Introduction to Optimization Theory
Homework Assignment 1

Chen Zhiyang, 2017011377

March 2019

Ex. 2.2

Proof:

For any $\boldsymbol{x}, \boldsymbol{y} \in S$, we have $f(\boldsymbol{x}), f(\boldsymbol{y}) \leq c$. f is convex so that for any $\lambda \in [0,1]$, we have

$$f(\lambda \cdot \boldsymbol{x} + (1-\lambda) \cdot \boldsymbol{y}) \leq \lambda f(\boldsymbol{x}) + (1-\lambda) f(\boldsymbol{y}) \leq \lambda \cdot c + (1-\lambda) \cdot c = c,$$

which means $\lambda \cdot \boldsymbol{x} + (1-\lambda) \cdot \boldsymbol{y} \in S$. Therefore, S is convex.

Ex. 2.4

The argument is wrong in that extreme points are representation-dependent. When we convert an arbitrary LP problem to an equivalence in standard form, we might add new variables and new constraints, which may create extreme points.

For example, polyhedron $P = \left\{(x,y) \middle| 0 \leq x \leq 1\right\}$ does not have an extreme point, while its standard form $P' = \left\{(x_0, x_1, y_0, y_1) \middle| x_0 - x_1 = 1, (x_0, x_1, y_0, y_1) \geq \mathbf{0}\right\}$ $(x_0 = x, x_1 = x - 1, y = y_0 - y_1)$ has extreme points.

Ex. 2.6

(a)

Proof:

For any $\boldsymbol{y} \in C$, i.e. $\boldsymbol{y} = \sum_{i=1}^{n} \lambda_i \boldsymbol{A}_i, (\lambda_1, \lambda_2, \ldots, \lambda_n) \geq \mathbf{0}$, we could define polyhedron

$$P = \left\{ (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{R}^n \middle| \sum_{i=1}^{n} \lambda_i \boldsymbol{A}_i = y, (\lambda_1, \lambda_2, \ldots, \lambda_n) \geq \mathbf{0} \right\}.$$

It's obvious that P is non-empty. Notice that P is a polyhedron in standard form, so P does not contain a line, which means P has at least one extreme point (basic feasible solution) $\boldsymbol{\Lambda}^* = (\lambda_1^*, \lambda_2^*, \ldots, \lambda_n^*)$. For a basic feasible solution, there are at most m non-zero components, which means

$$\boldsymbol{y} = \sum_{i=1}^{n} \lambda_i^* \boldsymbol{A}_i$$

is a representation with at most m non-zero coefficients.

(b)

Proof:

Similar to **(a)**, for any $\boldsymbol{y} = \sum_{i=1}^{n} \lambda_i \boldsymbol{A}_i, \sum_{i=1}^{n} \lambda_i = 1, (\lambda_1, \lambda_2, \ldots, \lambda_n) \geq \mathbf{0}$, we could define polyhedron

$$\Lambda = \left\{ (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{R}^n \middle| \sum_{i=1}^{n} \lambda_i \boldsymbol{A}_i = y, \sum_{i=1}^{n} \lambda_i = 1, (\lambda_1, \lambda_2, \ldots, \lambda_n) \geq \mathbf{0} \right\}$$

with $m + 1$ equation constraints. Again, we know that Λ has at least one basic feasible solution, with at most $m+1$ non-zero components, which is a representation of $\boldsymbol{y}$ with at most $m+1$ non-zero coefficients.

Ex. 2.7

Proof:

That $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_m$ span $\mathbb{R}^n$ means there exist n linearly independent vectors in $\left\{ \boldsymbol{a}_i \middle| i = 1, 2, \ldots, m \right\}$ (i.e. $\operatorname{rank} \boldsymbol{A} = \operatorname{rank} [\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_m] = n$), which means the polyhedron does not contain a line. Therefore, there exist n linearly independent vectors that span $\mathbb{R}^n$ in $\left\{ \boldsymbol{g}_i \middle| i = 1, 2, \ldots, k \right\}$, too.

Ex. 2.8

Proof:

$\Rightarrow$:

If a basis is associated with the basic solution $\boldsymbol{x}$, for any i s.t. the i-th column is not in the basis, we set $x_i = 0$. So if $x_i \geq 0$, the i-th column has to be in the basis.

$\Leftarrow$:

If every column $\boldsymbol{A}_i, i \in J$ is in the basis, all columns $\{x_i\}$ that are not in the basis have $x_i = 0$. Therefore, the basis is associated with the solution.

Ex. 2.9

(a)

Proof:

If two different bases lead to the same basic solution, more than $n - m$ variables are zero, which means more than n constraints are active at here.

(b)

False.

Degenerate basic solution may have only one basis, e.g. consider polyhedron in standard form $P = \left\{(x, y) \middle| x = 0, y = 0, (x, y) \geq 0\right\}$. The polyhedron contains only one point, and obviously there's only one choice of basis.

(c)

False.

The counterexample is the same as **(b)**. There's no adjacent basic solution.