ariaDNE: Algorithm Descriptions

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For a continuous 2D manifold embedded in $\mathbb{R}^3$, $f : \Omega \to \mathbb{R}^3$, DNE is defined by

$$\int_\Omega K_1^2 + K_2^2 \; dA,$$

where K_1 and K_2 are the principal curvatures. Given a triangular mesh G with N vertices, the discrete version of DNE is then defined by summing the local curvature across G,

$$\sum_{j=1}^{N} (K_{1,j}^2 + K_{2,j}^2) \cdot \text{Area}(V_j),$$

where the area of V_j ($\text{Area}(V_j)$) is obtained by taking a weighted average of its adjacent face areas.

To estimate the local curvature of V_j, we conduct a weighted PCA. Fix a weight function $f(x)$, for example, the Gaussian kernel $f(x) = e^{-x^2/\epsilon^2}$.

1. Let $\vec{p_j} \in \mathbb{R}^3$ denotes the x, y, z coordinates of V_j. Define the covariance matrix,

   $$C = \sum_{j=1}^{N} (\vec{p}_i - \vec{p}_j) w_{ij} (\vec{p}_i - \vec{p}_j)^T,$$

   where $w_{ij} = f(||\vec{p}_i - \vec{p}_j||)$.

2. Compute eigenvectors and eigenvalues of C,

   $$C\vec{v}_k = \lambda_k \vec{v}_k, k \in \{0, 1, 2\}.$$

   Then $\{\vec{v}_k\}_{k=0}^2$ are the three principal components and $\{\lambda_k\}_{k=0}^2$ are the three principal component scores.

3. Compute the normal of V_j, and call it $\vec{n}_j^0$. Find the principal component that is closest to $\vec{n}_j^0$, say $\vec{v}_{\text{chosen}}$. The span of the other two principal components approximates the plane tangent to the surface at V_j, and the principal component score of $\vec{v}_{\text{chosen}}$ measures how much the surface deviates from that tangent plane, therefore estimating local curvature.

4. The local curvature of V_j is then $\sigma = \lambda_{\text{chosen}}/(\lambda_0 + \lambda_1 + \lambda_2)$.

Finally, the total DNE on the surface is computed by summing the local curvature across the surface:

$$E = \sum_{j=1}^{N} \sigma_j^2 \cdot \mathrm{Area}(V_j)$$