

Theorem 1 (CTL in the iid case). X_1, X_2, \dots iid random variables, $\mathbb{E}X_1 = a$, $\text{Var}(X_1) = \sigma^2 \in (0, \infty)$, then $\frac{X_1 + \dots + X_n - na}{\sqrt{n}\sigma} \implies \mathcal{N}(0, 1)$.

Corollary 2 (de Moivre-Laplace). $X_n \sim \text{Bin}(n, p)$, then $\frac{X_n - np}{\sqrt{np(1-p)}} \implies \mathcal{N}(0, 1)$.

Definition 3 (triangular array). $(X_{n,k})_{n=1,2,\dots; 1 \leq k \leq k_n}$ is a *triangular array of random variables* if $\forall_n X_{n,1}, \dots, X_{n,k_n}$ are independent.

Theorem 4 (CLT – Lindeberg’s version). $(X_{n,k})$ a triangular array, $S_n = X_{n,1} + \dots + X_{n,k_n}$,

1. $\mathbb{E}S_n \rightarrow m$

2. $\text{Var}(S_n) \rightarrow \sigma^2$

3. Lindeberg’s condition: $\forall_{\varepsilon > 0} \sum_{k=1}^{k_n} \mathbb{E} \left(|X_{n,k} - \mathbb{E}X_{n,k}|^2 \mathbb{1}_{\{|X_{n,k} - \mathbb{E}X_{n,k}| > \varepsilon\}} \right) \rightarrow 0$

Then $S_n \implies \mathcal{N}(m, \sigma^2)$.

Lemma 5. Lindeberg’s condition implies

- $\forall_{t > 0} \lim_{n \rightarrow \infty} \mathbb{P} \left(\max_{1 \leq k \leq k_n} |X_{n,k}| \geq t \right) = 0$,
- $\lim_{n \rightarrow \infty} \max_{1 \leq k \leq k_n} \text{Var}(X_{n,k}) \rightarrow 0$.

Theorem 6 (Feller). Suppose that $(X_{n,k})$ is a triangular array, $\sum_{k=1}^{k_n} \mathbb{E}X_{n,k} \rightarrow m$, $\sum_{k=1}^{k_n} \text{Var}(X_{n,k}) \rightarrow$

σ^2 , and $S_n = \sum_{k=1}^{k_n} X_{n,k} \implies \mathcal{N}(m, \sigma^2)$, then if $\max_{1 \leq k \leq k_n} \text{Var}(X_{n,k}) \rightarrow 0$, then Lindeberg’s condition is satisfied.