Definition 1 (filtration)**.** $(\mathcal{F}_t)_T$, for T a segment in $\mathbb{Z}$, is a *filtration*, if $\mathcal{F}_t \subset \mathcal{F}$ is a σ-body and $\forall_{t \leqslant s} \mathcal{F}_t \subset \mathcal{F}_s$.

Definition 2 (stopping time)**.** $\tau : \omega \to T \cup \{\infty\}$ is a *stopping time*, if $\forall_{t \in T} \{\tau \leqslant t\} \in \mathcal{F}_t$ ($\iff \forall_{t \in T} \{\tau = t\} \in \mathcal{F}_t\}$).

Definition 3. Let $(\mathcal{F}_t)$ be a filtration, τ be a stopping time, then define

$$\mathcal{F}_\tau = \{A \in \mathcal{F} : A \cap \{\tau \leqslant t\} \in \mathcal{F}_t\}.$$

Proposition 4. $\mathcal{F}_\tau = \{A \in \mathcal{F} : A \cap \{\tau = t\} \in \mathcal{F}_t\}$.

Proposition 5. *τ_1, τ_2 stopping times, then $\tau_1 \wedge \tau 2 = \min(\tau_1, \tau_2)$ and $\tau_1 \vee \tau_2 = \max(\tau_1, \tau_2)$ are too.*

$\tau = t$ is a stopping time.

$\tau_1 \leqslant \tau_2$ stopping times $\implies \mathcal{F}_{\tau_1} \subset \mathcal{F}_{\tau_2}$

τ jest $\mathcal{F}_\tau$-mierzalne.

Definition 6 (adapted process)**.** $(X_t)_{t \in T}$ is *adapted to the filtration* $(\mathcal{F}_t)_{t \in T}$ or just $(\mathcal{F}_t)$-*adapted*, if $\forall_t \ X_t$ is $\mathcal{F}_t$-measurable.

Proposition 7. *$(\mathcal{F}_t)$ filtration, (X_t) is $(\mathcal{F}_t)$-adapted, τ a stopping time, then $\tau < \infty \implies X_\tau$ is $\mathcal{F}_\tau$-measurable.*

More generally, X_τ is $\mathcal{F}_\tau$-measurable on the set $\{\tau < \infty\}$, i.e. $\forall_{B \in \mathcal{B}(\mathbb{R})} \{X_\tau \in B\} \cap \{t < \infty\} \in \mathcal{F}_\tau$.

Definition 8 (martingale)**.** (X_t) is a *martingale* (resp. submartingale, supermartingale) with respect to a foltration $(\mathcal{F}_t)$, if

- $\forall_{t \in T} \ X_t$ is $\mathcal{F}_t$-measurable,
- $\forall_{t \in T} \mathbb{E}|X_t| < \infty$,
- $\forall_{s \leqslant t, s, t \in T} \mathbb{E}(X_t | \mathcal{F}_s) = X_s$ a.s. (resp. $\geqslant$, $\leqslant$).