Example 1. (X, Y) has density $g_{(X,Y)}(x, y)$. Then

$$\mathbb{E}(X|Y) = \varphi(y) = \frac{\int x g(x, y)\, dx}{\int g(x, y)\, dy}.$$

Proposition 2. *Conditional Expected Value is linear.*

Proposition 3. $\mathbb{E}(\mathbb{E}(X|\mathcal{G})) = \mathbb{E}X$

Proposition 4. $X_1 \geqslant X_2 \implies \mathbb{E}(X_1|\mathcal{G}) \geqslant \mathbb{E}(X_2|\mathcal{G})$ *almost surely.*

Proposition 5. $|\mathbb{E}(X|\mathcal{G})| \leqslant \mathbb{E}(|X|\,|\mathcal{G})$ *a.s., in particular CEV is a contraction in* L^1 *because* $\mathbb{E}|\mathbb{E}(X|\mathcal{G})| \leqslant \mathbb{E}|X|$.

Proposition 6. X *is* $\mathcal{G}$*-measurable, then* $\mathbb{E}(X|\mathcal{G}) = X$ *a.s.*

Proposition 7. $X \perp Y \implies \mathbb{E}(X|\mathcal{G}) = \mathbb{E}X$ *a.s.*
In particular $\mathbb{E}(X|\{\varnothing, \Omega\}) = \mathbb{E}X$.

Proposition 8 (Lebesgue monotone convergence thm)**.** $0 \leqslant X_n \nearrow X$, X_n, X *integrable, then* $\mathbb{E}(X_n|\mathcal{G}) \nearrow \mathbb{E}(X|\mathcal{G})$ *a.s.*

Proposition 9 (Lebesgue dominated convergence thm)**.** $|X_n| \leqslant Y$, $\mathbb{E}Y < \infty$, $X_n \to X$ *a.s., then* $\lim_{n\to\infty} \mathbb{E}(X_n|\mathcal{G}) = \mathbb{E}(X|\mathcal{G})$ *a.s.*

Proposition 10 (Fatou lemma)**.** X_n *integrable, then* $\liminf_{n\to\infty} \mathbb{E}\left(X_n|\mathcal{G}\right) \geqslant \mathbb{E}\left(\liminf_{n\to\infty} X_n|\mathcal{G}\right)$.

Proposition 11. X *is* $\mathcal{G}$*-measurable,* $\mathbb{E}|Y| < \infty, \mathbb{E}|XY| < \infty$*, then* $\mathbb{E}(XY|\mathcal{G}) = X\mathbb{E}(Y|\mathcal{G})$ *a.s.*

Proposition 12. X *integrable,* $\mathcal{G}_1 \subset \mathcal{G}_2$*, then* $\mathbb{E}(X|\mathcal{G}_1) = \mathbb{E}(\mathbb{E}(X|\mathcal{G}_2)|\mathcal{G}_1) = \mathbb{E}(\mathbb{E}(X|\mathcal{G}_1)|\mathcal{G}_2)$.

Proposition 13 (Jensen inequality)**.** $\varphi : \mathbb{R} \to \mathbb{R}$ *convex,* $\mathbb{E}|X| < \infty$*,* $\mathbb{E}|\varphi(X)| < \infty$*, then* $\mathbb{E}(\varphi(X)|\mathcal{G}) \geqslant \varphi(\mathbb{E}(X|\mathcal{G}))$.
In particular $\mathbb{E}\varphi(X) \geqslant \varphi(\mathbb{E}X)$.

Corollary 14. $\mathbb{E}|X|^p < \infty$, $p \geqslant 1$, then $|\mathbb{E}(X|\mathcal{G})|^p \leqslant \mathbb{E}\left(|X|^p|\mathcal{G}\right)$.
In particular $\mathbb{E}(\bullet|\mathcal{G})$ is a contraction in L^p.

Filtrations and stopping times
$T \subset \mathbb{Z}$

Definition 15 (filtration)**.** A *filtration* is a sequence of σ-bodies $(\mathcal{F}_t)_{t\in T}$ such that $\mathcal{F}_t \subset \mathcal{F}$ and $\mathcal{F}_t \subset \mathcal{F}_s$ for $t < s$.

Definition 16 (stopping time)**.** A *stopping time* with respect to filtration $(\mathcal{F}_t)_{t\in T}$ is a random variable $\tau : \Omega \to T \cup \{\infty\}$ such that $\forall_{t\in T}\{\tau \leqslant t\} \in \mathcal{F}_t$.

Proposition 17. $\tau : \Omega \to T \cup \{\infty\}$ *is a stopping time iff* $\forall_{t\in T}\{\tau = t\} \in \mathcal{F}_t$.