TTT4120 Digital Signal Processing

Problem Set 8

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Contents

Problem 1 (2 points)

A random signal $x(n)$ is generated by filtering white Gaussian noise $w(n)$ with variance $\sigma_w^2 = \frac{3}{4}$ by a causal filter with transfer function

$$H(z) = \frac{1}{1 + \frac{1}{2}z^{-1}}.$$

Note that this is the same signal as in Problem 2 of Problem Set 7.

(a) Which type of process is $x(n)$? Justify your answer.

This is an autoregressive (AR) process as the filter filter $H(z)$ only a pole and no zeros, therefore $x(n)$.

(b) Find the coefficients of the optimal first- and second order predictor.

We start with defining a autoregressive model

$$\hat{x}(n) = -\sum_{k=1}^{p} a_k \cdot x(n-k)$$

giving us the first order:

$$\hat{x}(n) = -a \cdot x(n-1)$$

we start by elaborating the mean square error (MSE)

$$\begin{aligned}
\sigma_f^2 &= E\left\{(X[n] - \hat{X}[n])^2\right\} = E\left\{(X[n] + a_1 X[n-1])^2\right\} \\
&= E\left\{X[n]^2 + a_1 X[N] X[n-1] + a_1^2 X[n-1]^2\right\} \\
&= \gamma_{XX}[0] + 2a_1\gamma_{XX}[1] + a_1^2\gamma_{XX}[0]
\end{aligned}$$

• MSE is minimum if we choose a_1 such that

$$\frac{d\sigma_f^2}{da_k} = 0$$

this gives

$$\frac{d\sigma_f^2}{da_k} = 2\gamma_{xx}[1] + 2a_1\gamma_{xx}[o] = 0$$

this is 0 when

$$a_1 = -\frac{\gamma_{xx}[1]}{\gamma_{xx}[0]}$$

From problem set 7 we have calculated that

$$\gamma_{xx}(l) = \left(\frac{1}{2}\right)^l$$

giving us

$$a_1 = -\left(\frac{1}{2}\right)^1 = -\frac{1}{2}$$

the second order

$$\hat{x}(n) = -2 \cdot x(n-1) - a_2 \cdot x(n-2)$$

this gives us

$$\begin{aligned}
\sigma_f^2 &= E\left\{(X[n] - \hat{X}[n])^2\right\} = E\left\{(X[n] + a_1X[n-1] + a_2X[n-2])^2\right\} \\
&= E\{X[n]^2 + 2a_1X[n]X[n-1] + a_1^2X^2[n-1] + a_2X[n]X[n-2] \\
&+ 2a_2^2X^2[n-2] + 2a_1x[n-1]a_2x[n-2]\} \\
&= \gamma_{XX}[0] + 2a_1\gamma_{XX}[1] + a_1^2\gamma_{XX}[0] + 2a_2\gamma_{XX}[2] + a_2^2\gamma_{XX}[0] + 2a_1a_2\gamma_{xx}[1]
\end{aligned}$$

using

$$\frac{d\sigma_f^2}{da_2} = 0$$

we get

$$\frac{d\sigma_f^2}{da_2} = 2\gamma_{xx}[2] + 2a_2\gamma_{xx}[0] + 2a_1\gamma_{xx}[1] = 2\gamma_{xx}[2] + 2a_2\gamma_{xx}[0] - \gamma_{xx}[1] = 0$$

$$
\begin{aligned}
\Rightarrow a_2 &= -\frac{\gamma_{xx}[2] - \gamma_{xx}[1]}{2} \\
&= \frac{\left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^1}{2} \\
&= \frac{3}{8}
\end{aligned}
$$

Problem 2 (3 points)

A random process $x(n)$ is given by

$$x(n) = w(n) - 0.4w(n-1),$$

where $w(n)$ is white Gaussian noise with variance $\sigma_w^2 = 1$.

(a) Which type of process is $x(n)$? Justify your answer.

This is an autoregressive (AR) process as it is described in time domain as

$$X[n] = -\sum_{k=1}^{p} a_k X[n-k] + W[n]$$

where $P = 1$

(b) Find the autocorrelation function $\gamma_{xx}(l)$ and the power density spectrum $\Gamma_{xx}(f)$ for this process.

Autocorrelation function

$$\begin{aligned}
\gamma_{xx}(l) = E[x(n)x(n-l)] &= E\left[\sum_{k=-\infty}^{\infty} h(k)w(n-k) \sum_{m=-\infty}^{\infty} h(m)w(n-l-m)\right] \\
&= \sum_{k=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} h(k)h(m)E[w(n-k)w(n-l-m)] \\
&= \sum_{k=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} h(k)h(m)\gamma_{ww}(l+m-k) \\
&= \sum_{n=-\infty}^{\infty} \left(\sum_{m=-\infty}^{\infty} h(m)h(m+n)\right)\gamma_{ww}(l-n) \\
&= \sum_{n=-\infty}^{\infty} r_{hh}(n)\gamma_{ww}(l-n) = \gamma_{ww}(l) * r_{hh}(l)
\end{aligned}$$

Where $\gamma_{ww}(l)$ is the autocorrelation of $w[n]$ and $r_{hh}(l)$ is the autocorrelation of $h(n)$. From the given equation we can conclude that the system's impulse response $h(n)$ is

$$h(n) = \delta[n] - 0.4\delta[n-1]$$

this gives us the autocorrelation

$$r_{hh}(l) = h(l) * h(-l)$$

as we have the delta function, we will only get a value for $n = 0$ and $n = 1$

$$\begin{aligned} r_{hh}(0) &= h(0) \cdot h(0) + h(1) \cdot h(1) \\ &= 1^2 + (-0.4)^2 = 1 + 0.16 = 1.16 \end{aligned}$$

$$\begin{aligned} r_{hh}(1) &= h(0) \cdot h(-1) + h(1) \cdot h(0) \\ &= -0.4 \end{aligned}$$

The autocorrelation of the white noise process $\gamma_{ww}(l)$ is given by:

$$\gamma_{ww}(l) = \sigma_w^2 \delta(l)$$

Since $\sigma_w^2 = 1, \gamma_{ww}(l) = \delta(l).$ Now, we can find the autocorrelation function $\gamma_{xx}(l)$ by convolving $r_{hh}(l)$ with $\gamma_{ww}(l)$:

$$\gamma_{xx}(l) = \gamma_{ww}(l) * r_{hh}(l)$$

This simplifies to:

$$\gamma_{xx}(l) = \delta(l) * r_{hh}(l)$$

Since convolving with a delta function is equivalent to sampling the function at the point where the delta is located, we get:

$$\gamma_{xx}(l) = r_{hh}(l)$$

So, the autocorrelation function $\gamma_{xx}(l)$ for $l = 0, \pm 1$ is:

$$\begin{aligned} &\gamma_{xx}(0) = 1.16 \\ &\gamma_{xx}(1) = \gamma_{xx}(-1) = -0.4 \\ &\gamma_{xx}(l) = 0, \text{ for } |l| > 1 \end{aligned}$$

Power density spectrum

$$\begin{aligned}\Gamma_{xx}(f) &= \text{DTFT}\left\{\gamma_{xx}(l)\right\} \\ &= \sum_{l=-1}^{1} \gamma_{xx}(l) e^{-j2\pi f l} \\ &= 1.16 - 0.4e^{-j2\pi f} - 0.4e^{j2\pi f} \\ &= 1.16 - 1\left(\frac{0.8e^{2\pi f} + 0.8e^{2\pi f}}{2}\right) \\ &= 1.16 - \cos(2\pi f)\end{aligned}$$

(c) Calculate the coefficients of the optimal first, second and third order predictor for the process $x(n)$. (You can use Matlab to solve the Yule-Walker equations). Compute also the corresponding prediction error variances. Comment on the results.

Order	Coefficients	Prediction Error Variance
1	$a_1 = -0.3448$	$\sigma^2 = 1.0221$
2	$a_1 = -0.3914, a_2 = -0.1350$	$\sigma^2 = 1.0035$
3	$a_1 = -0.3986, a_2 = -0.1560, a_3 = -0.0538$	$\sigma^2 = 1.0006$

Table 1: Coefficients and Prediction Error Variances for AR Predictors

We can se that when the order of predictor increases, the prediction error variance reduces as wellwe do howewer see that the there is a larger decrease from 1. to 2. order than 2. to 3. order, indicating that we will get diminishing returns for for each time we increase the order of predictors.

(d) Write an expression for the power density spectrum estimate based on an AR [p] model. Plot the calculated power density spectrum together with its estimates based on AR[1], AR[2] and AR[3] models. (Use Matlab functions freqz.) Which of the AR models gives the best approximation of the MA process? Justify your answer.

The power density spectrum (PDS) of an AR(p) process can be estimated using the following expression:

$$\hat{\Gamma}_{xx}(f) = \frac{\sigma^2}{\left|1 - \sum_{k=1}^{p} a_k e^{-j2\pi f k}\right|^2}$$

this gives us

$$\hat{\Gamma}_{xx}^{(}1)(f) = \frac{1.0221}{\left|1 - 0.3448e^{-j2\pi f}\right|^2}$$
$$\hat{\Gamma}_{xx}^{(}2)(f) = \frac{1.0035}{\left|1 - 0.3914e^{-j2\pi f} - 0.1350e^{-j4\pi f}\right|^2}$$
$$\hat{\Gamma}_{xx}^{(}3)(f) = \frac{1.0006}{\left|1 - 0.3986e^{-j2\pi f} - 0.1560e^{-j4\pi f} - 0.0538e^{-j6\pi f}\right|^2}$$

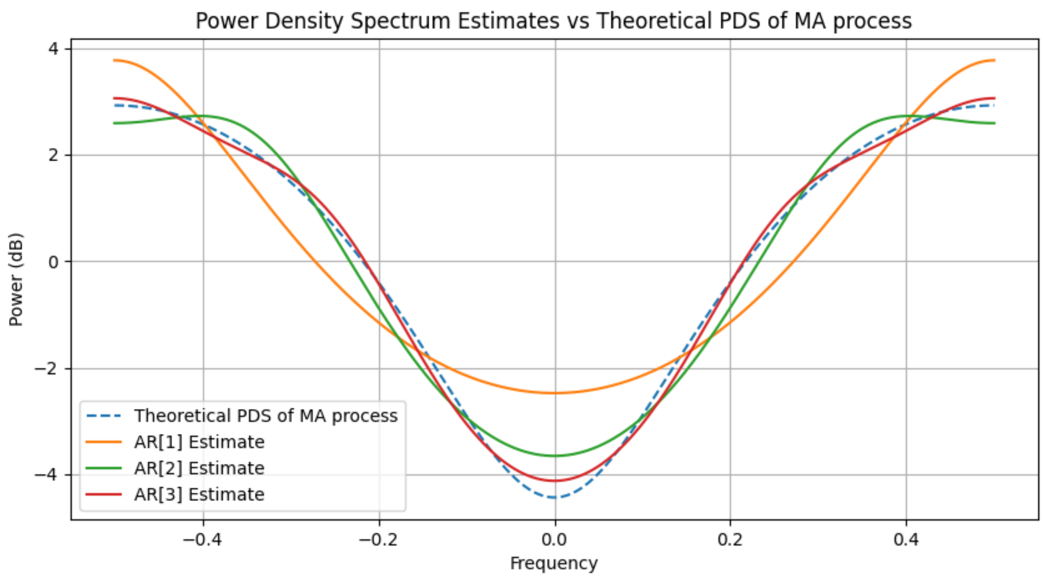

Figure 1: Power density spectrum estimate vs theoretical PDS of MA process

as figure 1 shows, the AR[3] model gives us the best approximation of the MA process.

Problem 3 (5 points)

In this problem you should make a vowel transformer that lets you record a vowel, and then transforms it to any desired Norwegian vowel pronounced in your own voice. This problem requires the use of headphones and microphone. One sample of each Norwegian vowel can be downloaded from It's learning. Start by converting one of those vowels to other vowels. When recording your own vowels you should use the sampling frequency $F_s = 8\text{kHz}$, and make sure to extract only the file part containing the vowel before proceeding. Hint: Vowels can be regarded as stationary signals, and are well modeled as $\mathrm{AR}[10]$ processes. Useful Matlab functions are audioread, audiorecorder, lpc, filter and sound.