Proposition 1 (1.1)**.** $(\delta_{x_n} \implies \delta_x) \iff x_n \to x$

Proposition 2 (1.2)**.** $\frac{1}{n}\sum_1^n \delta_{k/n} \implies \lambda$, *$\lambda$ Lebesgue measure on* $[0,1]$.

Proposition 3 (1.3)**.** *If $X_n \to X$ a.s., then $X_n \implies X$.*
If $X_n \xrightarrow{\mathbb{P}} X$, then $X_n \implies X$.
If $X_n \implies c$, then $X_n \xrightarrow{\mathbb{P}} c$.

Proposition 4 (1.7)**.** *If $np_n \to \lambda$, then* $\mathrm{Bin}(p_n, n) \implies \mathrm{Poiss}(\lambda)$.

Proposition 5 (1.12)**.** $\mathcal{N}(a_n, \sigma_n^2) \implies \mathcal{N}(a, \sigma^2)$ *iff $a_n \to a$ and $\sigma_n^2 \to \sigma^2$.*

Proposition 6 (2.1)**.** *Let X have density and $a_n, a \geqslant 0$, then $a_n X + b_n \implies aX + b$ iff $a_n \to a$ and $b_n \to b$.*

Proposition 7 (2.2)**.** *If f continuous, $X_n \implies X$, then $f(X_n) \implies f(X)$.*

Theorem 8 (Scheffe; problem 2.4)**.** *If $g_n \to g$ a.s., then $\mu_n \implies \mu$.*

Proposition 9 (2.7)**.** *$X_n \implies X$, X has continuous distribution, then $F_{X_n} \rightrightarrows F_X$.*

Proposition 10 (3.4)**.** *Convex combinations of characteristic functions are characteristic functions.*

Proposition 11 (3.7)**.** *If φ_X has second derivative in* 0, *then* $\mathbb{E}X^2 < \infty$.

Proposition 12 (5.9)**.** *If $X_n \implies X, Y_n \implies c$, then $X_n + Y_n \implies c + X$ and $X_n Y_n \implies cX$.*

Proposition 13 (6.5)**.** *$X_1, X_2, \ldots$ iid, $\mathbb{E}X_1 = 0$, $\mathrm{Var}X_1 = 1$, (a_n) bounded sequence, $s_n = \sqrt{a_1^2 + \ldots + a_n^2} \to \infty$, then $s_n^{-1}(a_1X_1 + \ldots + a_nX_n) \implies \mathcal{N}(0,1)$.*

Proposition 14 (6.7)**.** *$X \sim \mathcal{N}(a, B)$ has density iff B is nondegenerate and then* $g_X(x) = \dfrac{\sqrt{\det C}}{(2\pi)^{d/2}} \exp\left(\dfrac{\langle B^{-1}(x-a), x-a\rangle}{2}\right)$.

Proposition 15 (7.6)**.** *X, Y independent, f Borel function, $\mathbb{E}|f(X,Y)| < \infty$, then $\mathbb{E}(f(X,Y)|Y) = g(Y)$ a.s. where $g(y) = \mathbb{E}(f(X,y))$.*

Proposition 16 (7.7)**.** *X, Y integrable iid, then $\mathbb{E}(X|X+Y) = \mathbb{E}(Y|X+Y) = \frac{1}{2}(X+Y)$.*

Proposition 17 (8.1)**.** *τ, σ are stopping times for $(\mathcal{F}_n)$. Then $\tau + \sigma$ is too. Notice $\sigma = 1$ is a stopping time, so $\tau + 1$ is too. Notice $\tau - 1$ in not a stopping time in general.*

Proposition 18 (8.2)**.** *Let X_n be $(\mathcal{F}_n)$-adapted, B a borel set. Then $\tau_1 = \inf\{n : X_n \in B\}$ is a stopping time. And $\tau_k = \inf\{n > \tau_{k-1} : X_n \in B\}$ is also.*

Proposition 19 (8.3)**.** *τ, σ stopping times with respect to $(\mathcal{F}_n)$, then $\{\tau < \sigma\}, \{\tau \leqslant \sigma\}, \{\tau = \sigma\} \in \mathcal{F}_\tau \cap \mathcal{F}_\sigma$ and $\mathcal{F}_\tau \cap \mathcal{F}_\sigma = \mathcal{F}_{\tau \wedge \sigma}$.*

Proposition 20 (8.5)**.** *$X_1, X_2, \dots$ iid, $\mathbb{E}X_1 = 0, \mathrm{Var}X_1 < \infty$, $S_n = X_1 + \ldots + X_n$. Then S_n, $S_n^2 - \mathrm{Var}(S_n)$ are martingales with respect to the filtration generated by (X_n).*

Proposition 21 (9.1)**.** *Let X_n be a martingale.*
Then $|X_n|^p$ for $p \geqslant 1$ is a submartingale.
And $X_n \wedge a$ is a supermartingale.
And $X_n \vee a$ is a submartingale.

Proposition 22 (normal distribution)**.** $X \sim \mathcal{N}(\mu, \sigma^2)$

$$\mathbb{E}X = \mu \qquad \mathrm{Var}X = \sigma^2 \qquad g(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \qquad \varphi(t) = e^{i\mu t - \frac{1}{2}\sigma^2 t^2}$$

Proposition 23 (binomial distribution)**.** $X \sim \mathrm{Bin}(n, p)$

$$\mathbb{E}X = pn \qquad \mathrm{Var}X = np(1-p) \qquad g(x) = \binom{n}{k} p^k (1-p)^{n-k} \qquad \varphi(t) = (1 - p + pe^{it})^n$$

Proposition 24 (geometric distribution – 1st success)**.** $X \sim \mathrm{Geom}(p)$

$$\mathbb{E}X = \frac{1}{p} \qquad \mathrm{Var}X = \frac{1-p}{p^2} \qquad g(x) = (1-p)^{k-1}p \qquad \varphi(t) = \frac{pe^{it}}{1-(1-p)e^{it}}$$

Proposition 25 (Poisson distribution)**.** $X \sim \mathrm{Poiss}(\lambda)$

$$\mathbb{E}X = \lambda \qquad \mathrm{Var}X = \lambda \qquad g(x) = \frac{\lambda^k}{k!} e^{-\lambda} \qquad \varphi(t) = e^{\lambda(e^{it}-1)}$$

Proposition 26 (exponential distribution)**.** $X \sim \mathrm{Exp}(\lambda)$

$$\mathbb{E}X = \frac{1}{\lambda} \qquad \mathrm{Var}X = \frac{1}{\lambda^2} \qquad g(x) = \lambda e^{-\lambda x} \qquad \varphi(t) = \left(1 - \frac{it}{\lambda}\right)^{-1} = \frac{\lambda}{\lambda - it}$$

Proposition 27 (Cauchy distribution)**.** $X \sim \mathrm{Cauchy}(h)$
Not integrable!

$$g(x) = \frac{h}{\pi(h^2 + x^2)} \qquad \varphi(t) = e^{-h|t|}$$