TTT4120 Digital Signal Processing

Problem Set 5

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Contents

Problem 1 (4 points)

(a) Derive the energy density spectrum $S_{xx}(f)$ of the signal

$$x(n) = \begin{cases} a^n & n \geq 0, \quad |a| < 1 \\ 0 & \quad\quad n < 0. \end{cases}$$

To find the energy density spectrum we need to first compute the Discrete Fourier Transform of the signal $x[n]$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

$$X(e^{j\omega}) = \sum_{n=0}^{\infty} a^n e^{-j\omega n}$$

as this is the sum of an infinite geometric series we can use the formula

$$S = \sum_{n=0}^{\infty} ar^n = |\frac{a}{1-r}|$$

if we rewrite $a^n e^{-j\omega n}$ to $(ae^{-j\omega})^n$ then we get

$$X(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}}$$

AS the energy density spectrum $S_{xx}(f)$ is given by

$$S_{xx}(f) = \left|X(e^{j\omega})^2\right|$$

we get

$$\begin{aligned} S_{xx}(f) &= \frac{1}{(1 - ae^{-j\omega})(1 - ae^{j\omega})} \\ &= \frac{1}{1 - a(e^{-j\omega} + e^{j\omega})\, a^2 e^{-j\omega} e^{j\omega}} \end{aligned}$$

Using Euler's formula, where $e^{j\omega} = \cos(\omega) + j\sin(\omega)$ and $e^{-j\omega} = \cos(\omega) - j\sin(\omega)$, and the fact that $e^{-j\omega}e^{j\omega} = 1$:

$$S_{xx}(f) = \frac{1}{1 - a2\cos(2\pi f) + a^2}$$

(b) Derive the autocorrelation $r_{xx}(l)$ of the signal given in 1a. Use $r_{xx}(l)$ to verify the expression for $S_{xx}(f)$ found in 1a.

The autocorrelation of $x(n)$ is defined as the sequence

$$r_{xx}(l) = \sum_{n=-\infty}^{\infty} x(n+l)x(n)$$

This gives us

$$r_{xx}(l) = \sum_{n=0}^{\infty} a^{n+l}a^n$$

this can be rewritten as $a^{2n}a^l$, this gives us

$$r_{xx}(l) = \frac{a^l}{1-a^2},$$

as $r_{xx}(l) = r_{xx}(-l)$

$$r_{xx}(l) = \frac{a^{|l|}}{1-a^2},$$

$$r_{xx}[l] = x[l] * x[-l] \overset{\mathcal{F}}{\leftrightarrow} S_{xx}(\omega) = X(\omega)X^*(\omega) = |X(\omega)|^2$$

By taking the Discrete Fourier Transform of the autocorrelation $r_{xx}(l)$ we get

$$S_{xx}(f) = \sum_{l=-\infty}^{\infty} \frac{a^l}{1-a^2}e^{-j\omega l}$$

$$= \sum_{l=-\infty}^{\infty} \frac{a^l}{1-a^2}e^{-j\omega l}$$

$$= \frac{1}{1-a^2} \sum_{l=-\infty}^{\infty} a^l e^{-j\omega l}$$

$$= \frac{1}{1-a^2} \left(\sum_{l=1}^{\infty} a^l e^{j\omega l} + \sum_{l=0}^{\infty} a^l e^{-j\omega l} \right)$$

$$= \frac{1}{1-a^2} a e^{j\omega l} \left(\sum_{l=0}^{\infty} a^l e^{j\omega l} + \sum_{l=0}^{\infty} a^l e^{-j\omega l} \right)$$

using the formula for infinite geometric series we get

$$S_{xx}(f) = \frac{1}{1-a^2} \left(\frac{ae^{j\omega}}{1-ae^{j\omega}} + \frac{1}{1-ae^{-j\omega}} \right)$$

$$= \frac{1}{1-a^2} \cdot \frac{ae^{j\omega} - a^2 + 1 - ae^{j\omega}}{(1-ae^{j\omega})(1-ae^{-j\omega})}$$

$$= \frac{1}{(1-ae^{j\omega})(1-ae^{-j\omega})}$$

$$= \frac{1}{1+a^2-2a\cos\omega}$$

which is the same as in 1b

(c) Plot $x(n), r_{xx}(l)$ and $S_{xx}(f)$ for $a = 0.4, a = 0.95$, and $a = -0.95$. Let $n \in [0, 50], l \in [-50, 50]$ and $f \in [-0.5, 0.5]$ in your plots.

Compare the plots for the three different values of a.
Write down your observations and explain them. What properties of the autocorrelation function can you see from the plots?

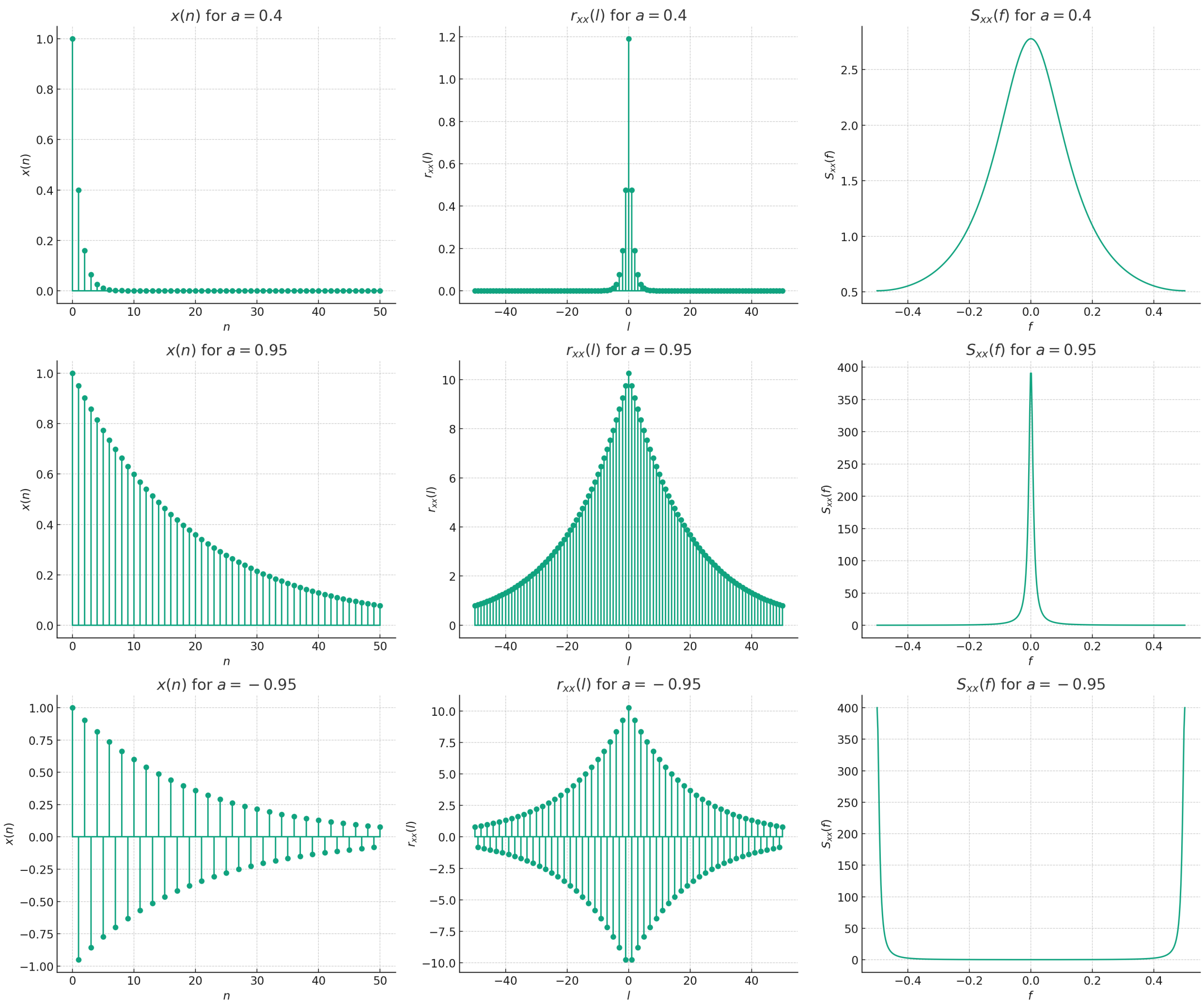

Figure 1: $x(n), r_{xx}(l)$ and $S_{xx}(f)$ for $a = 0.4$, $a = 0.95$, and $a = -0.95$.

In figure 1 we can se that for $a > 0$ the signal decays exponentialy where the lower the value the mor exponential decay, when $a < 0$ we can observe that the signal decays exponentially while oscilating the the same time. When looking at the autocoerrelation we can observe that for a lower magnitude of $|a|$ the less correlation we get at a higher lag. When $a < 0$ we can se that the autocorrelation is negative for when l is an odd number and high for an even l this makes sense as the signal is oscilating from positiv to negative for each incrementation in n.

(d) Find the energy of the signal x(n).

The energy of sequences $x[n]$ is given by

$$E_x = \sum_{n=-\infty}^{\infty} x^2[n] = r_x[0] \geq 0$$

this gives us

$$E_x = r_{xx}[0] = \frac{1}{1-a^2}$$

(e) At this point, the signal $x(n)$ is first passed through the filter $h_1(n)$ and then the result is fed to the filter $h_2(n)$, where the first filter is given in terms of its impulse response and the second one in terms of its frequency response as follows. The final result is denoted by $y(n)$.

$$h_1(n) = \delta(n) - a\delta(n-1)$$
$$H_2(f) = \begin{cases} \cos(2\pi f) & |f| \leq \frac{1}{4} \\ 0 & \frac{1}{4} < |f| \leq \frac{1}{2} \end{cases}$$

Find the energy density spectrum of the output signal, $S_{yy}(f)$. Compare the result to $S_{xx}(f)$ and comment. Find the total energy in the output signal.

$$S_{yy}(f) = |H(f)|^2 S_{xx}(f)$$

the overall transfer function of the two systems is

$$H_1(f) = \sum_{n=-\infty}^{\infty} h_1(n) e^{-j2\pi f n}$$

$$H_1(f) = \sum_{n=-\infty}^{\infty} \delta(n) e^{-j2\pi f n} - a\delta(n-1) e^{-j2\pi f n}$$

$$H_1(f) = \delta(0) e^{-j2\pi f \cdot 0} - a\delta(1-1) e^{-j2\pi f \cdot 1}$$
$$H_1(f) = 1 - ae^{-j2\pi f}$$

This gives us

$$S_{y_1 y_1}(f) = \frac{\left|1 - ae^{-j2\pi f}\right|^2}{1 + a^2 - 2a\cos\omega}$$

$$S_{y_1y_1}(f) = \frac{\left(1 - ae^{-j\omega}\right)\left(1 - ae^{j\omega}\right)}{1 + a^2 - 2a\cos\omega}$$

"same" as 1a

$$S_{y_1y_1}(f) = 1$$

then by adding the second filter we get

$$S_{yy}(f) = |H_2(f)|^2 \, S_{y_1y_1}(f) = \begin{cases} \cos^2(2\pi f) & |f| \leq \frac{1}{4} \\ 0 & \frac{1}{4} < |f| \leq \frac{1}{2} \end{cases}$$

As the filter has a cutoff at $|f| \leq \frac{1}{4}$ we can get the total energy by

$$E_y = \int_{-\frac{1}{4}}^{\frac{1}{4}} \cos^2(2\pi f) df = \frac{1}{4}$$

Problem 2 (3 points)

The emitted signal will be attenuated and contaminated by noise as it travels through the air, so the received signal is given by

$$y(n) = \begin{cases} \alpha x(n-D) + w(n), & \textbf{if an object is hit} \\ w(n), & \textbf{if no object is hit} \end{cases}$$

where α is the attenuation factor, D is the delay, and $w(n)$ is the noise.

(a) Plot the signals $x(n)$ and $y(n)$. Can you determine reliably whether an object has been hit by the emitted signal from these two plots only? Explain.

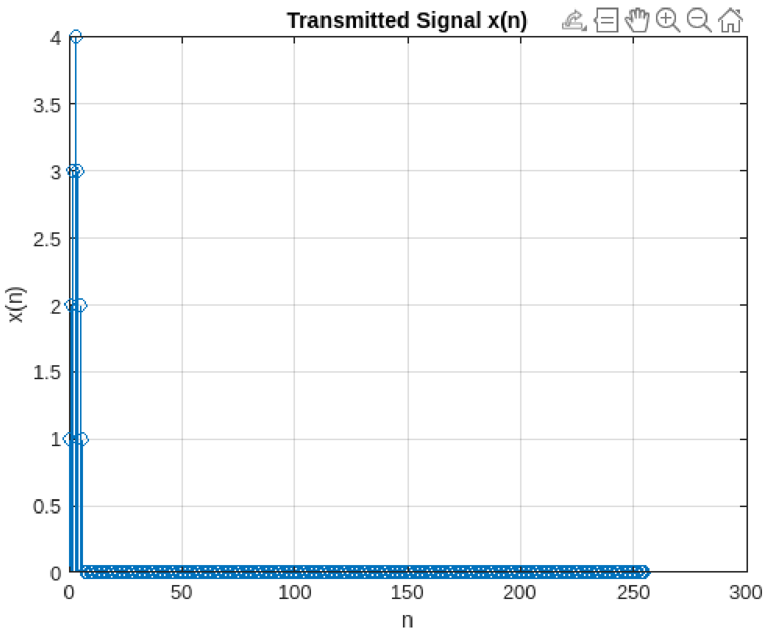

Figure 2: $x(n)$

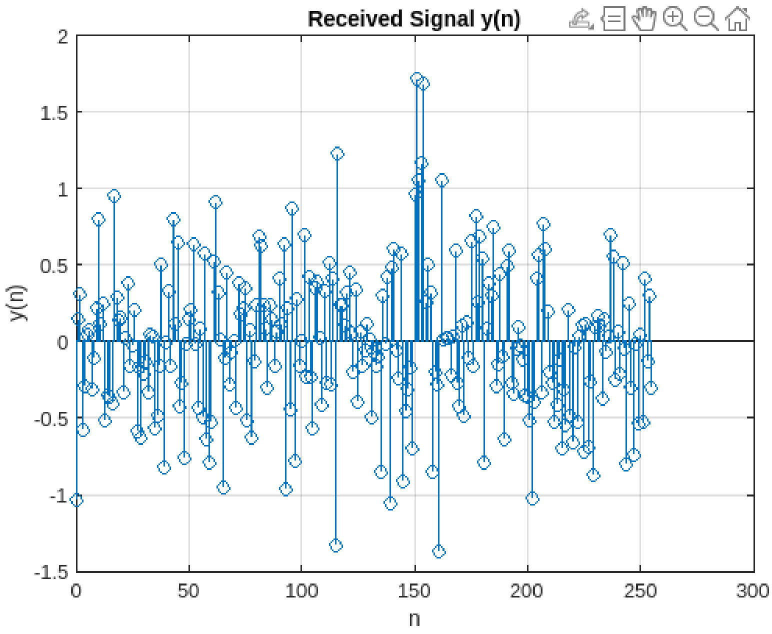

Figure 3: $y(n)$

From the two plots we can't relibly determine whenever an object has hit the emitted signal. It is difficult to determine whenever the recieved signal is noise or the attenuated signal. My best guess is at $n = 150$ aas this is the highest peak.

(b) Find the crosscorrelation function $r_{yx}(l)$ by using the Matlab function xcorr.

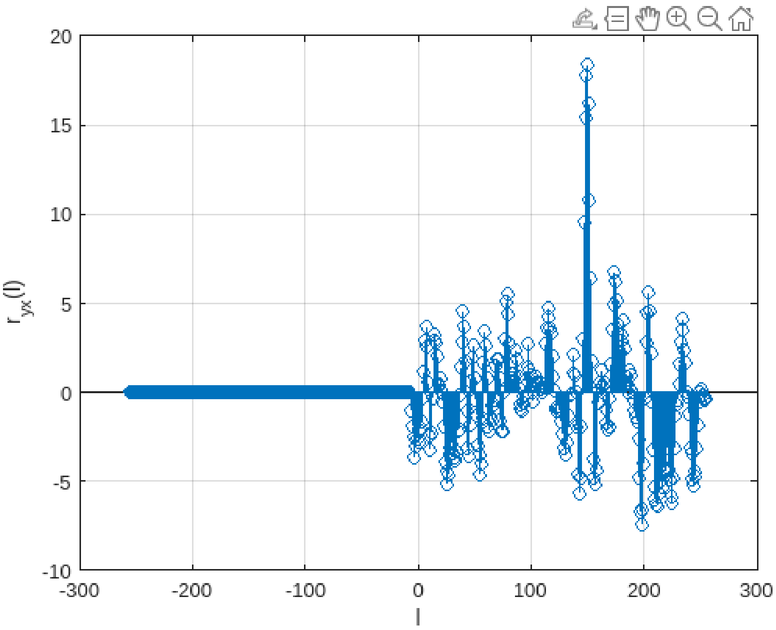

Figure 4: $r_{yx}(l)$

(c) Find the crosscorrelation function $r_{yx}(l)$ by using the Matlab function conv.

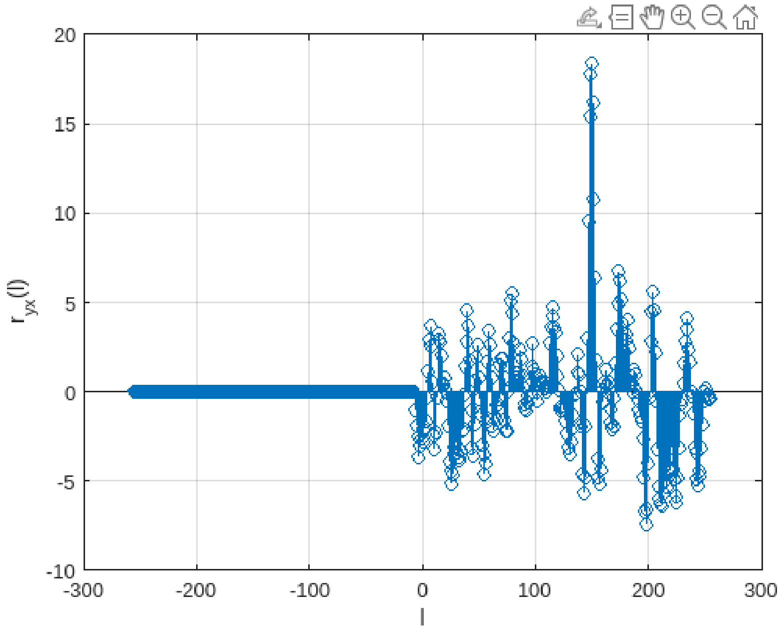

Figure 5: $r_{yx}(l)$

(d) Based on the plot of $r_{yx}(l)$, find out whether an object has been hit by the emitted signal, and f so, determine the value of the delay D. Is this result more reliable than that of the direct comparisson of the signals $x(n)$ and $y(n)$ in 2a) ?

I would say whe use of crosscorrelation is a much more reliable way of finding the attenuated signal rather than the direct comparisson of the signals $x(n)$ and $y(n)$ in 2a)

Problem 3 (2 points)

Consider the simple filter structure in figure 6. The output signal $y[n]$ is composed of the input signal $x[n]$ and a delayed (by R samples) and scaled (by the factor α) replica of the input.

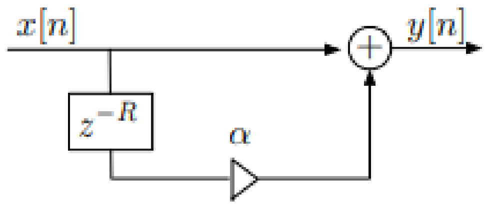

Figure 6: Filter structure of single-echo-generating filter

Find the transfer function $H(z)$ for the filter in figure 6.

$$
\begin{aligned}
Y(z) &= X(z) + \alpha z^{-R} X(z) \\
&= X(z)\left(1 + \alpha z^{-R}\right) \\
H(z) &= \frac{Y(z)}{X(z)} = 1 + \alpha z^{-R}
\end{aligned}
$$

Given a sample frequency of 22050 Hz for the input signal. What is the delay in seconds for a given delay R ?

We can find the delay in seconds by dividing the delay with the sampling frequency.

Implement the filter in MATLAB, plot the unit sample response and the frequency response. (Hint: You can use the MATLAB functions impz and freqz)

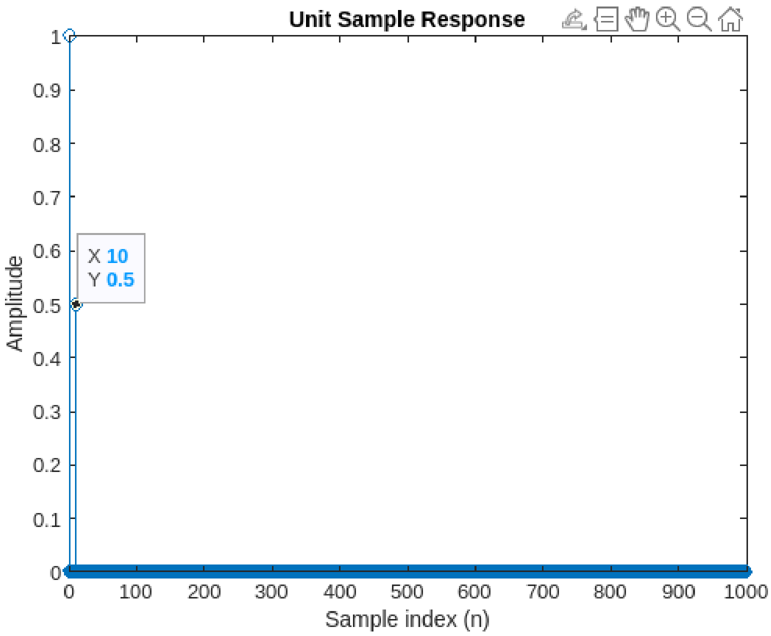

Figure 7: unit sample response

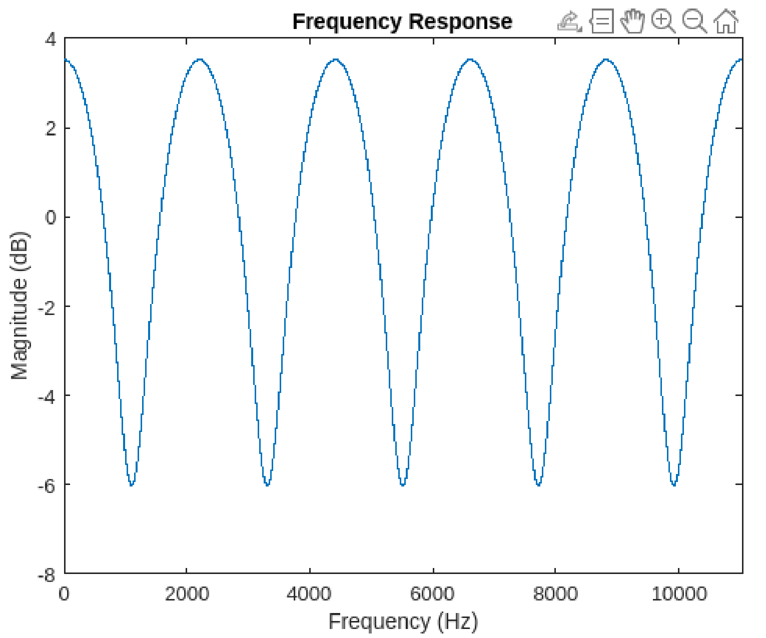

Figure 8: frequency response

Load the sound file piano.wav. The sample frequency is 22050 Hz. (Available on It's learning). Pass it through the filter and observe the effect for different values of R and α. Comment!

The echo gets more pronounced with an increased R and α

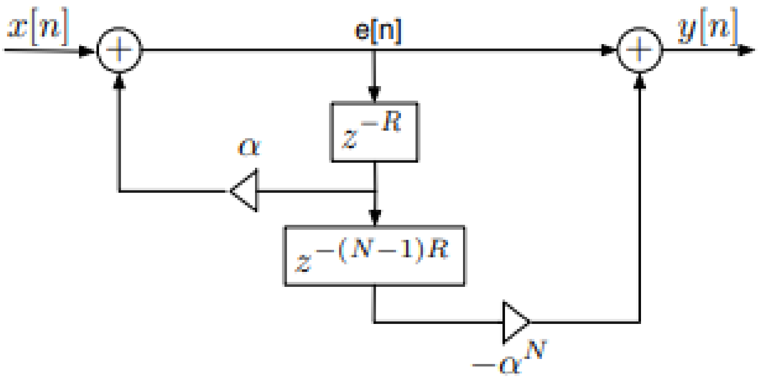

Figure 9: Filter structure for generating *N* echoes.

Implement the filter in figure 2 in MATLAB. Vary the parameters R, α and N (where R is the spacing between echoes and N is the number of echoes, α as above).

sounds more like in a hall

Notice the difference in impulse response compared to the single-echo case.

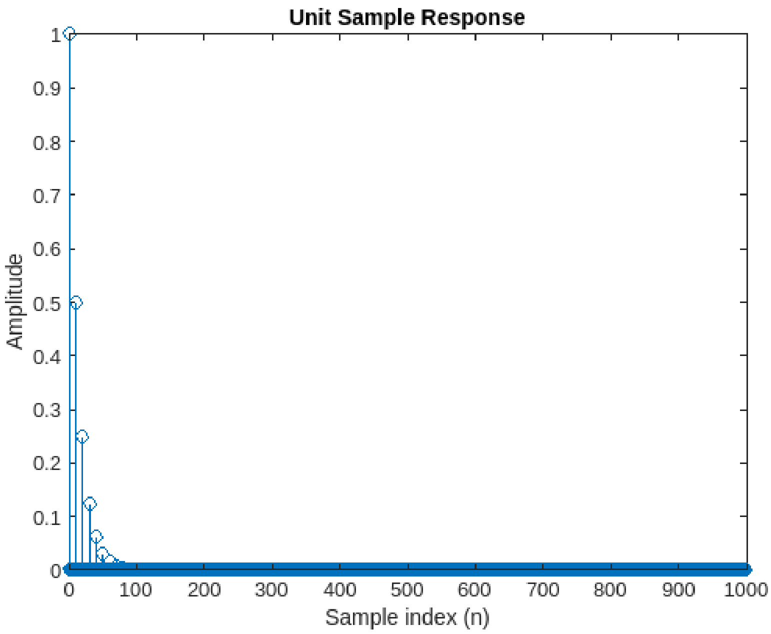

Figure 10: unit sample response filter 2

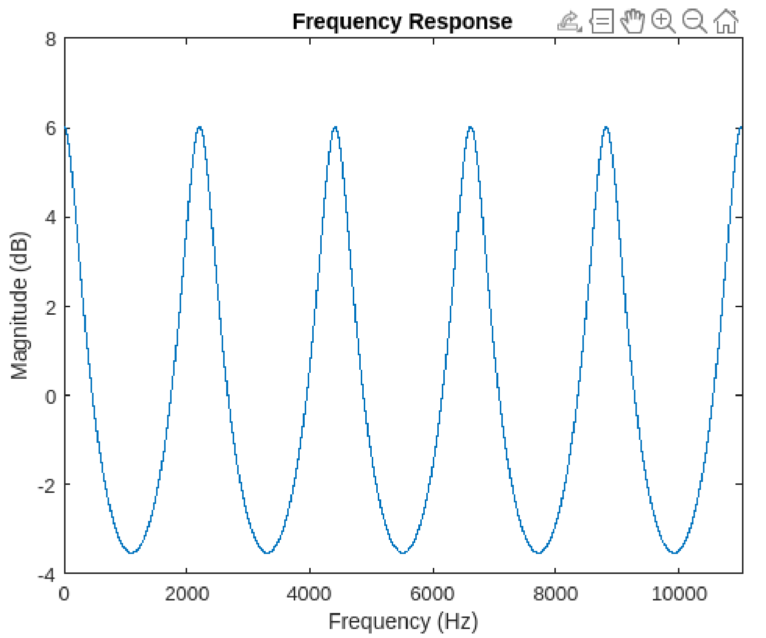

Figure 11: frequency response filter 2