A One-Period Crowdfunding Model

June 2, 2023

Suppose a startup designs a new product, yet its limited cash flow or a high fixed cost precludes it from manufacturing the goods. Hence it initiates a crowdfunding campaign, by which it collects money from customers first and then produce the goods. There are several essential features of a reward-based crowdfunding campaign:

- If one decides to back the project, she pays the price immediately.
- The campaign lasts for a time period and the production also takes time. Therefore the value of the reward is discounted and backing a project involves liquidity cost.
- The project may fail to reach its goal. In this case, no reward is given and the money is refunded. Hence the failure of a campaign does not incur any monetary cost for the backers.

A natural question one may ask is how can a producer st the price and the goal to raise as much money as possible. To answer this question, let us first consider a toy model of one-period crowdfunding.

1 Model Setting

There are N consumers and one firm in the market. The firm has constant marginal cost c and fixed cost F if it decides to produce goods. A consumer i's valuation of the good V_i is random, which has probability q to be $\overline{v}$ and $1-q$ to be $\underline{v}$. We assume $\overline{v} > \underline{v} > c > 0$.

The firm first chooses price p and goal G to initiate a crowdfunding campaign. Consumers then choose whether to back. We denote the number of backers by N_B. If a consumer chooses to quit (denoted by Q), she incurs no cost. If she chooses to back (denoted by B), she is bound to incurs a liquidity cost $\underline{L} < \underline{v}$ and gain $V_i - p$ if $N_B p \geq G$ and 0 otherwise.

2 Solving the Model

We first solve the distribution of demand given a price p and a goal G, and then solve the firm's optimization problem.

Notice that $N_B p \geq G$ is equivalent to $N_B \geq ceil(\frac{G}{p})$. Hence for convenience we denote $ceil(\frac{G}{p})$ by $\underline{N}$, the minimum number of backers needed to reach the goal. Obviously, choosing G is equivalent to choosing $\underline{N}$.

The consumer's decisions given a campaign is essentially a Bayesian game $< N, V, (A_i), (T_i), (\tau_i), (P_i), (u_i) >$, where $V = \times_{i \in N}\{\overline{v}, \underline{v}\}$, $A_i = \{B, Q\}$, $T_i = \{\overline{v}, \underline{v}\}$, $\tau_i(V) = V_i$, $P_i = P_j = P$, the product measure over V with $P(V_i = \overline{v}) = q$. And the utility function is:

$$u_i(a, V) = \begin{cases} (V_i - p)\mathbb{1}_{\sum_{j \in N} \mathbb{1}_{a_j = B} \geq \underline{N}} - L & \text{if } a_i = B \\ 0 & \text{if } a_i = Q \end{cases} \tag{1}$$

We first consider the case when $\underline{v} - p > L$, that is, every consumer would back the project if the reward is guaranteed.

Two equilibria are salient: $a_i \equiv B \quad \forall i$ or $a_i \equiv Q \quad \forall i$

These results are not interesting and are unlikely to happen in reality: At least some people back a project and there are consumers complaining that the uncertainty of the campaign prevents them from backing it even if they are fond of the good.

Intuition suggests the following separating equilibrium:

$$a_i = \begin{cases} B & \text{if } V_i = \overline{v} \\ Q & \text{if } V_i = \underline{v} \end{cases} \tag{2}$$

This means that consumers with low (not enough) WTP quit due to the uncertainty and those with high WTP back the project albeit the risk.

The existence of such equilibrium requires:

$$\begin{cases} P(\mathbb{1}_{\sum_{j \in -i} \mathbb{1}_{V_j = \overline{v}} \geq \underline{N} - 1})(\overline{v} - p) - L \geq 0 \\ P(\mathbb{1}_{\sum_{j \in -i} \mathbb{1}_{V_j = \overline{v}} \geq \underline{N} - 1})(\underline{v} - p) - L < 0 \end{cases} \tag{3}$$

The tricky part is $P(\mathbb{1}_{\sum_{j \in -i} \mathbb{1}_{V_j = \overline{v}} \geq \underline{N} - 1})$, the probability of a random variable following binomial distribution greater than or equal to some number. Of course we can expand it as a discrete sum. Yet it has a neater representation: This probability is equal to the probability of the $(\underline{N} - 1)$th variable in ascending order of $N - 1$ independent random variables $X_1, X_2, \ldots X_{N-1}$following uniform distribution on $[0, 1]$. This probability involving ordered statistics can be expressed by beta distribution:

$$\begin{aligned} P(\mathbb{1}_{\sum_{j \in -i} \mathbb{1}_{V_j = \overline{v}} \geq \underline{N} - 1}) &= P(X_{[\underline{N} - 1]} \leq q) \\ &= I_q(\underline{N} - 1, N - \underline{N} + 1) \\ &= \frac{B(q; \underline{N} - 1, N - \underline{N} + 1)}{B(\underline{N} - 1, N - \underline{N} + 1)} \\ &= \frac{\Gamma(N)}{\Gamma(\underline{N} - 1)\Gamma(N - \underline{N} + 1)} \int_0^q t^{\underline{N} - 2}(1 - t)^{N - \underline{N}} dt \end{aligned}$$

The function $I_q(\underline{N}-1, N-\underline{N}+1)$ is called the regularized incomplete beta function. It is the CDF of beta distribution.

By solving the above inequalities, we get:

$$\frac{L}{\overline{v}-p} \leq I_q(\underline{N}-1, N-\underline{N}+1) < \frac{L}{\underline{v}-p} \tag{4}$$

Now back to the firm's optimization problem assuming this condition holds. If the firm is maximizing its expected profit, it solves:

$$\begin{aligned} \max_{p,\underline{N}} \quad & E[\pi] = E[N_B(p-c)\mathbb{1}_{N_b \geq \underline{N}} - F\mathbb{1}_{N_b \geq \underline{N}}] \\ & \qquad = (p-c)E[N_B\mathbb{1}_{N_B \geq \underline{N}}] - FE[\mathbb{1}_{N_B \geq \underline{N}}] \end{aligned} \tag{5}$$

$$s.t. \quad \frac{L}{\overline{v}-p} \leq I_q(\underline{N}-1, N-\underline{N}+1) < \frac{L}{\underline{v}-p} \tag{6}$$

$$where \quad N_B \sim Bin(N, q) \tag{7}$$

By previous result,

$$\begin{aligned} E[\mathbb{1}_{N_B \geq \underline{N}}] &= P(N_B \geq \underline{N}) \\ &= I_q(\underline{N}, N-\underline{N}+1) \end{aligned}$$

And

$$\begin{aligned} & E[N_B\mathbb{1}_{N_B \geq \underline{N}}] \\ & = \sum_{n_b=\overline{N}}^{N} n_B \binom{N}{n_B} q^{n_B}(1-q)^{N-n_B} \\ & = Nq \sum_{i=\overline{N}-1}^{N-1} \binom{N-1}{i} q^i(1-q)^{N-1-i} \\ & = NqI_q(\underline{N}-1, N-\underline{N}+1) \end{aligned}$$

So

$$E\pi(p, \underline{N}) = Nq(p-c)I_q(\underline{N}-1, N-\underline{N}+1) - FI_q(\underline{N}, N-\underline{N}+1) \tag{8}$$

Since $\underline{N}$ can only take finite values, we can compute the optimal $E\pi$ given each $\underline{N}$ and then compare them.

For a fixed $\underline{N}$, $E\pi$ is maximized when p takes the greatest possible value, which is $\overline{v} - \frac{L}{I_q(\underline{N}-1,N-\underline{N}+1)}$. This yields

$$\begin{aligned} E\pi^*(\underline{N}) = & \ Nq(\overline{v} - \frac{L}{I_q(\underline{N}-1, N-\underline{N}+1)} - c)I_q(\underline{N}-1, N-\underline{N}+1) \\ & - FI_q(\underline{N}, N-\underline{N}+1) \\ & = Nq(\overline{v}-c)I_q(\underline{N}-1, N-\underline{N}+1) - LNq - FI_q(\underline{N}, N-\underline{N}+1) \end{aligned} \tag{9}$$

Take difference with respect to $\underline{N}$ yields:

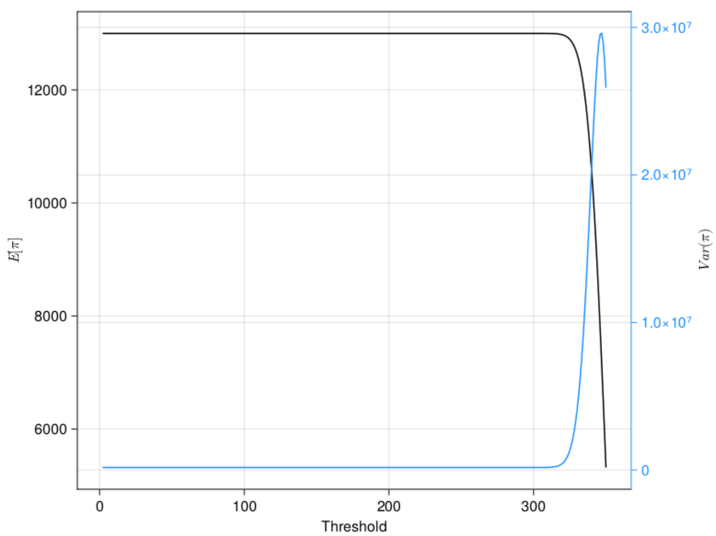
$$\begin{aligned}
&E\pi^*(\underline{N}+1) - E\pi^*(\underline{N}) \\
&= Nq(\overline{v}-c)[I_q(\underline{N}, N-\underline{N}) - I_q(\underline{N}-1, N-\underline{N}+1)] \\
&- F[I_q(\underline{N}+1, N-\underline{N}) - I_q(\underline{N}, N-\underline{N}+1)]
\end{aligned} \tag{10}$$

, for $2 \leq \underline{N} \leq N-1 \quad \underline{N} \in \mathbb{N}$ If this difference is a monotonic function of $\underline{N}$, then the optimal level of threshold is the largest $\overline{N}$ that makes this difference positive.

3 Initial Result

However, neither hand computing nor symbolic computing software managed to give a closed form of this problem, therefore I resorted to numeric simulation (`simu_version_0_1_0.qmd`). With a reasonable set of parameters ($N = 500$, $q = 0.7$, $\overline{v} = 100$, $c = 50$, $F = 1000$, $L = 10$), I got a result that had never been expected: I had conjectured that $N * q$ might be the optimal level, since this is the expected number of high WTP consumers. However, as Fig 3 indicates, if $\underline{N}$ is below a certain level, then both the expectation and the variance of the firm's profit almost remains unchanged! According to my naive theory, the project initiator should not set a goal at all! This result makes sense, though: For any realization of WTP distribution, a project with a lower goal must attract at least as many consumers as a project with a higher goal.

Figure 1: Expectation and Variance of Profit as $\underline{N}$ increases



Then why does this not happen in reality? The most possible reason is that the firm not only cares about the expected profit, but the minium profit as well, since it needs backers to cover its fixed cost. Plus, firms do not know in practice the total number of consumers and their preference distribution, thus may adopt a ‘maximin’-like strategy to offset this uncertainty.

It seems that I still have a long way to go...