ELEC5650 Homework1

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Problem 1

(a) For $X \geq Y > 0$, we have for any non-zero vector z, $z^T Y z > 0$ and $z^T(X-Y)z \geq 0$ holds. Then $z^T Y z + z^T(X-Y)z = z^T X z > 0$, which means X is PD. If we set $\{z_k = 0 | 1 \leq k < i, j < k \leq n\}$, then we have

$$z^T X z = \sum_{u=0}^{n}\sum_{v=0}^{n} z_u z_v x_{uv} = \sum_{u=i}^{j}\sum_{v=i}^{j} z_u z_v x_{uv} = z[i:j]^T X[i:j, i:j] z[i:j] > 0 \tag{1}$$

for any non zero subvector $z[i:j]$, so $X[i:j, i:j]$ is PD, $X[i:j, i:j] > 0$.

(b) We have PD matrix X and PSD matrix $X - Y$, denote $y = C^T z$ which could be any vector including zero vector, $z^T C X C^T z = y^T X y \geq 0$, so CXC^T is PSD. The same came be proven, $C(X-Y)C^T$ is PSD, so $CXC^T \geq CYC^T$.

(c) Sufficiency: C has full row rank, then $y = C^T z$ is nonzero for any nonzero z, so that CXC^T is PD.

Necessity: CXC^T is PD means for any nonzero z, $z^T C X C^T z > 0$. Because X is PD, $y = C^T z$ should also be nonzero, then C is full row rank.

(d) In (a) we proved that X is PD, then it is equivalent to $Det(X) > 0$ and $Tr(X) > 0$. From (a) we know $X[i:j, i:j]$ is PD, when $i = j$ holds, which means all diagonal elements are positive, so the trace is positive. Since diagonal elements are positive, and all the principle minors are PD, the determinate is positive.

(e) We know $X - Y \geq 0$, and X, Y are invertible, if we multiply X^{-1} and Y^{-1} on both side we have $Y^{-1} - X^{-1} \geq 0$, then $Y^{-1} \geq X^{-1}$.

For any nonzero, $y = Xz$,

$$y^T X^{-1} y = z^T X X^{-1} X z = z^T X z > 0 \tag{2}$$

so X^{-1} is also PD, $Y^{-1} \geq X^{-1} > 0$ proven.

Problem 2

(a) If A has no multiplicity eigenvalue, then z_i can span the whole space, so that any nonzero vector can be a linear combination of z_i, $z = \sum_{i=0}^{n} a_i z_i$,

$$z^T A^T B A z - z^T B z = \sum_{i=0}^{n} a_i^2 (\lambda_i^2 - 1) z_i^T B z_i \geq 0 \tag{3}$$

which is true.

To give a toy example, we set

$$A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, X = I, Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \Longrightarrow AXA^T + Q - X = \begin{bmatrix} 2 & -1 \\ -1 & 0 \end{bmatrix} \tag{4}$$

which has eigenvalues -0.4142 and 2.4142, so $AXA^T + Q - X \geq 0$ is invalid.

(b) We have both $(AA^T - I)$ and X are PSD, then

$$\begin{aligned} Tr(h(X)) - Tr(X) = Tr(AXA^T + Q - X) &= Tr(A^T A X - X) + Tr(Q) \\ \geq Tr((A^T A - I)X) \geq 0 \end{aligned} \tag{5}$$

Problem 3

Necessity: If $\rho(A) < 1$, there exist a $B > 0$, such that $B - A^T B A > 0$.

$\rho(A) < 1$ means there exist a nonsingular matrix P to perform similarity transformation or model decomposition, such that $||P^{-1}AP||_2 < 1$, which is equivalent to $(P^{-1}AP)^T(P^{-1}AP) < 1$, if we set $B = (PP^T)^{-1}$ which is PD, then we have

$$(P^{-1}AP)^T(P^{-1}AP) < 1 \Longrightarrow A^T(PP^T)^{-1}A - (PP^T)^{-1} < 0 \Longrightarrow B - A^T B A > 0 \tag{6}$$

Sufficiency: If a $B > 0$, has property $B - A^T B A > 0$, then $\rho(A) < 1$.

If A has a eigenvalue $|\lambda_i| \geq 1$ with its corresponding eigenvector z_i,

$$z_i^T (B - A^T B A) z_i = z_i^T B z_i - z_i^T A^T B A z_i = (1 - \lambda_i^2) z_i B z_i \leq 0 \tag{7}$$

which contradicts with our assumption, so $\rho(A) < 1$ is proven.