Definition 1. μ_n, μ probability measures on $(E, \mathcal{B}(E))$, where (E, ρ) is a metric space. We say that $\mu_n \implies \mu$, that is μ_n *converges in law/in distribution/weakly* to μ, iff $\forall_{f \in C_b(E)} \int_E f\, d\mu_n \to \int_E f\, d\mu$.

Theorem 2. *The following are equivalent:*

1. $\mu_n \implies \mu$,
2. $\forall_f$ *if f is uniformly continuous and bounded on E, then* $\int_E f\, d\mu_n \to \int f\, d\mu$,
3. $\forall_{G \subset E \text{ open}} \liminf_{n\to\infty} \mu_n(G) \geqslant \mu(G)$,
4. $\forall_{F \subset E \text{ closed}} \limsup_{n\to\infty} \mu_n(F) \leqslant \mu(F)$,
5. $\forall_{A \in \mathcal{B}(E)}\ \mu(\partial A) = 0 \implies \lim_{n\to\infty} \mu_n(A) = \mu(A)$.

Definition 3. μ probability measure on $\mathbb{R}^d$, then its distribution function is $F_\mu(t) = \mu((-\infty, t_1] \times \ldots \times (-\infty, t_n])$.

Theorem 4. $\mu_n \implies \mu$ *iff* $\forall_t$ *if F_μ is continuous at t, then* $\lim_{n\to\infty} F_{\mu_n}(t) = F_\mu(t)$.

Lemma 5. *μ_n, μ probability measures on $(E, \mathcal{B}(E))$, $\mathcal{A}$ a π-system of Borel sets (i.e. $\forall_{A,B\in\mathcal{A}} A \cap B \in \mathcal{A}$) such that any open set in E is a union of at most countably many sets in $\mathcal{A}$, and $\forall_{A\in\mathcal{A}} \mu_n(A) \to \mu(A)$, then $\mu_n \implies \mu$.*

Proposition 6. *μ_n, μ probability measures on $\mathbb{R}^d$, then $\mu_n \implies \mu$ iff* $\forall_{f \in C_c(\mathbb{R}^d)} \int_E f\, d\mu_n \to \int_E f\, d\mu$.

Definition 7. X_n, X random variables with values in metric space (E, ρ), then X_n converges to X *in law (or in distribution, or weakly)* iff $\mu_{X_n} \implies \mu_X$. We write $X_n \xrightarrow{\mathcal{L}} X$ or $X_n \xrightarrow{d} X$ or $X_n \implies X$.

Remark 8. $\int_E f(t)\, \mu_X(t) = \mathbb{E} f(X)$, so $X_n \implies X$ iff $\forall_{f \in C_b(E)} \mathbb{E} f(X_n) \to \mathbb{E} f(X)$.

Remark 9. X_n and X may be defined on distinct spaces.

Remark 10. X_n may have the same law and be very different.

Remark 11. Convergence in probability implies convergence in law, but not conversely.

Proposition 12. *X_n, Y_n, X random variables with values in (E, ρ), $X_n \implies X$, $\rho(X_n, Y_n) \xrightarrow{\mathbb{P}} 0$.*
Then $Y_n \implies X$.

Corollary 13. $Y_n \xrightarrow{\mathbb{P}} X$ implies $Y_n \implies X$.

Definition 14. $\{\mu_i\}_{i\in I}$ a family of probability measures on (E,ρ), we say that this family is *tight* iff $\forall_{\varepsilon>0}\exists_{K \text{ compact in } E}\forall_{i\in I}\mu_i(K) \geqslant 1-\varepsilon$.

Definition 15. $\{X_i\}$ tight iff $\{\mu_{X_i}\}$ tight.

Example 16. $E=\mathbb{R}, \mu_n=\delta_n$ not tight.

Remark 17. $E=\mathbb{R}, \{\mu_i\}$ tight iff $\forall_{\varepsilon>0}\exists_M\forall_{i\in I}\mu_i([-M,M]) \geqslant 1-\varepsilon$.

Remark 18. X_i random variables in $\mathbb{R}$, $p>0$, $\sup_i \mathbb{E}|X_i|^p < \infty \implies \{X_i\}_{i\in I}$ is tight.

Theorem 19 (Prokhorov)**.** *$\{\mu_i\}_{i\in I}$ probability measures on $\mathbb{R}^d$. This family is tight iff for any sequence of measures μ_k in the family one may choose a weakly convergent subsequence μ_{k_l}.*