Introduction to Optimization Theory
Homework Assignment 2

Chen Zhiyang, 2017011377

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Ex. 3.3

Proof

$\Rightarrow$:

If $\boldsymbol{d}$ is a feasible direction at $\boldsymbol{x}$, there exists a scalar $\lambda > 0$, s.t. $\boldsymbol{x} + \lambda \boldsymbol{d} \in P$.
Therefore,

$$\boldsymbol{A}(\boldsymbol{x} + \lambda \boldsymbol{d}) = \boldsymbol{A}\boldsymbol{x} + \lambda \boldsymbol{A}\boldsymbol{d} = \boldsymbol{0} \Rightarrow \boldsymbol{A}\boldsymbol{d} = \boldsymbol{0},$$

and

$$\boldsymbol{x} + \lambda \boldsymbol{d} \geq 0 \Rightarrow \text{for each } i \text{ s.t. } x_i = 0, x_i + \lambda d_i \geq 0 \Rightarrow d_i \geq 0.$$

$\Leftarrow$:

Let

$$\lambda = \min_{x_i \neq 0} -\frac{d_i}{x_i},$$

obviously we have $\boldsymbol{x} + \lambda \boldsymbol{d} \geq \boldsymbol{0}$, and $\boldsymbol{A}\boldsymbol{d} = \boldsymbol{0} \Rightarrow \boldsymbol{A}\boldsymbol{x} + \lambda \boldsymbol{A}\boldsymbol{d} = \boldsymbol{A}(\boldsymbol{x} + \lambda \boldsymbol{d}) = \boldsymbol{0}$, which means $\boldsymbol{d}$ is a feasible direction.

Ex. 3.4

Proof

If $\boldsymbol{d}$ is a feasible direction, there exists a scalar $\lambda > 0$ s.t. $\boldsymbol{x}^* + \lambda \boldsymbol{d} \in P$. We have

(1) $\boldsymbol{A}(\boldsymbol{x} + \lambda \boldsymbol{d}) = \boldsymbol{0} \Rightarrow \boldsymbol{A}\boldsymbol{x} + \lambda \boldsymbol{A}\boldsymbol{d} = \boldsymbol{0} \Rightarrow \boldsymbol{A}\boldsymbol{d} = \boldsymbol{0}$.

(2) $\boldsymbol{D}\boldsymbol{x}^* = \boldsymbol{f}, \boldsymbol{D}(\boldsymbol{x}^* + \lambda \boldsymbol{d}) \leq \boldsymbol{f} \Rightarrow \lambda \boldsymbol{D}\boldsymbol{d} < \boldsymbol{0} \Rightarrow \boldsymbol{D}\boldsymbol{d} < \boldsymbol{0}$.

If $\boldsymbol{Ad} = \mathbf{0}, \boldsymbol{Dd} \leq \mathbf{0}$, let

$$\lambda = \min_{(\boldsymbol{Ed})_i > 0} \frac{(\boldsymbol{g} - \boldsymbol{E}\boldsymbol{x}^*)_i}{(\boldsymbol{Ed})_i}.$$

Note that $\lambda > 0$ in that $\boldsymbol{E}\boldsymbol{x}^* < \boldsymbol{g}$ (if the set $\{i : (\boldsymbol{Ed})_i > 0\}$ is empty, let λ be any positive number).
We have

(1) $\boldsymbol{A}(\boldsymbol{x}^* + \lambda \boldsymbol{d}) = \boldsymbol{A}\boldsymbol{x}^* + \lambda \boldsymbol{Ad} = \mathbf{0}$.

(2) $\boldsymbol{D}(\boldsymbol{x}^* + \lambda \boldsymbol{d}) = \boldsymbol{D}\boldsymbol{x}^* + \lambda \boldsymbol{Dd} \leq \boldsymbol{f}$.

(3) $\boldsymbol{E}(\boldsymbol{x}^* + \lambda \boldsymbol{d}) \leq \boldsymbol{g}$.

Therefore, $\boldsymbol{d}$ is a feasible direction iff $\boldsymbol{Ad} = \mathbf{0}, \boldsymbol{Dd} \leq \mathbf{0}$.

Ex. 3.5

Let a direction $\boldsymbol{d} = (d_1, d_2, d_3)^T$. $\boldsymbol{d}$ is feasible at $\boldsymbol{x}$ if there exists some $\lambda > 0$ s.t. $\boldsymbol{x} + \lambda \boldsymbol{d} \in P$. To require this, we have to make $\lambda(d_1 + d_2 + d_3) + 1 = 1$, which means $d_3 = -d_1 - d_2$. Also, $(\lambda d_1, \lambda d_2, 1 + \lambda(-d_1 - d_2)) \geq \mathbf{0} \Rightarrow d_1, d_2 \geq 0$.
Therefore, the set of feasible directions at $\boldsymbol{x}$ is $\left\{(d_1, d_2, -d_1 - d_2) : d_1, d_2 \geq 0\right\}$.

Ex. 3.6

(a)

Proof
Suppose $\boldsymbol{y}$ is an arbitrary feasible solution other than $\boldsymbol{x}$, let $\boldsymbol{d} = \boldsymbol{y} - \boldsymbol{x}$, then $\boldsymbol{Ad} = \boldsymbol{Ay} - \boldsymbol{Ax} = \mathbf{0}$. Let N be the set of nonbasic indices associated with $\boldsymbol{B}$. Rewrite this as

$$\boldsymbol{B}\boldsymbol{d}_B + \sum_{i \in N} \boldsymbol{A}_i d_i = \mathbf{0}.$$

Since $\boldsymbol{B}$ is invertible, we have $\boldsymbol{d}_B = -\sum_{i \in N} \boldsymbol{B}^{-1} \boldsymbol{A}_i d_i$. Therefore,

$$\boldsymbol{c}^T \boldsymbol{d} = \boldsymbol{c}_B^T \boldsymbol{d}_B + \sum_{i \in N} c_i d_i = \sum_{i \in N} (c_i - \boldsymbol{c}_B^T \boldsymbol{B}^{-1} \boldsymbol{A}_i) d_i = \sum_{i \in N} \bar{c}_i d_i.$$

If for all $i \in N$, $y_i = 0$, we have $\boldsymbol{Ay} = \boldsymbol{B}\boldsymbol{y}_B = \boldsymbol{b}$, which means $\boldsymbol{x} = \boldsymbol{y}$. Therefore, there exists $j \in N$ s.t. $y_j > 0$, and

$$\boldsymbol{c}^T \boldsymbol{d} = \sum_{i \in N} \bar{c}_i d_i = \sum_{i \in N} \bar{c}_i y_i \geq \bar{c}_j y_j > 0.$$

(b)

Proof
Suppose $\bar{c}_j \leq 0, j \in N$. Let $\boldsymbol{d}$ be the j-th basic direction. As $\boldsymbol{x}$ is non-degenerate, we know $\boldsymbol{d}$ is always a feasible direction. There exists $\lambda > 0$, s.t. $\boldsymbol{x} + \lambda\boldsymbol{d} \in P$. Then $\boldsymbol{c}^T\boldsymbol{d} = \bar{c}_j d_j = \bar{c}_j \leq 0$. By choosing sufficiently small $\lambda > 0$ so that $\boldsymbol{x}+\lambda\boldsymbol{d} \in P$, we get $\lambda\boldsymbol{c}^T\boldsymbol{d} \leq 0$, which means the cost of $\boldsymbol{x}+\lambda\boldsymbol{d}$ is not larger than the cost of $\boldsymbol{x}$. However, we know $\boldsymbol{x}$ is the unique optimal solution, which leads to a contradiction.

Ex. 3.7

Proof
That the new LP problem has an optimal cost of zero means for all $\boldsymbol{d}$, s.t. $\boldsymbol{A}\boldsymbol{d} = \boldsymbol{0}, d_i \geq 0, i \in Z$, we have $\boldsymbol{c}^T\boldsymbol{d} \geq 0$. This is equivalent to at $\boldsymbol{x}$, for any feasible direction $\boldsymbol{d}$, $\boldsymbol{c}^T\boldsymbol{d} \geq 0$, which means $\boldsymbol{x}$ is optimal.

Ex. 3.12

(a)

We add two artificial variables y_1, y_2. The constraints become:

$$\begin{aligned}
x_1 - x_2 + y_1 &= 2, \\
x_1 + x_2 + y_2 &= 6, \\
x_1, x_2, y_1, y_2 &\geq 0.
\end{aligned}$$

It's easy to see $(x_1, x_2, y_1, y_2) = (0, 0, 2, 6)$ is a basic feasible solution.

(b)

In this problem we have

$$\boldsymbol{A} = \begin{bmatrix} 1 & -1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}, \boldsymbol{b} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}, \boldsymbol{c} = \begin{bmatrix} -2 \\ -1 \\ 0 \\ 0 \end{bmatrix}.$$

The initial basic variables are y_1 and y_2, so at first $\boldsymbol{B} = \boldsymbol{I}_2$.
Now we get the initial tableau.

0	-2	-1	0	0
2	1	-1	1	0
6	1	1	0	1

Let x_1 enter the basis and y_1 exit the basis.

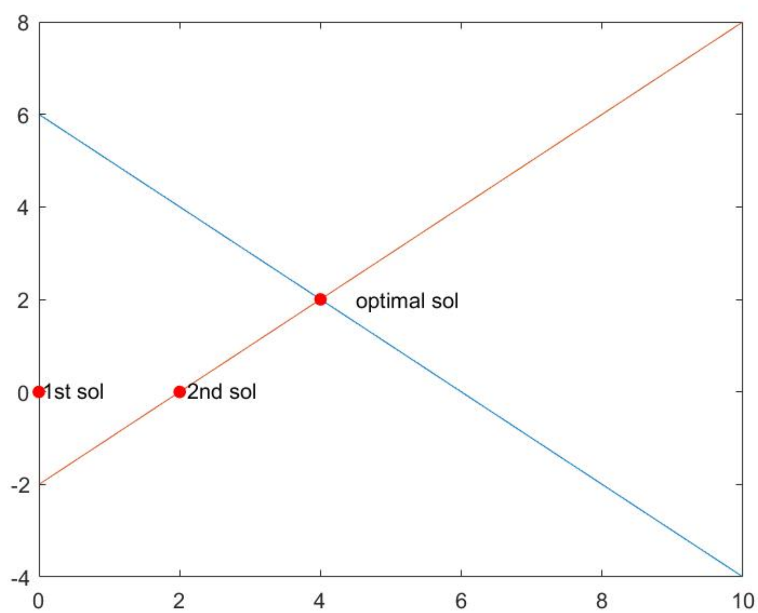
4	0	-3	2	0
2	1	-1	1	0
4	0	2	-1	1

Let x_2 enter the basis and y_2 exit the basis.

10	0	0	0.5	1.5
4	1	0	0.5	0.5
2	0	1	-0.5	0.5

Now all the components of $\bar{\boldsymbol{c}}$ is non-negative. We get the optimal solution $(x_1, x_2) = (4, 2)$, and the optimal cost is -10.

(c)



Ex. 3.17

Phase I:
We transform the original LP problem into:

$$\text{minimize} \;\; y_1 + y_2 + y_3,$$

$$\text{subj. to} \begin{cases} x_1 + 3x_2 + 4x_4 + x_5 + y_1 = 2, \\ x_1 + 2x_2 - 3x_4 + x_5 + y_2 = 2, \\ -x_1 + 4x_2 + 3x_3 + y_3 = 1, \\ x_1, x_2, \ldots, x_5, y_1, y_2, y_3 \geq 0. \end{cases}$$

We begin with a BFS $\boldsymbol{x} = (0,0,0,0,0,2,2,1)^T$.

$$\begin{array}{|c|cccccccc|}
\hline
-5 & -1 & -1 & -3 & -1 & -2 & 0 & 0 & 0 \\
\hline
2 & 1 & 3 & 0 & 4 & 1 & 1 & 0 & 0 \\
2 & 1 & 2 & 0 & -3 & 1 & 0 & 1 & 0 \\
1 & -1 & -4 & 3 & 0 & 0 & 0 & 0 & 1 \\
\hline
\end{array}$$

Let x_3 enter the basis and y_3 exit the basis.

$$\begin{array}{|c|cccccccc|}
\hline
-4 & -2 & -5 & 0 & -1 & -2 & 0 & 0 & 1 \\
\hline
2 & 1 & 3 & 0 & 4 & 1 & 1 & 0 & 0 \\
2 & 1 & 2 & 0 & -3 & 1 & 0 & 1 & 0 \\
\frac{1}{3} & -\frac{1}{3} & -\frac{4}{3} & 1 & 0 & 0 & 0 & 0 & \frac{1}{3} \\
\hline
\end{array}$$

Let x_1 enter the basis and y_1 exit the basis.

$$\begin{array}{|c|cccccccc|}
\hline
0 & 0 & 1 & 0 & 7 & 0 & 2 & 0 & 1 \\
\hline
2 & 1 & 3 & 0 & 4 & 1 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 & -7 & 0 & -1 & 1 & 0 \\
1 & 0 & -\frac{1}{3} & 1 & -\frac{4}{3} & \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\
\hline
\end{array}$$

Let x_2 enter the basis and y_2 exit the basis.

$$\begin{array}{|c|cccccccc|}
\hline
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
\hline
2 & 1 & 0 & 0 & -17 & 1 & -2 & 3 & 0 \\
0 & 0 & 1 & 0 & 7 & 0 & 1 & -1 & 0 \\
1 & 0 & 0 & 1 & \frac{11}{3} & \frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & \frac{1}{3} \\
\hline
\end{array}$$

Now we get an initial BFS $\boldsymbol{x} = (2,0,1,0,0)^T$.
Phase II:

-7	0	0	0	3	-5
2	1	0	0	-17	1
0	0	1	0	7	0
1	0	0	1	$\frac{11}{3}$	$\frac{1}{3}$

Let x_5 enter the basis and x_1 exit the basis.

3	5	0	0	-82	0
2	1	0	0	-17	1
0	0	1	0	7	0
$\frac{1}{3}$	$-\frac{1}{3}$	0	1	$\frac{28}{3}$	0

Let x_4 enter the basis and x_2 exit the basis.

3	5	$\frac{82}{7}$	0	0	0
2	1	$\frac{17}{7}$	0	0	17
0	0	$\frac{1}{7}$	0	1	0
$\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{4}{3}$	1	$\frac{28}{3}$	0

Now we get the optimal solution $\boldsymbol{x} = (0, 0, \frac{1}{3}, 0, 2)^T$, and the optimal cost is -3.

Ex. 3.18

(a)

False. In one iteration of the simplex method, we always choose some j s.t. $\bar{c}_j < 0$. If the solution does move a positive solution, the cost must decrease.

(b)

True. When we implement the full tableau method, we find some j s.t. $\bar{c}_j < 0$. Assume the k-th variable exits the basis. We know in the tableau the element in the pivot row and the pivot column must be positive. Therefore, we must add a positive multiple of the pivot row to the zero-th row. The element in the k-th column and the pivot row is 1, which means after the iteration, $\bar{c}_k$ must be positive, so it cannot reenter in the next iteration.

(c)

False. Consider the example demonstrated in class:

		x_1	x_2	x_3	x_4	x_5	x_6
	0	-10	-12	-12	0	0	0
$x_4 =$	20	1	2	2	1	0	0
$x_5 =$	20	2*	1	2	0	1	0
$x_6 =$	20	2	2	1	0	0	1

Let x_1 enter the basis and x_5 exit the basis. We get:

		x_1	x_2	x_3	x_4	x_5	x_6
	100	0	-7	-2	0	5	0
$x_4 =$	10	0	1.5	1*	1	-0.5	0
$x_1 =$	10	1	0.5	1	0	0.5	0
$x_6 =$	0	0	1	-1	0	-1	1

In class, the example let x_4 exit the basis. However, we can notice that the ratio $x_{B(1)}/u_1$ and $x_{B(2)}/u_2$ are both 10. We can let x_1 exit the basis just after it entered the basis.

(d)

False. Consider the LP problem:

$$\text{minimize } \; x_1 + x_2 + x_3,$$

$$\text{subj. to } \; x_1 + x_2 + x_3 = 1, x_1, x_2, x_3 \geq 0.$$

$\boldsymbol{A}_1, \boldsymbol{A}_2, \boldsymbol{A}_3$ are all nondegenerate basis, but $\boldsymbol{x} = (1,0,0)^T, (0,1,0)^T$ or $(0,0,1)^T$ are all optimal solutions.

(e)

False. Consider the same example as **(d)**. $\boldsymbol{x} = (0.5, 0.5, 0)^T$ is an optimal solution while it has two positive components.