993).

These theories justify the assertion that the extended Minkowski and curvature measures satisfy, analogously to Formula (1.78), the relation

$$\begin{aligned}\Phi_k(A, B) = &\sum_i \Phi_k(K_i, B) \\ &- \sum_{i_1 < i_2} \sum \Phi_k(K_{i_1} \cap K_{i_2}, B) + \cdots + (-1)^{n-1} \Phi_k(K_1 \cap \cdots \cap K_n, B) \quad (7.42)\end{aligned}$$

for any Borel set $B$, if $A$ is given in the form $A = \bigcup_{i=1}^n K_i$ where the $K_i$ are convex bodies.

Note that the measures $\Phi_k(A, \cdot)$ are finite *signed* measures for $k = 0, 1, \ldots, d-2$; they are not necessarily positive.

As in the convex case, for any Borel set $B$,

$$\Phi_d(A, B) = V_A(B), \tag{7.43}$$

and for any regular closed $A$,

$$2\Phi_{d-1}(A, B) = S_A(B). \tag{7.44}$$

Consider now the measure $\Phi_0$. For convex $K$ it is

$$\Phi_0(K, B) = \frac{1}{db_d} h_{d-1}\big(\sigma(K, B)\big), \tag{7.45}$$

where $\sigma(K, B)$ is the 'spherical image' of $K$, that is, the set of all outward-pointing unit normal vectors of $B \cap \partial K$ in the sphere $S^{d-1}$; see Schneider (1993).

In the planar case the value of $\Phi_0(A, B)$ for a sufficiently regular polyconvex set $A$ (e.g. having a boundary of a finite number of vertices and being smooth between the vertices) is obtained as follows for a regular set $B$ (e.g. a rectangle). As with all curvature measures $\Phi_k$ with $k < d$, it is concentrated on the boundary $\partial A$ of $A$. Because of the assumptions on $A$, the curvature $\kappa(x)$ is finite at almost all points $x$, and at almost all points of $\partial A$ the interior of

$A$ is on one side of $\partial A$. The inner angle at a vertex $x$ is denoted by $\omega(x)$. Then it holds

$$2\pi\Phi_0(A, B) = \int_{B\cap\partial A} \kappa(x)\, l(\mathrm{d}x) + \sum_{\text{vertices } x \text{ of } \partial A \text{ in } B} \big(\pi - \omega(x)\big), \tag{7.46}$$

where the integral is taken over all curved segments of $\partial A$ in $B$, with $l(\cdot)$ being the corresponding length measure. The orientation of the curves is taken so that the interior of $A$ is on the left-hand side of an observer moving in an anticlockwise direction along the curves of $\partial A$. Thus for a circle belonging to $\partial A$ and lying completely in $B$, the integration yields the value $2\pi$. If $\partial A$ has a vertex at the point $x$, then $x$ gives a separate contribution to the integral. It is determined by the inner angle $\omega(x)$ of $A$ at $x$, which will lie between 0 and $2\pi$. If $0 < \omega(x) < \pi$ then at $x$ there is an outward vertex; if $\pi < \omega(x) < 2\pi$ then there is an inward vertex. The contribution is then $\pi - \omega(x)$, and is positive if the vertex is outward.

The measures $\Phi_k$ are homogeneous of degree $k$, that is,

$$\Phi_k(\alpha A, \alpha B) = \alpha^k \Phi_k(A, B) \qquad \text{for } \alpha > 0,$$

and they are *motion-covariant*, that is,

$$\Phi_k(\boldsymbol{m}A, \boldsymbol{m}B) = \Phi_k(A, B) \qquad \text{for all rigid motions } \boldsymbol{m}.$$

Random $\mathbb{S}$-sets

The local character of the curvature measures enables to extend their definition to sets in the extended convex ring $\mathbb{S}$, that is, to sets $A$ with the property that $A \cap K \in \mathcal{R}$ for every convex body $K$. For such a set $A$ and any bounded Borel set $B$ define

$$\Phi_{A,k}(B) = \Phi_k\big(A \cap B(o, R), B\big), \tag{7.47}$$

where $R$ is large enough that $B$ is contained in the open ball $B^{\mathrm{int}}(o, R)$, and by definition, $A \cap B(o, R)$ belongs to the convex ring $\mathcal{R}$.

In the same way the random curvature measures $\Phi_{\Xi,k}$ are defined for random $\mathbb{S}$-sets $\Xi$:

$$\Phi_{\Xi,k}(B) = \Phi_k\big(\Xi \cap B(o, R), B\big) \qquad \text{for all Borel sets } B. \tag{7.48}$$

Federer (1959) defined curvature measures for sets of positive reach. Zähle (1984a,b, 1986, 1987a) and Weiss and Zähle (1988) extended this to unions and to locally finite unions of such sets.

*Positive or absolute curvature measures*

There is still another extension of the curvature measures to the polyconvex sets and $\mathbb{S}$-sets, which leads to proper (positive) measures. However, this other extension is not additive. These absolute curvature measures were introduced by Matheron (1975, p. 119) and, in a refined form, by Schneider (1979a,b, 1980).

In the following the construction of Matheron is sketched for any $A \in \mathcal{R}$.

Let $\Pi_A(x)$ be the set of all projections of $x$ in $\mathbb{R}^d$ onto $A$, that is, the set of all points $x' \in A$ for which there exists an open set $O$ with $x' \in O$ and

$$\|x - y\| > \|x - x'\|$$

for all $y \in O \cap A$ with $y \neq x'$. This set is finite. (If $A$ is convex then $\Pi_A$ is the singleton set $\{p(x, A)\}$, with $p(x, A)$ defined as on p. 290.)

Then

$$n(A, B, r; x) = \sum_{z \in \Pi_A(x) \cap B^{\text{int}}(x,r)} \mathbf{1}_B(z) \qquad \text{for } r \geq 0 \text{ and } B \text{ a Borel set}$$

is the number of distinct points in $\Pi_A(x) \cap B^{\text{int}}(x, r) \cap B$, and satisfies

$$\int_{\mathbb{R}^d} n(A, B, r; x)\, \mathrm{d}x < \infty \qquad \text{for any bounded Borel set } B. \tag{7.49}$$

Matheron (1975) showed that there exist measures $W_k^+(A, \cdot)$ called *positive Minkowski measures* yielding the expansion

$$\int_{\mathbb{R}^d} n(A, B, r; x)\, \mathrm{d}x = \sum_{k=0}^{d} \binom{d}{k} W_k^+(A, B) r^k \tag{7.50}$$

for all Borel sets $B$ and $k = 0, 1, \ldots, d$. These can be transformed in *positive curvature measures* $\Phi_k^+(A, \cdot)$ by

$$b_{d-k} \Phi_k^+(A, \cdot) = \binom{d}{k} W_{d-k}^+(A, \cdot). \tag{7.51}$$

The positive curvature measure $\Phi_0^+$ is related to the so-called convexity number; see Matheron (1975, p. 122) and Schneider (1993, p. 225), who defined positive curvature measures directly.

In the three-dimensional case the measure $\Phi_1^+$ has a natural interpretation in terms of curvature. Let $\partial A^+$ be the subset of the boundary $\partial A$ in which all curvatures are nonnegative. Since $A$ is a polyconvex set, $\partial A \setminus \partial A^+$ consists of curves generated by the intersections of the surfaces of the constituent convex bodies; see Figure 7.1. Then, if $m(x)$ is the mean curvature at $x$, it can be shown that

$$\pi \Phi_1^+(A, B) = \int_{B \cap \partial A^+} m(x)\, \mathrm{d}S \qquad \text{for all Borel sets } B, \tag{7.52}$$

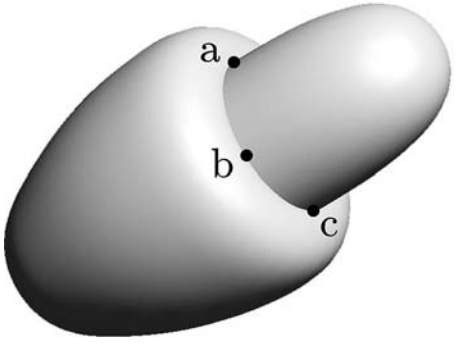


**Figure 7.1** Intersection curve ( — a — b — c —) of two convex bodies.

where $\mathrm{d}S$ corresponds to the surface area. Hence, $\Phi_1^+(A, \cdot)$ is called the *mean positive curvature measure*.

A further (positive) curvature measure is related to the *absolute curvature* described in Baddeley and Averback (1983). The absolute curvature $m^a(x)$ of a smooth surface at point $x$ is given by

$$m^a(x) = \begin{cases} |m(x)|, & \text{if } \kappa_1(x) \text{ and } \kappa_2(x) \\ & \text{have the same sign,} \\ \left(\dfrac{4}{\pi}\theta_0(x) - 1\right) m(x) + \dfrac{1}{\pi}\big(\kappa_1(x) - \kappa_2(x)\big) \sin 2\theta_0(x), & \text{otherwise,} \end{cases}$$

where $\kappa_1(x)$ and $\kappa_2(x)$ are the principal curvatures and

$$\theta_0(x) = \tan^{-1} \sqrt{\frac{-\kappa_1(x)}{\kappa_2(x)}},$$

choosing the branch of arctangent so that $0 \le \theta_0(x) \le \pi/2$; see Baddeley (1980). The definition of the absolute curvature measure is entirely analogous to that of the mean positive curvature measure of (7.52), but the integration is performed over $B \cap \partial A$ rather than $B \cap \partial A^+$ and $m(x)$ is replaced by $m^a(x)$.

Absolute curvature measures can also be defined for $\mathbb{S}$-sets. Such measures for unions of sets of positive reach are studied in Zähle (1989) and Rother and Zähle (1992).

The random positive curvature measures belonging to a random $\mathbb{S}$-set $\Xi$ are denoted by $\Phi^+_{\Xi,k}$. For stationary (motion-invariant) $\Xi$ the random measures are also stationary (motion-invariant).

*Intensities of stationary curvature measures*

If the underlying random closed set $\Xi$ is stationary, then so are the corresponding curvature measures. Thus it makes sense to introduce the corresponding intensities $v_k$ and $v_k^+$, $k = 0, 1, \ldots, d$. These intensities play an important rôle in practical applications as first-order characteristics of $\Xi$, in particular in the planar and spatial case. Therefore a special notation is reserved for them: in $\mathbb{R}^3$ $v_3$ is denoted as $V_V$ and $2v_2$ as $S_V$. On p. 249 there is a complete list of the symbols for the $v_k$. For the null-dimensional intrinsic volumes these are

$$N_A = v_0 \quad \text{and} \quad N_A^+ = v_0^+ \quad \text{in the planar case} \tag{7.53}$$

and

$$N_V = v_0 \quad \text{and} \quad N_V^+ = v_0^+ \quad \text{in the spatial case.} \tag{7.54}$$

It is natural to call $N_A$ the *specific planar connectivity number* and $N_A^+$ the *specific planar convexity number*. Furthermore, it is convenient to use $N_A^-$ to denote the intensity of $\Phi_0^+$ of the closure of the complement of $\Xi$. It is

$$N_A = N_A^+ - N_A^-. \tag{7.55}$$

Formula (7.46) suggests the equation

$$C_A = 2\pi N_A \tag{7.56}$$

for the *integral of mean curvature per area unit*. The intensities of the curvature measures for $d = 3$ are $M_V$ for $\Phi_{\Xi,2}$ and $M_V^+$ for $\Phi_{\Xi,2}^+$.

Note that the intensities discussed here coincide with the densities introduced in Formulae (3.37) and (6.71). Thus, for example, $S_V$ can also be understood as

$$S_V = \lim_{n\to\infty} \frac{\mathbf{E}\big(S(\partial\Xi \cap W_n)\big)}{\nu_3(W_n)}$$

where $\{W_n\}$ is a convex averaging sequence of windows.

To aid understanding, some formulae are given for germ–grain models of non-intersecting grains. Suppose the germ point process has intensity $\lambda$. If $\overline{\chi}$ is the mean connectivity number of the typical grain, then

$$N_A = \lambda\overline{\chi}, \quad \text{and likewise} \quad N_V = \lambda\overline{\chi}.$$

If $\overline{M}$ is the mean of the integral of mean curvature of the typical grain, then

$$M_V = \lambda\overline{M}.$$

Finally, $C_A$ satisfies

$$C_A = 2\pi\lambda\overline{\chi}.$$

In the planar case, $N_A$ can be determined by tangent count. Adler and Taylor (2007, Theorem 6.1.2) gave the theoretical foundation. This can be used statistically in a method suggested already in 1967 by Cahn, DeHoff and Haas *et al.*, which is described as follows.

In the stationary and isotropic case, $N_A^+$ is the intensity of the point process of lower convex tangent points; see Figure 7.2. The value $N_A^+$ can be estimated by counting all those lower tangent points that fall in a given window $W$ of observation. This count $n^+(W)$ yields

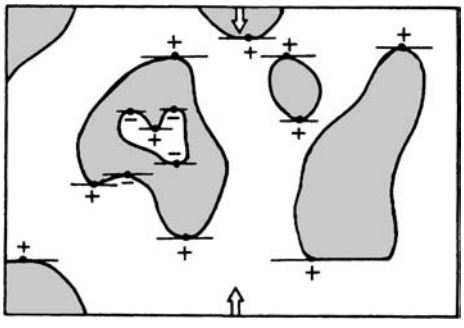

**Figure 7.2** Tangent count with respect to the bottom line. There are six lower convex tangent points corresponding to tangents parallel to the bottom line and three lower concave tangent points; there are four upper convex tangent points and one upper concave tangent point. Thus, assuming that the set shown belongs to a planar stationary and isotropic random closed set and that the window area $\nu_2(W) = 1$, one obtains $\hat{N}_A^+ = 5$, $\hat{N}_A^- = 2$ and $\hat{N}_A = 3$.

the estimator

$$\hat{N}_A^+ = \frac{n^+(W)}{\nu_2(W)}. \tag{7.57}$$

It is useful to determine $\hat{N}_A^+$ for various orientations (instead of using only the upward direction) and then to average. Baddeley (1984) considered the accuracy of the tangent count method, which is of course sensitive to image noise.

Analogously, estimates of $N_A^-$ are obtained by counting the lower concave tangents (or upper convex tangents of the complement); see also Figure 7.2. Then finally $N_A$ is estimated via

$$\hat{N}_A = \hat{N}_A^+ - \hat{N}_A^-. \tag{7.58}$$

The estimation of the other intensities $v_k$ is described in Section 6.5.7.