

**Theorem 1** (Lévy-Cramer simplified).  $X_n \implies X$  iff  $\varphi_{X_n} \rightarrow \varphi_X$  pointwise.

**Corollary 2.**  $X_n, X$  are random vectors in  $\mathbb{R}^d$ , then  $X_n \implies X$  iff  $\forall_{t \in \mathbb{R}^d} \langle t, X_n \rangle \implies \langle t, X \rangle$ .

*Remark 3.* We know  $X \sim \mathcal{N}(a, \sigma^2)$ , for which  $\varphi_{\mathcal{N}(a, \sigma^2)}(t) = e^{ita - \frac{\sigma^2}{2}t^2}$  and  $X \sim a + \sigma Y$  for  $Y \sim \mathcal{N}(0, 1)$ .

**Definition 4** (canonical Gaussian distribution). The canonical  $d$ -dimensional Gaussian distribution on  $\mathbb{R}^d$  is the probability measure  $\gamma_d$  with the density  $\frac{d\gamma_d(x)}{dx} = \frac{1}{(2\pi)^{\frac{d}{2}}} e^{-\frac{|x|^2}{2}}$ .

Equivalently, we say that a  $d$ -dimensional random vector  $X$  has the canonical Gaussian distribution if  $X \sim \gamma_d$ , i.e.  $X_1, \dots, X_d$  are independent with  $\mathcal{N}(0, 1)$ -distribution. We write  $X \sim \gamma_d$  or  $X \sim \mathcal{N}(0, I_d)$ .

**Definition 5** (pushforward).  $\mu$  – a measure on  $\mathbb{R}^m$ ,  $F : \mathbb{R}^m \rightarrow \mathbb{R}^d$  measurable,  $F_{\#}\mu$  – a pushforward of  $\mu$  is the measure on  $\mathbb{R}^d$  given by  $F_{\#}\mu(A) = \mu(F^{-1}(A))$ .

**Definition 6** (Gaussian vector). A probability measure  $\mu$  on  $\mathbb{R}^d$  is called *Gaussian* or *normal* iff there exists affine  $U : \mathbb{R}^m \rightarrow \mathbb{R}^d$  such that  $\mu = U_{\#}\gamma_m$ .

A  $d$ -dimensional random vector  $X$  is called Gaussian if its law is Gaussian, i.e.  $X \sim UY$ ,  $Y \sim \gamma_m$ .

*Remark 7.*  $X \sim UY = AY + b$ , then  $\mathbb{E}X = b$ ,  $\text{Cov}(X) = AA^T$ .

*Remark 8.*  $X \sim AY + b$ , then  $\varphi_X(t) = e^{i\langle \mathbb{E}X, t \rangle - \frac{\langle \text{Cov}(X)t, t \rangle}{2}}$

**Definition 9** (Gaussian again). We say that a random  $d$ -dimensional vector  $X$  (respectively, a probability measure  $\mu$  on  $\mathbb{R}^d$ ) is *Gaussian* iff  $\varphi_X(t) = e^{i\langle b, t \rangle - \frac{\langle Ct, t \rangle}{2}}$ , where  $b \in \mathbb{R}^d$ ,  $C \in M_{d \times d}$ ,  $C = C^T$ ,  $C \geq 0$ .

**Proposition 10.** *These definitions are equivalent.*

**Corollary 11.** Every  $d$ -dimensional Gaussian vector is an affine image of  $d$ -dimensional canonical Gaussian vector.

*Remark 12.* Equivalently,  $\langle t, X \rangle$  is Gaussian for any  $t \in \mathbb{R}^d$ .

**Corollary 13.**  $X_1 \sim \mathcal{N}(b_1, C_1)$ ,  $X_2 \sim \mathcal{N}(b_2, C_2)$  independent Gaussian, then  $X_1 + X_2 \sim \mathcal{N}(b_1 + b_2, C_1 + C_2)$  also Gaussian.

**Corollary 14.** Affine image of a Gaussian vector is a Gaussian vector.

*Remark 15.*  $X \sim \mathcal{N}(b, C)$  has the density iff  $\det(C) \neq 0$  and then  $g(x) = \frac{1}{(2\pi)^{\frac{d}{2}} \sqrt{\det C}} e^{-\frac{\langle C^{-1}(x-b), (x-b) \rangle}{2}}$ .

**Theorem 16.** If  $X$  is a Gaussian vector in  $\mathbb{R}^d$ , then coordinates of  $X$  are independent iff they are uncorrelated (i.e.  $X_1, \dots, X_d$  independent iff  $\text{Cov}(X_j, X_k) = 0$  for  $j \neq k$ ).

*Remark 17.* In the theorem it is important that the whole vector is Gaussian, not only its coordinates!

**Theorem 18** (CTL in the iid case).  $X_1, X_2, \dots$  iid random variables,  $\mathbb{E}X_1 = a$ ,  $\text{Var}(X_1) = \sigma^2 \in (0, \infty)$ , then  $\frac{X_1 + \dots + X_n - na}{\sqrt{n}\sigma} \implies \mathcal{N}(0, 1)$ .