Introduction to Optimization Theory
Homework Assignment 4

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ILO Ex. 8.5

Proof

(a)⇒(b)

If P is full-dimensional, there exists an interior point $\boldsymbol{x}$ s.t. $\boldsymbol{x} \in P$ and all the constraints are not active at $\boldsymbol{x}$, i.e., $\boldsymbol{Ax} > \boldsymbol{b}$.

(b)⇒(c)

Suppose not, i.e., all the extreme points lie on the same hyperplane. Let $\boldsymbol{d}$ be a vector which is vertical to the hyperplane, since the polyhedra is the convex hull of all extreme points, for any $\lambda \in \mathbb{R}$, we have $\boldsymbol{x} + \lambda \boldsymbol{d} \notin P$, which means at least one constraint is active at $\boldsymbol{x}$ and we get a contradiction.

(c)⇒(a)

If there are $n+1$ extreme points of P that do not lie on a common hyperplane, note that the convex hull of the extreme points is a subset of the polyhedra, and the convex hull has positive volume, which means the polyhedra is full-dimensional.

CO Ex. 3.4

Proof

$$\begin{aligned}
\int_0^1 f(x+\lambda(y-x))d\lambda &\le \int_0^1 ((1-\lambda)f(x)+\lambda f(y))d\lambda \\
&= f(x) + (f(y)-f(x))\int_0^1 \lambda d\lambda \\
&= f(x) + \frac{1}{2}(f(y)-f(x)) \\
&= \frac{f(x)+f(y)}{2}.
\end{aligned}$$

CO Ex. 3.7

Suppose not. There exists $\boldsymbol{x}, \boldsymbol{y}$ s.t. $f(\boldsymbol{x}) \neq f(\boldsymbol{y})$. WLOG, assume $f(\boldsymbol{x}) < f(\boldsymbol{y})$. The function f is convex indicates that $g(t) = f(\boldsymbol{x} + t(\boldsymbol{y} - \boldsymbol{x}))$ is convex, which means $g(t) > (g(1) - g(0))t + g(0), \forall t > 1$. Notice that $g(1) = f(\boldsymbol{y}) > f(\boldsymbol{x}) = g(0)$, which means $(g(1) - g(0))t + g(0)$ is unbounded. Therefore, g is unbounded, which leads to a contradiction.

CO Ex. 3.11

Proof

For any $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$, we have

$$f(\boldsymbol{y}) \ge f(\boldsymbol{x}) + (\nabla f(\boldsymbol{x}))^T(\boldsymbol{y}-\boldsymbol{x}) \ge f(\boldsymbol{y}) + (\nabla f(\boldsymbol{y}))^T(\boldsymbol{x}-\boldsymbol{y}) + (\nabla f(\boldsymbol{x}))^T(\boldsymbol{y}-\boldsymbol{x}).$$

Decrease by $f(\boldsymbol{y})$ in both sides, we get

$$(\nabla f(\boldsymbol{y}) - \nabla f(\boldsymbol{x}))^T(\boldsymbol{y}-\boldsymbol{x}) \ge 0.$$

The converse is false. Consider

$$g(\boldsymbol{x}) = g([x_1, x_2]^T) = [2x_1, 2x_1 + 2x_2]^T.$$

Note that

$$(\boldsymbol{x}-\boldsymbol{y})^T \begin{bmatrix} 2 & 0 \\ 2 & 2 \end{bmatrix} (\boldsymbol{x}-\boldsymbol{y}) = (\boldsymbol{x}-\boldsymbol{y})^T \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} (\boldsymbol{x}-\boldsymbol{y}) > 0, \forall \boldsymbol{x} \neq \boldsymbol{y},$$

in that

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

is a positive definite matrix. Therefore, $g(\boldsymbol{x})$ is monotone. However, there doesn't exist a f s.t. $\nabla f = g$ in that

$$\frac{\partial^2 f}{\partial x_1 \partial x_2} = 0, \frac{\partial^2 f}{\partial x_2 \partial x_1} = 2,$$

which is contradictory to that f is differentiable.

CO Ex. 3.12

Proof
Let $A = \left\{(\boldsymbol{x}, t)^T : t \geq f(\boldsymbol{x})\right\}, B = \left\{(\boldsymbol{x}, t)^T : g(\boldsymbol{x}) \geq t\right\}$. We know A and B are convex sets since f is convex and g is concave. By separating hyperplane theorem, we know there exists a hyperplane $\left\{(\boldsymbol{x}^T, t)^T : (\boldsymbol{x}^T, t)\boldsymbol{y} = b\right\}$ separating two convex sets, i.e. $\forall \boldsymbol{a} \in A, \boldsymbol{a}^T\boldsymbol{y} > b$ and $\forall \boldsymbol{a} \in B, \boldsymbol{a}^T\boldsymbol{y} < b$.
Assume $\boldsymbol{y} = (\boldsymbol{z}^T, y_0)^T$ (WLOG, assume $y_0 > 0$), $\forall (\boldsymbol{x}^T, t)^T \in A$, we have $\boldsymbol{x}^T\boldsymbol{z} + ty_0 > b \Leftrightarrow -\frac{\boldsymbol{z}^T}{y_0}\boldsymbol{x} - \frac{b}{y_0} < t$. Therefore, $-\frac{\boldsymbol{z}^T}{y_0}\boldsymbol{x} - \frac{b}{y_0} < f(\boldsymbol{x}), \forall \boldsymbol{x}$, i.e., $f(\boldsymbol{x})$ is lower bounded by the affine function. Similarly, $g(\boldsymbol{x})$ is upper bounded by the affine function.

CO Ex. 5.15

Proof
Since $f_i(\boldsymbol{x})$ is convex, $\forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n, \lambda \in (0, 1)$, we have $f_i(\lambda\boldsymbol{x} + (1-\lambda)\boldsymbol{y}) \leq \lambda f_i(\boldsymbol{x}) + (1-\lambda)f_i(\boldsymbol{y})$. Since $h_i(\boldsymbol{x})$ is increasing and convex, we have

$$h_i(f_i(\lambda\boldsymbol{x} + (1-\lambda)\boldsymbol{y})) \leq h_i(\lambda f_i(\boldsymbol{x}) + (1-\lambda)f_i(\boldsymbol{y})) \leq \lambda h_i(f_i(\boldsymbol{x})) + (1-\lambda)h_i(f_i(\boldsymbol{y})).$$

Therefore, $h_i(f_i(\boldsymbol{x}))$ is convex, which means $\phi(\boldsymbol{x}) = f_0(\boldsymbol{x}) + \sum h_i(f_i(\boldsymbol{x}))$ is convex.

Since $\tilde{\boldsymbol{x}}$ minimizes ϕ, we have $\nabla\phi(\tilde{\boldsymbol{x}}) = \nabla f_0(\tilde{\boldsymbol{x}}) + \sum h_i'(f_i(\tilde{\boldsymbol{x}}))\nabla f_i(\tilde{\boldsymbol{x}}) = 0$.
Let $\lambda_i = h_i'(f_i(\tilde{\boldsymbol{x}}))$, we have $\lambda_i > 0$ in that h_i is increasing. Therefore, $\boldsymbol{\lambda}$ is dual feasible, i.e., $g(\boldsymbol{\lambda}) = f_0(\tilde{\boldsymbol{x}}) + \sum \lambda_i f_i(\tilde{\boldsymbol{x}})$ is a lower bound to the primal problem.

CO Ex. 5.31

Proof
Since f_i is convex, we have $f_i(\boldsymbol{x}^*) + \nabla f_i(\boldsymbol{x}^*)^T(\boldsymbol{x} - \boldsymbol{x}^*) \leq f_i(\boldsymbol{x}) \leq 0$. Therefore,

$$\begin{aligned}
0 \geq& \sum_{i=1}^{m} \lambda_i^*(f_i(\boldsymbol{x}^*) + \nabla f_i(\boldsymbol{x}^*)^T(\boldsymbol{x} - \boldsymbol{x}^*)) \\
=& \sum_{i=1}^{m} \lambda_i^* f_i(\boldsymbol{x}^*) + \lambda_i^* \nabla f_i(\boldsymbol{x}^*)^T(\boldsymbol{x} - \boldsymbol{x}^*) \\
=& \sum_{i=1}^{m} \lambda_i^* \nabla f_i(\boldsymbol{x}^*)^T(\boldsymbol{x} - \boldsymbol{x}^*) \\
=& -\nabla f_0(\boldsymbol{x}^*)(\boldsymbol{x} - \boldsymbol{x}^*),
\end{aligned}$$

which is equivalent to $\nabla f_0(\boldsymbol{x}^*)(\boldsymbol{x} - \boldsymbol{x}^*) \geq 0$.

CO Ex. 9.1

(a)

Proof
Since $\boldsymbol{P}$ is not semi-definite positive, there exists $\boldsymbol{x}_0$ s.t. $\boldsymbol{x}_0^T \boldsymbol{P} \boldsymbol{x}_0 < 0$. Let $\boldsymbol{x} = k\boldsymbol{x}_0$, $f(\boldsymbol{x}) = \frac{1}{2}k^2 \boldsymbol{x}_0^T \boldsymbol{P} \boldsymbol{x}_0 + k\boldsymbol{q}^T \boldsymbol{x}_0 + r$, which is obviously unbounded below.

(b)

Proof
Rewrite $\boldsymbol{q}$ as $\boldsymbol{q}_0 + \boldsymbol{v}$ s.t. $\boldsymbol{q}_0 \in \mathrm{Col}(\boldsymbol{P}), \boldsymbol{v} \perp \mathrm{Col}(\boldsymbol{P})$, i.e., $\boldsymbol{P}\boldsymbol{v} = 0$. Let $\boldsymbol{x} = k\boldsymbol{v}$, we have

$$f(\boldsymbol{x}) = k\boldsymbol{q}^T \boldsymbol{v} + r = k\boldsymbol{v}^T \boldsymbol{v} + r = k\|\boldsymbol{v}\|_2^2 + r,$$

which is unbounded below.