TTT4120 Digital Signal Processing

Problem Set 3

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Contents

Problem 1 (2.5 points)

Consider the two causal analog filters shown in Figure 1.

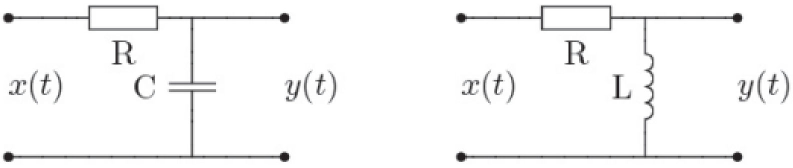


Figure 1: Analog filters

(a) Derive the differential equations and transfer functions for the two filters.

For the RC filter

$$
\begin{aligned}
v_R(t) &= Ri(t) \\
i(t) &= C\frac{dv(t)}{dt} \\
v_R(t) &= RC\frac{dv(t)}{dt}
\end{aligned}
$$

using KVL we get

$$
\begin{aligned}
x(t) &= v_R(t) + v_C(t) \\
x(t) &= RC\frac{dv(t)}{dt} + v_C(t) \\
y(t) &= x(t) - RC\frac{dv(t)}{dt}
\end{aligned}
$$

The transfer function $H(s) = \frac{Y(s)}{X(s)}$

$$
\begin{aligned}
\mathcal{L}\left\{x(t)\right\} &= RC \cdot \mathcal{L}\left\{\frac{dy(t)}{dt}\right\} + \mathcal{L}\left\{y(t)\right\} \\
X(s) &= RCsY(s) + Y(s)
\end{aligned}
$$

This gives us

$$
\begin{aligned}
H(s) &= \frac{Y(s)}{X(s)} = \frac{Y(s)}{RCsY(s) + Y(s)} \\
H(s) &= \frac{1}{RCs + 1}
\end{aligned}
$$

For the RL filter

$$
\begin{aligned}
v_R(t) &= Ri(t) \\
v_L(t) &= L\frac{di(t)}{dt}
\end{aligned}
$$

using KVL we get

$$
\begin{aligned}
x(t) &= Ri(t) + L\frac{di(t)}{dt} \\
y(t) &= x(t) - Ri(t)
\end{aligned}
$$

To find the transfer function we need to take the laplace transform:

$$
\begin{aligned}
\mathcal{L}\{x(t)\} &= R\mathcal{L}\{i(t)\} + L\mathcal{L}\left\{\frac{di(t)}{dt})\right\} \\
X(s) &= RI(s) + LsI(s) \\
\mathcal{L}\{y(t)\} &= \mathcal{L}\{x(t)\} - R\mathcal{L}\{i(t)\} \\
Y(s) &= LsI(s) + RI(s) - RI(s)
\end{aligned}
$$

$$
\begin{aligned}
H(s) &= \frac{Y(s)}{X(s)} = \frac{LsI(s)}{RI(s) + LsI(s)} \\
H(s) &= \frac{LsI(s)}{LsI(s) + RI(s)} \\
H(s) &= \frac{Ls}{Ls + R}
\end{aligned}
$$

(b) Determine frequency responses and filter types (i.e. lowpass, highpass, bandpass, bandstop) for the filters.

For the RL filter the capacitor will ground all high frequencies, therefore it is a lowpass filter. The RL filter will ground all low frequencies, therefore it is a highpass filter.

(c) Derive the unit pulse responses for the filters.

We can derive the unit pulse response for the filter by taking the inverse Laplace transform of the transfer function, $H(s)$. We can use this by using a inverce laplace table.

For the RC filter we have

$$\frac{1}{s-a} = e^{at}$$

$$h(t) = \mathcal{L}^{-1}\{H(s)\} = e^{-\frac{t}{RC}}u(t)$$

For the RL filter we have:

$$\begin{aligned} H(s) &= \frac{Ls}{Ls+R} \\ &= 1 - \frac{R}{Ls+R} \\ &= 1 - \frac{1}{\frac{R}{L}s+1} \end{aligned}$$

this using the same equation from the table we get

$$h(t) = \delta(t) - e^{-\frac{tR}{L}}u(t)$$

Problem 2 (2.5 points)

Find the region of convergence and unit pulse response $h[n]$ for the following digital filters.

(a) A causal filter with transfer function

$$H(z) = \frac{1}{1 - \frac{2}{3}z^{-1}}$$

We start by finding the pole

$$\begin{aligned} 1 - \frac{2}{3}z^{-1} &= 0 \\ z &= \frac{2}{3} \end{aligned}$$

Since this is a causal sequence, the ROC will be the region outside the circle with a radius equal to the magnitude of the largest pole which in this case is $|z| > \frac{2}{3}$

To find the unit pulse response we can use a simple inverse z-transform table

$$\frac{1}{1 - az^{-1}} = a^n u[n]$$

this gives us

$$h[n] = \left(\frac{2}{3}\right)^n u[n]$$

(b) A causal filters with transfer function

$$H(z) = \frac{1}{\left(1 + \frac{1}{2}z^{-1}\right)\left(1 - z^{-1}\right)}$$

To find the poles of the transfer function, we set the denominator to zero and solve for z:

$$\left(1+\frac{1}{2}z^{-1}\right)\left(1-z^{-1}\right)=0$$
$$\Rightarrow \quad 1+\frac{1}{2}z^{-1}=0 \quad \text{or} \quad 1-z^{-1}=0$$

Solving the first equation $1+\frac{1}{2}z^{-1}=0$:

$$\begin{aligned}\frac{1}{2}z^{-1}&=-1\\ z^{-1}&=-2\\ z&=-\frac{1}{2} \quad \text{(Pole 1)}\end{aligned}$$

Solving the second equation $1-z^{-1}=0$:

$$\begin{aligned}z^{-1}&=1\\ z&=1 \quad \text{(Pole 2)}\end{aligned}$$

Thus, the transfer function has poles at $z=-\frac{1}{2}$ and $z=1$.

Since the magnitde of the largest pole is 1 the ROC will be the region outside the circle with a radius equal to 1.

To find the impulse response, we need to perform partial fraction decomposition.

$$\begin{aligned}H(z)&=\frac{1}{\left(1+\frac{1}{2}z^{-1}\right)\left(1-z^{-1}\right)}=\frac{z}{\left(z+\frac{1}{2}\right)(z-1)}\\ &=\frac{A}{\left(z+\frac{1}{2}\right)}+\frac{B}{(z-1)}\end{aligned}$$

$$\begin{aligned}z&=A(z-1)+B(z+\frac{1}{2})\\ 0&=z-\left(A(z-1)+B\left(z+\frac{1}{2}\right)\right)\end{aligned}$$

this gives us $A = \frac{1}{3}$ and $B = \frac{2}{3}$

This gives us the inverse Z-transform

$$\frac{A}{1 - az^{-1}} = Aa^n u[n]$$

$$h[n] = \frac{1}{3}\left(-\frac{1}{2}\right)^n u[n] + \frac{2}{3}u[n]$$

(c) An anti-causal filter with transfer function

$$H(z) = \frac{z^{-1}}{\left(1 + \frac{3}{2}z^{-1}\right)\left(1 - 3z^{-1}\right)}$$

The poles of this filter is at $z = -\frac{3}{2}$ and $z = 3$. Since this is an anti-causal filter, the ROC will be inside a circle with the radius equal to the magnitude of the smallest pole $\frac{3}{2}$

To find the impulse response, we need to perform partial fraction decomposition.

$$H(z) = \frac{z^{-1}}{\left(1 + \frac{3}{2}z^{-1}\right)\left(1 - 3z^{-1}\right)} = \frac{1}{\left(z + \frac{3}{2}\right)(z - 3)}$$
$$H(z) = \frac{A}{\left(z + \frac{3}{2}\right)} + \frac{B}{(z - 3)}$$

$$1 = A(z - 3) + B\left(z + \frac{3}{2}\right)$$

This gives us $A = -\frac{2}{9}$ and $B = \frac{2}{9}$

This gives us the inverse Z-transform

$$\frac{1}{1 - az^{-1}} = -a^n u[-n - 1]$$

$$h[n] = -\frac{2}{9}\left(-\frac{3}{2}\right)^n u[-n - 1] + \frac{2}{9}3^n u[-n - 1]$$

(d) Which of the three systems are stable? Justify your answer.

The stability of a LTI system can be determined by the locations of the poles of its transfer function. The magnitudes of all the poles should be less than 1. Therefore the first system is the only stable one as the magnitude of the pole is less than 1.

Problem 3 (3 points)

Consider a linear time-invariant (LTI) system with unit pulse response $h[n]$ and input signal $x[n]$.

$$h[n] = \begin{cases} \frac{1}{2^n} & n \geq 0 \\ 0 & n < 0, \end{cases}$$
$$x[n] = \begin{cases} 1 & n \geq 2 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Determine the z-transform of the unit pulse response $h[n]$ and input signal $x[n]$ and their respective ROCs.

The Z-transform for a causal sequence is given by:

$$X(z) = \sum_{n=0}^{\infty} x[n] \cdot z^{-n}$$

For the unit pulse response $h[n] = \frac{1}{2^n}$ for $n \geq 0$, substituting into the Z-transform formula gives:

$$H(z) = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n \cdot z^{-n}$$

This is a geometric series with the first term $a = 1$ and the common ratio $r = \frac{1}{2} \cdot z^{-1}$. The sum of an infinite geometric series is given by:

$$S = \frac{a}{1 - r}$$

Substituting the values of a and r we get:

$$H(z) = \frac{1}{1 - \frac{1}{2} \cdot z^{-1}} = \frac{z}{z - \frac{1}{2}}$$

The Region of Convergence (ROC) for this Z-transform is $|z| > \frac{1}{2}$.

Given the signal $x[n]$ which is 1 for $n \geq 2$ and 0 otherwise, the Z-transform $X(z)$ is represented as:

$$X(z) = z^{-2} + z^{-3} + z^{-4} + \ldots = \frac{z^{-2}}{1 - z^{-1}} = \frac{1}{z^2(1 - z^{-1})}$$

The Z-transform of $x[n]$ is:

$$X(z) = \frac{1}{z^2(1 - z^{-1})}$$

The Region of Convergence (ROC) for $x[n]$ is $|z| > 0$.

(b) Derive an expression for the output signal $y[n]$ by performing the convolution in the time domain.

To find $y[n]$, we perform the convolution of $x[n]$ and $h[n]$:

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k] \cdot h[n-k]$$

Given $x[n] = 1$ for $n \geq 2$ and $h[n] = \frac{1}{2^n}$ for $n \geq 0$, we will have non-zero terms only when $k \geq 2$ and $n - k \geq 0$, so the lower limit for k in the summation will be 2 and the upper limit will be n.

$$y[n] = \sum_{k=2}^{n} \frac{1}{2^{n-k}} = \frac{1}{4}\left(1 + \frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^{n-2}}\right), \quad \text{for } n \geq 2$$

(c) Derive an expression for the output signal $y[n]$ using the z-transform.

To derive the expression for the output signal $y[n]$ using the z-transform, we multiply the Z-transforms of the input signal $x[n]$ and the unit pulse response $h[n]$ and then find the inverse Z-transform:

$$Y(z) = X(z) \cdot H(z) = \frac{1}{z^2(1 - z^{-1})} \cdot \frac{2z}{2z - 1} = \frac{2}{(z-1)(2z-1)}$$

Performing partial fraction decomposition on $Y(z)$, we get:

$$Y(z) = -\frac{4}{2z - 1} + \frac{2}{z - 1}$$

Thus, we have:

$$A = 0, \quad B = -4, \quad C = 2$$

The inverse Z-transform for each term separately is:

$$y[n] = -4\left(\frac{1}{2}\right)^n u[n] + 2u[n] = -2^n u[n] + 2u[n], \quad \text{for } n \geq 0$$

Problem 4 (2 points)

A digital filter is given by the following difference equation

$$y[n] = x[n] - x[n-2] - \frac{1}{4}y[n-2]$$

(a) Find the transfer function of the filter.

The transfer function $H(z)$ can be found by taking the Z-transform of the given difference equation:

$$Y(z) = X(z) - z^{-2}X(z) - \frac{1}{4}z^{-2}Y(z)$$

Rearranging terms to solve for $\frac{Y(z)}{X(z)}$:

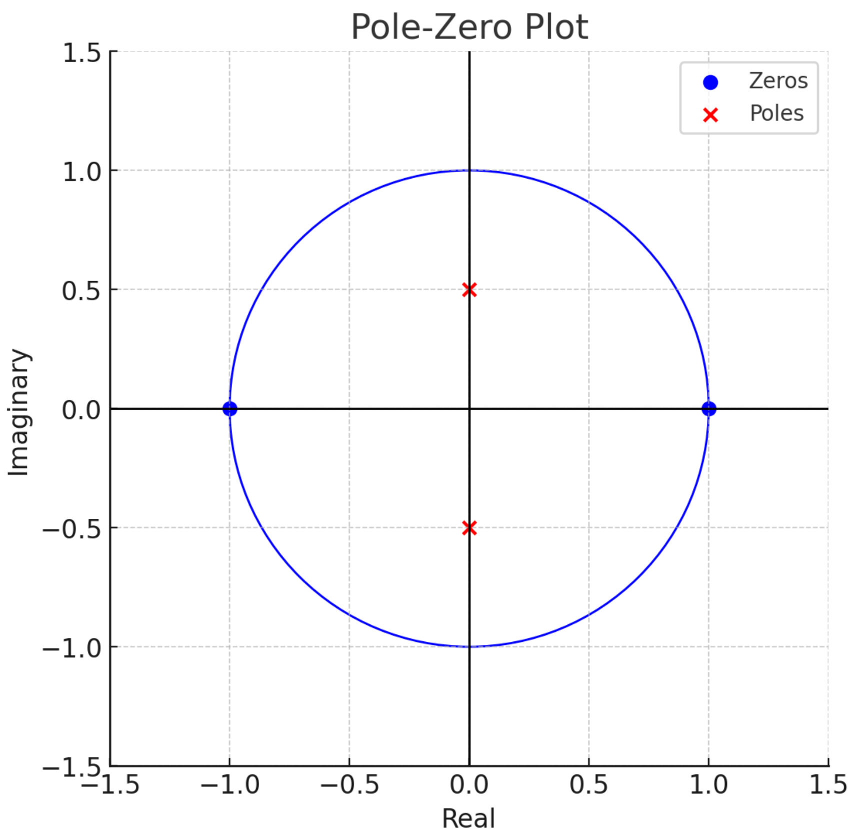
$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 - z^{-2}}{1 + \frac{1}{4}z^{-2}}$$

Simplifying:

$$H(z) = \frac{z^2 - 1}{z^2 + 0.25}$$

(b) Find the poles and zeros of the filter and sketch them in the z-plane.

The zeros of the transfer function are at $z = -1$ and $z = 1$. The poles of the transfer function are at $z = 0.5i$ and $z = -0.5i$. The poles and zeros are sketched in the z-plane as shown in the figure attached.



(c) Is the filter stable? Justify your answer based on the pole-zero plot.

The filter is stable as all the poles are inside the unit circle in the z-plane.

(d) Determine the filter type (i.e. HP, LP, BP or BS) based on the pole-zero plot.

The filter is a Bandpass (BP) filter. This can be inferred from the pole-zero plot as there are zeros at both $z = 1$ and $z = -1$, suggesting that it attenuates both the high and low-frequency components, allowing a certain band of frequencies to pass through.

Appendix