Definition 1 (filtration)**.** $(\mathcal{F}_t)_T$, for T a segment in $\mathbb{Z}$, is a *filtration*, if $\mathcal{F}_t \subset \mathcal{F}$ is a σ-body and $\forall_{t \leqslant s} \mathcal{F}_t \subset \mathcal{F}_s$.

Definition 2 (stopping time)**.** $\tau : \omega \to T \cup \{\infty\}$ is a *stopping time*, if $\forall_{t \in T} \{\tau \leqslant t\} \in \mathcal{F}_t$ ($\iff \forall_{t \in T} \{\tau = t\} \in \mathcal{F}_t$).

Definition 3. Let $(\mathcal{F}_t)$ be a filtration, τ be a stopping time, then define

$$\mathcal{F}_\tau = \{A \in \mathcal{F} : A \cap \{\tau \leqslant t\} \in \mathcal{F}_t\}.$$

Proposition 4. $\mathcal{F}_\tau = \{A \in \mathcal{F} : A \cap \{\tau = t\} \in \mathcal{F}_t\}$.

Proposition 5. *τ_1, τ_2 stopping times, then $\tau_1 \wedge \tau 2 = \min(\tau_1, \tau_2)$ and $\tau_1 \vee \tau_2 = \max(\tau_1, \tau_2)$ are too.*

$\tau = t$ is a stopping time.

$\tau_1 \leqslant \tau_2$ stopping times $\implies \mathcal{F}_{\tau_1} \subset \mathcal{F}_{\tau_2}$

τ jest $\mathcal{F}_\tau$-mierzalne.

Definition 6 (adapted process)**.** $(X_t)_{t \in T}$ is *adapted to the filtration* $(\mathcal{F}_t)_{t \in T}$ or just $(\mathcal{F}_t)$-*adapted*, if $\forall_t\ X_t$ is $\mathcal{F}_t$-measurable.

Proposition 7. *$(\mathcal{F}_t)$ filtration, (X_t) is $(\mathcal{F}_t)$-adapted, τ a stopping time, then $\tau < \infty \implies X_\tau$ is $\mathcal{F}_\tau$-measurable.*

More generally, X_τ is $\mathcal{F}_\tau$-measurable on the set $\{\tau < \infty\}$, i.e. $\forall_{B \in \mathcal{B}(\mathbb{R})} \{X_\tau \in B\} \cap \{t < \infty\} \in \mathcal{F}_\tau$.

Definition 8 (martingale)**.** (X_t) is a *martingale* (resp. submartingale, supermartingale) with respect to a foltration $(\mathcal{F}_t)$, if

- $\forall_{t\in T}\ X_t$ is $\mathcal{F}_t$-measurable,
- $\forall_{t\in T}\mathbb{E}|X_t| < \infty$,
- $\forall_{s\leqslant t, s,t\in T}\mathbb{E}(X_t|\mathcal{F}_s) = X_s$ a.s. (resp. $\geqslant$, $\leqslant$).

Remark 9. X_t is a martingale iff it is both a submartingale and a supermartingale.

Remark 10. (X_t) is a $(\mathcal{F}_t)$-martingale if X_t is $\mathcal{F}_t$-measurable, integrable and $\forall_{s<t, A\in\mathcal{F}_s}\mathbb{E}(X_s\mathbb{1}_A) = \mathbb{E}(X_t\mathbb{1}_A)$ (resp. $\leqslant$ for submartingale, $\geqslant$ for supermartingale).

Remark 11. For T a segment in $\mathbb{Z}$, (X_t) is $(\mathcal{F}_t)$-martingale iff X_t is $\mathcal{F}_t$-measurable, $\mathbb{E}|X_t| < \infty$, $\mathbb{E}(X_{s+1}|\mathcal{F}_s) = X_s$ a.s. (resp. $\geqslant$ for submartingale, $\leqslant$ for supermartingale).

Remark 12. X_t submartingale iff $-X_t$ supermartingale.

Remark 13. X_t, Y_t are $\mathcal{F}_t$-martingales, then $aX_t + bY_t$ also (for submartingale take $a, b \geqslant 0$).

Definition 14. $(F_n)_{n\geqslant 0}$ filtration generated by $(X_1, X_2, \ldots)$, $\mathcal{F}_0 = \{\varnothing, \Omega\}$, $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$.

Example 15. $X_1, X_2, \ldots$ independent random variables, $S_0 = 0$, $S_n = X_1 + \ldots + X_n$, $\mathcal{F}_n$ filtration generated by (X_n).

Then S_n is a martingale iff X_n are integrable and $\mathbb{E}X_n = 0$ ($\geqslant$ for submartingale, $\leqslant$ for supermartingale).

Example 16. X integrable random variable, $(\mathcal{F}_t)$ filtration, $X_t = \mathbb{E}(X|\mathcal{F}_t)$ is a $(\mathcal{F}_t)$-martingale.

Example 17. (X_t) is a $(\mathcal{F}_t)$-martingale, $\varphi : \mathbb{R} \to \mathbb{R}$ convex, $\mathbb{E}|\varphi(X_t)| < \infty$, then $(\varphi(X_t), \mathcal{F}_t)$ is a submartingale.

Corollary 18. $(X_t, \mathcal{F}_t)$ a martingale, $p \geqslant 1$, $\mathbb{E}|X_t|^p < \infty$, then $(|X_t|^p, \mathcal{F}_t)$ is a submartingale.

Corollary 19. $(X_t, \mathcal{F}_t)$ submartingale, then $(X_t \vee a, \mathcal{F}_t)$ submartingale.

Corollary 20. $(X_t, \mathcal{F}_t)$ martingale, then $(X_t^+, \mathcal{F}_t)$ and $(X_t^-, \mathcal{F}_t)$ submartingales (where $Y^+ = Y \wedge 0$, $Y^- = (-Y) \wedge 0$.

Example 21 (martingale transformation). $(X_n, \mathcal{F}_n)$ martingale, let $Y_n = X_0 + V_1(X_1 - X_0) + V_2(X_2 - X_1) + \ldots + V_n(X_n - X_{n-1})$ for V_n being $(\mathcal{F}_{n-1})$-measurable and bounded, then $(Y_n, \mathcal{F}_n)$ is a martingale.