TTT4120 Digital Signal Processing

Problem Set 7

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Contents

Problem 1 (4 points)

Given three different types of white noise with unit variance:

- White binary noise (each sample takes the value -1 or 1 with the same probability)
- White Gaussian noise (each sample has a normal distribution)
- White uniform noise

(a) For each noise type, generate and plot one realization of length 100 samples. (Useful Matlab functions: rand and randn)

- What do these noises have in common?
- How do they differ?

We start by describing the noises individualy:

White binary noise:

Binary noise consist of a signal consisting of only binary values, this means that the signal is either 1 or 0, when you have a long sequence of random sequence of 1s and 0s you will end up with a signal where all frequencies are equally represented.

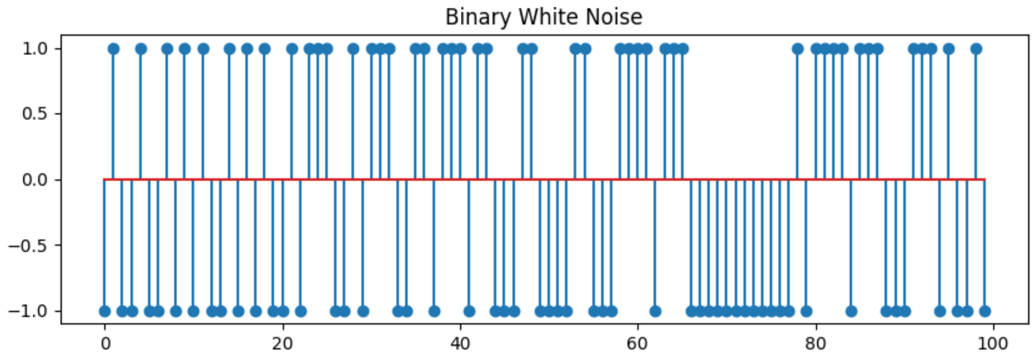

Figure 1: Plot of white binary noise with a length of 100

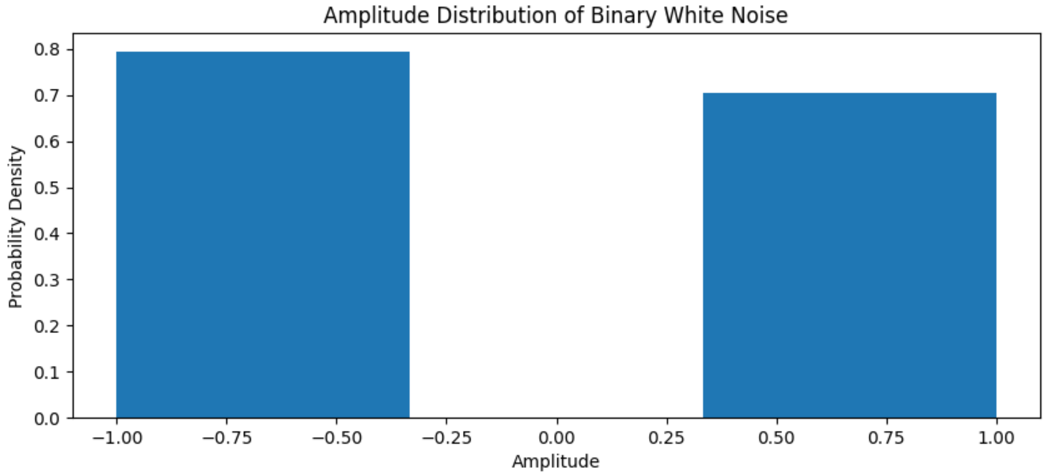

Figure 2: Plot of the distrobution of the white binary noise with a length of 100

White Gaussian noise:

This noise has amplitude values that are statistically distributed according to the Gaussian distrobution. It is a white noise because it has a flat spectrum, meaning all frequencies are equally likely

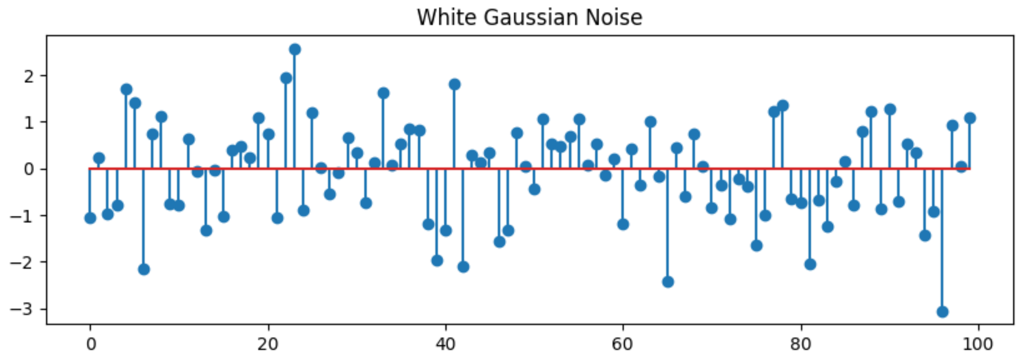

Figure 3: Plot of white Gaussian noise with a length of 100

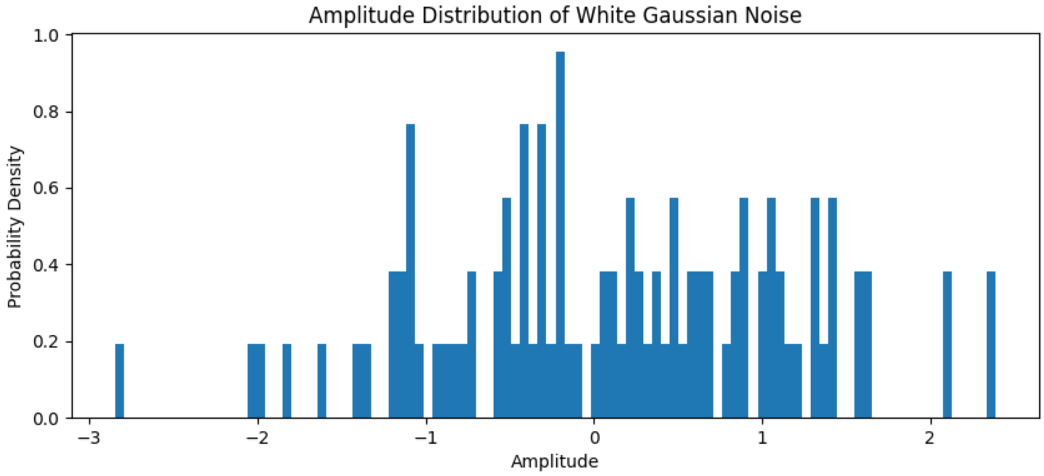

Figure 4: Plot of the distrobution of the white Gaussian noise with a length of 100

White uniform noise:

This noise has a amplitude value that are distributed uniformily across a vertain range

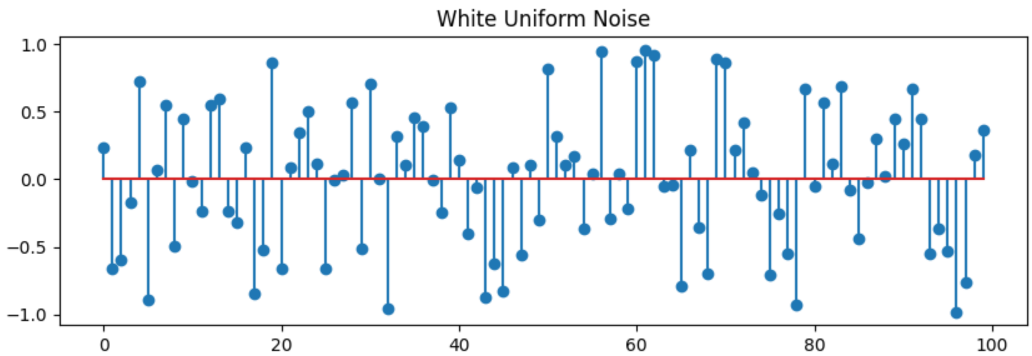

Figure 5: Plot of white uniform noise with a length of 100

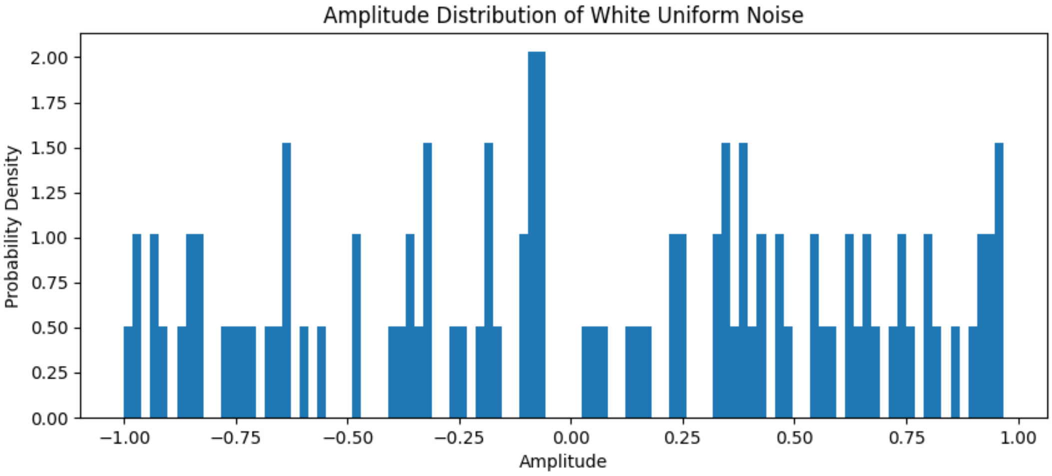

Figure 6: Plot of the distrobution of the white uniform noise with a length of 100

We can se in the figures 1, 3 and 5 That what they have in common is the sqeuence of positive and negative values are random, this is what gives the signals its "white" noise features.

They differ in terms of amplitude, from figure 2 we can si that binary are either 1 or 0, 4 we can se that the the distrobution of amplitude is normalised and in figure 6 we can se that the distrobution of amplitude is wait for it.... uniform! After playing a longer sequence of the diffrent noises, I was also able to observe that the binary was the loudest, then Gaussian, and then uniform being the most quiet. This was with binary having a random choice between 1 and -1, Gaussian having a normal distrobution with mean being 0 and standard deviation being 1, and lastly uniform having a uniform distrobution between 1 and -1.

(b) For each noise type

- write an expression for the probability distribution
- compute the mean value, autocorrelation function and power density spectrum

Probability distrobution of white binary noise:

$$P_x = \binom{n}{x} p^x q^{n-x}$$

Probability distrobution of white Gaussian noise:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\frac{x-\mu}{\sigma}}$$

where σ is the standard deviation, μ is mean and x being the amplitude.

Probability distrobution of white uniform noise:

$$f(x) = \frac{1}{b-a}$$

where a is the minimum amplitude and b is the maximum amplitude

When computing the mean value of the diffrent noises, I ended up with values approximately to 0:

- Mean Value of Binary White Noise: -0.08
- Mean Value of White Gaussian Noise: 0.11586957047161961
- Mean Value of White Uniform Noise: 0.03997036015289663

This is expected as a ideal mean for white noise should be zero

When plotting the autocorrelation we got:

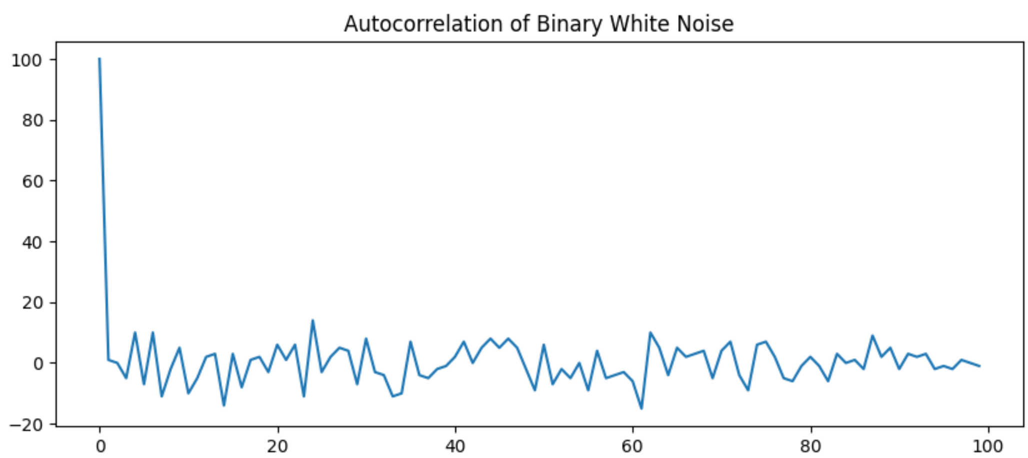

Figure 7: Autocorrelation of binary white noise

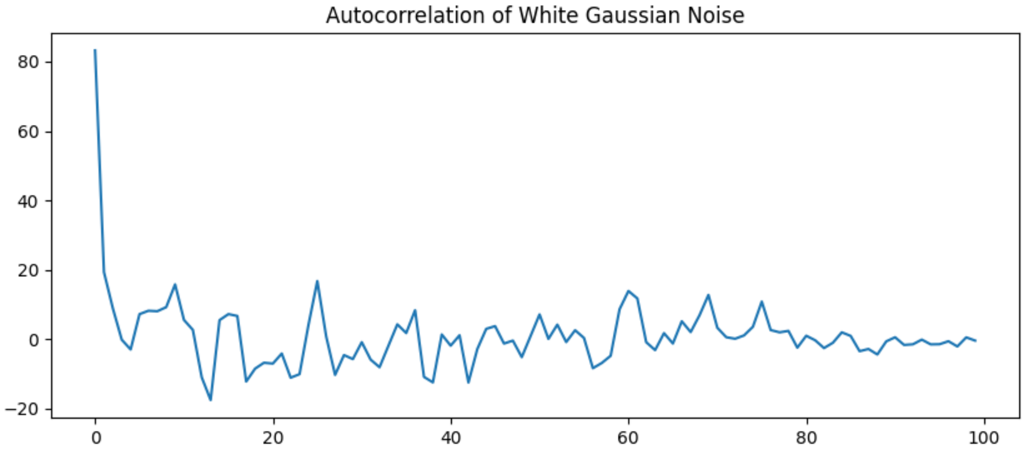

Figure 8: Autocorrelation of Gaussian white noise

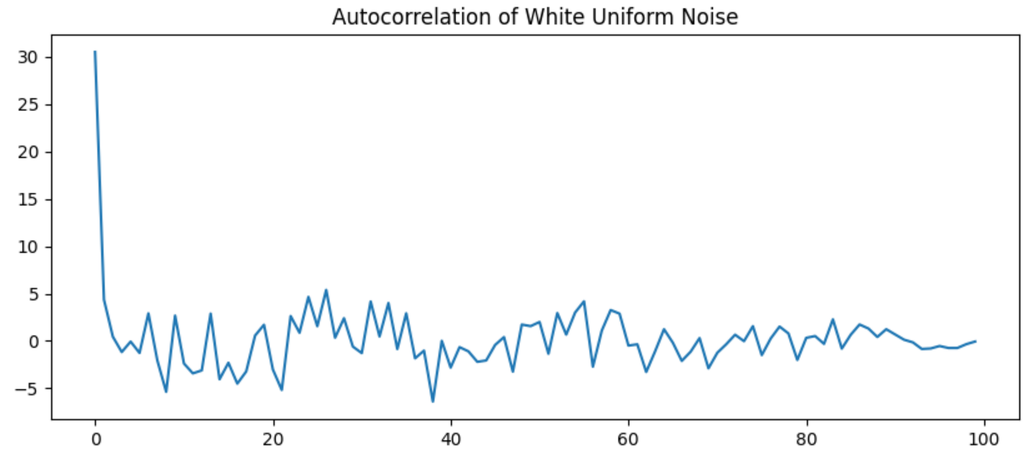

Figure 9: Autocorrelation of uniform white noise

When plotting the power deinsity spectrum we got:

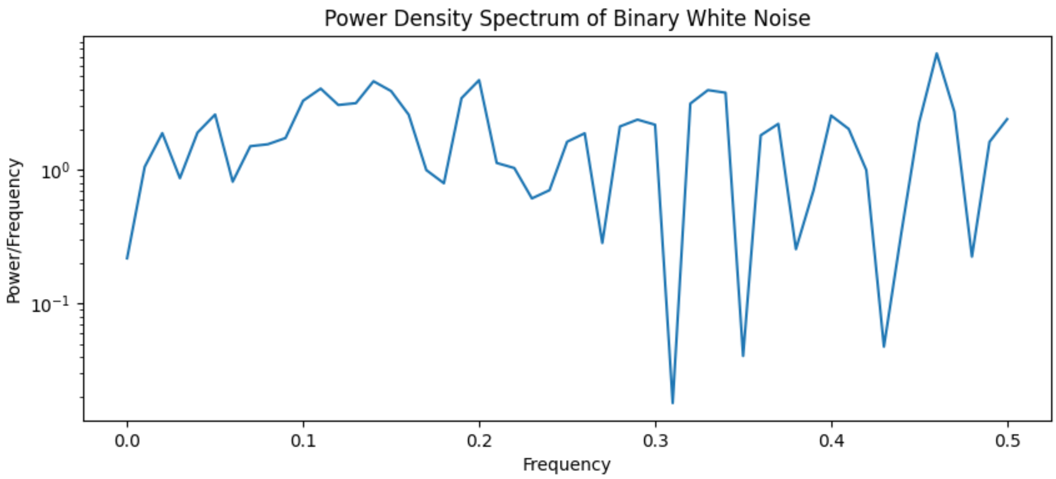

Figure 10: Power density spectrum of

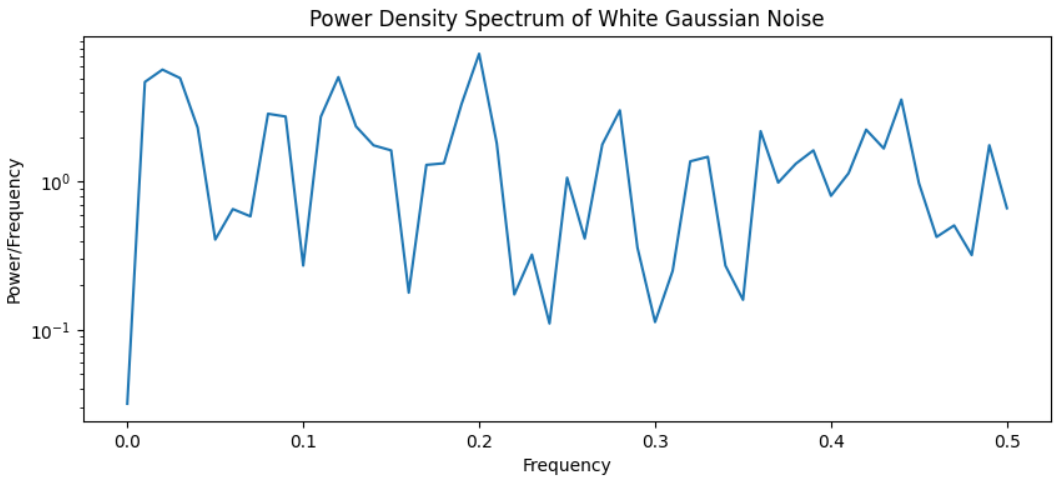

Figure 11: Power density spectrum of Gaussian white noise

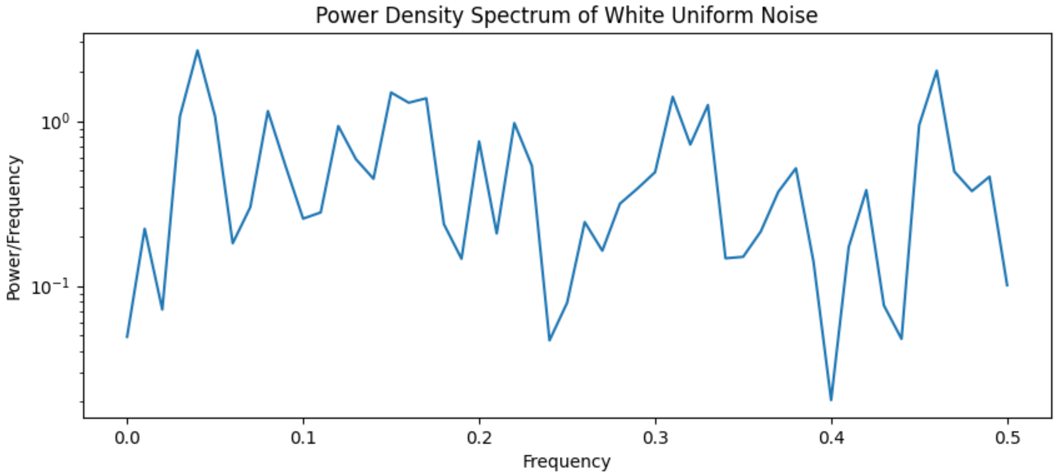

Figure 12: Power density spectrum of uniform white noise

(c) We would now like to estimate the statistical properties of the three noise types based on a noise segment.

- Generate a segment of 20000 samples for each noise type.
- Compute mean value estimates and compare to the theoretical values computed in (b).
- Compute estimates of the autocorrelation function. Plot them on the interval $[-10, 10]$, and compare to the theoretical values computed in (b).

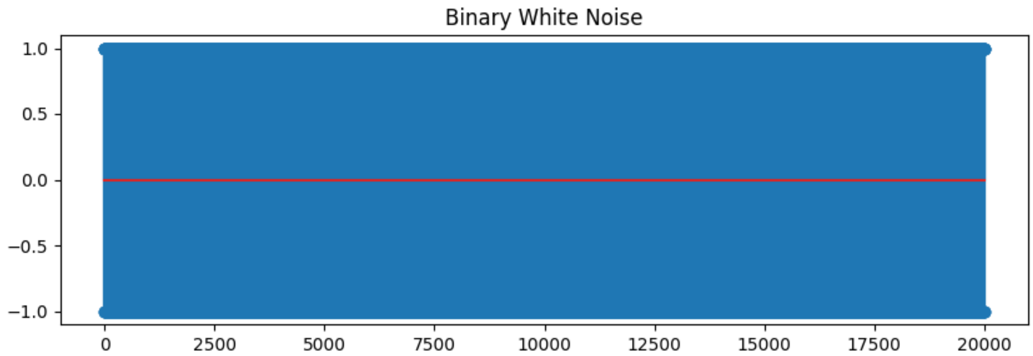

Figure 13: Plot of white binary noise with a length of 20000

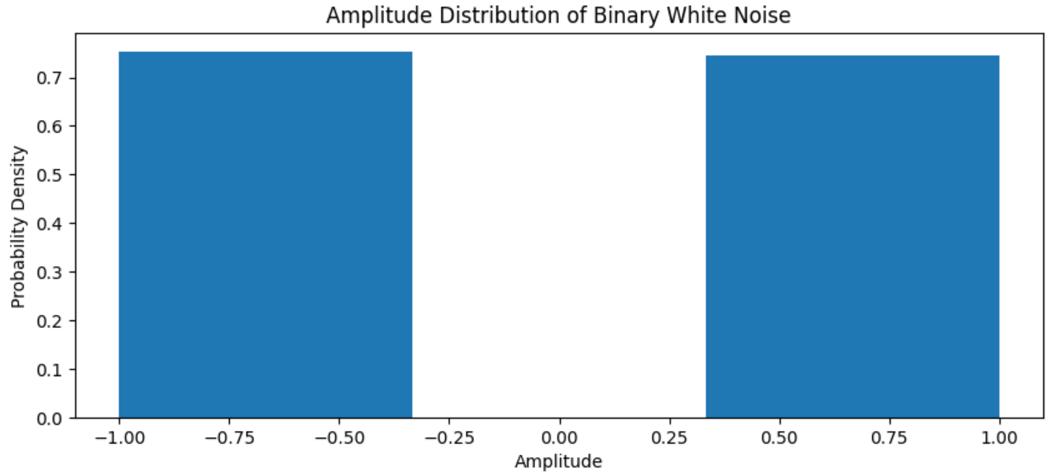

Figure 14: Plot of the distrobution of the white binary noise with a length of 20000

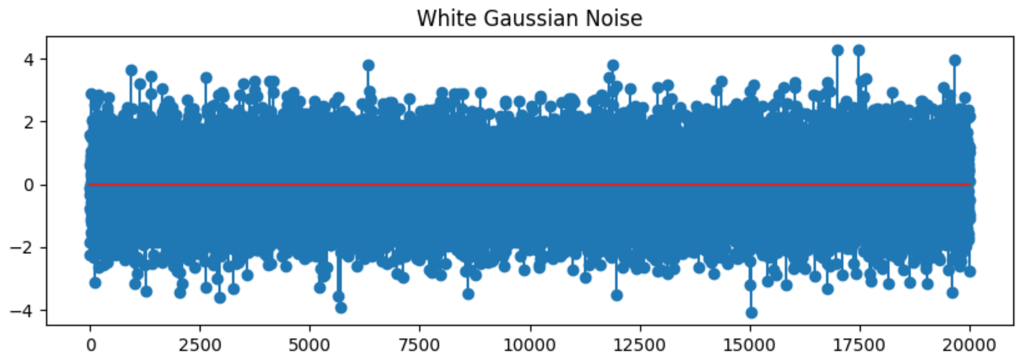

Figure 15: Plot of white Gaussian noise with a length of 20000

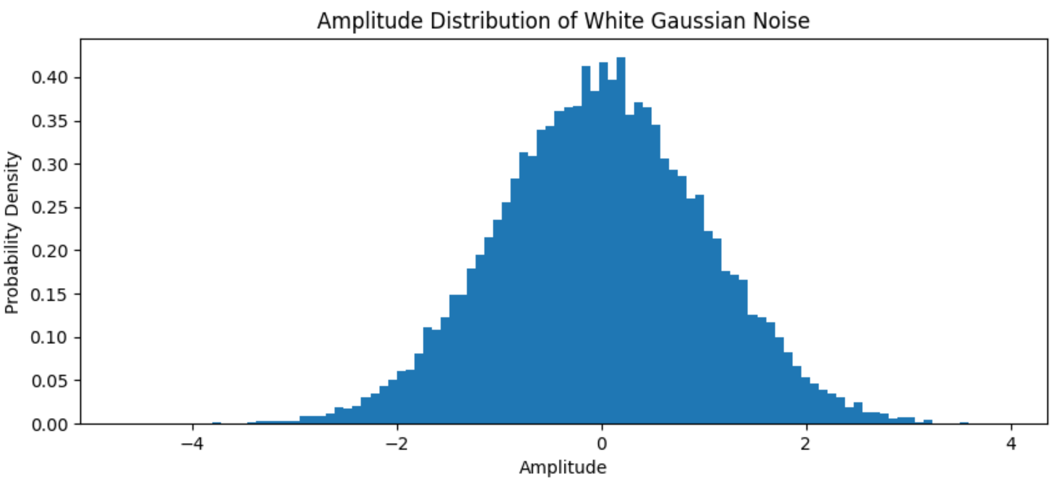

Figure 16: Plot of the distrobution of the white Gaussian noise with a length of 20000

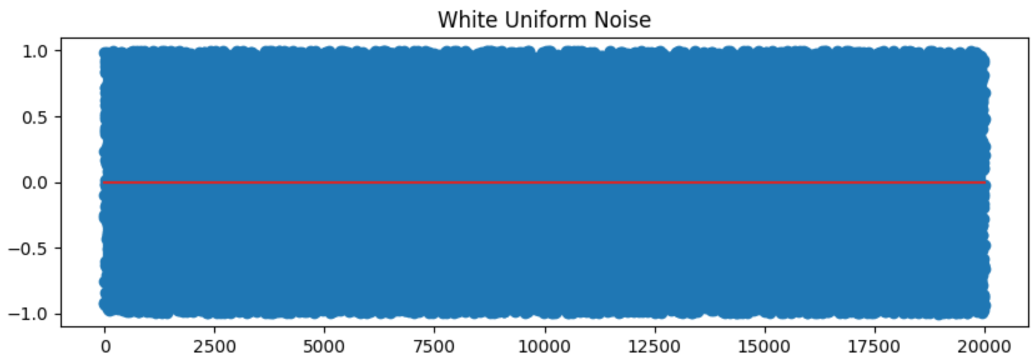

Figure 17: Plot of white uniform noise with a length of 20000

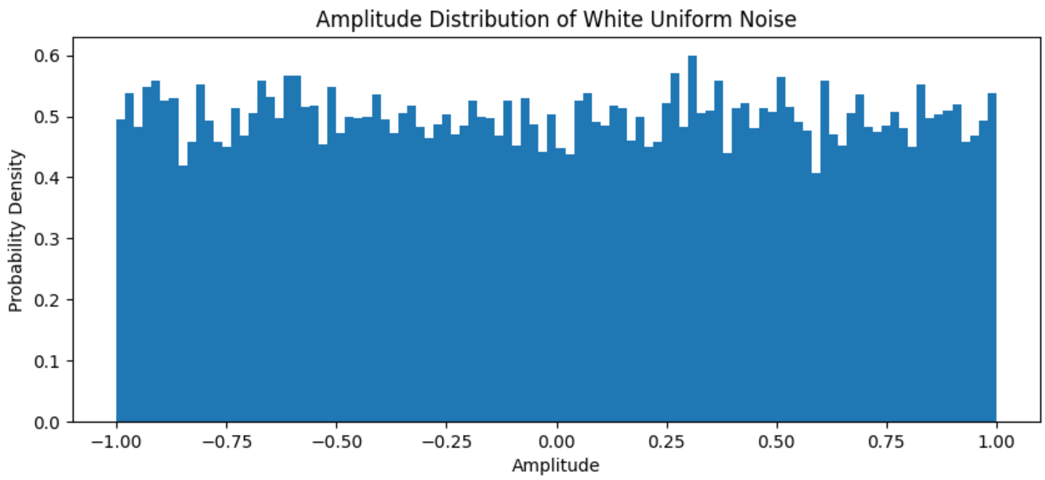

Figure 18: Plot of the distrobution of the white uniform noise with a length of 20000

When computing the mean value of the diffrent noises we got:

- Mean Value of Binary White Noise: -0.0036
- Mean Value of White Gaussian Noise: 0.0025846854929424063
- Mean Value of White Uniform Noise: 0.00016222991807518508

This is also expected as a longer sequence will be able have "more random" sequences and therefor be better at represent all the frequencies

When plotting the autocorrelation we got:

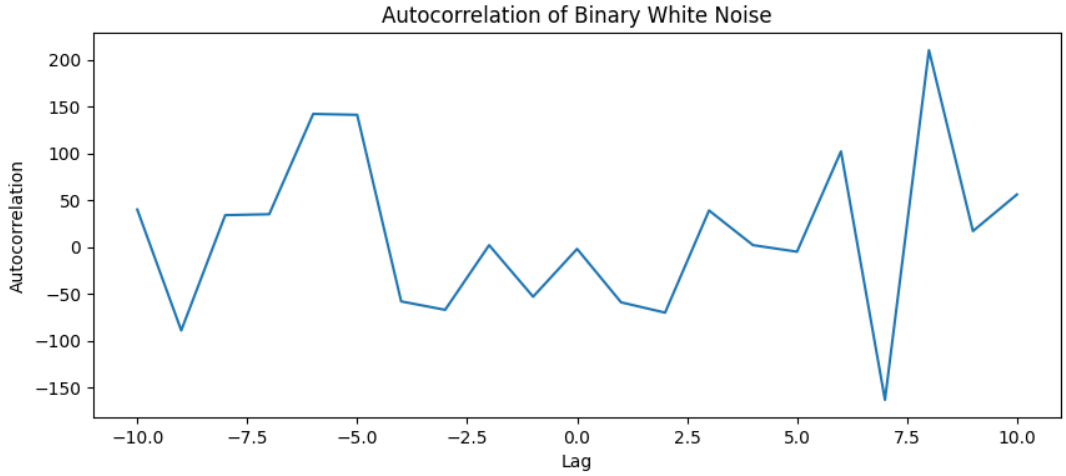

Figure 19: Autocorrelation of binary white noise

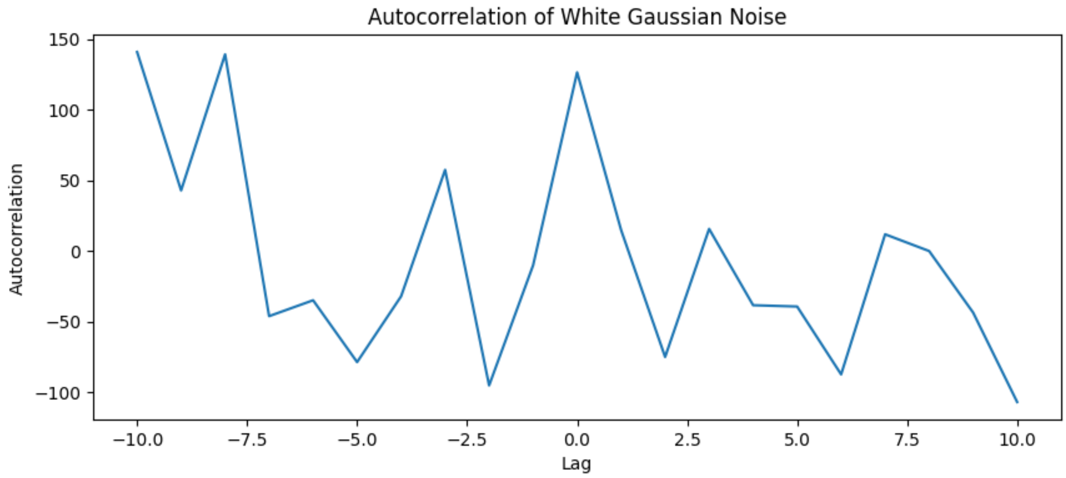

Figure 20: Autocorrelation of Gaussian white noise

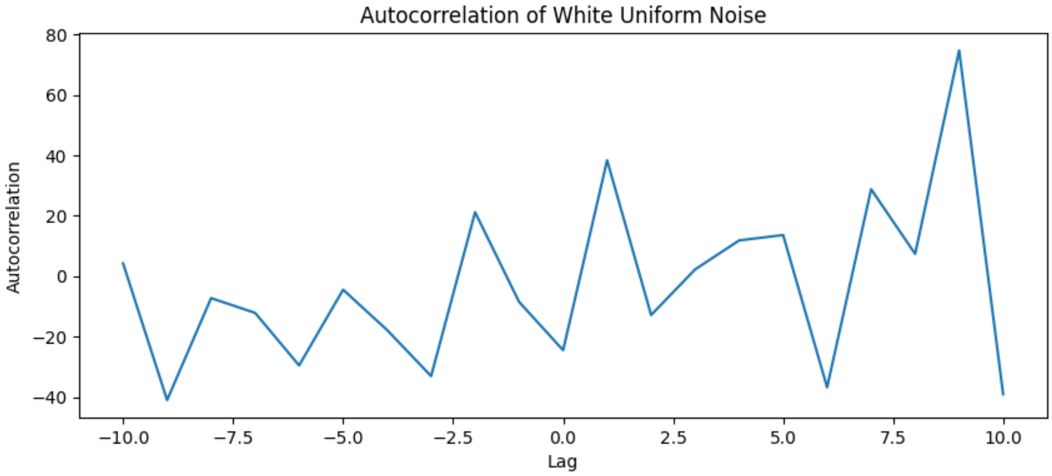

Figure 21: Autocorrelation of uniform white noise

We se that the autocorrelation is greater for a higher sequence, this is because there is a greater chance of a similar sequence more places on on the signal when the sequence length increases.

When plotting the power deinsity spectrum we got:

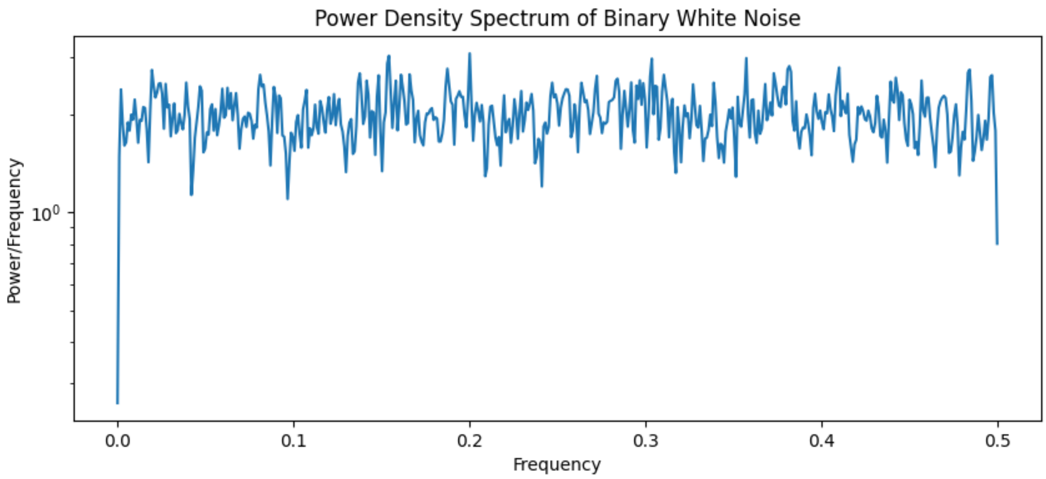

Figure 22: Power density spectrum of

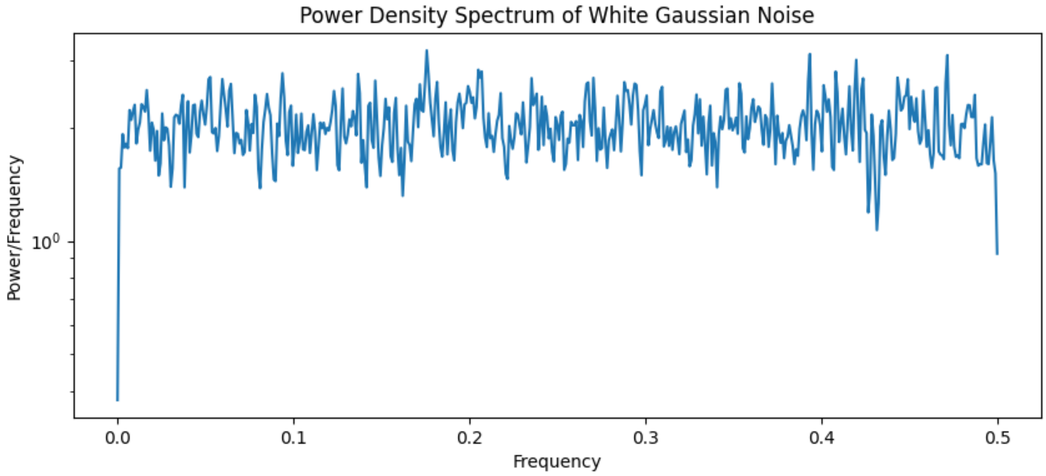

Figure 23: Power density spectrum of Gaussian white noise

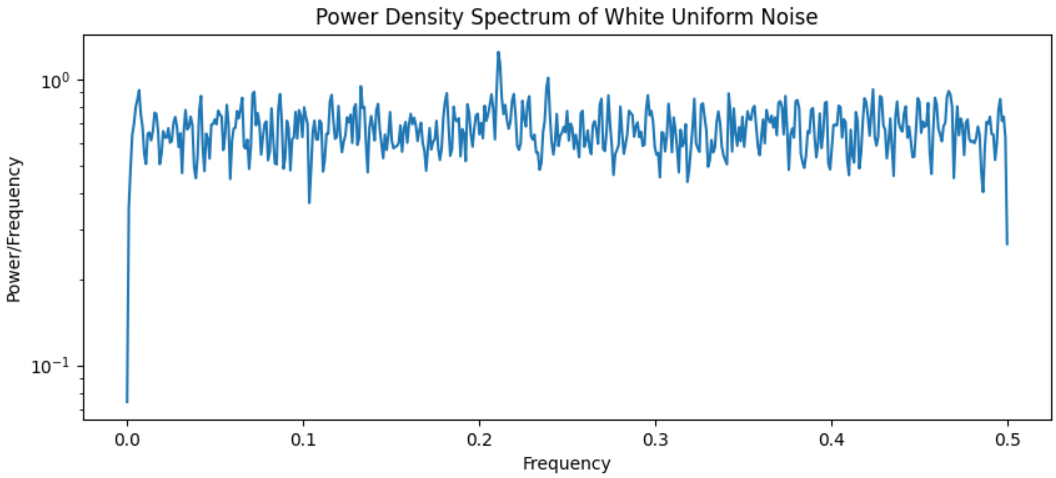

Figure 24: Power density spectrum of uniform white noise

Problem 2 (3 points)

A random signal $x[n]$ is generated by filtering white Gaussian noise $w[n]$ with variance $\sigma_w^2 = \frac{3}{4}$ by a causal filter with transfer function

$$H(z) = \frac{1}{1 + \frac{1}{2}z^{-1}}.$$

(a) Calculate the following statistic properties for $x[n]$:

- mean
- autocorrelation function
- power density spectrum
- power

Mean

$$\begin{aligned} m_x = E[y(n)] &= E\left[\sum_{k=-\infty}^{\infty} h(k)w(n-k)\right] \\ &= \sum_{k=-\infty}^{\infty} h(k)E[w(n-k)] \\ &= m_w \sum_{k=-\infty}^{\infty} h(k) = m_w H(0) \end{aligned}$$

As m_w equals to 0, we get that the mean of m_x also becomes 0.

Autocorrelation function

$$\begin{aligned}
\gamma_{xx}(l) = E[x(n)x(n-l)] &= E\left[\sum_{k=-\infty}^{\infty} h(k)w(n-k) \sum_{m=-\infty}^{\infty} h(m)w(n-l-m)\right] \\
&= \sum_{k=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} h(k)h(m)E[w(n-k)w(n-l-m)] \\
&= \sum_{k=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} h(k)h(m)\gamma_{ww}(l+m-k) \\
&= \sum_{n=-\infty}^{\infty} \left(\sum_{m=-\infty}^{\infty} h(m)h(m+n)\right)\gamma_{ww}(l-n) \\
&= \sum_{n=-\infty}^{\infty} r_{hh}(n)\gamma_{ww}(l-n) = \gamma_{ww}(l) * r_{hh}(l)
\end{aligned}$$

Where $\gamma_{ww}(l)$ is the autocorrelation of $w[n]$ and $r_{hh}(l)$ is the autocorrelation of hn

We start by taking the invers z-transform of $H(z)$

$$\begin{aligned}
H(z) &= \frac{1}{1 + \frac{1}{2}z^{-1}} \\
&= \frac{z}{z + \frac{1}{2}}
\end{aligned}$$

From inverse z-transform table

$$\frac{Az}{z-a} \stackrel{\mathcal{Z}^{-1}}{\Longrightarrow} Aa^n \cdot u[n]$$

this gives us

$$h[n] = \left(\frac{1}{2}\right)^n \cdot u[n]$$

$$r_{hh} = \sum_{n=-\infty}^{\infty} h[n] \cdot h[n-l]$$

as $h[n]$ consist of the unit step function we only need to sum from 0, this gives us

$$\begin{aligned}
r_{hh} &= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n} \cdot \left(\frac{1}{2}\right)^{n-l} \\
&= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{2n-l} \\
&= \left(\frac{1}{2}\right)^{-l} \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{2n} \\
&= \left(\frac{1}{2}\right)^{-l} \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^{n} \\
&= \left(\frac{1}{2}\right)^{-l} \cdot \frac{1}{1-\frac{1}{4}} \\
&= \left(\frac{1}{2}\right)^{-l} \cdot \frac{4}{3}
\end{aligned}$$

we end up with

$$\begin{aligned}
\gamma_{xx}(l) &= \frac{3}{4}\delta(l) * \left(\frac{1}{2}\right)^{-l} \cdot \frac{4}{3} \\
&= \delta(l) * \left(\frac{1}{2}\right)^{-l} \\
&= \left(\frac{1}{2}\right)^{-l}
\end{aligned}$$

Power density spectrum:

$$\Gamma_{xx}(f) = \text{DTFT}\left\{\gamma_{xx}(l)\right\}$$

$$
\begin{aligned}
X(f) &= \sum_{l=-\infty}^{\infty} \left(\frac{1}{2}\right)^{-l} e^{-j2\pi f l} \\
&= \sum_{l=-\infty}^{\infty} 2^{-l} e^{-j2\pi f l} \\
&= \sum_{l=0}^{\infty} 2^{l} e^{-j2\pi f l} + \sum_{l=1}^{\infty} 2^{-l} e^{-j2\pi f l} \\
&= \sum_{l=0}^{\infty} \left(2e^{-j2\pi f}\right)^{l} + \sum_{l=1}^{\infty} \left(\frac{1}{2} e^{-j2\pi f}\right)^{l} \\
&= \frac{1}{1-\left(2e^{-j2\pi f}\right)} + \frac{1}{1-\frac{1}{2}e^{-2\pi f}} - 1
\end{aligned}
$$

Power

$$P_x = \gamma_{xx}(0)$$

this gives us

$$\gamma_{xx}(0) = \left(\frac{1}{2}\right)^{0} = 1$$

(b) Write the expressions for the estimators of the statistical properties in (a) based on a signal segment of length N.

Mean estimator:

$$\hat{m}_x = \frac{1}{N} \sum_{n=0}^{N-1} x[n]$$

Autocorrelation function estimator:

$$\hat{\gamma}_{xx}(l) = \frac{1}{N-|l|} \sum_{n=0}^{N-|l|-1} x[n] \cdot x[n+|l|]$$

Power density spectrum estimator:

$$\hat{\Gamma}_{XX}(f) = \sum_{l=-\infty}^{\infty} \hat{\gamma}_{XX}[l]e^{-j2\pi fl}$$

where $\hat{\gamma}_{XX}[l] = \frac{1}{N}\sum_{n=0}^{N-|l|-1} x[n]x[n+|l|]$

Power estimator:

$$\hat{P} = \frac{1}{N}\sum_{n=0}^{N_1} |x[n]|^2$$

(c) Generate a segment of length N = 20000 samples of the signal $x[n]$.

- Use the estimators from (b) to compute the estimates based on this signal segment.
- Compare the result to the theoretical values computed in a). (You should plot both theoretical and estimated values of the autocorrelation function and power density spectrum in order to compare them. The autocorrelation function should be plotted on the interval $[-10, 10]$.

Mean estimator: 0.006558341682570523

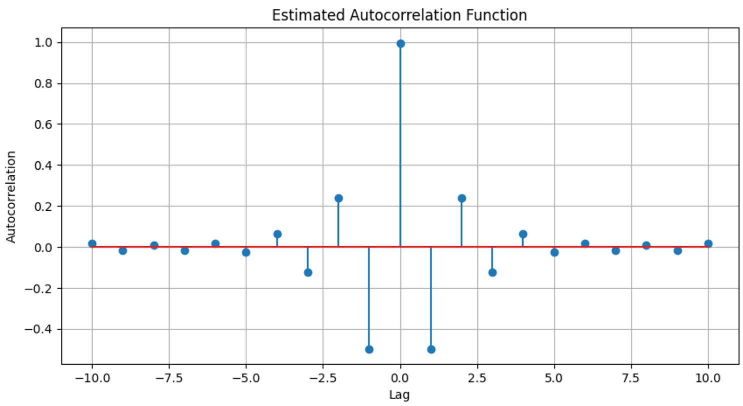

Figure 25: Estimated autocorrelation

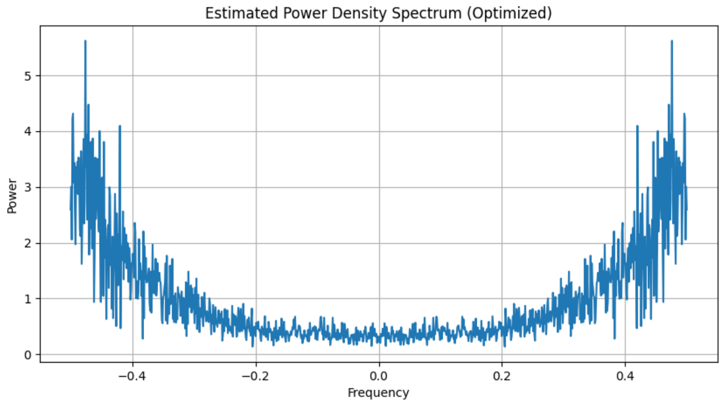

Figure 26: Estimated power density

Power estimator: 0.99596334378878824

The estimators and theoretical values does seen to fit.

(d) Write a Matlab program that estimates the power density spectrum using Bartlett method.

- Use the program to compute the estimate of the power density spectrum based on K = 10 and K = 100 nonoverlaping segments of the data generated in (c).
- Compare the estimate with the theoretical value and with the periodogram estimate.

(e) Repeat (c) and (d) several times to see how the obtained estimates vary when the different signal segments are used for the estimation.

Problem 3 (3 points)

This problem deals with the statistical properties of the mean estimator $\hat{m}_x$.

(a) Use the realization of signal $x[n]$ generated in Problem 2c), and compute 200 mean estimates based on nonoverlapping segments of length $K = 20$.

Writing the first 10 mean estimates: -0.09814215010018247, 0.3035875863738622, -0.20854327404628908, 0.08203965191883893, 0.16622584560587167, 0.20174441955570285, 0.1873038770870635, -0.04815623770506109, -0.007319351280463104, -0.002238756800624686.

(b) Use the Matlab function hist to plot the histogram of the mean estimates. Use 20 histogram bins.

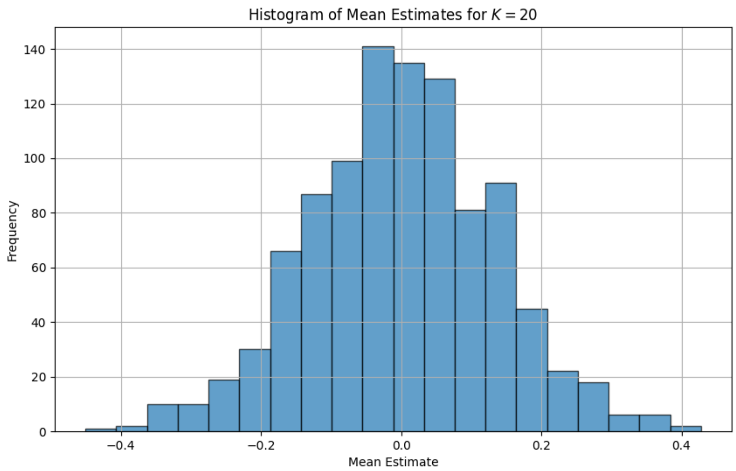

Figure 27: Histogram of the mean estimates

(c) Use the Matlab functions mean and var to estimate the mean and variance of $\hat{m}_x$.

- means k=20:-0.00012022523022167375
- variance k=20:0.019823887283148035
- means k=40:-0.00012022523022167286

- variance k=40:0.008598811045152263
- means k=100:-0.00012022523022167108
- variance k=100:0.003670062692836803

(e) Compare the results obtained with different values of K. Do they agree with the theoretical results on the statistical properties of the mean estimator $\hat{m}_x$? Explain!

The means stay the same, while the variance decreases. As the segment length K increases, the sample mean estimator becomes more consistent, as evidenced by the decreasing variance.

Problem 4: Fast Fourier Transform (FFT) [2 points]