Kalman Filter Application in Mean-Variance Portfolio Management

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Abstract

Mean-variance portfolio optimization one of the most powerful technique practiced by quantitative managers. To address the potential temporal fluctuation in conventional MV optimization, this brief report introduces a Kalman filter technique in mean-variance portfolio management based on several recent research papers. Kalman filter is combined with a well studied financial e Capital Asset Pricing Model and MV optimizer for prediction and measurement updates. Empirical results on 46 years of monthly returns in 17 industrial portfolio demonstrated that the proposed Kalman filter technique outperformed the MV optimization in both expected net return and lower annual portfolio standard deviation, the impact of Kalman filter on this improvement is further discussed.

1 Introduction

Mean-variance portfolio optimization first proposed by Markowitz [8] in 1952 targeted on selecting the best portfolio or asset distribution, out of the set of all portfolios being considered, according to some objective. The objective typically maximizes factors such as expected return, and minimizes costs like financial risk. The solution of this optimization problem have proved to be a reliable reference for institutions and investment professionals for portfolio distribution along the risky assets [5]. While in practical investment, mean-variance optimization is challenged by out-of-sample behavior of mean, variance-covariance which is unexpected. This dilemma force the investment professionals to recognize the importance of portfolio rebalancing.

Single period out-of-sample issue are well studied and addressed by many approaches [9, 1]. For example, Disatnik *et al.* [4] proposed a shrinkage method to ensure a better estimator of sample covariance matrix, Yang *et al.* [11] applied Random Matrix Theroy (RMT) to further improve the robustness of the shrinkage covariance estimator.

Potential fluctuations of portfolio weights in time series also is another practical aspects for rebalancing portfolio, when taking transaction cost into consideration. Many financial literature tried to optimize this rebalancing problem by proposing different methods, for example, The universal portfolio (UP), the successive constant rebalanced portfolio (SCRP) and the mixture successive constant rebalanced portfolio (MSCRP). In this aspect, we utilized Kalman filter (KF) model, which demonstrated its efficiency in real time series engineering problems. Kalman filter constists of a transition model and a measurement model, transition equation is used to update the predicted states, and the measurement equation updates the observation states. Disturbance assumptions are made on both transition model and measurement model which calibrate both models expected errors with a set of parameters. Kalman filter also has a long history in financial applications, Harvey *et al.* [7] decribed the Kalman filter application for maximum likelihood estimation in forecasting financial market. Godbey *et al.* [6] proved that Kalman filter can generate better parameters in multi-period hedging. In this report what we are interested in is, comparing to a comparable mean-variance method, if a Kalman filter generate higher net returns when transaction costs are considered in different risk level.

Recently, the application using Kalman filter coupled with optimization problem like mean-variance portfolio optimization becomes an active research topic. James [3] proposed a Kalman filter control technique in MV portfolio management, the Kalman filter's transition model is deducted by an ordinary least square (OLS) model, and the estimation model is replaced by a MV optimizer. Alternatively, Nkomo *et al.* [10] share the same idea of using a MV optimizer as estimation model, while they took the advantage of Cyclically Adjusted Price Relative (CAPR), another Capital Asset Pricing Model (CAPM) which is well invested and applied in financial research. In this report, we followed the main clue of [3, 10], re-formulated the Kalman filter model, and discuss the insight of the improvement due to Kalman filter from the emprical results.

This short report is organized in the following way, we begin by stating the mean-variance optimization problem, then we introduced the one of the typical Kalman filter model using three-factor model as KF's transition model and its corresponding measurement model. At last, some empirical results on historical portfolio data is demonstrated, and the effectiveness and the influence of proposed KF model is discussed.

2 Mean-variance Optimization

We define the portfolio ω_k is the distribution of the investor's asset distribution on the market X_k at time k to $k+1$, usually we normalized the portfolio ω_k to be $\sum_i^n \omega_k^i = 1$ where n is the number of portfolio, and ω_k^i is the proportion of total asset the investor put in stock i at time k. Then the relative return is defined as

$$r_k = \frac{X_{k+1} - X_k}{X_k} \tag{1}$$

Thus the expected return of this portfolio is

$$R = r_k \omega_k \tag{2}$$

Mean-variance portfolio optimization proposed by Markowitz [8] minimizes the exposure to risk under a certain level of expected returns, or a trade-off between risk and return expectations. Suppose we want to allocate the asset using a portfolio ω_k, then the expected mean μ_k and covariance of return σ_k^2 are

$$\begin{aligned} \mu_k &= (H_{k-1} + 1)\omega_k - 1 \\ \sigma_k^2 &= \omega_k{}' S_k \omega_k \end{aligned} \tag{3}$$

where $H_{k-1} = [r_1(t_{k-1})\ r_2(t_{k-1})\ \ldots\ r_n(t_{k-1})]$ is row vector of average returns over n risky assets up to time t_{k-1}, S_k is the historical sample covariance matrix of expected returns from n-risky assets observed up to time t_k.

The optimization problem of MV Portfolio Optimization can be written as

$$\begin{aligned} &\arg\min_{\omega_k}\ (1-\lambda)\sigma_k^2 - \lambda\mu_k \\ &\text{subject to} \sum_i^n \omega_k^i = 1, \omega_k^i \geq 0 \end{aligned} \tag{4}$$

where $0 \geq \lambda \geq 1$ is a trade-off parameter reflecting risk level that the investor willing to take. Noted that this formulation is equivalent to the formulation from Markowitz [8].

When we make the $\omega_k^i \geq 0$ constraint means short-selling is prohibited, investor never borrows stocks from broker and sell them. Short-selling is a useful financial technique in the investment, and it makes the optimization problem mathematically simpler, for example, when $\lambda = 0$, the optimal solution can be directly written as the following close-form equation,

$$\omega_k^\star = \frac{\mathbf{1}' S_k^{-1}}{\mathbf{1}' S_k^{-1}\mathbf{1}} \tag{5}$$

while we just discuss the constrained case in the whole report.

Even though mean-variance optimization can achieve the optimal solution in a single period of time, but in a time sequence, this optimization approach may cause potential fluctuations, and may reduce the net return of investment when transaction cost is included.

3 Kalman Filter

We assume the time series portfolio optimization problem as a discrete time linear dynamic system, considering the following state-space form

$$\begin{aligned} x_{k+1} &= A_k x_k + w_k \\ y_k &= x_k + v_k \\ w_k \sim N(0, Q_k)&, v_k \sim N(0, R_k) \\ x_0 &= \frac{1}{n}\mathbf{1}, P_0 = 0.1\mathbf{I} \end{aligned} \tag{6}$$

where A_k depends on the Capital Asset Pricing Model (CAPM) we use, we will discuss the deduction of A_k in the Sec. 3.1, B_k and C_k are directly assume to be identity. Both process noise and observation noise are normally distributed with corresponding covariance Q_k and R_k. The portfolio is initialized as uniform portfolio and an small identity initial covariance P_0.

As discussed in Sec. 1, James [3] and Nkomo *et al.* [10] share the same idea of constructing Kalman filter model, but with different Capital Asset Pricing Model (CAPM) in prediction update. In this report, we follow [3] as an example to re-formulate the Kalman filter's transition in Sec. 3.1and measurement models in Sec. 3.2.

3.1 Transition Model

Following the definition of Kalman filter, the transition model or prediction update is

$$\begin{aligned} \hat{x}_{k|k-1} &= A_{k-1}\hat{x}_{k-1|k-1} \\ P_{k|k-1} &= A_{k-1}P_{k-1|k-1}A'_{k-1} \end{aligned} \tag{7}$$

We begin by assuming that the previous M-months of m-daily portfolio returns can be estimated by an ordinary least squares (OLS) model

$$\begin{aligned} z_{k-1} &= G_{k-1}F_{k-1} + \epsilon_{k-1} \\ G_{k-1} &= [1_M \; r_{MKT}(t_{k-1}) \; r_{SMB}(t_{k-1}) \; r_{HML}(t_{k-1})] \\ F_{k-1} &= [\alpha \; \beta_{MKT} \; \beta_{SMB} \; \beta_{HML}] \end{aligned} \tag{8}$$

where F_k is the coefficient we want to estimate, $\beta_{MKT}, \beta_{SMB}, \beta_{HML}$ are parameters defined as market, small minus big, high minus low, so this regression model is also called three factor model. G_k is provided by Ken French's dataset [1].

From historical data, the actual portfolio return can also be written as

$$z_{k-1} = H_{k-1}\hat{x}_{k-1|k-1} = [r_1(t_{k-1}) \; r_2(t_{k-1}) \; ... \; r_n(t_{k-1})]\hat{x}_{k-1|k-1} \tag{9}$$

So the estimated coefficients F_k is

$$F_{k-1} = G^+_{k-1}H_{k-1}\hat{x}_{k-1|k-1} \tag{10}$$

where G^+_k is pseudoinverse of G_k, and $G^+_k = (G'_kG_k)^{-1}G'_k$.

Using the previous period weights, the prediction prediction of portfolio returns is found as a column vector of length of additional days predicted in the next month, which is

$$\hat{z}_k = G_kF_k = [1_{M+1} \; r_{MKT}(t_k) \; r_{SMB}(t_k) \; r_{HML}(t_k)] \tag{11}$$

So the predicted portfolio weight $\hat{x}_{k|k}$ is solved by

$$\hat{z}_k = H_k\hat{x}_{k-1|k-1} \tag{12}$$

This yields,

$$\hat{x}_{k|k-1} = H^+_k\hat{z}_k = [H^+_kG_kG^+_{k-1}H_{k-1}]\hat{x}_{k-1|k-1} \tag{13}$$

which means A_k has the following formulation is this CAPM

$$A_k = H^+_kG_kG^+_{k-1}H_{k-1} \tag{14}$$

To conclude the prediction update, we have the following three steps:

- Determine regression coefficients F_k based on actual returns from previous portfolio $\hat{x}_{k-1|k-1}$;
- Apply regression coefficients to determine k predicted portfolio returns;
- Estimate $\hat{x}_{k|k-1}$ using ordinary least squares with knowledge of $\hat{z}_k$.

[1]Daily price histories for market, small minus big, and high minus low, are freely available at Ken French's website, located at http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data$_l$*ibrary.html*

3.2 Measurement Model

The second step in Kalman filter is the measurement update, here the optimal result from mean-variance optimization in Eq. (2) is directly applied as Kalman filter measurement, yeilding $y_k = \omega_k^\star$.

Thus, the measurement update is

$$\begin{aligned} K_k &= P_{k|k-1}[P_{k|k-1} + R]^{-1} \\ \hat{x}_{k|k} &= \hat{x}_{k|k-1} + K_k[y_k - \hat{x}_{k|k-1}] \\ P_{k|k} &= P_{k|k-1} - K_k P_{k|k-1} \end{aligned} \tag{15}$$

Noted that for all $\hat{x}_{k|k-1}, \hat{x}_{k|k}$, x should be re-normalized to 1 after update.

4 Evaluation and Discussion

To evaluate the effectiveness of proposed Kalman filter method against the comparable mean-variance optimization method, experiments on two different level are conducted, where $\lambda = 0$ means the MV optimizer force to take lowest level of risk and consider no expectation on potential return, $\lambda = 0.5$ means the investor willing to take a certain level of risk and at the same time pursue higher expected return. The experiments are conducted on a dataset which contains monthly relative return in 17 industries from 1927 till present, a period from 1965 to 2010, total 552 month are evaluated.

Transaction cost are also assumed as needed for direct comparison to optimization method, we assume the cost per transaction is 50 basis point (0.5%). Here we followed the definition turnover by DeMiguel *et al.* [2] to determine the amount of trading required for Kalman filter and optimizer

$$Turnover(k) = \sum_{i=1}^{n} |\omega_k^i - \omega_{k-1}^i| \tag{16}$$

We first evaluate the average monthly turnover at three different risk levels listed in Table 4. When $\lambda = 0$, the optimization method may have slightly smaller transaction cost, because of the optimization approach minimize the risk, while $\lambda = 0.5$, the Kalman filter have better performance in reducing turnovers.

Risk Level, λ	KF	OPT
0	12.1%	10.5%
0.5	31.3%	34.8%

Table 1: Turnover statistics. KF stands for proposed Kalman filter, OPT stands for comparable optimization method.

To prove Kalman filter's effectiveness on improving net return, cumulative returns, annualized returns, annualized standard deviations from 552 returns are calulated in Table 4. When investor select a low risk level $\lambda = 0$, the KF gross have higher cumulative and annual return, because applying Kalman filter sacrificed the minimum risk portfolio, and boosted expected return, while from statics, the net return is still higher than OPT method, despite the fact that OPT method has lower annual standard deviation. When $\lambda = 0.5$, the KF has comparable gross comparing to the optimization approach, while resulted in lower turnovers, and eventually obtained better net return performance. Specifically, the curve of turnovers in each month is illustrated in Fig. 4.

	Portfolio	Cumulative Return	Annual Return	Annual standard deviation
	KF gross	9762%	10.5%	13.8%
	OPT gross	7037%	9.72%	13.1%
$\lambda = 0$	KF net	6968%	9.70%	13.8%
	OPT net	5252%	9.04%	13.1%
	KF gross	21572%	12.40%	15.1%
	OPT gross	22223%	12.48%	14.9%
$\lambda = 0.5$	KF net	9026%	10.31%	15.2%
	OPT net	8421%	10.15%	15.0%

Table 2: Return at different risk level.KF gross and KF net represent the performance of the Kalman filtered managed portfolio, before and after transaction costs. OPT gross and OPT net represent performance of the optimally managed portfolio, before and after transaction costs.

To conclude, this short report discussed a set of recent proposed Kalman filter method in mean-variance portfolio management. Optimizer is considered as the measurement in Kalman filter and a CAPM is applied in the transition model to compute the Kalman filter's state-space model. From empirical evaluation, the effectiveness of Kalman filter to improve the net return and maintain a acceptable transaction frequency is well demonstrated on a risk adjusted basis.

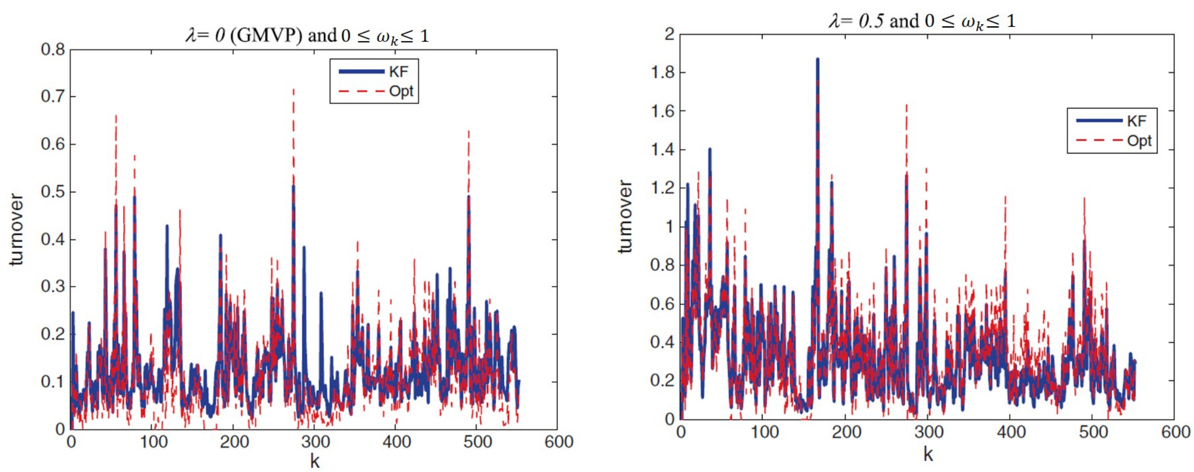

Figure 1: KF portfolio turnover and optimal portfolio turnover with different risk level

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