Theorem 1 (CLT Lindeberg-Levy). *Let $(X_{n,k})$ be a triangular array, $\sum_{k=1}^{k_n} X_{n,k} \to m$, $\sum_{k=1}^{k_n} \mathrm{Var}(X_{n,k}) \to \sigma^2$, and Lindeberg's condition be satisfied:*

$$\forall_{\varepsilon>0} \sum_{k=1}^{k_n} \mathbb{E}|X_{n,k} - \mathbb{E}X_{n,k}|^2 \mathbb{1}_{\{|X_{n,k}-\mathbb{E}X_{n,k}|>\varepsilon\}} \to 0.$$

Then

$$S_n = \sum_{k=1}^{k_n} X_{n,k} \implies \mathcal{N}(m, \sigma^2).$$

Proposition 2. *For $X_1, X_2, \ldots$ iid, $\mathrm{Var}(X_1) < \infty$, array $X_{n,k} = \dfrac{X_k - \mathbb{E}X_k}{\sqrt{n}}$ satisfies Lindeberg's condition.*

Proposition 3 (Lyapunov condition). *Lyapunov's condition*

$$\exists_{\delta>0} \sum_{k=1}^{k_n} \mathbb{E}|X_{n,k} - \mathbb{E}X_{n,k}|^{2+\delta} \to 0$$

implies Lindeberg's condition (under CLT's assumptions about $\mathbb{E}$ and Var of sums).

Corollary 4. For $\sum \mathbb{E}X_{n,k} \to 0, \sum \mathrm{Var}(X_{n,k}) \to 1$, it follows that

$$\sup_t \left| \mathbb{P}\left(\sum X_{n,k} \leqslant t \right) - \phi(t) \right| \to 0,$$

and in particular $\mathbb{P}(\sum X_{n,k} \leqslant t) \to \phi(t) = \mathbb{P}(\mathcal{N}(0,1) \leqslant t)$.

Theorem 5 (Berny-Esseen). *$X_1, X_2, \ldots, X_n$ independent, $\mathbb{E}X_i = 0, \sum_{k=1}^n \mathrm{Var}(X_k) = 1$, then*

$$\left| \mathbb{P}\left(\sum_{k=1}^n X_k \leqslant t \right) - \phi(t) \right| \leqslant C_1 \sum_{k=1}^n \mathbb{E}|X_k|^3.$$

Corollary 6. In particular, for $X_1, \ldots, X_n$ iid, $\mathbb{E}X_1 = m$, $\mathrm{Var}(X_1) = \sigma^2$, we have

$$\left| \mathbb{P}\left(\frac{\sum_{k=1}^n X_k - nm}{\sqrt{n}\sigma} \leqslant t \right) - \phi(t) \right| \leqslant C_2 \sum_{k=1}^n \mathbb{E} \left| \frac{X_k - \mathbb{E}X_k}{\sqrt{n}\sigma} \right|^3 = C_2 \frac{\mathbb{E}|X_1 - \mathbb{E}X_1|^3}{\sqrt{n}\sigma^3}.$$

Remark 7. Obviously $C_2 \leqslant C_1$, it is known that $C_1, C_2 \geqslant \frac{\sqrt{10}+e}{6\sqrt{2\pi}} \approx 0.4097$, and also $C_1 \leqslant 0.56, C_2 \leqslant 0.4748$.

Example 8 (de Moivre-Laplace). $S_n \sim \mathrm{Bin}(n,p)$, $\left| \mathbb{P}(S_n \leqslant t) - \phi\left(\frac{t-np}{\sqrt{np(1-p)}} \right) \right| \leqslant C_2 \frac{p^2+(1-p)^2}{\sqrt{np(1-p)}}$.

Definition 9. $\mathcal{G} \subset \mathcal{F}$ a σ-field, X random variable, $\mathbb{E}|X| < \infty$, then $Z = \mathbb{E}(X|\mathcal{G})$ is a random variable such that it is $\mathcal{G}$-measurable, $\forall_{A\in\mathcal{G}} \mathbb{E}(Z\mathbb{1}_A) = \mathbb{E}(X\mathbb{1}_A)$.

Proposition 10. *The above exists and is unique up to a set of probability 0.*

Proposition 11. *Y random variable, X measurable with respect to $\sigma(Y)$, then $X = h(Y)$ for some Borel function h.*

Definition 12. $\mathbb{E}(X|Y) = \mathbb{E}(X|\sigma(Y))$ for X, Y random variables, $\mathbb{E}|X| < \infty$.