

Example 1. (X, Y) has density $g_{(X,Y)}(x, y)$. Then

$$\mathbb{E}(X|Y) = \varphi(y) = \frac{\int x g(x, y) dx}{\int g(x, y) dy}.$$

Proposition 2. *Conditional Expected Value is linear.*

Proposition 3. $\mathbb{E}(\mathbb{E}(X|\mathcal{G})) = \mathbb{E}X$

Proposition 4. $X_1 \geq X_2 \implies \mathbb{E}(X_1|\mathcal{G}) \geq \mathbb{E}(X_2|\mathcal{G})$ almost surely.

Proposition 5. $|\mathbb{E}(X|\mathcal{G})| \leq \mathbb{E}(|X||\mathcal{G})$ a.s., in particular CEV is a contraction in L^1 because $\mathbb{E}|\mathbb{E}(X|\mathcal{G})| \leq \mathbb{E}|X|$.

Proposition 6. X is \mathcal{G} -measurable, then $\mathbb{E}(X|\mathcal{G}) = X$ a.s.

Proposition 7. $X \perp Y \implies \mathbb{E}(X|\mathcal{G}) = \mathbb{E}X$ a.s.

In particular $\mathbb{E}(X|\{\emptyset, \Omega\}) = \mathbb{E}X$.

Proposition 8 (Lebesgue monotone convergence thm). $0 \leq X_n \nearrow X$, X_n, X integrable, then $\mathbb{E}(X_n|\mathcal{G}) \nearrow \mathbb{E}(X|\mathcal{G})$ a.s.

Proposition 9 (Lebesgue dominated convergence thm). $|X_n| \leq Y$, $\mathbb{E}Y < \infty$, $X_n \rightarrow X$ a.s., then $\lim_{n \rightarrow \infty} \mathbb{E}(X_n|\mathcal{G}) = \mathbb{E}(X|\mathcal{G})$ a.s.

Proposition 10 (Fatou lemma). X_n integrable, then $\liminf_{n \rightarrow \infty} \mathbb{E}(X_n|\mathcal{G}) \geq \mathbb{E}\left(\liminf_{n \rightarrow \infty} X_n|\mathcal{G}\right)$.

Proposition 11. X is \mathcal{G} -measurable, $\mathbb{E}|Y| < \infty$, $\mathbb{E}|XY| < \infty$, then $\mathbb{E}(XY|\mathcal{G}) = X\mathbb{E}(Y|\mathcal{G})$ a.s.

Proposition 12. X integrable, $\mathcal{G}_1 \subset \mathcal{G}_2$, then $\mathbb{E}(X|\mathcal{G}_1) = \mathbb{E}(\mathbb{E}(X|\mathcal{G}_2)|\mathcal{G}_1) = \mathbb{E}(\mathbb{E}(X|\mathcal{G}_1)|\mathcal{G}_2)$.

Proposition 13 (Jensen inequality). $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ convex, $\mathbb{E}|X| < \infty$, $\mathbb{E}|\varphi(X)| < \infty$, then $\mathbb{E}(\varphi(X)|\mathcal{G}) \geq \varphi(\mathbb{E}(X|\mathcal{G}))$.

In particular $\mathbb{E}\varphi(X) \geq \varphi(\mathbb{E}X)$.

Corollary 14. $\mathbb{E}|X|^p < \infty$, $p \geq 1$, then $|\mathbb{E}(X|\mathcal{G})|^p \leq \mathbb{E}(|X|^p|\mathcal{G})$.

In particular $\mathbb{E}(\bullet|\mathcal{G})$ is a contraction in L^p .

Filtrations and stopping times

$T \subset \mathbb{Z}$

Definition 15 (filtration). A *filtration* is a sequence of σ -bodies $(\mathcal{F}_t)_{t \in T}$ such that $\mathcal{F}_t \subset \mathcal{F}$ and $\mathcal{F}_t \subset \mathcal{F}_s$ for $t < s$.

Definition 16 (stopping time). A *stopping time* with respect to filtration $(\mathcal{F}_t)_{t \in T}$ is a random variable $\tau : \Omega \rightarrow T \cup \{\infty\}$ such that $\forall_{t \in T} \{\tau \leq t\} \in \mathcal{F}_t$.

Proposition 17. $\tau : \Omega \rightarrow T \cup \{\infty\}$ is a stopping time iff $\forall_{t \in T} \{\tau = t\} \in \mathcal{F}_t$.