Theorem 1 (Lévy-Cramer simplified)**.** *$X_n \implies X$ iff $\varphi_{X_n} \to \varphi_X$ pointwise.*

Corollary 2. X_n, X are random vectors in $\mathbb{R}^d$, then $X_n \implies X$ iff $\forall_{t\in\mathbb{R}^d} \langle t, X_n\rangle \implies \langle t, X\rangle$.

Remark 3. We know $X \sim \mathcal{N}(a,\sigma^2)$, for which $\varphi_{\mathcal{N}(a,\sigma^2)}(t) = e^{ita-\frac{\sigma^2}{2}t^2}$ and $X \sim a + \sigma Y$ for $Y \sim \mathcal{N}(0,1)$.

Definition 4 (canonical Gaussian distribution)**.** The canonical d-dimensional Gaussian distribution on $\mathbb{R}^d$ is the probability measure γ_d with the density $\frac{d\gamma_d(x)}{dx} = \frac{1}{(2\pi)^{\frac{d}{2}}} e^{-\frac{|x|^2}{2}}$.

Equivalently, we say that a d-dimensional random vector X has the canonical Gaussian distribution if $X \sim \gamma_d$, i.e. $X_1, \ldots, X_d$ are independent with $\mathcal{N}(0,1)$-distribution. We write $X \sim \gamma_d$ or $X \sim \mathcal{N}(0, I_d)$.

Definition 5 (pushforward)**.** μ – a measure on $\mathbb{R}^m$, $F : \mathbb{R}^m \to \mathbb{R}^d$ measurable,
$F_\#\mu$ – *a pushforward of* μ is the measure on $\mathbb{R}^d$ given by $F_\#\mu(A) = \mu(F^{-1}(A))$.

Definition 6 (Gaussian vector)**.** A probability measure μ on $\mathbb{R}^d$ is called *Gaussian* or *normal* iff there exists affine $U : \mathbb{R}^m \to \mathbb{R}^d$ such that $\mu = U_\#\gamma_m$.

A d-dimensional random vector X is called Gaussian if its law is Gaussian, i.e. $X \sim UY$, $Y \sim \gamma_m$.

Remark 7. $X \sim UY = AY + b$, then $\mathbb{E}X = b, \mathrm{Cov}(X) = AA^T$.

Remark 8. $X \sim AY + b$, then $\varphi_X(t) = e^{i\langle \mathbb{E}X, t\rangle - \frac{\langle \mathrm{Cov}(x)t, t\rangle}{2}}$

Definition 9 (Gaussian again)**.** We say that a random d-dimensional vector X (respectively, a probability measure μ on $\mathbb{R}^d$) is *Gaussian* iff $\varphi_X(t) = e^{i\langle b,t\rangle - \frac{\langle Ct, t\rangle}{2}}$, where $b \in \mathbb{R}^d$, $C \in M_{d\times d}, C = C^T, C \geqslant 0$.

Proposition 10. *These definitions are equivalent.*

Corollary 11. Every d-dimensional Gaussian vector is an affine image of d-dimensional canonical Gaussian vector.

Remark 12. Equivalently, $\langle t, X\rangle$ is Gaussian for any $t \in \mathbb{R}^d$.

Corollary 13. $X_1 \sim \mathcal{N}(b_1, C_1), X_2 \sim \mathcal{N}(b_2, C_2)$ independent Gaussian, then $X_1 + X_2 \sim \mathcal{N}(b_1 + b_2, C_1 + C_2)$ also Gaussian.

Corollary 14. Affine image of a Gaussian vector is a Gaussian vector.

Remark 15. $X \sim \mathcal{N}(b, C)$ has the density iff $\det(C) \neq 0$ and then $g(x) = \frac{1}{(2\pi)^{\frac{d}{2}}\sqrt{\det C}} e^{-\frac{\langle C^{-1}(x-b),(x-b)\rangle}{2}}$.

Theorem 16. *If X is a Gaussian vector in $\mathbb{R}^d$, then coordinates of X are independent iff they are uncorrelated (i.e. $X_1, \ldots, X_d$ independent iff $\mathrm{Cov}(X_j, X_k) = 0$ for $j \neq k$).*

Remark 17. In the theorem it is important thet the whole vector is Gaussian, not only its coordinates!

Theorem 18 (CTL in the iid case)**.** *$X_1, X_2, \ldots$ iid random variables, $\mathbb{E}X_1 = a$, $\mathrm{Var}(X_1) = \sigma^2 \in (0,\infty)$, then $\frac{X_1+\ldots+X_n-na}{\sqrt{n}\sigma} \implies \mathcal{N}(0,1)$.*