



Generalized linear models, binary data

Regression models

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Key ideas

- Frequently we care about outcomes that have two values
 - Alive/dead
 - Win/loss
 - Success/Failure
 - etc
- Called binary, Bernoulli or 0/1 outcomes
- Collection of exchangeable binary outcomes for the same covariate data are called binomial outcomes.

Example Baltimore Ravens win/loss

Ravens Data

```
download.file("https://dl.dropboxusercontent.com/u/7710864/data/ravensData.rda"  
             , destfile="./data/ravensData.rda",method="curl")  
load("./data/ravensData.rda")  
head(ravensData)
```

	ravenWinNum	ravenWin	ravenScore	opponentScore
1	1	W	24	9
2	1	W	38	35
3	1	W	28	13
4	1	W	34	31
5	1	W	44	13
6	0	L	23	24

Linear regression

$$RW_i = b_0 + b_1 RS_i + e_i$$

RW_i - 1 if a Ravens win, 0 if not

RS_i - Number of points Ravens scored

b_0 - probability of a Ravens win if they score 0 points

b_1 - increase in probability of a Ravens win for each additional point

e_i - residual variation due

Linear regression in R

```
lmRavens <- lm(ravensData$ravenWinNum ~ ravensData$ravenScore)
summary(lmRavens)$coef
```

	Estimate	Std. Error	t value	Pr(> t)
(Intercept)	0.2850	0.256643	1.111	0.28135
ravensData\$ravenScore	0.0159	0.009059	1.755	0.09625

Odds

Binary Outcome 0/1

$$RW_i$$

Probability (0,1)

$$\Pr(RW_i | RS_i, b_0, b_1)$$

Odds (0, ∞)

$$\frac{\Pr(RW_i | RS_i, b_0, b_1)}{1 - \Pr(RW_i | RS_i, b_0, b_1)}$$

Log odds ($-\infty, \infty$)

$$\log\left(\frac{\Pr(RW_i | RS_i, b_0, b_1)}{1 - \Pr(RW_i | RS_i, b_0, b_1)}\right)$$

Linear vs. logistic regression

Linear

$$RW_i = b_0 + b_1 RS_i + e_i$$

or

$$E[RW_i | RS_i, b_0, b_1] = b_0 + b_1 RS_i$$

Logistic

$$\Pr(RW_i | RS_i, b_0, b_1) = \frac{\exp(b_0 + b_1 RS_i)}{1 + \exp(b_0 + b_1 RS_i)}$$

or

$$\log\left(\frac{\Pr(RW_i | RS_i, b_0, b_1)}{1 - \Pr(RW_i | RS_i, b_0, b_1)}\right) = b_0 + b_1 RS_i$$

Interpreting Logistic Regression

$$\log\left(\frac{\Pr(RW_i|RS_i, b_0, b_1)}{1 - \Pr(RW_i|RS_i, b_0, b_1)}\right) = b_0 + b_1 RS_i$$

b_0 - Log odds of a Ravens win if they score zero points

b_1 - Log odds ratio of win probability for each point scored (compared to zero points)

$\exp(b_1)$ - Odds ratio of win probability for each point scored (compared to zero points)

Odds

- Imagine that you are playing a game where you flip a coin with success probability p .
- If it comes up heads, you win X . If it comes up tails, you lose Y .
- What should we set X and Y for the game to be fair?

$$E[\text{earnings}] = Xp - Y(1 - p) = 0$$

- Implies

$$\frac{Y}{X} = \frac{p}{1 - p}$$

- The odds can be said as "How much should you be willing to pay for a p probability of winning a dollar?"
 - (If $p > 0.5$ you have to pay more if you lose than you get if you win.)
 - (If $p < 0.5$ you have to pay less if you lose than you get if you win.)

Visualizing fitting logistic regression curves

```
x <- seq(-10, 10, length = 1000)
manipulate(
  plot(x, exp(beta0 + beta1 * x) / (1 + exp(beta0 + beta1 * x)),
       type = "l", lwd = 3, frame = FALSE),
  beta1 = slider(-2, 2, step = .1, initial = 2),
  beta0 = slider(-2, 2, step = .1, initial = 0)
)
```

Ravens logistic regression

```
logRegRavens <- glm(ravensData$ravenWinNum ~ ravensData$ravenScore, family="binomial")
summary(logRegRavens)
```

Call:

```
glm(formula = ravensData$ravenWinNum ~ ravensData$ravenScore,
     family = "binomial")
```

Deviance Residuals:

Min	1Q	Median	3Q	Max
-1.758	-1.100	0.530	0.806	1.495

Coefficients:

	Estimate	Std. Error	z value	Pr(> z)
(Intercept)	-1.6800	1.5541	-1.08	0.28
ravensData\$ravenScore	0.1066	0.0667	1.60	0.11

(Dispersion parameter for binomial family taken to be 1)

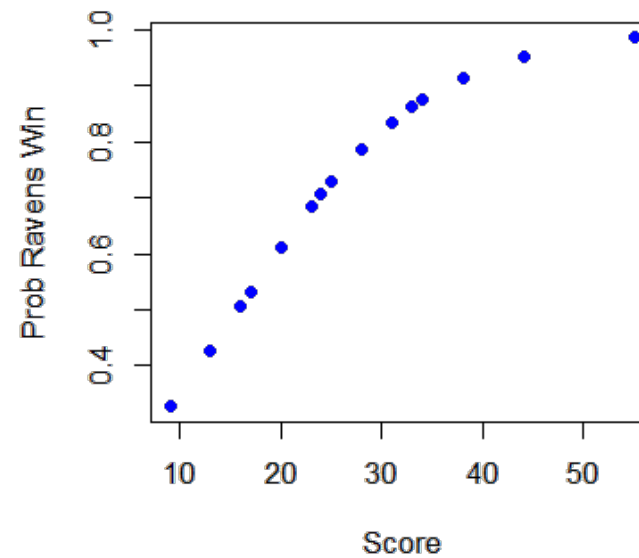
Null deviance: 24.435 on 19 degrees of freedom

Residual deviance: 20.895 on 18 degrees of freedom

AIC: 24.89

Ravens fitted values

```
plot(ravensData$ravenScore, logRegRavens$fitted, pch=19, col="blue", xlab="Score", ylab="Prob Ravens Win")
```



Odds ratios and confidence intervals

```
exp(logRegRavens$coeff)
```

```
(Intercept) ravensData$ravenScore  
0.1864      1.1125
```

```
exp(confint(logRegRavens))
```

```
                2.5 % 97.5 %  
(Intercept)      0.005675  3.106  
ravensData$ravenScore 0.996230 1.303
```

ANOVA for logistic regression

```
anova(logRegRavens, test="Chisq")
```

Analysis of Deviance Table

Model: binomial, link: logit

Response: ravensData\$ravenWinNum

Terms added sequentially (first to last)

	Df	Deviance	Resid.	Df	Resid.	Dev	Pr(>Chi)
NULL			19		24.4		
ravensData\$ravenScore	1	3.54	18		20.9	0.06	.

Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Interpreting Odds Ratios

- Not probabilities
- Odds ratio of 1 = no difference in odds
- Log odds ratio of 0 = no difference in odds
- Odds ratio < 0.5 or > 2 commonly a "moderate effect"
- Relative risk $\frac{\Pr(RW_i|RS_i=10)}{\Pr(RW_i|RS_i=0)}$ often easier to interpret, harder to estimate
- For small probabilities $RR \approx OR$ but **they are not the same!**

[Wikipedia on Odds Ratio](#)

Further resources

- [Wikipedia on Logistic Regression](#)
- [Logistic regression and glms in R](#)
- Brian Caffo's lecture notes on: [Simpson's paradox](#), [Case-control studies](#)
- [Open Intro Chapter on Logistic Regression](#)