2

Point processes I – The Poisson point process

2.1 Introduction

The basic ingredients of practical geometry are points. Similarly random point patterns, point fields or *point processes* (in mathematical terminology) play a fundamental rôle in stochastic geometry and they arise directly as results of investigations into nature and technology. Figures in Chapters 2, 4 and 5 display various point patterns that appear in this fashion.

This chapter begins the task of showing how to study them mathematically. Frequently they occur as systems of points connected with particles, pores, or other geometrical objects. For example, a system of particles gives rise to a point pattern generated by the particle centroids. This leads on to a second, more indirect, fashion in which random point patterns arise. In many problems of stochastic geometry it is helpful to interpret patterns of geometrical objects as systems of 'points' in suitable 'representative spaces'; a 'point' from the system represents a particular geometrical object in the original pattern. This is sketched in Section 4.8 and described in detail for the case of random lines in the plane in Section 8.2.2.

In the current chapter the simplest and most important random point pattern is studied: the Poisson (point) process. This study will be carried out in a heuristic fashion, generally avoiding use of the abstract theory of point processes. It is hoped that readers having less mathematical training will thus be encouraged to read further and study the general theory as presented in Chapter 4 and the discussion in Chapter 5 of further models for random point patterns. A detailed treatment of the Poisson process is given in Kingman (1993, 2006), Daley and Vere-Jones (2003) and Streit (2010).

An appreciation of the possible varieties of random point patterns leads to an understanding of the central position that the Poisson point process holds among theoretical models. A given random point pattern may well exhibit various kinds of interaction between its

Stochastic Geometry and its Applications, Third Edition.
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constituent points. For example, the points may occur in clusters (see Figures 5.4 and 5.6) or may exhibit regularity (see Figure 5.7). There may be a *hard-core distance* (a minimum inter-point distance). These features may occur together in the same pattern (see Figure 5.9) and in various scales. The subject of point process statistics aims to detect and to quantify such interactions. Models from the theory of point processes can be used both in comparison to the original point pattern and also in representation of it.

In the absence of any of the above interactions a point pattern can be thought of as completely random. A theoretical model of such a pattern is of importance as a basis for comparison, as a null-model. Calculations can be made of the extent of probable fluctuations in such a model. These fluctuations provide objective grounds for ignoring sufficiently small empirically observed interactions, on the basis that they are insignificant compared with features that might well be observed in a completely random point pattern. If intuitively appealing axioms are imposed (concerning homogeneity and lack of interaction) then completely random processes can be characterised as Poisson point processes. Thus one of the central rôles of the Poisson point process is to serve as a null hypothesis for statistical tests of interaction.

The other main rôle lies in the use of the Poisson process as a basic building block for other more complicated models; see, for example, Sections 3.1, 5.3 and 9.7. Theoretical definitions of varieties of point processes frequently make reference to Poisson point processes. Simulation procedures often include the construction of a Poisson point process, which is then modified into the form required.

A history of the concept of a Poisson process can be found in Guttorp and Thorarinsdottir (2012). They reported that the first recorded use of the Poisson process in spatial statistics appears to be that of Michell (1767), who studied distances of stars, followed by Clausius (1857), who applied it in the kinetic theory of heat, and Abbe (1879), who considered the problem of counting blood particles — of course all these scientists did not know the modern idea of a Poisson process.

2.2 The binomial point process

2.2.1 Introduction

The most elementary example of a point process is one that contains one point only. A random point *x uniformly distributed* in a compact set $W \subset \mathbb{R}^d$ is a random point such that

$$\mathbf{P}(x \in A) = \frac{\nu_d(A)}{\nu_d(W)} \tag{2.1}$$

for all Borel sets A contained in W. Many problems in geometric probability arise from the study of uniformly distributed random points and the determination of the volume ratio in (2.1) when A is defined by some geometrical construction. The progress of modern stochastic geometry can to some extent be summarised as systematic replacement of uniformly distributed random points by more general random point patterns.

A uniformly distributed random point is a rather trivial random pattern. However, n independent uniformly distributed random points can be superposed to form a new, more interesting random point pattern, a *binomial point process of n points*. Such a process is

formed by n independent points $x_1, \ldots, x_n$ uniformly distributed in the same compact set W. From Formula (2.1)

$$\begin{aligned}\mathbf{P}(x_1 \in A_1, \ldots, x_n \in A_n) &= \mathbf{P}(x_1 \in A_1) \cdot \ldots \cdot \mathbf{P}(x_n \in A_n) \\ &= \frac{\nu_d(A_1) \cdot \ldots \cdot \nu_d(A_n)}{\nu_d(W)^n} \end{aligned} \tag{2.2}$$

for Borel subsets $A_1, \ldots, A_n$ of W. This leads to an equivalent definition; the points $x_1, \ldots, x_n$ form a binomial point process in W (for $W \subset \mathbb{R}^d$) if the random vector $(x_1, \ldots, x_n)$ is uniformly distributed in W^n. Figure 2.1 (on p. 39) shows the result of a simulation of a binomial point process in $W = [0, 1]^2$ with $n = 100$ points.

If $x_1, \ldots, x_n$ form a binomial point process in W then the random pattern constituted by these points is denoted by $\Phi_{W^{(n)}}$. For most purposes the ordering of the points is irrelevant and $\Phi_{W^{(n)}}$ can be regarded as a random set.

It can also be thought of as a random counting measure; for a Borel set A let $\Phi_{W^{(n)}}(A)$ denote the number of points of $\Phi_{W^{(n)}}$ falling in A. Thus

$$\Phi_{W^{(n)}}(\emptyset) = 0, \qquad \Phi_{W^{(n)}}(W) = n,$$

and

$$\Phi_{W^{(n)}}(A_1 \cup A_2) = \Phi_{W^{(n)}}(A_1) + \Phi_{W^{(n)}}(A_2)$$

whenever A_1 and A_2 are disjoint subsets of W. The σ-additivity property also holds. Hence $\Phi_{W^{(n)}}$ is a random measure.

The binomial point process $\Phi_{W^{(n)}}$ is the first nontrivial example of a point process considered in this book. Notice that it can be viewed either as a point process, or as a random set, or as a random measure; these multiple views are important for the study of all (simple) random point processes. A point process is *simple* if no two points coincide.

2.2.2 Basic properties

The binomial point process $\Phi_{W^{(n)}}$ earns its name from a distributional property. If A is a Borel subset of W then $\Phi_{W^{(n)}}(A)$ has a binomial distribution with parameters $n = \Phi_{W^{(n)}}(W)$ and $p = p(A) = \nu_d(A)/\nu_d(W)$. Since the mean of the binomial distribution is np, the mean number of points per unit volume, or *intensity* of the binomial point process, is given by

$$\lambda = \frac{n}{\nu_d(W)} \tag{2.3}$$

and

$$\mathbf{E}\big(\Phi_{W^{(n)}}(A)\big) = \lambda \nu_d(A) \qquad \text{for any Borel set } A \subset W. \tag{2.4}$$

Numbers of points in different subsets of W are *not* independent even if the subsets are disjoint. This is clear since $\Phi_{W^{(n)}}(A) = m$ necessarily implies $\Phi_{W^{(n)}}(W \setminus A) = n - m$.

The distribution of $\Phi_{W^{(n)}}$ as a point process is completely characterised by the so-called *finite-dimensional distributions*:

$$\mathbf{P}(\Phi_{W^{(n)}}(A_1) = n_1, \ldots, \Phi_{W^{(n)}}(A_k) = n_k) \qquad \text{for } k = 1, 2, \ldots, \tag{2.5}$$

where $A_1, \ldots, A_k$ are arbitrary Borel sets and $n_1, \ldots, n_k$ are nonnegative integers satisfying $n_1 + \cdots + n_k \leq n$.

Obviously, the binomial process is simple; all points are isolated.

A mathematical theorem says that the distribution of a simple point process is determined by its so-called *void-probabilities* $v_K = \mathbf{P}(\Phi_{W^{(n)}}(K) = 0)$, where K is an arbitrary compact subset of W. Section 6.1 discusses this way of specifying distributions in the much more general context of random sets; see also the remarks on void-probabilities in Section 2.3.1.

The void-probabilities for the binomial point process are given by

$$v_K = \mathbf{P}(\Phi_{W^{(n)}}(K) = 0) = \frac{\big(\nu_d(W) - \nu_d(K)\big)^n}{\nu_d(W)^n}. \tag{2.6}$$

When $A_1, \ldots, A_k$ are disjoint Borel sets with $A_1 \cup \cdots \cup A_k = W$ and $n_1 + \cdots + n_k = n$, the finite-dimensional distributions are given by the multinomial probabilities

$$\begin{aligned} &\mathbf{P}(\Phi_{W^{(n)}}(A_1) = n_1, \ldots, \Phi_{W^{(n)}}(A_k) = n_k) \\ &\qquad = \frac{n!}{n_1! \cdot \ldots \cdot n_k!} \cdot \frac{\nu_d(A_1)^{n_1} \cdot \ldots \cdot \nu_d(A_k)^{n_k}}{\nu_d(W)^n}. \end{aligned} \tag{2.7}$$

As noted above, $\Phi_{W^{(n)}}(A)$ is of binomial distribution with parameters n and $p(A)$. The well-known Poisson limit theorem yields that if the total number of points n tends to infinity and the second parameter $p(A)$ tends to zero in such a way that the product $\lambda \cdot \nu_d(A)$ remains fixed then $\Phi_{W^{(n)}}(A)$ is asymptotically of Poisson distribution of mean $\lambda \cdot \nu_d(A)$. This limit can be obtained if the region W is enlarged to fill out all of $\mathbb{R}^d$ while n is allowed to tend to infinity. If the ratio $n/\nu_d(W)$ remains fixed as n increases and W is enlarged then the Poisson limit will hold for $\Phi_{W^{(n)}}(A)$ for any fixed bounded Borel set A. So if there is a limiting point process Φ then it should possess the property:

$$\Phi(A) \text{ is Poisson of mean } \lambda \cdot \nu_d(A) \text{ for each bounded Borel set } A,$$

where $\Phi(A)$ is the number of points of Φ in A. It is an implication of (2.7) that such a limiting process should be 'independently scattered':

$$\Phi(A_1), \ldots, \Phi(A_k) \text{ are independent if } A_1, \ldots, A_k \text{ are disjoint bounded Borel sets.}$$

These points are discussed again in Section 2.3.1.

2.2.3 Simulation

The simulation of a binomial point process follows easily from superposition once one knows how to simulate a random point uniformly distributed over the required region; the binomial point process can be obtained by n independent replications of simulating random points.

It is straightforward to simulate a random point uniformly distributed in $[0, 1]^2$. If $\{u_j\}$ is a sequence of independent random numbers uniformly distributed in $[0, 1]$ then the points

$$x_i = (u_{2i-1}, u_{2i}) \qquad \text{for } i = 1, 2, \ldots \tag{2.8}$$

form a sequence of independent random points uniformly distributed in $[0, 1]^2$. Figure 2.1 shows a sample of 100 points obtained in this way. The numbers $\{u_j\}$ were produced by a

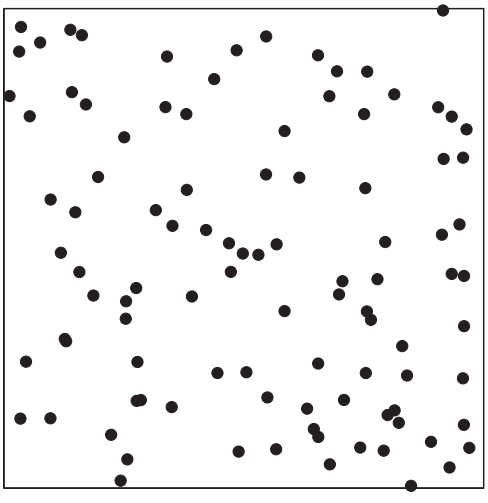

Figure 2.1 A pattern of 100 simulated random points uniformly distributed in a square.

random number generator; see Gentle (2003) and Kroese *et al.* (2011) for more on random number generation.

A sequence of random points uniformly distributed in the hypercube $[0, 1]^d$ is produced by

$$x_i = (u_{(i-1)d+1}, \ldots, u_{id}) \qquad \text{for } i = 1, 2, \ldots. \tag{2.9}$$

Translation and scale changes can be used to produce sequences of points uniformly distributed in any fixed rectangle or hypercube.

Such simulation procedures cover a large number of cases in practice, as the binomial point process to be simulated will frequently take place in a square or cube. However, other regions do arise and then further work must be done. Simulation of a uniform random point in a general bounded region W is tackled using one of three main techniques. (To be definite only the common planar case is discussed.)

(a) *Rejection sampling.* A rectangle R containing W is found and a sequence of independent uniform random points is simulated in R until a point first falls in W. This point will be uniformly distributed in W. To obtain a binomial point process this whole procedure is repeated until n points have fallen in W, and these n points constitute the sample for the binomial point process. To maximise efficiency R should be chosen to be as small as possible. Figure 2.2 illustrates the process.

(b) *Approximation.* The region W is replaced by a disjoint union of k open squares approximating W. A random point distributed uniformly over this union is simulated by first choosing a square with probability proportional to its area and then simulating a random point uniformly distributed over that square.

Exact simulations for complicated regions can be obtained by combining this technique with rejection sampling.

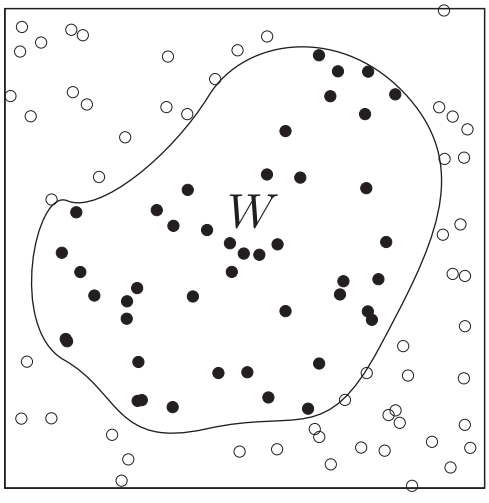

Figure 2.2 A simulation of 45 random points. The points • are uniformly distributed in the set W. The points ○ are those points of Figure 2.1 that do not lie in W.

(c) *Transformation of coordinates.* If the region W exhibits some symmetry then transformation of coordinates may be useful. For example, if W is the unit disc $B(o, 1)$ then a uniform random point can be described in polar coordinates

$$x = (r, \theta) \qquad \text{for } r \in [0, 1] \text{ and } \theta \in (0, 2\pi].$$

The random variables r and θ are independent, with θ uniform over $(0, 2\pi]$ and r satisfying the law

$$\mathbf{P}(r \leq t) = t^2 \qquad \text{if } 0 \leq t \leq 1.$$

Thus if u_1 and u_2 are independent random numbers uniform over $[0, 1]$ then the formulae

$$r = \sqrt{u_1} \qquad \text{and} \qquad \theta = 2\pi u_2 \tag{2.10}$$

provide a method of simulating $x = (r, \theta)$.

As explained in Section 2.3.1 paragraph (e) on p. 43 the binomial point process arises from the homogeneous Poisson point process by conditioning. Consequently it is not necessary to give statistical methods for testing the hypothesis that a given point pattern is a realisation of a binomial point process; one can use the methods for homogeneous Poisson point processes described in Section 2.6.4. On the other hand, the simulation procedures for homogeneous Poisson point processes and general Poisson point processes depend heavily on the simulation methods for binomial point processes described here.

2.3 The homogeneous Poisson point process

2.3.1 Definition and defining properties

Since a general treatment of point processes appears in Chapter 4, only a brief explanation of terminology is given here. As seen above, a random point pattern or *point process* Φ can be regarded either as a random sequence (more strictly a random set) $\Phi = \{x_1, x_2, \ldots\}$ or else as a random counting measure: for each Borel set B the symbol $\Phi(B)$ denotes the random number of points of Φ which lie in the set B.

As a random set, Φ can be intersected with other sets; if B is a Borel set then $B \cap \Phi$ is the random set of points of Φ that also belong to B. Because the random point patterns considered are all *locally finite* the random set Φ will always be closed and $B \cap \Phi$ will be finite whenever B is bounded.

A homogeneous Poisson point process Φ is characterised by two fundamental properties, which have already appeared as asymptotic properties in Section 2.2.2. They are:

(1) *Poisson distribution of point counts*. The random number of points of Φ in a bounded Borel set B has a Poisson distribution of mean $\lambda v_d(B)$ for some constant λ, that is,

$$\mathbf{P}\big(\Phi(B) = m\big) = \frac{\mu^m}{m!} \exp(-\mu) \qquad \text{for } m = 0, 1, 2, \ldots, \tag{2.11}$$

where

$$\mu = \lambda v_d(B). \tag{2.12}$$

(2) *Independent scattering*. The numbers of points of Φ in k disjoint Borel sets form k independent random variables, for arbitrary k.

Property (2) is also known as the 'completely random' or 'purely random' property.

The number λ occurring in property (1) is the characteristic parameter of the homogeneous Poisson point process. It gives the mean number of points in a set of unit volume, and satisfies

$$\lambda v_d(B) = \mathbf{E}\big(\Phi(B)\big) \qquad \text{for all bounded Borel sets } B. \tag{2.13}$$

It is called the *intensity* or *density* of the homogeneous Poisson point process Φ. The following always assumes that λ is positive and finite. If $\lambda = 0$ then the point pattern contains no points while an infinite λ corresponds to a pathological case.

The intensity can also be interpreted in infinitesimal terms. If B is a set of small Lebesgue measure then by property (1) the following asymptotics hold:

$$\mathbf{P}\big(\Phi(B) = 0\big) = 1 - \lambda v_d(B) + o\big(v_d(B)\big),$$

$$\mathbf{P}\big(\Phi(B) = 1\big) = \lambda v_d(B) + o\big(v_d(B)\big),$$

$$\mathbf{P}\big(\Phi(B) > 1\big) = o\big(v_d(B)\big).$$

Let Φ be a homogeneous Poisson point process of intensity λ. From properties (1) and (2) the whole distribution of the homogeneous Poisson point process can be determined once the intensity λ is known. The following summarises some basic properties of Φ.

(a) *Finite-dimensional distributions.* It can be shown directly from properties (1) and (2) that if $B_1, \ldots, B_k$ are disjoint bounded Borel sets then $\Phi(B_1), \ldots, \Phi(B_k)$ are independent Poisson random variables with means $\lambda v_d(B_1), \ldots, \lambda v_d(B_k)$. Thus

$$\begin{aligned}&\mathbf{P}(\Phi(B_1) = n_1, \ldots, \Phi(B_k) = n_k)\\ &\quad = \frac{\lambda^{n_1+\cdots+n_k}\big(v_d(B_1)\big)^{n_1} \cdot \ldots \cdot \big(v_d(B_k)\big)^{n_k}}{n_1! \cdot \ldots \cdot n_k!} \exp\left(-\sum_{i=1}^{k} \lambda v_d(B_i)\right). \qquad (2.14)\end{aligned}$$

From this formula the joint probabilities $\mathbf{P}(\Phi(B_1) = n_1, \ldots, \Phi(B_k) = n_k)$ can be evaluated for general (possibly overlapping) $B_1, \ldots, B_k$.

(b) *Stationarity and isotropy.* For a point process $\Phi = \{x_n\}$ to be stationary the translated process $\Phi_x = \{x_n + x\}$ must have the same distribution for all x in $\mathbb{R}^d$. A point process is isotropic if the same is true for all rotated processes $\boldsymbol{r}\Phi = \{\boldsymbol{r}x_n\}$ where $\boldsymbol{r}$ is a rotation about the origin. A process is motion-invariant if it possesses both of these properties. The homogeneous Poisson point process Φ is defined by properties (1) and (2) above and the specification of the intensity λ. These properties and the characteristic λ are clearly invariant under rotation and translation. Therefore the homogeneous Poisson point process Φ must be stationary and isotropic; that is to say, motion-invariant.

This may be verified directly by noting that the finite-dimensional distributions above remain the same whether one uses a homogeneous Poisson point process Φ or its translation Φ_x or its rotation $\boldsymbol{r}\Phi$.

More generally, the Poisson process has the following conservation property. Let $\mathbf{A}$ be a nonsingular linear mapping from $\mathbb{R}^d$ to $\mathbb{R}^d$. If Φ is a homogeneous Poisson process of intensity λ then $\mathbf{A}\Phi = \{\mathbf{A}x : x \in \Phi\}$ is also a homogeneous Poisson process and its intensity is $\lambda|\det(\mathbf{A}^{-1})|$, where $\det(\mathbf{A}^{-1})$ is the determinant of the inverse of $\mathbf{A}$.

This book uses the term 'homogeneous' Poisson process instead of 'stationary' Poisson process, in contrast to the usage in SKM95 and in many mathematical texts on stochastic geometry. The use of the term 'homogeneous' Poisson process follows long traditions and may make understanding for non-mathematicians easier. See also the discussion of the terms 'stationary' and 'homogeneous' in Section 4.1.1.

(c) *Void-probabilities.* The *void-probabilities* of a point process are the probabilities of there being no point of the process in given test sets B:

$$v_B = \mathbf{P}\big(\Phi(B) = 0\big).$$

In the case of Φ being a homogeneous Poisson point process,

$$v_B = \exp\big(-\lambda v_d(B)\big). \qquad (2.15)$$

(d) *Contact distribution functions.* The *contact distribution functions* are closely associated with the void-probabilities. If B is a Borel set with $v_d(B) > 0$ having the property that $r_1 \leq r_2$ implies $r_1 B \subset r_2 B$ then the contact distribution function $H_B(r)$ (with respect to B) is given by

$$H_B(r) = 1 - v_{rB} = 1 - \mathbf{P}(\Phi(rB) = 0) \qquad \text{for } r \geq 0. \qquad (2.16)$$

It is easy to see that $H_B(r)$ is really a distribution function: in particular $0 \cdot B = \{o\}$ and $\Phi(\{o\}) = 0$ and thus $H_B(0) = 0$; $r_1 B \subset r_2 B$ and thus $H_B(r_1) \leq H_B(r_2)$ holds if $r_1 \leq r_2$; '$\Phi(\infty B) = \infty$' and thus '$\mathbf{P}(\Phi(\infty B) = 0) = 0$' and $H_B(\infty) = 1$.

A particularly important case uses $B = B(o, 1)$ the unit ball, yielding the *spherical contact distribution* or *first contact distribution function*

$$H_s(r) = 1 - \mathbf{P}\big(\Phi(B(o, r)) = 0\big) = 1 - \exp(-\lambda b_d r^d) \qquad \text{for } r \geq 0. \tag{2.17}$$

This can be reinterpreted as the distribution function of the distance from o to the nearest point of Φ.

(e) *Conditioning and binomial point processes.* If Φ is a homogeneous Poisson point process then one can consider the restriction of Φ to a compact set W under the condition that $\Phi(W) = n$, that is, that in W there are exactly n points. This conditioning yields a finite point process, which is the binomial point process in W with n points.

This assertion can be proved by showing that the finite-dimensional distributions of the two processes coincide. In fact, it is sufficient to consider the void-probabilities. If K is a compact subset of W then the void-probability for K of the conditioned homogeneous Poisson point process is given by

$$\begin{aligned}\mathbf{P}\big(\Phi(K) = 0 | \Phi(W) = n\big) &= \mathbf{P}\big(\Phi(K) = 0, \Phi(W) = n\big)/\mathbf{P}\big(\Phi(W) = n\big) \\ &= \mathbf{P}\big(\Phi(K) = 0\big)\mathbf{P}\big(\Phi(W \setminus K) = n\big)/\mathbf{P}\big(\Phi(W) = n\big) \\ &= \frac{\big(v_d(W) - v_d(K)\big)^n}{v_d(W)^n}\end{aligned}$$

after substitution and cancellation. This formula coincides with Formula (2.6) for the void-probability of the binomial point process.

The general theory of simple point processes contains a theorem asserting equality of the distributions of point processes if their systems of void-probabilities are equal. However, in the case just considered it is straightforward to directly compute the finite-dimensional distributions of the conditioned process (in a manner similar to the above) and to check that these are equal to the finite-dimensional distributions of the binomial point process as given by Formula (2.7).

2.3.2 Characterisation of the homogeneous Poisson point process

The properties (1) and (2) on p. 41 of the homogeneous Poisson point process are not logically independent; see Kingman (1993) and Daley and Vere-Jones (2003). Rényi (1967) shows that (1) implies (2); the Poisson distribution property forces the point process to have the independently scattered property. Indeed this follows from the remark after paragraph (e) above. However, property (2) does *not* follow if property (1) holds only for the class of all convex subsets of $\mathbb{R}^d$. Moreover, Moran (1976) shows that it is not sufficient to assume the independence and Poisson distribution of counts of points in k disjoint convex sets for some fixed k. Thus if property (1) is weakened then property (2) may be necessary for characterisation of the homogeneous Poisson point process.

Another set of properties also characterises the Poisson process, and should be borne in mind as it frequently provides a *prima facie* case for assuming that an empirical point pattern

is a realisation of a homogeneous Poisson point process. This characterisation asserts that a process must be a homogeneous Poisson point process if the following three properties are satisfied:

(I) *Simplicity*. No two points coincide, so that the process is a simple point process. That is to say, there are no multiple points.

(II) *Stationarity*. This is as defined above.

(III) *Independent scattering*. This is property (2).

In stochastic geometry the property of simplicity generally holds by definition. Usually it is evident from the nature of the considered point pattern whether or not it is simple.

The stationarity assumption has greater significance. It implies that the statistical properties of the point process do not depend on the location of the observer. In some cases, some aspects of stationarity or homogeneity can be tested statistically but often it must be assumed provisionally in order to make progress in the analysis of the point pattern.

The most important assumption is that of independent scattering. In effect it asserts that there is no interaction between the points of the pattern. It can be tested statistically. In some cases, it is suggested by underlying biological or physical theories. Plausible or not, it frequently provides the starting point for statistical analysis, even if only as a null hypothesis.

2.3.3 Moments and moment measures

Just as moments such as mean and variance are important characteristics for random variables, so are the corresponding entities for the Poisson process and for general point processes. However, there is increasing complexity in moving from a random variable to a random point process. While random variables have moments that are real numbers, a point process has moments that are measures. The following describes these measures for the specific case of the homogeneous Poisson point process. In this case there are explicit formulae of relatively simple forms. Section 4.3 discusses moment measures for general point processes.

Consider the homogeneous Poisson point process Φ, regarded as a random measure on the Borel sets in $\mathbb{R}^d$. If B is a Borel set then $\Phi(B)$ is a random variable with first moment or mean

$$\Lambda(B) = \mathbf{E}\big(\Phi(B)\big). \tag{2.18}$$

If B_1 and B_2 are two Borel sets then $\Phi(B_1)$ and $\Phi(B_2)$ are two random variables with a noncentred covariance

$$\mu^{(2)}(B_1 \times B_2) = \mathbf{E}\big(\Phi(B_1)\Phi(B_2)\big). \tag{2.19}$$

Both Λ and $\mu^{(2)}$ are moments of random variables but depend on Borel sets (B) or products of Borel sets ($B_1 \times B_2$). The nature of this dependence is σ-additive (this follows from Φ being a random measure) and so they can both be represented as measures.

The first-order quantity Λ has a straightforward form. By property (1) the random variable $\Phi(B)$ is Poisson of mean $\lambda \cdot \nu_d(B)$ and so

$$\Lambda(B) = \mathbf{E}\big(\Phi(B)\big) = \lambda \nu_d(B) \qquad \text{for all Borel sets } B. \tag{2.20}$$

Thus in the case of the homogeneous Poisson point process the *first moment measure* or *intensity measure* Λ is a constant multiple of Lebesgue measure and the multiple is given by the intensity of the Poisson process.

The second-order quantity $\mu^{(2)}(B_1 \times B_2)$ given by (2.19) is evaluated by using both properties (1) and (2). To evaluate it one notes that both B_1 and B_2 can be decomposed into disjoint unions

$$B_1 = (B_1 \cap B_2) \cup (B_1 \setminus B_2), \qquad B_2 = (B_1 \cap B_2) \cup (B_2 \setminus B_1).$$

Using property (2), and the fact that Φ is a random measure, one can establish

$$\begin{aligned}
\mu^{(2)}(B_1 \times B_2) &= \mathbf{E}\big(\Phi(B_1)\Phi(B_2)\big) \\
&= \mathbf{E}\big(\Phi(B_1 \setminus B_2)\big)\mathbf{E}\big(\Phi(B_2 \setminus B_1)\big) + \mathbf{E}\big(\Phi(B_1 \cap B_2)\big)\mathbf{E}\big(\Phi(B_2 \setminus B_1)\big) \\
&\quad + \mathbf{E}\big(\Phi(B_1 \setminus B_2)\big)\mathbf{E}\big(\Phi(B_1 \cap B_2)\big) + \mathbf{E}\Big(\big(\Phi(B_1 \cap B_2)\big)^2\Big) \\
&= \mathbf{E}\big(\Phi(B_1)\big)\mathbf{E}\big(\Phi(B_2)\big) + \mathbf{E}\Big(\big(\Phi(B_1 \cap B_2)\big)^2\Big) - \Big(\mathbf{E}\big(\Phi(B_1 \cap B_2)\big)\Big)^2.
\end{aligned}$$

Property (1) shows that $\Phi(B_1 \cap B_2)$ is Poisson of mean and variance $\Lambda(B_1 \cap B_2)$. So with the aid of (2.20) the final formula can be derived:

$$\begin{aligned}
\mu^{(2)}(B_1 \times B_2) &= \Lambda(B_1)\Lambda(B_2) + \Lambda(B_1 \cap B_2) \\
&= \lambda^2 v_d(B_1)v_d(B_2) + \lambda v_d(B_1 \cap B_2). \qquad (2.21)
\end{aligned}$$

Thus the *second moment measure* $\mu^{(2)}$, which is a measure on $\mathbb{R}^d \times \mathbb{R}^d$, can be expressed in terms of λ and Lebesgue measure.

Variances and covariances can be calculated directly from the second moment measure. The relevant formulae are

$$\mathbf{var}\big(\Phi(B)\big) = \mu^{(2)}(B \times B) - \big(\Lambda(B)\big)^2 \qquad (2.22)$$

and

$$\mathbf{cov}\big(\Phi(B_1), \Phi(B_2)\big) = \mu^{(2)}(B_1 \times B_2) - \Lambda(B_1) \cdot \Lambda(B_2) \qquad (2.23)$$

for all Borel sets B, B_1 and B_2. These formulae follow immediately from the definitions of variance and covariance. By using (2.21) these simplify to

$$\mathbf{var}\big(\Phi(B)\big) = \lambda v_d(B) \qquad (2.24)$$

and

$$\mathbf{cov}\big(\Phi(B_1), \Phi(B_2)\big) = \lambda v_d(B_1 \cap B_2). \qquad (2.25)$$

As a random measure, Φ counts the number of points falling in the set which is the argument. Therefore $\mu^{(2)}(B_1 \times B_2)$ can also be expressed as the expectation of a sum:

$$\begin{aligned}\mu^{(2)}(B_1 \times B_2) &= \mathbf{E}\big(\#\{(x_1, x_2) : x_1 \in \Phi \cap B_1, x_2 \in \Phi \cap B_2\}\big) \\ &= \mathbf{E}\left(\sum_{x_1,\, x_2 \in \Phi} \mathbf{1}_{B_1}(x_1)\mathbf{1}_{B_2}(x_2)\right), \end{aligned} \tag{2.26}$$

where Φ, as noted before, is also used to denote the random set of the point process. The two terms in (2.21) correspond to the dissection of this sum into the sum over distinct pairs of points $x_1, x_2 \in \Phi$ and the sum over equal points $x_1 = x_2 \in \Phi$.

For some purposes it is convenient to subtract out the second of these terms. The result is the *second-order factorial moment measure* $\alpha^{(2)}$:

$$\begin{aligned}\alpha^{(2)}(B_1 \times B_2) &= \mathbf{E}\big(\#\{(x_1, x_2) : x_1 \in \Phi \cap B_1, x_2 \in \Phi \cap B_2, x_1 \neq x_2\}\big) \\ &= \mathbf{E}\left(\sum_{x_1,\, x_2 \in \Phi}^{\neq} \mathbf{1}_{B_1}(x_1)\mathbf{1}_{B_2}(x_2)\right). \end{aligned} \tag{2.27}$$

Here $\sum^{\neq}$ stands for summation over all pairs (x_1, x_2) such that $x_1 \neq x_2$.

The difference between $\mu^{(2)}(B_1 \times B_2)$ and $\alpha^{(2)}(B_1 \times B_2)$ lies in the expectation of the sum

$$\sum_{\substack{x_1,\, x_2 \in \Phi \\ x_1 = x_2}} \mathbf{1}_{B_1}(x_1)\mathbf{1}_{B_2}(x_2) = \sum_{x \in \Phi} \mathbf{1}_{B_1}(x)\mathbf{1}_{B_2}(x) = \sum_{x \in \Phi} \mathbf{1}_{B_1 \cap B_2}(x).$$

Hence

$$\mu^{(2)}(B_1 \times B_2) = \alpha^{(2)}(B_1 \times B_2) + \Lambda(B_1 \cap B_2). \tag{2.28}$$

In the case of the homogeneous Poisson point process, the following elegant formula is obtained:

$$\alpha^{(2)}(B_1 \times B_2) = \lambda^2 \nu_d(B_1)\nu_d(B_2) = \Lambda(B_1)\Lambda(B_2). \tag{2.29}$$

So in this case the second-order factorial moment measure $\alpha^{(2)}$ is simply a constant multiple of Lebesgue measure on $\mathbb{R}^d \times \mathbb{R}^d$.

The density of $\alpha^{(2)}$ with respect to Lebesgue measure is known as the *second-order product density* $\varrho^{(2)}$ and the above shows that for a homogeneous Poisson process

$$\varrho^{(2)}(x_1, x_2) \equiv \lambda^2 \qquad \text{for } x_1, x_2 \in \mathbb{R}^d, \tag{2.30}$$

and normalising $\varrho^{(2)}(x_1, x_2)$ by λ^2 gives the *pair correlation function* $g(x_1, x_2)$:

$$g(x_1, x_2) = \varrho^{(2)}(x_1, x_2)/\lambda^2. \tag{2.31}$$

For a stationary and isotropic process both $\varrho^{(2)}(x_1, x_2)$ and $g(x_1, x_2)$ depend solely on the distance $\|x_1 - x_2\| = r$ and in this case it is more convenient to write

$$\varrho^{(2)}(r) = \varrho^{(2)}(x_1, x_2) \quad \text{and} \quad g(r) = g(x_1, x_2).$$

For a homogeneous Poisson process,

$$g(r) \equiv 1. \tag{2.32}$$

Note that $\mu^{(2)}$ does not have a density with respect to Lebesgue measure. This is the principal motivation for using $\alpha^{(2)}$. The following holds

$$\mathbf{E}\big(\Phi(B_1)\Phi(B_2)\big) = \int_{B_2}\int_{B_1} \varrho^{(2)}(x_1, x_2)\mathrm{d}x_1\mathrm{d}x_2.$$

The second-order product density has an infinitesimal interpretation. If B_1 and B_2 are two infinitesimally small disjoint Borel sets of volumes $\mathrm{d}x_1$ and $\mathrm{d}x_2$, respectively, and if $x_1 \in B_1$ and $x_2 \in B_2$ then

$$\varrho^{(2)}(x_1, x_2)\mathrm{d}x_1\mathrm{d}x_2 = \lambda^2\mathrm{d}x_1\mathrm{d}x_2 \tag{2.33}$$

is the probability that Φ places a point in each of B_1 and B_2.

Note that the 'first-order product density' has already been defined: by analogy with the above it is simply the intensity λ.

Just as higher moments can be defined for random variables so higher moment measures can be defined for point processes. The *n^{th} moment measure* $\mu^{(n)}$ is a measure on $\mathbb{R}^{nd}$ defined by

$$\begin{aligned}\mu^{(n)}(B_1 \times \cdots \times B_n) &= \mathbf{E}\big(\Phi(B_1) \cdot \ldots \cdot \Phi(B_n)\big) \\ &= \mathbf{E}\left(\sum_{x_1, \ldots, x_n \in \Phi} \mathbf{1}_{B_1}(x_1) \cdot \ldots \cdot \mathbf{1}_{B_n}(x_n)\right)\end{aligned} \tag{2.34}$$

for Borel sets $B_1, \ldots, B_n$. The *n^{th}-order factorial moment measure* $\alpha^{(n)}$ is a measure on $\mathbb{R}^{nd}$ given by

$$\alpha^{(n)}(B_1 \times \cdots \times B_n) = \mathbf{E}\left(\sum_{x_1, \ldots, x_n \in \Phi}^{\neq} \mathbf{1}_{B_1}(x_1) \cdot \ldots \cdot \mathbf{1}_{B_n}(x_n)\right) \tag{2.35}$$

where $\sum^{\neq}$ stands for summation over n-tuples $(x_1, \ldots, x_n)$ of *distinct* points. The *n^{th}-order product density* $\varrho^{(n)}$ is the density of $\alpha^{(n)}$ with respect to Lebesgue measure on $\mathbb{R}^{nd}$. In the case of homogeneous Poisson point process these product densities are given by the simple formula

$$\varrho^{(n)}(x_1, \ldots, x_n) \equiv \lambda^n \tag{2.36}$$

and $\alpha^{(n)}$ has the form

$$\alpha^{(n)}(B_1 \times \cdots \times B_n) = \lambda^n \nu_d(B_1) \cdot \ldots \cdot \nu_d(B_n). \tag{2.37}$$

For $n = 3$ the third moment measure is given by

$$\begin{aligned}\mu^{(3)}(B_1 \times B_2 \times B_3) = {} & \lambda^3 \nu_d(B_1)\nu_d(B_2)\nu_d(B_3) \\ & + \lambda^2 \nu_d(B_1)\nu_d(B_2 \cap B_3) + \lambda^2 \nu_d(B_2)\nu_d(B_1 \cap B_3) \\ & + \lambda^2 \nu_d(B_3)\nu_d(B_1 \cap B_2) + \lambda \nu_d(B_1 \cap B_2 \cap B_3)\end{aligned}$$

for Borel sets B_1, B_2 and B_3. For larger n the relationships between $\alpha^{(n)}$ and $\mu^{(n)}$ are quite complicated.

There is an infinitesimal interpretation for $\varrho^{(n)}(x_1, \ldots, x_n)$ which follows that of $\varrho^{(2)}(x_1, x_2)$.

2.3.4 The Palm distribution of a homogeneous Poisson point process

Many problems in point process theory require the consideration of an arbitrary 'typical' point of a point process Φ. It can be viewed informally as the result of a random selection procedure in which every point of the process has the same chance to be selected. For example, the nearest-neighbour distance distribution function $D(r)$ describes the distribution of the random distance from a typical point x of Φ to the nearest other point in the process, that is, in $\Phi \setminus \{x\}$.

The idea of a typical point is heuristically clear but needs to be made mathematically precise. Points sampled by some systematic method (such as by choosing the nearest to a given origin) are *not* typical just because they *have* been so sampled. Palm distribution theory makes precise the notion of a typical point. In intuitive terms the Palm distribution probabilities are the conditional probabilities of point process events given that a point (the typical point) is observed at a specific location.

Two heuristic approaches to the Palm distribution are discussed in this section, and calculations are made for the case of a homogeneous Poisson point process. One approach is essentially local and involves conditioning the distribution of the point process on there being a point at a given position x. The other approach, which is described on p. 50, is more global, and applies only to stationary point processes. It is related to statistical methods for estimation of Palm probabilities.

(a) Local approach

To discuss the local approach some notation is introduced here. It is necessary to discuss probabilities which involve conditioning of Φ on the event that Φ contains a point at x. If Y is some configuration set, describing some point process property (for example, the property that $\Phi(B) = n$ for some fixed Borel set B) then let

$$\mathbf{P}(\Phi \text{ has property } Y \parallel x) = \mathbf{P}(\Phi \text{ has property } Y \mid x \in \Phi).$$

This conditional probability will be understood as the *Palm distribution of* Φ and the point at x is the *typical point*. In accordance with notation introduced in Chapter 4, the event 'Φ has property Y' is written as '$\Phi \in Y$', interpreting Y as a configuration set, a member of a σ-algebra of sets of point-process realisations. By stationarity

$$\mathbf{P}(\Phi \in Y \parallel x) = \mathbf{P}(\Phi_{-x} \in Y \parallel o),$$

in which Φ_{-x} denotes the shifted point process $\{x_1 - x,\ x_2 - x,\ \ldots\}$, where Φ is the set $\{x_1, x_2,\ \ldots\}$.

In this notation the nearest-neighbour distance distribution function $D(r)$ for a homogeneous point process is given by

$$D(r) = \mathbf{P}\big(\Phi(B(o,r)) > 1 \parallel o\big) = 1 - \mathbf{P}\big(\Phi(B(o,r)) = 1 \parallel o\big),$$

since the distance from o (at which the typical point stands) to its nearest neighbour is not greater than r if the ball of radius r centred at o contains points of Φ.

The conditioning event $\{o \in \Phi\}$ always has probability zero if Φ is a homogeneous point process. This means that the conditioning must be carefully defined. Chapter 4 uses the Radon–Nikodym theorem, which frequently illuminates the meaning of conditional probability constructions in general probability theory. However, in the case of homogeneous Poisson point process the Palm probabilities can be calculated by a limit procedure, which is demonstrated below.

Calculation of the nearest-neighbour distance distribution function for the homogeneous Poisson point process

First consider a similar conditional distribution function, in which the conditioning is on a point being *near* to o. The conditional probability

$$D_\varepsilon(r) = 1 - \mathbf{P}\Big(\Phi\big(B(o,r) \setminus B(o,\varepsilon)\big) = 0 \mid \Phi\big(B(o,\varepsilon)\big) = 1\Big)$$

is well-defined for positive ε smaller than r, since the probability that the small ball $B(o,\varepsilon)$ contains a point, $\mathbf{P}\big(\Phi(B(o,\varepsilon)) = 1\big) = \lambda b_d \varepsilon^d \exp(-\lambda b_d \varepsilon^d)$ [where (2.11) is used], is positive. Using the definition of conditional probability

$$\mathbf{P}(A \mid B) = \mathbf{P}(A \cap B)/\mathbf{P}(B)$$

and property (2) of the homogeneous Poisson point process, one obtains

$$\begin{aligned} D_\varepsilon(r) &= 1 - \frac{\mathbf{P}\big(\Phi\big(B(o,r) \setminus B(o,\varepsilon)\big) = 0\big)\mathbf{P}\big(\Phi\big(B(o,\varepsilon)\big) = 1\big)}{\mathbf{P}\big(\Phi\big(B(o,\varepsilon)\big) = 1\big)} \\ &= 1 - \mathbf{P}\big(\Phi\big(B(o,r) \setminus B(o,\varepsilon)\big) = 0\big) \\ &= 1 - \exp\big(-\lambda\big(v_d(B(o,r)) - v_d(B(o,\varepsilon))\big)\big). \end{aligned}$$

It is reasonable to think of the nearest-neighbour distance distribution function as the limit of the above as $\varepsilon \to 0$. Setting $D(r) = \lim_{\varepsilon \to 0} D_\varepsilon(r)$ yields the result

$$D(r) = 1 - \exp\big(-\lambda v_d(B(o,r))\big) = 1 - \exp(-\lambda b_d r^d) \qquad \text{for } r \geq 0. \tag{2.38}$$

This result can be established rigorously with the ideas of Section 4.4.4 using Palm distributions.

The right-hand sides of (2.38) and (2.17) are equal. This shows that for a homogeneous Poisson process the spherical contact distribution function and nearest-neighbour distance distribution are equal,

$$D(r) = H_s(r) \qquad \text{for } r \geq 0. \tag{2.39}$$

The mean and variance of the random variable corresponding to the distribution function $D(r)$ are given by

$$\mu_D = \begin{cases} \dfrac{1}{2\sqrt{\lambda}} & \text{for } d = 2, \\ \left(\dfrac{3}{4\pi\lambda}\right)^{1/3} \Gamma\left(\dfrac{4}{3}\right) \approx \dfrac{0.554}{\lambda^{1/3}} & \text{for } d = 3, \end{cases} \tag{2.40}$$

and

$$\sigma_D^2 = \begin{cases} \dfrac{1}{\lambda}\left(\dfrac{1}{\pi} - \dfrac{1}{4}\right) \approx \dfrac{0.0683}{\lambda} & \text{for } d = 2, \\ \left(\dfrac{3}{4\pi\lambda}\right)^{2/3} \left(\Gamma\left(\dfrac{5}{3}\right) - \Gamma^2\left(\dfrac{4}{3}\right)\right) \approx \dfrac{0.0405}{\lambda^{2/3}} & \text{for } d = 3. \end{cases} \tag{2.41}$$

Illian *et al.* (2008, p. 76) presented formulae for the distributions of the distances to the second-nearest, third-nearest, ... neighbours.

Arguing as for (2.38) leads to the result

$$\mathbf{P}\big(\Phi(B) = n \parallel o\big) = \mathbf{P}\big(\Phi(B) = n\big)$$

for $n = 0, 1, 2, \ldots$ and for any compact set B not containing o. This suggests that the Palm distribution for the homogeneous Poisson point process is essentially the distribution of the original process together with a point adjoined at o. This is the content of the *Slivnyak–Mecke theorem*:

$$\mathbf{P}(\Phi \text{ has property } Y \parallel o) = \mathbf{P}(\Phi \cup \{o\} \text{ has property } Y). \tag{2.42}$$

In the σ-algebra language of Chapter 4,

$$\mathbf{P}(\Phi \in Y \parallel o) = \mathbf{P}(\Phi \cup \{o\} \in Y). \tag{2.43}$$

A proof of the Slivnyak–Mecke theorem is given in Section 4.4.4.

(b) Global approach

The second, more global, approach to Palm probabilities examines all points of Φ that fall within some fixed bounded Borel test set B with $\nu_d(B) > 0$. Denote by $N(\Phi, B, Y)$ the number of those points x in B such that Φ_{-x} has property Y. Then one can put

$$\begin{aligned} \mathbf{P}(\Phi \text{ has property } Y \parallel o) &= \mathbf{P}(\Phi \in Y \parallel o) \\ &= \mathbf{E}\big(N(\Phi, B, Y)\big)/\lambda\nu_d(B). \end{aligned} \tag{2.44}$$

Since $\lambda\nu_d(B)$ is the expected number of points falling in B, this equation exhibits the Palm probability for the property Y as the fraction of points x expected to fall in B such that Φ_{-x} has the property Y. By the stationarity of Φ this definition does not depend on B.

It can be shown that the two approaches yield the same Palm distribution.

So as $D(r)$, another important distributional point process characteristic, the *reduced second moment function* $K(r)$, is also related to Palm distribution theory. This will be examined at greater length in Section 4.5. A brief indication of the nature of the relationship follows from the definition

$$\lambda K(r) = \mathbf{E}\big(\Phi(B(o,r)) \parallel o\big) - 1, \tag{2.45}$$

which means that $\lambda K(r)$ is the mean number of further points in a ball of radius r and centred at the typical point, which itself is not counted in the mean. The Slivnyak–Mecke theorem (2.42) yields

$$\lambda K(r) = \mathbf{E}\big(\Phi(B(o,r)) + 1\big) - 1 = \mathbf{E}\big(\Phi(B(o,r))\big). \tag{2.46}$$

Thus for the homogeneous Poisson process

$$K(r) = b_d r^d \qquad \text{for } r \geq 0. \tag{2.47}$$

2.4 The inhomogeneous and general Poisson point process

The homogeneous Poisson point process has a constant point density λ and an intensity measure which is proportional to Lebesgue measure. The mean number of its points per unit area does not vary over space. However, many point patterns arising in applications exhibit fluctuations that make such a lack of spatial variation implausible. For example, they may display an apparent trend of increase in a certain direction, as can be observed in the pattern of locations of trees in some forests. Suppose that the forest is in an area for which height above sea level increases, or soil becomes progressively drier, in a given direction. Then the mean number of trees per unit area may correspondingly decrease. Figures 2.3 and 2.4 on p. 52 show point patterns which clearly exhibit such directional dependence.

In such situations, it makes sense to consider a point process model with an *intensity function* $\lambda(x)$. It has an appealing and intuitive infinitesimal interpretation; $\lambda(x)\mathrm{d}x$ is the infinitesimal probability that there is a point of Φ in a region of infinitesimal volume $\mathrm{d}x$ situated at x. The corresponding intensity measure Λ is given by

$$\Lambda(B) = \int_B \lambda(x)\mathrm{d}x \qquad \text{for Borel sets } B. \tag{2.48}$$

Still a further generalisation is possible. Assume that there is a diffuse (without atoms) Radon measure Λ on $\mathbb{R}^d$. It can be used to construct a *general Poisson point process* Φ *with intensity measure* Λ as a point process possessing the two following properties:

(1′) *Poisson distribution of point counts*: the number of points in a bounded Borel set B has a Poisson distribution with mean $\Lambda(B)$

$$\mathbf{P}(\Phi(B) = n) = \frac{\Lambda(B)^n}{n!} \exp\big(-\Lambda(B)\big) \qquad \text{for } n = 0, 1, 2, \ldots. \tag{2.49}$$

(2′) *Independent scattering*: the numbers of points in k disjoint Borel sets form k independent random variables.

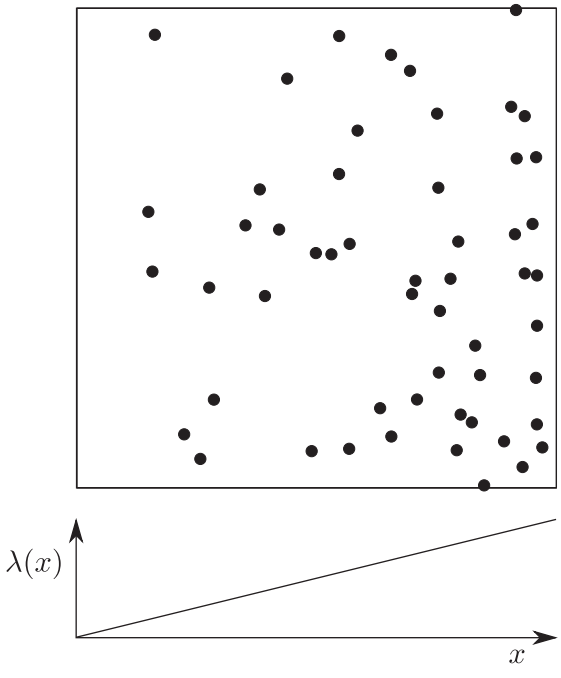

Figure 2.3 A simulated sample of a Poisson process which has a linearly increasing intensity function $\lambda(x)$. The pattern has been generated from that of Figure 2.1 by the thinning procedure described in Section 2.5.2.

It is clear from property (1′) that such a process Φ is *not* stationary in general. Without the assumption that Λ is diffuse, the process could have multiple points at the positions of the atoms.

The measure Λ is called the *intensity measure* of Φ and corresponds to the intensity measure previously introduced for the homogeneous Poisson point process. A point process

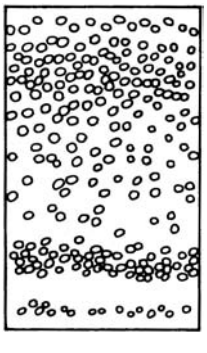

Figure 2.4 Cell nuclei developing in the pyramidal layer of the cerebral cortex of an embryo. The diagram is taken from von Economo and Koskinas (1925), see also Stephan (1975).

possessing an intensity function is called *inhomogeneous Poisson process*. For example, such a function does not exist when the points are randomly scattered on a deterministic system of lines. (If the lines are random, one is confronted with a Cox process; see Section 5.2.)

As for the homogeneous Poisson point process, here one can also define *moment measures*

$$\begin{aligned}\mu^{(n)}(B_1 \times \cdots \times B_n) &= \mathbf{E}\big(\Phi(B_1) \cdot \ldots \cdot \Phi(B_n)\big) \\ &= \mathbf{E}\left(\sum_{x_1, \ldots, x_k \in \Phi} \mathbf{1}_{B_1}(x_1) \cdot \ldots \cdot \mathbf{1}_{B_k}(x_k)\right)\end{aligned}$$

and *factorial moment measures*

$$\alpha^{(n)}(B_1 \times \cdots \times B_n) = \mathbf{E}\left(\sum_{x_1, \ldots, x_k \in \Phi}^{\neq} \mathbf{1}_{B_1}(x_1) \cdot \ldots \cdot \mathbf{1}_{B_k}(x_k)\right).$$

The factorial moment measures are given by

$$\alpha^{(n)}(B_1 \times \cdots \times B_n) = \Lambda(B_1) \cdot \ldots \cdot \Lambda(B_n). \tag{2.50}$$

Consequently in the case of an inhomogeneous Poisson process with intensity function $\lambda(x)$ the *product densities* $\varrho^{(n)}$ are given by

$$\varrho^{(n)}(x_1, \ldots, x_n) = \lambda(x_1) \cdot \ldots \cdot \lambda(x_n). \tag{2.51}$$

Further generalisations of the Poisson process are considered in Sections 5.2 and 5.3.

2.5 Simulation of Poisson point processes

2.5.1 Simulation of a homogeneous Poisson point process

The starting point for simulating a homogeneous Poisson point process is property (e) on p. 43: conditioning on the total number of points in a compact set produces a binomial point process. Thus the simulation of a homogeneous Poisson point process in a compact region W falls naturally into two stages. First the number of points in W is determined by simulating a Poisson random variable, and then the positions of the points in W are determined by simulating a binomial point process in W with that number of points. The second stage has already been covered in Section 2.2.3. There are various possibilities for the first stage, simulation of a Poisson random variable. Two of them are described now. Which of the two methods here is appropriate for simulating a Poisson random variable depends on its mean $\mu = \lambda v_d(W)$.

The usual method for small μ is to simulate a linear Poisson process, exploiting the fact that its inter-point distances are independent exponential random variables; see for example Kingman (1993, p. 39). Random variables e_i are generated to be independent and exponential of mean 1. This can be achieved by the transform method; if u_i is uniform on $[0, 1]$ then $e_i = -\ln(u_i)$ is as required. The desired Poisson random variable is the smallest n for which

$$\sum_{i=1}^{n+1} e_i > \mu. \tag{2.52}$$

Since addition of logarithms is equivalent to multiplication, the Poisson variable can be determined also as the smallest n satisfying

$$\prod_{i=1}^{n+1} u_i < \exp(-\mu). \tag{2.53}$$

An advantage of this method is its flexibility. It is an appropriate method to meet successive requirements to generate Poisson random variables of different (small) means.

In the case when μ is large then some form of acceptance/rejection technique should be used; see Gentle (2003, p. 188) and Kroese *et al.* (2011, pp. 100–1). Alternatively, the central limit theorem may simply be exploited. It states that for large μ, a Poisson random variable with mean μ approximately follows a Gaussian distribution with mean μ and variance μ. (This equality is because mean and variance coincide for a Poisson distribution.) Thus a Gaussian random number may be generated by well-known methods and then rounded to an integer.

In some cases, for example for a simulation of the typical Poisson-Voronoi polyhedron, it is useful to generate samples of a homogeneous Poisson process of intensity λ in a radial way, that is, as a sequence of points with increasing distance from the origin o. This method, called 'radial generation', is described in Quine and Watson (1984) for the d-dimensional case; here the formulae for $d = 2$ and $d = 3$ are given.

A point $x = (x_1, x_2)$ or $x = (x_1, x_2, x_3)$ is represented in polar coordinates by (r, θ) or (r, θ, θ') as

$$x_1 = r\cos\theta, \qquad x_2 = r\sin\theta,$$

or

$$x_1 = r\cos\theta\cos\theta', \qquad x_2 = r\sin\theta\cos\theta', \qquad x_3 = r\sin\theta'.$$

The simulation yields a sequence of points $\{(r_i, \theta_i)\}$ or $\{(r_i, \theta_i, \theta_i')\}$ with

$$r_i = \left((-\lambda b_d)^{-1} \sum_{j=1}^{i} \ln u_{ij}\right)^{1/d}$$

and

$$\theta_i = 2\pi u_i', \qquad \theta_i' = \sin^{-1}(1 - u_i''),$$

where u_{ij}, u_i' and u_i'' are independent uniform random numbers in $[0, 1]$. The sequence $\{r_i^d / b_d\}$ is a sample of a linear Poisson process with intensity λ.

2.5.2 Simulation of an inhomogeneous Poisson point process

For simulating a sample of an inhomogeneous Poisson point process with intensity function $\lambda(x)$ a thinning procedure can be used, as suggested by Lewis and Shedler (1979); see also Ogata (1981). Section 5.1 discusses thinning operations for point processes in a general setting.

The procedure, which assumes that the intensity function is bounded above by a number λ^*, falls into two stages. Firstly, a homogeneous Poisson point process of intensity λ^* is simulated as above. Secondly, the resulting point pattern is thinned by deleting each point x

independently of the others with probability $1 - \lambda(x)/\lambda^*$. If the points of the homogeneous Poisson point process pattern are $\{x_1, x_2, \ldots\}$ then this thinning can be performed with the aid of an independent sequence $u_1, u_2, \ldots$ of random numbers uniformly distributed over $[0, 1]$. The point x_k is deleted if $u_k > \lambda(x_k)/\lambda^*$. In the terminology of Section 5.1 the location-dependent thinning probability is $p(x) = \lambda(x)/\lambda^*$. Formula (5.2) ensures that the point pattern of undeleted points is indeed a sample of an inhomogeneous Poisson point process of intensity function $\lambda(x)$. The pattern in Figure 2.3 was generated in this manner.

2.6 Statistics for the homogeneous Poisson point process

2.6.1 Introduction

The discussion of the central rôle of the homogeneous Poisson point process (in Section 2.1) establishes the importance of statistical methods for this process, including the important problem of deciding whether or not a given point pattern is Poisson. This section is a brief survey of such methods. It considers mainly the case of *planar* point patterns, which is most studied in the literature: The methods presented carry over to point patterns on the line, in space, and on a sphere. In particular, there is special literature on the case of Poisson processes on the line; see, for example, Cox and Lewis (1966), Snyder (1975), Brillinger (1978) and Karr (1991). References for the planar case are Diggle (1983, 2003), Kutoyants (1998) and Illian *et al.* (2008).

2.6.2 Estimating the intensity

A fundamental statistical question for the homogeneous Poisson point process concerns the estimation of the intensity λ when the process is observed through a window W. A general unbiased estimator for the intensity λ of a homogeneous point process is given by

$$\hat{\lambda} = \frac{\Phi(W)}{\nu_d(W)}, \tag{2.54}$$

which quite naturally means that the observed point number in the window W is divided by the window's volume. This estimator is the most important also in the non-Poisson process case.

Distance methods, that is, methods employing inter-point distances or distances between points and chosen locations, are discussed in for example Diggle (1983, 2003) and Illian *et al.* (2008).

Confidence intervals for λ can be based on (2.54) since $\hat{\lambda} \cdot \nu_d(W) = \Phi(W)$ has a Poisson distribution. In the case of large $\Phi(W)$ Krebs (1999) and Armitage *et al.* (2002) give simple approximate $100(1 - \alpha)$ % confidence intervals for λ, employing the normal approximation and a continuity correction. For example,

$$\left(\frac{z_{\alpha/2}}{2} - \sqrt{\Phi(W)}\right)^2 \leq \lambda \nu_d(W) \leq \left(\frac{z_{\alpha/2}}{2} + \sqrt{\Phi(W) + 1}\right)^2. \tag{2.55}$$

Here the $z_{\alpha/2}$ are quantiles of the standard normal distribution:

$$z_{\alpha/2} = 1.645,\ 1.96,\ 2.576 \qquad \text{for } \alpha = 0.10,\ 0.05,\ 0.01, \text{ respectively.}$$

This confidence interval is useful in simple planning of the size of window required for a given accuracy of estimation. If δ is the desired width of the confidence interval and α is the required confidence level then

$$\delta \cdot \nu_d(W) \approx \left(\frac{z_{\alpha/2}}{2} + \sqrt{\lambda \nu_d(W) + 1}\right)^2 - \left(\frac{z_{\alpha/2}}{2} - \sqrt{\lambda \nu_d(W)}\right)^2,$$

from which follows the specification

$$\nu_d(W) \approx \frac{4 z_{\alpha/2}^2 \lambda}{\delta^2},$$

where λ must itself be estimated by a preliminary analysis or by using *a priori* information.

The next statistical questions to be considered concern the model assumptions – those of homogeneity and of the Poisson process hypothesis.

2.6.3 Testing the hypothesis of homogeneity

If the point pattern is a homogeneous Poisson point process then the quantity F given below has a probability distribution that is approximately an F-distribution of $2n_1 + 1$ and $2n_2 + 1$ degrees of freedom:

$$F = \frac{\nu_d(W_1)(2n_2 + 1)}{\nu_d(W_2)(2n_1 + 1)}. \tag{2.56}$$

Here n_1 and n_2 are the point numbers in two disjoint subregions W_1 and W_2 of the window W. It is assumed that the order of the indices '$_1$' and '$_2$' is chosen to make F greater than one. The homogeneity hypothesis is rejected at the significance level α if

$$F > F_{2n_1+1,\, 2n_2+1;\, \alpha/2}. \tag{2.57}$$

This test is strictly appropriate only if the subregions W_1 and W_2 are chosen *a priori* to investigate specific possibilities of trend. For example, such a trend might relate the intensity of a Poisson process of tree locations to some measure of soil dryness, which is *a priori* known. It should also be noted that the significance calculations above depend on the Poisson nature of the point process and that homogeneity has many aspects, which are difficult to test.

There are other statistics for comparing the point counts in two subregions; see Chiu (2010) for a comparison. Chiu and Wang (2009) discussed the statistics for the case of more than two subregions. For model-free statistics for testing the hypothesis of a constant intensity, see Guan (2008) and Chiu and Liu (2013).

2.6.4 Testing the Poisson process hypothesis

There is a vast range of tests to which one can subject the hypothesis that a given point pattern is Poisson. In the statistical literature one speaks about the CSR hypothesis, with CSR stands for complete spatial randomness. Naturally, there can be no single criterion to give the 'best' test to apply; see the discussion in Diggle (1983) and Illian *et al.* (2008). Which test is most appropriate depends on the nature of the alternative hypothesis envisaged and on limitations imposed by measurement methods. Here only two cases are considered.

The L-test

When the point pattern has been mapped exhaustively, so that measurements of the locations of all points in the pattern are available, then it becomes worthwhile to use Ripley's K-function for a goodness-of-fit test. Formula (2.47) gives this function

$$K(r) = b_d r^d \qquad \text{for } r \geq 0.$$

If an estimate of $K(r)$ deviates considerably from this simple form then the Poisson hypothesis is thrown into doubt. Statistical experience since Ripley (1977) suggests that tests using this idea have good statistical properties (are rather powerful), are the 'best' tests available; see also Myles *et al.* (1995) and Illian *et al.* (2008). Moreover, comparison of the theoretical form with an estimate can do much to reveal the structure of the point pattern, and the comparison can be refined and improved by using estimates of the pair correlation function. Section 4.7.4 contains a brief discussion of estimators for $\lambda^2 K(r)$ for general point processes.

In practice, usually the L-function is used rather than $K(r)$, where

$$L(r) = \sqrt[d]{\frac{K(r)}{b_d}} \qquad \text{for } r \geq 0. \tag{2.58}$$

The root transform stabilises estimation variance, and $L(r)$ has the simple theoretical linear form

$$L(r) = r \qquad \text{for } r \geq 0. \tag{2.59}$$

It is then natural to consider the *deviation test* statistics

$$\tau = \max_{r \leq r_{\max}} |\hat{L}(r) - r| \tag{2.60}$$

with

$$\hat{L}(r) = \sqrt[d]{\frac{\hat{K}(r)}{b_d}}. \tag{2.61}$$

If τ is large, then the Poisson hypothesis has to be rejected. (The alternative hypothesis is 'not Poisson point process' without further specification.) If $\hat{K}(r)$ is obtained via Ripley's estimator,

$$\hat{K}(r) = \frac{\hat{\kappa}_{\text{iso}}(r)}{\hat{\lambda}^2}, \tag{2.62}$$

where $\hat{\kappa}_{\text{iso}}(r)$ is defined in Formula (4.111) in Section 4.7.4, then in the planar case a critical value of τ for the significance level $\alpha = 0.05$ is

$$\tau_{0.05} = \frac{1.45\sqrt{a}}{n}, \tag{2.63}$$

where a is the window area and n the number of points observed (Ripley, 1988, p. 46). This value was obtained by simulations and can be used if $nlr_{\max}^3/a^2$ (where l is the boundary length of W) is small for a 'wide range of' $r_{\max}$ (and $r_{\max} = 1.25/\sqrt{n}$ was recommended for W being a unit square). Many statisticians believe that this $\tau_{0.05}$ can be used also in the case of other estimators of the K-function.

To obtain the value of $\tau_{0.01}$, the factor 1.45 has to be replaced by 1.75; see Chiu (2007).

If Ripley's approximation appears to be inappropriate then simulation can be employed to provide Monte Carlo deviation tests. In such tests a suitable value of $r_{\max}$ is chosen, for example the half length of the diagonal for a rectangular window. Then k independent binomial process samples with n points are simulated in the window. For each of the samples the L-function is estimated and the value

$$\tau_i = \max_{r \le r_{\max}} |\hat{L}_i(r) - r| \qquad \text{for } i = 1, 2, \ldots, k,$$

is determined. These values plus the corresponding value τ_{emp} for the empirical data are sorted in ascending order. If the rank of τ_{emp} in the series is too large, the CSR hypothesis is rejected. If, for example, $k = 999$, then ranks larger than 950 (respectively 990) lead to rejection for $\alpha = 0.05$ (respectively $\alpha = 0.01$). The p-value of the test is (approximately) given by

$$\hat{p} = \frac{s+1}{k+1}, \tag{2.64}$$

where s is the number of τ_i larger than or equal to τ_{emp}.

Instead of using the L^∞-norm to define τ_{emp} and τ_i, also the L^2-norm can be used, leading to

$$\tau = \int_0^{r_{\max}} \left(\hat{L}(r) - r\right)^2 \mathrm{d}r \tag{2.65}$$

and the corresponding τ_i from simulated patterns. It depends on *a priori* knowledge which τ-definition is used; see Example 2.3.

In applications, the choice of $r_{\max}$ may be crucial. For a pattern of n points observed in a unit square, Ripley (1979) suggest $r_{\max} = 1.25/\sqrt{n}$ and Diggle (2003, p. 87) recommend that $r_{\max}$ should not be bigger than 0.25. Ho and Chiu (2006) show empirically that if $\hat{\lambda}^2$ in Formula (2.62) is some distance-adapted intensity estimator (Stoyan and Stoyan, 2000), the power of the L-test will be robust against different choices of $r_{\max}$. See also Ho and Chiu (2009) for another suggestion of modification.

Quadrat count methods

The case when the point pattern is sampled by means of counts of points falling in several subregions of the window W is intermediate between the case where one can usefully estimate the reduced second moment function and the case where one is restricted to distance methods. The sampling window W is divided into several subregions (often *quadrats*, squares or cubes) of equal area or volume $\nu_d(Q)$. Under the hypothesis of a homogeneous Poisson point process the number of points in each quadrat is Poisson of mean $\lambda \cdot \nu_d(Q)$ and counts in disjoint quadrats are independent. Statistical tests can be based on these distributional properties.

The simplest test is the index-of-dispersion test; the Greig-Smith method below is a refined variant which investigates the extent to which the independence properties are valid.

The *index-of-dispersion I* is defined by

$$I = \frac{(k-1)s^2}{\bar{x}},$$

where k is the number of quadrats, $\bar{x}$ is the mean number of points per quadrat, and s^2 is the sample variance of the number of points per quadrat. This test statistic is exactly that of

a χ^2 goodness-of-fit test of the hypothesis that the n points are independent and uniformly distributed in W. Consequently, the index I follows approximately a χ^2-distribution of $k-1$ degrees of freedom provided that $k > 6$ and $\lambda v_d(Q) > 1$ (Diggle, 1983, 2003). So if I exceeds $\chi^2_{k-1;\alpha}$ or is smaller than $\chi^2_{k-1;1-\alpha}$ then the test rejects the homogeneous Poisson point process hypothesis at the significance level α. In the first case the alternative hypothesis is that the variability in the process is greater than that for the Poisson process, typically clustering. Analogously, in the second case the alternative hypothesis is that there is some regularity in the point pattern as the variability is smaller.

Of course, quadrat count tests have the disadvantage that quite different processes may yet have quadrat counts of similar joint distribution.

The *Greig-Smith method* (see Greig-Smith, 1952, 1983) uses not only the quadrat counts on their own but also counts grouped by neighbouring quadrats. This enables detection of clustering at different scales. Example 2.2 below illustrates this.

It is assumed that the number of quadrats is $n = 2^q$ for some integer q. The quantities $s_1, s_2, \ldots$ are calculated:

$$s_1^2 = \sum_j \begin{pmatrix} \text{count in the} \\ j^{\text{th}} \text{ quadrat} \end{pmatrix}^2 - \frac{1}{2} \sum_k \begin{pmatrix} \text{count in the } k^{\text{th}} \\ \text{pair of quadrats} \end{pmatrix}^2,$$

$$s_2^2 = \sum_j \begin{pmatrix} \text{count in the } j^{\text{th}} \\ \text{pair of quadrats} \end{pmatrix}^2 - \frac{1}{2} \sum_k \begin{pmatrix} \text{count in the } k^{\text{th}} \\ \text{foursome of quadrats} \end{pmatrix}^2,$$

and so on.

In the case $q = 6$ the quadrats can be numbered in chessboard notation, since $2^q = 64$. The pairs of quadrats are then the pairs $(a1, a2)$, $(a3, a4)$, $\ldots$, $(b1, b2)$, $(b3, b4)$, $\ldots$. The foursomes of quadrats are then $(a1, a2, b1, b2)$, $(a3, a4, b3, b4)$, $\ldots$.

The quantities $s_j^2/2^{q-j}$ are unbiased estimators for the variance of the random number of points in a quadrat. Thus the index-of-dispersion formula can be generalised, calculation made of the indices

$$I_j = s_j^2/\bar{x}$$

and the χ^2-test used on each index. Here $\bar{x}$ is as in the index-of-dispersion test. The degrees of freedom for the χ^2-statistics $I_1, I_2, \ldots$ are $2^{q-1}, 2^{q-2}, \ldots$.

The results of the tests should be interpreted as follows. The hypothesis of a homogeneous Poisson point process can be accepted if all the indices I_j lie between two-sided critical values of the appropriate χ^2-statistics. Otherwise, the values of the indices indicate the nature of the deviation from the hypothesis. Thus if $I_3/2^{q-3}$ is significantly greater than one then there is strong evidence of clustering on the scale of groups of four quadrats, etc.

Example 2.1. *Longleaf pine data*

Figure 2.5 on p. 60 shows the locations of 584 longleaf pine trees in a 200 m $\times$ 200 m window of an old-growth forest in Thomas County, Georgia in 1979.

The empirical L-function, given in Figure 2.6(a), shows clearly that the mean number of pines in a disc centred at an arbitrarily chosen one is larger than expected in a homogeneous Poisson process. Such an empirical L-function is typical for a clustered pattern.

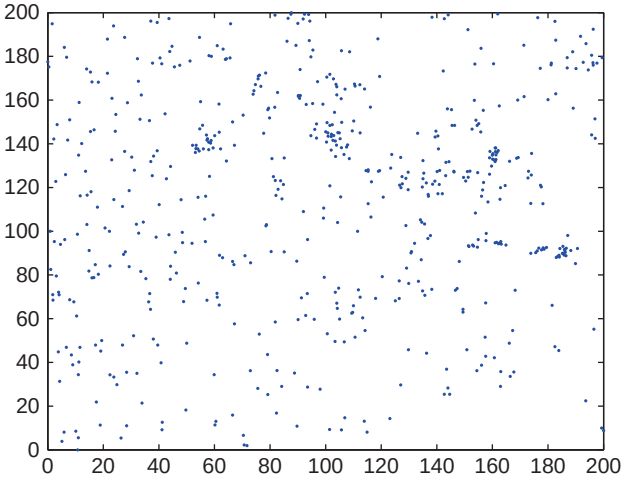

Figure 2.5 The longleaf pine data in a 200 m × 200 m window. (Data source: Cressie, 1993, Table 8.1.)

Using $r_{\max} = 10\,\text{m}$ in (2.60) leads to $\tau = 3.48$, which is much larger than $\tau_{0.01} = 0.60$ obtained by (2.63) but replacing the factor 1.45 by 1.75 there, and so the CSR hypothesis is rejected at the 1% level.

The empirical pair correlation function shown in Figure 2.6(b), which has the typical form of that of a cluster process, reveals that points with short inter-point distance are strongly positive correlated, that is, pines of short inter-point distance are more likely than in the Poisson case, indicating clustering in the pattern. It is plausible that $g(r)$ has a pole at $r = 0$, and Stoyan and Stoyan (1996) and Ghorbani (2012) showed that some generalisations of

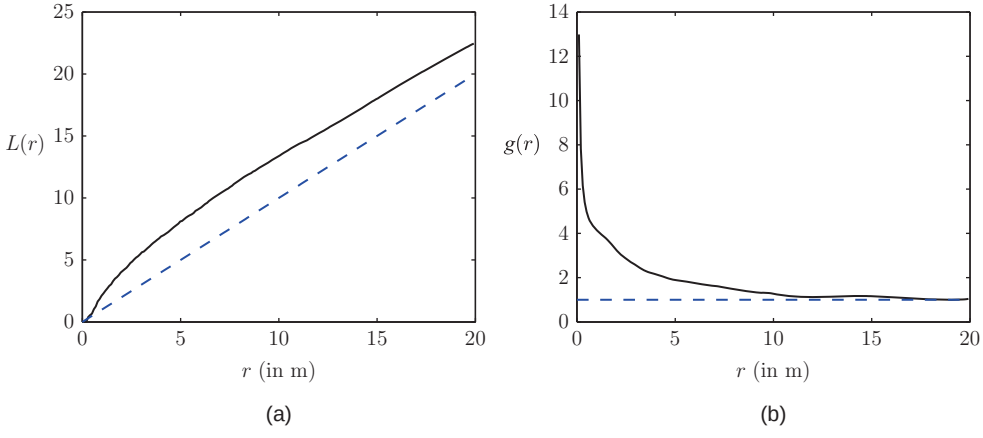

Figure 2.6 Estimate (solid line) for (a) $L(r)$ and (b) the pair correlation function $g(r)$ for the pattern of longleaf pine illustrated in Figure 2.5 in comparison with the theoretical line (dashed line) in the case of the homogeneous Poisson process, for which $L(r) = r$ and $g(r) \equiv 1$.

the Thomas process (see Section 5.3) give better fits to the pattern than the Thomas model suggested by Cressie (1993, p. 667).

Example 2.2. *Midpoints of large Martian craters*

Midpoints of large craters on Mars (where here 'large' is interpreted as having diameter greater than 200 km) form a point pattern on the sphere which is the surface of Mars. Lipskij *et al.* (1977) presented these data. Because this is a point pattern on the sphere rather than the plane, the theory above does not strictly apply; one must speak of the *homogeneous* Poisson point process hypothesis in the sense of homogeneity under rotations of the sphere. However, the modifications to the theory are entirely routine.

The surface of Mars being divided into 32 regions of equal area, the point counts of the regions yielded the numbers in Table 2.1. In this case, the Greig-Smith method can be applied with $q = 5$ and $\bar{x} = 0.97$. Table 2.1 also indicates the means by which the quadrats were grouped for the application of the Greig-Smith method. The dispersion indices are presented in Table 2.2 and compared with the χ^2-values at $\alpha = 0.05$. For $i = 1$, 3 and 5 the index values exceed the critical values. This implies that the differences between North and South hemispheres ($i = 5$), between quarter-spheres ($i = 3$), and between $\frac{1}{32}$-spheres ($i = 1$) are greater than is acceptable for a homogeneous Poisson point process, and thus clustering at these scales can be assumed.

Clearly the results depend on the choice of the division of the planet's surface.

The homogeneous Poisson point process hypothesis is also rejected by the index-of-dispersion test. The index-of-dispersion $I = 31 \times 2.10/0.97 = 67.06$, while the corresponding χ^2-value is 43.77 at $\alpha = 0.05$. When used to compare North and South hemispheres the

Table 2.1 Counts of large Martian craters in 32 segments of the planet's surface.

0°	90°	180°	270°	360°		
0	0	0	1		48.5°	
0	1	0	0			Northern
0	1	0	2		30°	hemisphere
1	0	0	2		14.5°	
					0°	———
0	0	0	4		14.5°	
0	0	1	0			Southern
0	2	1	2		30°	hemisphere
6	4	2	1		48.5°	

Table 2.2 Test statistics referred to the homogeneous Poisson point process hypothesis for the Martian crater data.

i	Degrees of freedom	$\chi^2_{0.95}$	I_i	$\chi^2_{0.05}$
1	16	8.0	32.5	26.3
2	8	2.7	6.9	15.5
3	4	0.7	20.0	9.5
4	2	0.1	0.3	6.0
5	1	0.0	7.2	3.8

homogeneity test yields $n_1 = 8, n_2 = 23$ and $F = 2.76$. Thus the hypothesis of equal intensity on both hemispheres must be rejected at the 5% level, as $F_{17,47;0.025} \approx 2.1$.

All these deviations from the homogeneous Poisson point process hypothesis presumably result either from the fact that the crater pattern originated by cause of showers of meteors or else from modifications of the original pattern by areological processes. Marcus (1972) gives a detailed study of point processes on planetary and lunar surfaces; see also Greeley (1987) and Malin *et al.* (2006).

Example 2.3. *Nodes of a network adapted to the pores of Fontainebleau sandstone*

Figure 2.7 shows the three-dimensional pattern of nodes of a network that is adapted to a sample of Fontainebleau sandstone; see for more details Tscheschel and Stoyan (2003).

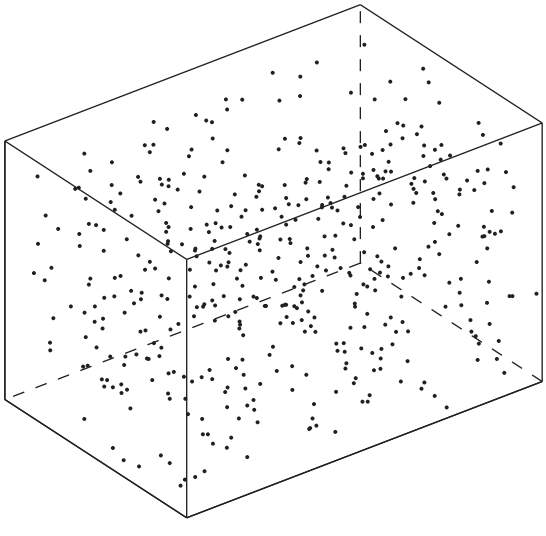

Figure 2.7 Nodes of a network adapted to the pores of a sample of Fontainebleau sandstone. The cuboid has side lengths 2 mm × 2 mm × 3 mm, the number of nodes is 470.

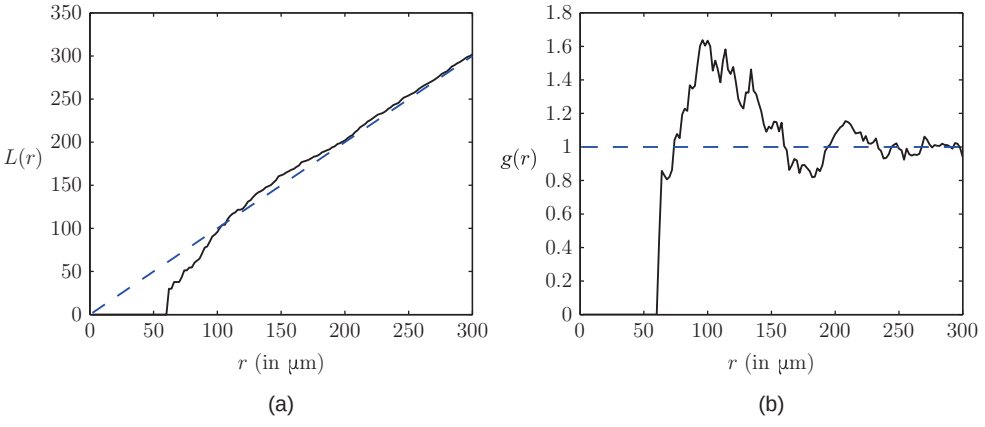

Figure 2.8 Estimate (solid line) for (a) $L(r)$ and (b) the pair correlation function $g(r)$ for the pattern of nodes illustrated in Figure 2.7 in comparison with the theoretical line (dashed line) in the case of the homogeneous Poisson process, for which $L(r) = r$ and $g(r) \equiv 1$.

There are pros and cons for the hypothesis that the pattern is of CSR type:

Pro: The pattern looks as a rather random pattern, and networks with nodes that form a Poisson process are indeed possible; see Section 9.11.5.

Con: The way in which the network was constructed by Sok *et al.* (2002) is perhaps not compatible with the idea of CSR, because pores close together may be considered as one large pore.

Thus it is of some interest to test the CSR hypothesis for the point pattern in Figure 2.7.

Figure 2.8(a) shows the empirical L-function of the pattern. For small r it deviates clearly from the theoretical form of a Poisson process, where $L(r) = r$. There is a hard-core distance r_0 of about 60 μm, that is, $L(r) = 0$ for $r \leq 60$ μm. However, for larger r, $L(r)$ comes quite close to r; thus, globally, the Poisson process seems to be a good model.

Under these conditions a Monte Carlo test was chosen with defining τ by (2.60). With $r_{\max} = 300$ μm as suggested by visual inspection of Figure 2.8(a), the value $\tau_{\text{emp}} = 61.644$ μm was obtained (which is just the hard-core distance) and the p-value obtained by (2.64) with $k = 9999$ was 0.0002. Thus the CSR hypothesis was rejected.

Some explanation for the rejection is given in Figure 2.8(b), which shows the empirical pair correlation function of the point pattern. There is a hard-core radius r_0 and inter-point distances around 80 μm are very frequent, much more frequent than in a Poisson process pattern. These two effects are canceled out in the cumulative function $L(r)$, resulting in $L(r) \approx r$ for $r \geq 250$ μm.

By the way, the hard-core distance in the point pattern is so large that a test suggested in Ripley and Silverman (1978), which is based only on the minimum inter-point distance, rejects the CSR hypothesis quite clearly.