TTT4120 Digital Signal Processing

Problem Set 2

Peter Pham

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Contents

Problem 1 (3 points)

Two signals $x[n]$ and $y[n]$ are given by

$$x[n] = \begin{cases} 2 & n = 0 \\ 1 & n = \pm 1 \\ 0 & \text{otherwise,} \end{cases} \qquad y[n] = \begin{cases} 1 & -M \leq n \leq M \\ 0 & \text{otherwise} \end{cases}$$

(a) Show that the Fourier transform of $x[n]$ is given by... and sketch it for $\omega \in [-\pi, \pi]$.

$$X(\omega) = 2 + 2\cos\omega$$

We have that

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

In our case we can write

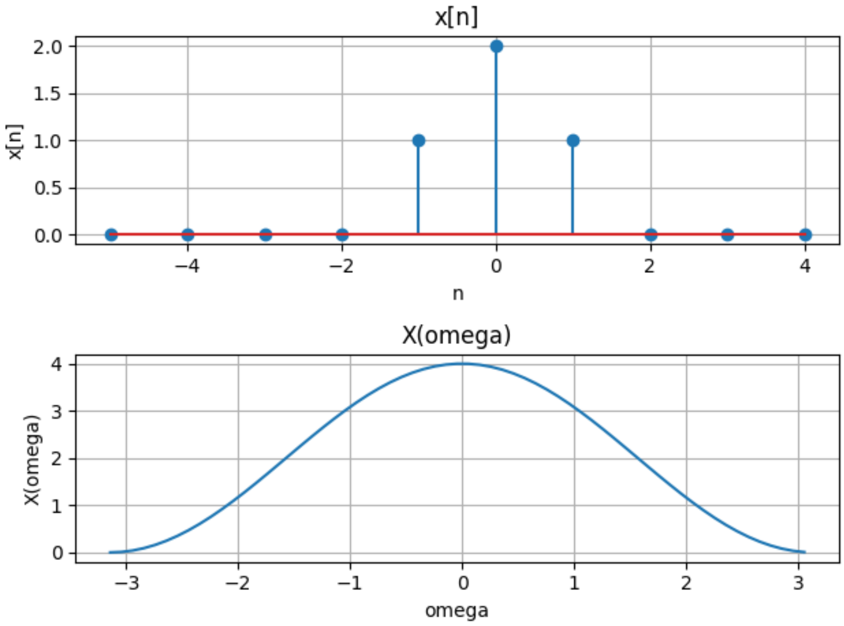
$$X(\omega) = \sum_{n=-1}^{1} x[n]e^{-j\omega n} = -e^{j\omega} + 2 + e^{-j\omega}$$

From Euler's formula we get that

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}$$

by combining these two we get:

$$X(\omega) = 2 + 2\cos\omega$$



(b) Show that the Fourier transform of y[n] is given by

$$Y(\omega) = \frac{\sin\left(\omega\left(M+\frac{1}{2}\right)\right)}{\sin\left(\frac{\omega}{2}\right)}$$

and sketch it for $M = 10$ **and** $\omega \in [-\pi, \pi]$

$$Y(\omega) = \sum_{n=-M}^{M} e^{-j\omega n}$$

This is a finite geometric series. By using the general formula for a geometric series

$$\sum_{k=m}^{n} ar^k = \begin{cases} a(n-m+1) & \text{if } r = 1 \\ \frac{a\left(r^m - r^{n+1}\right)}{1-r} & \text{if } r \neq 1 \end{cases}$$

we get

$$Y(\omega) = \frac{e^{j\omega M} - e^{-j\omega(M+1)}}{1 - e^{-j\omega}}$$

by multiplying the numerator and the denominator by $e^{\frac{j\omega}{2}}$.

$$Y(\omega) = \frac{e^{j\omega M} e^{j\omega/2} - e^{-j\omega(M+1)} e^{j\omega/2}}{(1 - e^{-j\omega})\, e^{j\omega/2}}$$
$$= \frac{e^{j\omega\left(M+\frac{1}{2}\right)} - e^{-j\omega\left(M+\frac{1}{2}\right)}}{e^{j\omega/2} - e^{-j\omega/2}}$$

Since we have expresions of the form $e^{j\Theta} - e^{-j\Theta}$, which can be simplidied using Euler's formula

$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$
$$e^{-j\theta} = \cos(-\theta) + j\sin(-\theta) = \cos(\theta) - j\sin(\theta)$$

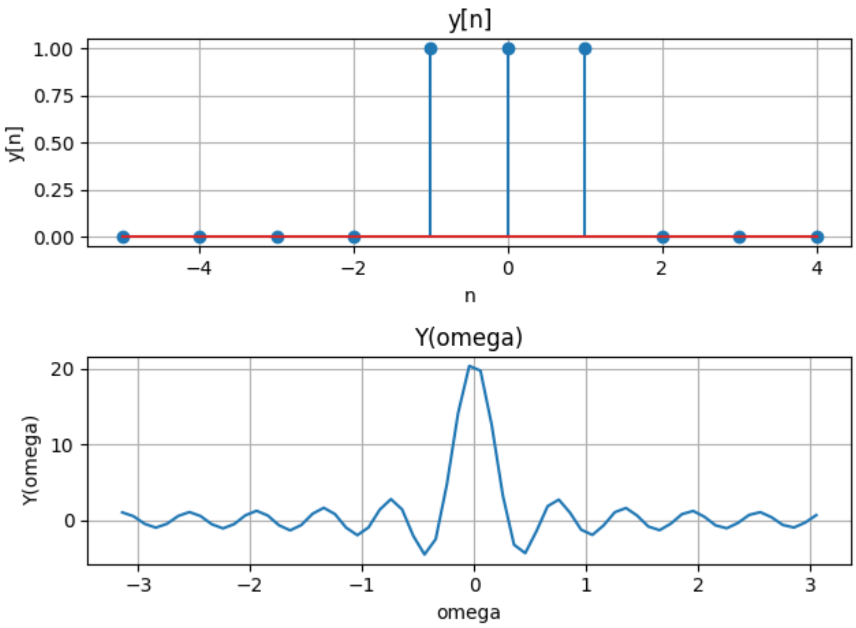
Therefore:

$$e^{j\Theta} - e^{-j\Theta} = 2j\sin\Theta$$

By substituing this back into the equation, we get

$$Y(\omega) = \frac{2j\sin\left(\omega\left(M+\frac{1}{2}\right)\right)}{2j\sin\left(\frac{\omega}{2}\right)}$$

$$Y(\omega) = \frac{\sin\left(\omega\left(M+\frac{1}{2}\right)\right)}{\sin\left(\frac{\omega}{2}\right)}$$



(c) Explain why the signals $x[n]$ and $y[n]$ have real valued spectra.

The signals have a real valued spectra because the sequencences $x[n]$ and $y[n]$ are real and symmetric.

(d) Let the signal

$$z[n] = \sum_{l=-\infty}^{\infty} x[n - lN]$$

be the periodic extension of $x[n]$. Assume that N is greater than the length of the signal, i.e. $N > 3$.
Sketch the signal $z[n]$.
Find the Fourier series coefficients $\{c_k\}$ of $z[n]$.
Sketch $\{c_k\}$ as a function of $\omega = 2\pi\frac{k}{N} \in [-\pi, \pi]$ for $N = 10$.

For a discrete-time periodic signal, the Fourier Series coefficients $\{c_k\}$ can be defined as:

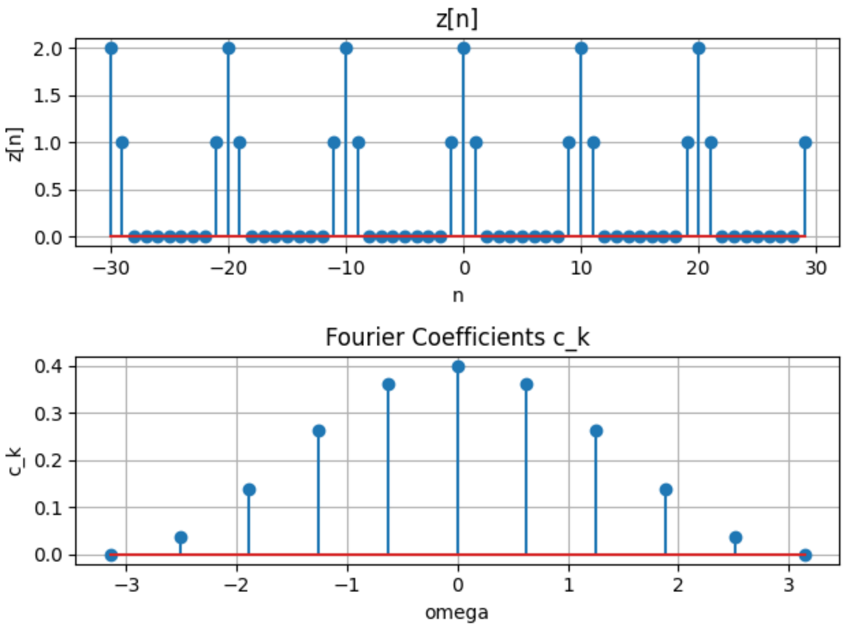
$$c_k = \frac{1}{N} \sum_{n=0}^{N-1} z[n] e^{-j(2\pi kn)/N}$$

For $z[n]$, you can see that most terms in this sum will be zero, except for the few terms that correspond to the non-zero values of $x[n]$. Thus:

$$c_k = \frac{1}{N} \left(2 + e^{-j(2\pi k)/N} + e^{j(2\pi k)/N}\right)$$

Simplifying this using Euler's formula $e^{j\theta} + e^{-j\theta} = 2\cos(\theta)$, we get:

$$c_k = \frac{1}{N} \left(2 + 2\cos\left(\frac{2\pi k}{N}\right)\right)$$
$$c_k = \frac{2}{N} \left(1 + \cos\left(\frac{2\pi k}{N}\right)\right)$$



(e) Compare the spectra of $x[n]$ and $z(n)$, i.e. $X(\omega)$ and $\{c_k\}$. What is the relationship between the spectra?

$X(\omega)$ and $\{c_k\}$ are essentially two different ways of representing the frequency content of a signal. $X(\omega)$ is used for the original aperiodic $x[n]$, while $\{c_k\}$ is used for the periodic $z[n]$, which is formed by periodically repeating $x[n]$.

Problem 2 (1.5 points)

Let $x[n]$ be a signal with the Fourier transform $X(\omega)$. Find the fourier tranforms of the following signals: We have that

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

(a) $x_1[n] = x[n+3]$

The Fourier Transform of a time-shifted signal $x[n-m]$ is given by:

$$\mathcal{F}\{x[n-m]\} = e^{-j\omega m}X(\omega)$$

$$\Rightarrow X_1(\omega) = e^{-j\omega m}X(\omega) = e^{j3\omega}X(\omega)$$

(b) $x_2[n] = x[-n]$

The Fourier Transform of a time-reversed signal $x[-n]$ is given by:

$$\mathcal{F}\{x[-n]\} = X(-\omega)$$

$$\Rightarrow X_2(\omega) = X(-\omega)$$

(c) $x_3[n] = x[3-n]$

Here we have both time-shift and time-reversed

$$X_3(\omega) = e^{-j\omega m}X(\omega) = e^{j3\omega}X(-\omega)$$

(d) $x_4[n] = x[n] * x[n]$

Covolution in the time domain corresponds to multiplication in the frequency domain

$$X_4(\omega) = X(\omega) \cdot X(\omega) = X^2(\omega)$$

Problem 3 (3 points)

Two systems (from Problem Set 1) are given by the following difference equations

$$y[n] = x[n] + 2x[n-1] + x[n-2]$$
$$y[n] = -0.9y[n-1] + x[n].$$

(a) Find the frequency responses of these two systems.

To find the frequency response $H(\omega)$ of a system described by a difference equation, you can take the Fourier Transform of both sides of the equation and solve for $H(\omega)$, which is essentially $\frac{Y(\omega)}{X(\omega)}$.

For the first system:

The difference equation is:

$$y[n] = x[n] + 2x[n-1] + x[n-2]$$

Taking the Fourier Transform of both sides:

$$Y(\omega) = X(\omega) + 2e^{-j\omega}X(\omega) + e^{-2j\omega}X(\omega)$$

Simplifying:

$$Y(\omega) = X(\omega)(1 + 2e^{-j\omega} + e^{-2j\omega})$$

The frequency response $H_1(\omega)$ is:

$$H_1(\omega) = \frac{Y(\omega)}{X(\omega)} = 1 + 2e^{-j\omega} + e^{-2j\omega}$$

For the second system:

The difference equation is:

$$y[n] = -0.9y[n-1] + x[n]$$

Taking the Fourier Transform of both sides:

$$Y(\omega) = -0.9e^{-j\omega}Y(\omega) + X(\omega)$$

Solving for $Y(\omega)$:

$$Y(\omega) = \frac{X(\omega)}{1 + 0.9e^{-j\omega}}$$

The frequency response $H_2(\omega)$ is:

$$H_2(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{1}{1 + 0.9e^{-j\omega}}$$

Find the magnitude and phase responses of the two systems. Are they even or odd functions?

To find the magnitude and phase of a frequency response $H(\omega)$, we can express $H(\omega)$ in polar form: $H(\omega) = |H(\omega)|e^{j\angle H(\omega)}$ where $|H(\omega)|$ is the magnitude and $\angle H(\omega)$ is the phase.

System 1

The difference equation is $y[n] = x[n] + 2x[n-1] + x[n-2]$.

The Fourier Transform of the system, known as the frequency response $H_1(\omega)$, is given by:

$$H_1(\omega) = 1 + 2e^{-j\omega} + e^{-2j\omega}$$

To find the magnitude $|H_1(\omega)|$:

$$\begin{aligned}
|H_1(\omega)| &= \sqrt{\mathrm{Re}[H_1(\omega)]^2 + \mathrm{Im}[H_1(\omega)]^2} \\
&= \sqrt{(1 + 2\cos(\omega) + \cos(2\omega))^2} \\
&= \sqrt{2 + 4\cos(\omega) + 4\cos^2(\omega) + 2\cos(2\omega)} \\
&= \sqrt{2(1 + 2\cos(\omega) + 2\cos^2(\omega) + \cos(2\omega))}
\end{aligned}$$

To find the phase $\angle H_1(\omega)$:

$$\angle H_1(\omega) = 0$$

System 2

The difference equation is $y[n] = -0.9y[n-1] + x[n]$.

The frequency response $H_2(\omega)$ is:

$$H_2(\omega) = \frac{1}{1 + 0.9e^{-j\omega}}$$

To find the magnitude $|H_2(\omega)|$:

$$\begin{aligned}
|H_2(\omega)| &= \left| \frac{1}{1 + 0.9(\cos(-\omega) - j\sin(-\omega))} \right| \\
&= \frac{1}{\sqrt{(1 - 0.9\cos(\omega))^2 + (0.9\sin(\omega))^2}} \\
&= \frac{1}{\sqrt{1 - 1.8\cos(\omega) + 0.81}}
\end{aligned}$$

To find the phase $\angle H_2(\omega)$:

$$\angle H_2(\omega) = -\text{atan2}(0.9\sin(\omega), 1 - 0.9\cos(\omega))$$

Problem 4 (2 points)

Problem 5 (2 points)

Appendix