ELEC5650 Homework2

CHENG, Wei 20292233

May 22, 2019

Problem 1

According to the conclusion we derived in the lecture, we have $R_1 \leq R_2 \Rightarrow g1 \leq g2$; and $\forall X, Y$, and $0 \leq X \leq Y$, then $g(X) \leq g(Y)$ for both g_1 and g_2. We denote

$$\phi_k = E[P_k] = \alpha_k \phi_{k-1} \circ g_1 + (1 - \alpha_k)\phi_{k-1} \circ g_2$$

where $\phi_0 = I$ is the identity map. We define $\delta_k = \alpha_k - \alpha \geq 0$. We use mathematical induction to prove the problem, together with the statement $\phi_k(X) \leq \phi_k(Y)$, $\forall\ 0 \leq X \leq Y$.

When $k = 0$, $\phi_0 = f_0$, both statements holds.

When $k = 1$, $\phi_1 - f_1 = \delta_1(g_1 - g_2) \leq 0$;

$$\phi_1(X) = \alpha_1 g_1(X) + (1 - \alpha_1)g_2(X) \leq \alpha_1 g_1(Y) + (1 - \alpha_1)g_2(Y) = \phi_1(Y)$$

so both statements holds.

When $k \geq 2$, we assume $\phi_{k-1} \leq f_{k-1}$, and $\forall X \leq Y, \phi_{k-1}(X) \leq \phi_{k-1}(Y)$, obviously we have

$$\begin{aligned}\phi_k(X) &= \alpha_k \phi_{k-1} \circ g_1(X) + (1 - \alpha_k)\phi_{k-1} \circ g_2(X) \\ &\leq \alpha_k \phi_{k-1} \circ g_1(Y) + (1 - \alpha_k)\phi_{k-1} \circ g_2(Y) = \phi_1(Y)\end{aligned}$$

$$\begin{aligned}\phi_k - f_k &= \alpha_k \phi_{k-1} \circ g_1 + (1 - \alpha_k)\phi_{k-1} \circ g_2 - \alpha f_{k-1} \circ g_1 - (1 - \alpha) f_{k-1} \circ g_2 \\ &= \alpha(\phi_{k-1} \circ g_1 - f_{k-1} \circ g_1) + (1 - \alpha)(\phi_{k-1} \circ g_2 - f_{k-1} \circ g_2) + \delta_k(\phi_{k-1} \circ g_1 - \phi_{k-1} \circ g_2) \\ &\leq 0\end{aligned}$$

First two term hold because of the first assumption, and the last term holds due to our second assumption.

Therefore, we have $\phi_k \leq f_k$, for all k.

Problem 2

1) In our lecture, we proved that $\forall X \le Y$, $h(X) \le h(Y), g(X) \le g(Y)$. To prove $g^i h^i \le h^i g^i$, we just need to prove $g^2 h \le hg^2$ and $gh^2 \le h^2 g$, omitting X.

$$gh \le hg \Longrightarrow gh \circ h \le hg \circ h \Longrightarrow gh \circ h \le h \circ gh \le h \circ hg \Longrightarrow gh^2 \le h^2 g$$

$$gh \le hg \Longrightarrow gh \circ g \le hg \circ g \Longrightarrow h \circ gg \ge g \circ hg \ge g \circ gh \Longrightarrow g^2 h \le hg^2$$

2) To find a sufficient condition of the previous statement, we should relax both size towards the statement and irrelevant to X.

$$gh(X) = \tilde{g} \circ hh(X) = [hh(X)^{-1} + BR^{-1}B']^{-1} \le [BR^{-1}B']^{-1}$$

$$hg(X) = A'g(X)A + Q \ge Q$$

Noted that the matrix inversion lemma is used in $\tilde{g}$. After eliminating X related terms, we can easily find a sufficient condition for $gh \le hg$, which is $[BR^{-1}B']^{-1} \le Q$

Problem 3

Omitting θ, we first define

$$J_k(x) = x'Qx$$

$$f(x_k, u_k, \gamma_k) = x_k'Qx_k + \gamma_k u_k' R u_k$$

$$g(x_k, u_k, \gamma_k) = Ax_k + \gamma_k B u_k$$

Then we can solve the schedule optimal by solving the equivalent dynamic programming problem

$$\begin{aligned}
J_{k-1}^\star(x_{k-1}) &= \min_{u_{k-1}} \{ f(x_{k-1}, u_{k-1}, \gamma_{k-1}) + J_k^\star(g(x_{k-1}, u_{k-1}, \gamma_{k-1})) \} \\
&= \min_{u_{k-1}} \{ x_{k-1}'Qx_{k-1} + \gamma_{k-1} u_{k-1}' R u_{k-1} + (Ax_{k-1} + \gamma_{k-1} B u_{k-1})' Q (Ax_{k-1} + \gamma_{k-1} B u_{k-1}) \} \\
&= \min_{u_{k-1}} \{ x_{k-1}'Qx_{k-1} + \gamma_{k-1} u_{k-1}' R u_{k-1} + x_{k-1}'A'QAx_{k-1} + \gamma_{k-1} x_{k-1}' A'QABu_{k-1} \\
&\quad + \gamma_{k-1} u_{k-1}' B'QAx_{k-1} + \gamma_{k-1}^2 u_{k-1}' B'QBu_{k-1} \}
\end{aligned}$$

When $\gamma_{k-1} = 0$, we have

$$J_{k-1}^\star(x_{k-1}) = \min_{u_{k-1}} \{ x_{k-1}'Qx_{k-1} + x_{k-1}'A'QAx_{k-1} \} = x_{k-1}'(A'QA + Q)x_{k-1}$$

When $\gamma_{k-1} = 1$, according matrix calculus we have

$$\begin{aligned}
u_{k-1}^\star &= (R + B'QB)^{-1} B'QAx_{k-1} \\
J_{k-1}^\star(x_{k-1}) &= x_{k-1}'Qx_{k-1} - x_{k-1}'A'QB[B'QB + R]^{-1} BQAx_{k-1} \\
&= x_{k-1}'(Q - A'QB[B'QB + R]^{-1} BQA)x_{k-1}
\end{aligned}$$

If we define $P_k = Q$, then we can have

$$J^\star_{k-1}(x_{k-1}) = x'_{k-1}P_{k-1}x_{k-1}$$

$$u^\star_{k-1} = -L_{k-1}x_{k-1} = (R + B'QB)^{-1}B'QAx_{k-1}$$

$$P_{k-1} = \begin{cases} A'P_kA + Q = h(P_k) & \text{,if } \gamma_{k-1} = 0 \\ Q - A'P_kB[B'P_kB + R]^{-1}BP_kA = g(P_k) & \text{,if } \gamma_{k-1} = 1 \end{cases}$$

So the problem is equivalent to solve

$$J^\star_0(x_{k-1}) = x'_0P_0x_0$$

$$P_0 = f_0 \circ f_1 \circ ... \circ f_k(P_k)$$

$$f_i = \begin{cases} h & \text{,if } \gamma_i = 0 \\ g & \text{,if } \gamma_i = 1 \end{cases}$$

From Problem 2, we proved that for d times g and $T - d$ times h

$$g^d h^{T-d} \leq f_0 \circ f_1 \circ ... \circ f_k \leq h^{T-d} g^d$$

so it is trivial to find the optimal schedule is

$$\gamma_i(\theta) = \begin{cases} 0 & \text{,if } i = d, d+1, ..., T-1 \\ 1 & \text{,if } i = 0, 1, ..., d-1 \end{cases}$$