Proposition 1 (1.1). $(\delta_{x_n} \implies \delta_x) \iff x_n \to x$

Proposition 2 (1.2). $\frac{1}{n} \sum_1^n \delta_{k/n} \implies \lambda$, λ *Lebesgue measure on* $[0, 1]$.

Proposition 3 (1.3). *If* $X_n \to X$ *a.s., then* $X_n \implies X$.
    If $X_n \xrightarrow{\mathbb{P}} X$, *then* $X_n \implies X$.
    If $X_n \implies c$, *then* $X_n \xrightarrow{\mathbb{P}} c$.

Proposition 4 (1.7). *If* $np_n \to \lambda$, *then* $\mathrm{Bin}(p_n, n) \implies \mathrm{Poiss}(\lambda)$.

Proposition 5 (1.12). $\mathcal{N}(a_n, \sigma_n^2) \implies \mathcal{N}(a, \sigma^2)$ *iff* $a_n \to a$ *and* $\sigma_n^2 \to \sigma^2$.

Proposition 6 (2.1). *Let* X *have density and* $a_n, a \geqslant 0$, *then* $a_n X + b_n \implies aX + b$ *iff* $a_n \to a$ *and* $b_n \to b$.

Proposition 7 (2.2). *If* f *continuous,* $X_n \implies X$, *then* $f(X_n) \implies f(X)$.

Theorem 8 (Scheffe; problem 2.4). *If* $g_n \to g$ *a.s., then* $\mu_n \implies \mu$.

Proposition 9 (2.7). $X_n \implies X$, X *has continuous distribution, then* $F_{X_n} \rightrightarrows F_X$.

Proposition 10 (3.4). *Convex combinations of characteristic functions are characteristic functions.*

Proposition 11 (3.7). *If* φ_X *has second derivative in* 0, *then* $\mathbb{E}X^2 < \infty$.

Proposition 12 (5.9). *If* $X_n \implies X, Y_n \implies c$, *then* $X_n + Y_n \implies c + X$ *and* $X_n Y_n \implies cX$.

Proposition 13 (6.5). $X_1, X_2, \ldots$ *iid,* $\mathbb{E}X_1 = 0$, $\mathrm{Var} X_1 = 1$, (a_n) *bounded sequence,* $s_n = \sqrt{a_1^2 + \ldots + a_n^2} \to \infty$, *then* $s_n^{-1}(a_1 X_1 + \ldots + a_n X_n) \implies \mathcal{N}(0, 1)$.

Proposition 14 (6.7). $X \sim \mathcal{N}(a, B)$ *has density iff* B *is nondegenerate and then* $g_X(x) = \frac{\sqrt{\det C}}{(2\pi)^{d/2}} \exp\left(\frac{\langle C(x - a), x - a \rangle}{2}\right)$ *for* $C = B^{-1}$.

Proposition 15 (7.6). X, Y *independent,* f *Borel function,* $\mathbb{E}|f(X, Y)| < \infty$, *then* $\mathbb{E}(f(X, Y)|Y) = g(Y)$ *a.s. where* $g(y) = \mathbb{E}(f(X, y))$.

Proposition 16 (7.7). X, Y *integrable iid, then* $\mathbb{E}(X|X + Y) = \mathbb{E}(Y|X + Y) = \frac{1}{2}(X + Y)$.

Proposition 17 (8.1). τ, σ *are stopping times for* $(\mathcal{F}_n)$. *Then* $\tau + \sigma$ *is too. Notice* $\sigma = 1$ *is a stopping time, so* $\tau + 1$ *is too. Notice* $\tau - 1$ *in not a stopping time in general.*

Proposition 18 (8.2). *Let* X_n *be* $(\mathcal{F}_n)$*-adapted,* B *a borel set. Then* $\tau_1 = \inf\{n : X_n \in B\}$ *is a stopping time. And* $\tau_k = \inf\{n > \tau_{k-1} : X_n \in B\}$ *is also.*

Proposition 19 (8.3). τ, σ *stopping times with respect to* $(\mathcal{F}_n)$,
then $\{\tau < \sigma\}, \{\tau \leqslant \sigma\}, \{\tau = \sigma\} \in \mathcal{F}_\tau \cap \mathcal{F}_\sigma$ *and* $\mathcal{F}_\tau \cap \mathcal{F}_\sigma = \mathcal{F}_{\tau \wedge \sigma}$.

Proposition 20 (8.5). $X_1, X_2, \ldots$ *iid,* $\mathbb{E}X_1 = 0, \mathrm{Var}X_1 < \infty$, $S_n = X_1 + \ldots + X_n$.
    Then S_n, $S_n^2 - \mathrm{Var}(S_n)$ *are martingales with respect to the filtration generated by* (X_n).

Proposition 21 (9.1). *Let* X_n *be a martingale.*
    Then $|X_n|^p$ *for* $p \geqslant 1$ *is a submartingale.*
    And $X_n \wedge a$ *is a supermartingale.*
    And $X_n \vee a$ *is a submartingale.*

Proposition 22 (normal distribution). $X \sim \mathcal{N}(\mu, \sigma^2)$

$$\mathbb{E}X = \mu \qquad \mathrm{Var}X = \sigma^2 \qquad g(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-\mu)^2}{2\sigma^2}} \qquad \varphi(t) = e^{i\mu t - \frac{1}{2}\sigma^2 t^2}$$

Proposition 23 (binomial distribution). $X \sim \mathrm{Bin}(n, p)$

$$\mathbb{E}X = pn \qquad \mathrm{Var}X = np(1-p) \qquad g(k) = \binom{n}{k}p^k(1-p)^{n-k} \qquad \varphi(t) = (1 - p + pe^{it})^n$$

Proposition 24 (geometric distribution – 1st success). $X \sim \mathrm{Geom}(p)$

$$\mathbb{E}X = \frac{1}{p} \qquad \mathrm{Var}X = \frac{1-p}{p^2} \qquad g(k) = (1-p)^{k-1}p \qquad \varphi(t) = \frac{pe^{it}}{1-(1-p)e^{it}}$$

Proposition 25 (Poisson distribution). $X \sim \mathrm{Poiss}(\lambda)$

$$\mathbb{E}X = \lambda \qquad \mathrm{Var}X = \lambda \qquad g(x) = \frac{\lambda^k}{k!}e^{-\lambda} \qquad \varphi(t) = e^{\lambda(e^{it}-1)}$$

Proposition 26 (exponential distribution). $X \sim \mathrm{Exp}(\lambda)$

$$\mathbb{E}X = \frac{1}{\lambda} \qquad \mathrm{Var}X = \frac{1}{\lambda^2} \qquad g(x) = \lambda e^{-\lambda x} \qquad \varphi(t) = \left(1 - \frac{it}{\lambda}\right)^{-1} = \frac{\lambda}{\lambda - it}$$

Proposition 27 (Cauchy distribution). $X \sim \mathrm{Cauchy}(h)$
    Not integrable!

$$g(x) = \frac{h}{\pi(h^2 + x^2)} \qquad \varphi(t) = e^{-h|t|}$$