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Introduction to cohomology theory of Lie groups and Lie algebras

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Abstract

Technically, this paper is a brief summary of what I have read on Chevalley and Eilenberg's famous thesis [1], as a report of self-learning on the course Lie Groups instructed by Professor WANG Song. The main purpose of Chevalley's thesis is to give a systematic approach by which topological questions concerning compact Lie groups may be reduced to algebraic questions concerning the correspondent Lie algebras.

Contents

1 Manifolds And Cohomology Groups

1.1 Differential Forms on A Manifold

Let M be a smooth manifold of dimension m. Given any vector space V of dimension k over $\mathbb{R}$, we denote by $\Omega^n(p, V)$ the vector space of all n-linear alternating functions defined on T_pM with values in V. By definition $\Omega^0(p, V) = V$.

A V-differential form of order n on M is a function ω which to each $p \in M$ assigns an element $\omega(p) \in \Omega^n(p, V)$. The space of all V-differential form of order n on M is denoted as $\Omega^n(M, V)$. The direct sum $\Omega^*(M,V) := \bigoplus_n \Omega^n(M,V)$ forms a graed ring in an obvioius way. If $V = \mathbb{R}$, it coincides with our classical terminology as differential forms.

We select a basis $v_1, \cdots, v_k$ for V. The V-form ω can then be written as $\omega = \omega^i v_i$ (Here and afterwards we adopt the famous Einstein summation convention for convenience), where ω^i are differential forms. Define $\mathrm{d}\omega = \mathrm{d}\omega^i v_i$. Clearly, $\mathrm{d}\omega$ is a V-form of order $n+1$ independent of the choice of the basis.

V-forms ω such that $\mathrm{d}\omega = 0$ are called *closed*. The vector space of all closed V-forms of order n is denoted as $\mathrm{Z}^n(M,V)$. Those of the form $\omega = \mathrm{d}\theta$ where θ is a V-form of one lower order are called *exact*. The vector space of all exact V-forms of order n is denoted as $\mathrm{B}^n(M,V)$. Since $\mathrm{dd}\omega = 0$, every exact V-form is closed. The quotient space $\mathrm{H}^n(M,V) := \mathrm{Z}^n(M,V)/\mathrm{B}^n(M,V)$ will be called the n-dimensional *cohomology group* of M obtained using V-fomrs.

Let $T : V \to V$ be a linear transformation. For every $f \in \Omega^n(p,V)$ we then have the composite function $T \circ f \in \Omega^n(p,V)$ and the correspondence $f \mapsto T \circ f$ is a linear transformation $T : \Omega^n(p,V) \to \Omega^n(p,V)$. Hence for each V-form ω on M we may define a V-form $T\omega$ by setting $(T\omega)(p) = T \circ (\omega(p))$. It is easy to see that $T\omega$ is smooth if ω is smooth and that 1) $\mathrm{d}(T\omega) = T\mathrm{d}\omega$; 2) $T_1(T_2\omega) = (T_1T_2)\omega,\ \forall T_1, T_2 \in \mathrm{End}(V)$; 3) $T(r_1\omega_1 + r_2\omega_2) = r_1 T\omega_1 + r_2 T\omega_2,\ \forall r_1, r_2 \in \mathbb{R}$.

Consider two manifolds M_1, M_2, a smooth map $F : M_1 \to M_2$ and a V-form ω on M_2. If $p \in M_1$ then F defines a linear mapping of the tangent vector spaces $F_{p*} : T_pM_1 \to T_{F(p)}M_2$. If $f \in \Omega^n(p,V)$ then $F_p^* f$ defined by

$$F_p^* f(v_1, \cdots, v_k) = f\big(F_{p*}v_1, \cdots, F_{p*}v_k\big), \qquad v_1, \cdots, v_k \in V$$

it further induces a map $F^* : \Omega^*(M_2,V) \to \Omega^*(M_1,V)$ defined by

$$\big(F^*\omega\big)(p) = F_p^*\Big(\omega\big(F(p)\big)\Big)$$

If ω is smooth then so is $F^*\omega$. The following properties of $F^*\omega$ is easily obtained: 1) $\mathrm{d}(F^*\omega) = F^*(\mathrm{d}\omega)$; 2) $F^*(r_1\omega_1 + r_2\omega_2) = r_1F^*\omega_1 + r_2F^*\omega_2,\ \forall r_1, r_2 \in \mathbb{R}$; 3) $TF^*\omega = F^*(T\omega),\ \forall T \in \mathrm{End}(V)$; 4) $(F_1 \circ F_2)^*\omega = F_2^*F_1^*\omega$, where $F_1 : M_1 \to M_2$ and $F_2 : M_2 \to M_3$, ω is a V-form on M_3.

1.2 Equivariant Forms

We shall assume that a topological group G is acting as a group of diffeomorphisms on M. By this we mean that for each $g \in G$ a diffeomorphism $F_g : M \to M$ is given that 1) $F : G \times M \to M$ is continuous; 2) $F_{g_1g_2} = F_{g_1}F_{g_2}$; 3) $F_e = \mathrm{Id}_M$, where e denotes the unit of G.

We shall assume that a representation (ρ, V) of G with a finite-dimensional vector space V over $\mathbb{R}$ as representation space is given. A smooth V-form ω on M will be called *equivariant* provided that

$$\rho(g)\omega = F_g^*\omega$$

for each $g \in G$. It is easily seen that if ω is equivariant then $\mathrm{d}\omega$ is equivariant and if ω_1 and ω_2 are equivariant, so is $r_1\omega_1 + r_2\omega_2$ for any $r_1, r_2 \in \mathbb{R}$.

A V-form which is the differential of an equivariant form will be called equivariantly exact. The quotient space of the closed equivariant V-forms of order n on M by the subspace of equivariantly exact V-forms will be denoted by $\mathrm{E}^n(M,\rho)$. We shall refer to $\mathrm{E}^n(M,\rho)$ as the n-dimensional cohomology group of M obtained using equivariant V-forms. It follows as before that the direct sum of the spaces $\mathrm{E}^n(M,\rho)$ forms a ring $\mathrm{E}^*(M,\rho)$.

If $V = \mathbb{R}$ and ρ is the trivial representation we shall write $\mathrm{E}^n(M)$ instead of $\mathrm{E}^n(M,\rho)$

$\mathrm{E}^n(M,\rho)$ form a linear subspace of $\Omega^n(M,V)$. This leads to a natural homomorphism of the cohomology groups

$$\pi : \mathrm{E}^n(M,\rho) \to \mathrm{H}^n(M,V). \tag{1}$$

If $V = \mathbb{R}$ and ρ is trivial, then π is a ring homomorphism. The following theorems will be proved in an appropriate time.

Theorem 1.1. *If G is compact then π maps $\mathrm{E}^n(M,\rho)$ isomorphically into a subspace of $\mathrm{H}^n(M,V)$.*

A more detailed analysis of (1) follows from the decomposition of ρ into irreducible components. Such a decomposition always exists if G is compact. The proof is a little bit tricky to consider averaging process with Haar measure on compact groups. Let then $V = V_1 \oplus \cdots \oplus V_k$ be a decomposition of V into irreducible invariant subspaces and let ρ_i be the corresponding representations of G in V_i. Every V-form ω then decomposes uniquely as $\omega = \omega_1 \oplus \cdots \oplus \omega_k$ with ω_i being a V_i-form. These result direct sum decompositions $\mathrm{E}^n(M,\rho) = \bigoplus_i \mathrm{E}^n(M,\rho_i)$, $\mathrm{H}^n(M,V) = \bigoplus_i \mathrm{H}^n(M,V_i)$ and an appropriate decomposition of (1). Hence we may concentrate on irreducible representations.

In the following two theorems it is assumed that G is compact and connected.

Theorem 1.2. *If the representation ρ of G is irreducible and nontrivial then $\mathrm{E}^n(M,\rho) = \{0\}$.*

Theorem 1.3. *If $V = \mathbb{R}$ and ρ is trivial, the correspondence (1) is a ring isomorphism onto $\mathrm{E}^*(M) \cong \mathrm{H}^*(M)$.*

1.3 The Averaging Process

We assume that G is compact. This implies the existence of a Haar measure with the measure of G being 1. Given a continuous V-form ω of order n on M consider the family of V-forms

$$\omega^g := F_g^*\big(\rho(g^{-1})\omega\big),$$

for $g \in G$. For each $p \in M$, $\omega^g(p)$ is a continuous function on G with values in the vector space $\Omega^n(M,V)$. Hence the integral

$$(I\omega)(p) := \int_G \omega^g(p)\mathrm{d}g$$

is a well defined element of $\Omega^n(M,V)$. It can be easily checked that the V-form $I\omega$ obtained has the following properties 1) $\mathrm{d}(I\omega) = I\mathrm{d}\omega$; 2) $I\omega$ is equivariant; 3) if ω is equivariant then $I\omega = \omega$.

Remark 1.1. Property 1) involves the commutativity of differential and integral which is correct concerning here G is compact. Property 2) is just a direct calculation:

$$\begin{aligned}
\rho(h)I\omega(p) &= \rho(h)\int_G \omega^g(p)\mathrm{d}g = \int_G \rho(h)F_g^*\big(\rho(g^{-1})\omega\big)(p)\mathrm{d}g = \int_G F_g^*\big(\rho(h)(\rho(g^{-1})\omega)\big)(p)\mathrm{d}g \\
&= \int_G F_h^* F_{gh^{-1}}^*\big(\rho(hg^{-1})\omega\big)(p)\mathrm{d}(gh^{-1}) = F_h^*\int_G F_t^*\big(\rho(t^{-1})\omega\big)(p)\mathrm{d}t = F_h^* I\omega(p)
\end{aligned}$$

for all $h \in G$. And property 3) is due to $\int_G \mathrm{d}g = 1$.

In the following we shall use the lemma:

Lemma 1.1. If ω is a closed form such that $\displaystyle\int_z \omega = 0$ for every homology class, then ω is exact.

There is no explicit proof of the above lemma in the literature. De Rham's original proof is valid for closed manifolds M carrying a simplicial decomposition of a rather special kind.

Proof of Theorem 1.1. We only have to prove that if an equivariant closed V-form ω is exact, then it is equivariant exact. First, $\omega = \mathrm{d}\theta$ for some V-form θ. Therefore we have $\mathrm{d}(I\theta) = I(\mathrm{d}\theta) = I\omega = \omega$. □

Before we proceed with the proofs of Theorem 1.2 and Theorem 1.3, we prove the following lemma:

Lemma 1.2. If G is connected then for every $g \in G$ and every homology class z in M, $F_g z = z$

Proof. Assume M can be simplicially decomposited and let K be a subcomplex of M containing a cycle of the homology class z. Let K_1 be a complex containing K in its interior. We may then find a neighborhood U of the identity e in G such that $F_g(K) \subseteq K_1, \forall g \in U$. Consider the family of mappings $F_g : K \to K_1$, $g \in U$. It follows that there is a neighborhood $U_1 \subseteq U$ of the identity such that $F_g : K \to K_1$, with $g \in U_1$ is homotopic with the identity map $F_e : K \to K_1$. Hence $F_g z = z$ for $g \in U_1$. Since G is connected, this holds for any $g \in G$. □

Proof of Theorem 1.2. Let ω be a closed equivariant V-form on M. Since $\rho(g)\omega = F_g^*\omega$, $\forall g \in G$. For each homology class z,

$$\rho(g)\int_z \omega = \int_z \rho(g)\omega = \int_z F_g^*\omega = \int_{F_g z} \omega = \int_z \omega, \quad \forall g \in G.$$

Since ρ is irreducible and nontrivial, $\int_z \omega = 0$. Since this holds for every homology class z, by Lemma 1.1, ω is exact and Theorem 1.1 implies ω is equivariantly exact. □

Proof of Theorem 1.3. We have already shown that $\pi : \mathrm{E}^k(M) \to \mathrm{H}^k(M)$ is an isomorphism into. It is therefore sufficient to prove that $\mathrm{E}^k(M)$ is mapped onto $\mathrm{H}^k(M)$. Let ω be a closed form of order k on M. Consider the integral $\int_z I\omega$ over a k-dimensional homology class z. We have

$$\int_z I\omega = \int_z \int_G \omega^g \mathrm{d}g = \int_z \int_G F_g^*\omega \overset{(*)}{=} \int_G \int_z F_g^*\omega \mathrm{d}g = \int_G \int_{F_g z} \omega \mathrm{d}g = \int_G \int_z \omega \mathrm{d}g = \int_z \omega.$$

Hence $\int_z (\omega - I\omega) = 0$ and it follows from Lemma 1.1 that $\omega - I\omega$ is exact. This completes the proof. □

1.4 Double Equivariance

Let G and H be two groups having actions on M. Here we assume $V = \mathbb{R}$. A differential form ω on M will be called doubly equivariant provided $F_g^*\omega = \omega = F_h^*\omega$ for all $g \in G$ and $h \in H$. As before we may define cohomology groups $\tilde{\mathrm{E}}(M)$ using doubly equivariant forms only. As before we have a natural ring homomorphism $\pi : \tilde{\mathrm{E}}^*(M) \to \mathrm{H}^*(M)$.

Theorem 1.4. *If G and H are compact and connected and if the transformations F_g and F_h commute for all $g \in G$ and $h \in G$, then $\pi : \tilde{\mathrm{E}}^*(M) \to \mathrm{H}^*(M)$ is a ring isomorphism.*

Proof. Consider the direct product $G \times H$. For $(g, h) \in G \times H$ define $F_{(g,h)}^* = F_g^* F_h^*$. It follows from our assumptions that $G \times H$ is a compact and connected group operating on M. Let ω be a form equivariant relative to $G \times H$, then $F_g^* F_h^* \omega = \omega$. Taking $g = e_g$ we find $F_h^*\omega = \omega$ and similarly $F_g^*\omega = \omega$ so that ω is doubly equivariant. Conversely every doubly equivariant form is equivariant relative to $G \times H$. Thus this theorem is a consequence of Theorem 1.3. □

2 Lie Groups

2.1 The Transitive Case

We shall assume now that a compact group G operates on the connected manifold M transitively. Further we assume $\bigcap \mathrm{Stab}_G(p) = \{e\}$. Let $p \in G$, and $H = \mathrm{Stab}_G(p)$. Then the mapping $gH \to F_g(p)$ is then a $1:1$ continuous mapping of G/H onto M. And since G/H is compact, this is a homeomorphism. On the other hand, it follows from a theorem of Montgomery that G is in this case a Lie group. Bochner and Montgomery have alos proved that the mapping $gH \mapsto F_g(p)$ and its converse are both of class C^2. Therefore we may assume without less of generality that M is identical with G/H. and that $F_{g_1}(g_2 H) = (g_1 g_2)H$.

In the following, G will be an arbitrary Lie group and H a closed subgroup of G. For the moment we shall study the simple case when H is the trivial subgroup and $M = G$.

Let $\mathfrak{g}$ be the Lie algebra of the group G and $\mathfrak{g}$ identically isomorphic to the space of left invariant vector fields, denoted by $\widetilde{\mathfrak{X}}(G)$. Therefore $\Lambda^n \mathfrak{g} \cong \Lambda^n \widetilde{\mathfrak{X}}(G)$. Given any V-form ω of order n on G. We define $\widetilde{\omega} \in \widetilde{\Omega}^n(G)$ as the image of ω_e under the map $\Lambda^n \mathfrak{g}^* \xrightarrow{\simeq} \widetilde{\Omega}^n(G)$. The correspondence $\omega \mapsto \widetilde{\omega}$ is obviously linear. Moreover if ω is equivariant then $\widetilde{\omega} = \omega$. The passage from ω to $\widetilde{\omega}$ will be referred to as *localization*.

2.2 Left Invariant Forms And Invariant Forms

Here we assume $V = \mathbb{R}$ and the representation is trivial. The equivariant condition becomes a condition of *left invariant*: $L_g^*\omega = \omega$ where $L_g : h \mapsto gh$. It can be easily seen that $\widetilde{\omega_1 \wedge \omega_2} = \widetilde{\omega}_1 \wedge \widetilde{\omega}_2$.

Theorem 2.1. *Let ω be a left invariant differential n-form. Then we have*

$$\widetilde{\mathrm{d}\omega}(v_1, \cdots, v_{n+1}) = \sum_{k<l} (-1)^{k+l} \widetilde{\omega}\big([v_k, v_l], v_1, \cdots, \widehat{v}_k, \cdots, \widehat{v}_l, \cdots, x_{n+1}\big), \qquad \forall\, v_i \in \widetilde{\mathfrak{X}}(G). \tag{2}$$

Proof. Let ω be a differential n-form and we have the following formula by [3, III.3. Remark 7.]

$$\begin{aligned} \mathrm{d}\omega\,(v_1, \cdots, v_{n+1}) &= \sum_{i=1}^{n+1} (-1)^{i+1} v_i\, \omega\big(v_1, \cdots, \widehat{v}_i, \cdots, v_{n+1}\big) \\ &\quad + \sum_{i<j} (-1)^{i+j} \omega\big([v_i, v_j], v_1, \cdots, \widehat{v}_i, \cdots, \widehat{v}_j, \cdots, v_{n+1}\big), \qquad \forall\, v_i \in \mathfrak{X}(G). \end{aligned} \tag{3}$$

Now assume ω is left invariant and we claim that $X\omega(v_1, \cdots, v_n) = 0$, $\forall\, X, v_i \in \tilde{\mathfrak{X}}(G)$. Indeed,

$$\begin{aligned} 0 = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \omega(v_1, \cdots, v_n) &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \big(L^*_{\exp(tX)}\omega\big)(v_1, \cdots, v_n) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} L^*_{\exp(tX)} \Big(\omega(L_{\exp(tX)*}v_1, \cdots, L_{\exp(tX)*}v_n)\Big) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} L^*_{\exp(tX)} \big(\omega(v_1, \cdots, v_n)\big) = X\omega(v_1, \cdots, v_n) \end{aligned}$$

Left invariant differential forms ω implies $\mathrm{d}\omega$ is also left-invariant. Therefore $\omega = \widetilde{\omega}$ and □

Similarly, a differential form ω is called right invariant if $R_g^*\omega = \omega$, where $R_g : h \mapsto hg$. A differential form is called *invariant* if it is both left and right invariant. It is clear that if ω is invariant then $\mathrm{d}\omega$ is invariant. In the following context, we assume Lie group G is compact and connected.

Theorem 2.2. *Every invariant form is closed.*

Proof. Theorem 1.3 implies that an invariant form ω is closed if $\mathrm{d}\omega(v_1, \cdots, v_{n+1}) = 0$ for all $v_i \in \widetilde{\mathfrak{X}}(G)$. $L_g^*\omega = R_g^*\omega$ implies $\mathrm{Ad}_g^*\omega = \omega$. For arbitrary left invariant vector field X, $\mathrm{Ad}^*_{\exp(tX)}\omega = \omega$. Notice that $[v, w] = ad_v W = \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0} \mathrm{Ad}_{\exp(tv)} w$, we have the following equations:

$$\begin{aligned} 0 = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \omega(V_1, \cdots, v_n) &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \big(\mathrm{Ad}^*_{\exp(tX)}\omega\big)(v_1, \cdots, v_n) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \mathrm{Ad}^*_{\exp(tX)} \big(\omega(\mathrm{Ad}_{\exp(tX)_*} v_1, \cdots, \mathrm{Ad}_{\exp(tX)_*} v_n)\big) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \mathrm{Ad}^*_{\exp(tX)} \big(\omega(v_1, \cdots, v_n)\big) \\ &\quad + \sum_{i=1}^{n} \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \omega\big(v_1, \cdots, \mathrm{Ad}_{\exp(tX)_*} v_i, \cdots, v_n\big) \\ &= X\omega(v_1, \cdots, v_n) + \sum_{i=1}^{n} \omega(v_1, \cdots, [X, v_i], \cdots, v_n) \\ &= \sum_{i=1}^{n} (-1)^i \omega([X, v_i], v_1, \cdots, \widehat{v}_i, \cdots, v_n) \qquad \forall\, v_i \in \widetilde{\mathfrak{X}}(G). \end{aligned} \tag{4}$$

It then follows from Theorem 2.1 that $\mathrm{d}\omega(v_1, \cdots, v_{n+1}) = 0$ for arbitrary $v_i \in \widetilde{\mathfrak{X}}(G)$, which implies $\mathrm{d}\omega = 0$. □

Combining this with Theorem 1.4 we have the following

Theorem 2.3. *Let G be a compact and connected Lie group. Every cohomology class of differential forms on G contains precisely one invariant form. The invariant forms span a ring isomorphic with the cohomology ring of the manifold G.*

Proof. Assume $[\omega_1] = [\omega_2]$, both ω_1 and ω_2 are invariant forms. It shows that $\omega_1 - \omega_2 = \mathrm{d}\theta$ for some equivariant (in this case, invariant) differential form θ. Therefore $\omega_1 - \omega_2 = 0$ by Theorem 2.2. □

Remark 2.1. If in addition to being compact and connected G is also semi-simple then G possesses a Riemannian metric invariant with respect to both left and right translations. Moreover Hodge([4]) has shown that the harmonic forms with respect to this Riemannian metric coincide with the forms that are invariant. Thus Theorem 2.3 gives a relatively simple proof of Hodge's theoremson harmonic forms on a Riemannian manifold in the case where this manifold is the group manifold of a semi-simple compact group.

3 Lie Algebras

3.1 The Cohomology Ring of A Lie Algebra

Definition 3.1. The differential complex of a Lie algebra $\mathfrak{g}$ is defined as follows:

$$0 \to \mathbb{R} =: \mathfrak{g}_0^* \xrightarrow{\delta_0} \mathfrak{g}^* \xrightarrow{\delta} \Lambda^2\mathfrak{g}^* \to \cdots \to \Lambda^{n-1}\mathfrak{g}^* \xrightarrow{\delta} \Lambda^n\mathfrak{g}^* \to \cdots \tag{5}$$

$[\cdot,\cdot] : \Lambda^2\mathfrak{g} \to \mathfrak{g}$ as $a \wedge b \mapsto [a,b]$ induces a linear map $\delta : \mathfrak{g}^* \to \Lambda^2\mathfrak{g}^*$. Higher degree differentials are defined by induction that satisfies the relation: $\delta(a \wedge b) = \delta a \wedge b + (-1)^{|a|} a \wedge \delta b$. The element of $\Lambda^*\mathfrak{g}$ is called *cochain*. An element $\omega \in \Lambda^*\mathfrak{g}$ is called *cocycle*, if $\delta\omega = 0$. ω is called *coboundary* if $\omega = \delta\theta$ for some cochain θ.

Remark 3.1. The map δ is indeed a differential (i.e., $\delta^2 = 0$). We check it by induction on the order. First, we can formulate the explicit expression in lower orders. Assume $\alpha \in \mathfrak{g}^*$, then $\delta\alpha(a,b) = \alpha([a,b])$, $\forall a,b \in \mathfrak{g}$ by definition. If $\omega = \alpha \wedge \beta \in \Lambda^2\mathfrak{g}$, then $\delta\omega = \delta\alpha \wedge \beta - \alpha \wedge \delta\beta$. For any $a,b,c \in \mathfrak{g}$

$$\begin{aligned}
\delta\alpha \wedge \beta(a,b,c) &= \frac{1}{2}\Big(\delta\alpha(a,b)\beta(c) - \delta\alpha(b,a)\beta(c) + \delta\alpha(b,c)\beta(a) \\
&\quad -\delta\alpha(c,b)\beta(a) + \delta\alpha(c,a)\beta(b) - \delta\alpha(a,c)\beta(b)\Big) \\
&= \delta\alpha(a,b)\beta(c) + \delta\alpha(c,a)\beta(b) + \delta\alpha(b,c)\beta(a) \\
&= \alpha\big([a,b]\big)\beta(c) + \alpha\big([c,a]\big)\beta(b) + \alpha\big([b,c]\big)\beta(a)
\end{aligned}$$

For the same reason, $\alpha \wedge \delta\beta(a,b,c) = \beta\big([a,b]\big)\alpha(c) + \beta\big([c,a]\big)\alpha(b) + \beta\big([b,c]\big)\alpha(a)$. Therefore

$$\delta\omega(a,b,c) = \omega\big([a,b],c\big) + \omega\big([c,a],b\big) + \omega\big([b,c],a\big) \tag{6}$$

in the case $\omega = \alpha \wedge \beta$. We know that every element of $\Lambda^2\mathfrak{g}^*$ is the finite sum of such form, which implies this formula (6) is corret for any $\omega \in \Lambda^2\mathfrak{g}^*$. Then we shall verify that $\delta^2 = 0$. If n=1, then every $f \in \mathfrak{g}^*$, we have

$$\begin{aligned}
(\delta^2 f)(a,b,c) &= \delta(\delta f)(a,b,c) = \delta f([a,b],c) + \delta f\big([c,a],b\big) + \delta f\big([b,c],a\big) \\
&= f\Big(\big[[a,b],b\big] + \big[[c,a],b\big] + \big[[b,c],a\big]\Big) = 0 \qquad \text{by Jacobi's identity.}
\end{aligned}$$

For $n > 1$, we proceed by induction. Assume $\omega = \omega_1 \wedge \omega_2$, where $\omega_1 \in \mathfrak{g}^*$ and $\omega_2 \in \Lambda^{n-1}\mathfrak{g}^*$. Then we have $\delta\omega = \delta\omega_1 \wedge \omega_2 - \omega_1 \wedge \delta\omega_2$ and $\delta^2\omega = \delta^2\omega_1 \wedge \omega_2 + \delta\omega_1 \wedge \delta\omega_2 - \delta\omega_1 \wedge \delta\omega_2 + \omega_1 \wedge \delta^2\omega_2 = 0$ by induction.

In this sense, we define the cohomology groups and their graded ring in the usual way, denoted by $\mathrm{H}^*(\Lambda^*\mathfrak{g}^*)$. Notice that we always have $\mathrm{H}^0(\Lambda^*\mathfrak{g}^*) = \mathbb{R}$.

Proposition 3.1. *Assume $\mathfrak{g}$ is a Lie algebra, then $\mathrm{H}^1(\Lambda^*\mathfrak{g}^*)$ is the dual space of $\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$.*

Proof. Notice that $\delta_0 = 0$, then $\mathrm{H}^1(\Lambda^*\mathfrak{g}^*) = \mathrm{Z}^1(\Lambda^*\mathfrak{g}^*)$. For any $f \in \mathrm{Z}^1(\Lambda^*\mathfrak{g}^*)$, $\delta f(a,b) = f([a,b]) = 0$ for all $a,b \in \mathfrak{g}$. Therefore f is an element of $\big(\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]\big)^\vee$. The converse is also clear. This completes the proof. □

Having the above definition in mind, combining Theorem 1.2 and Theorem 1.3 we have

Theorem 3.1. *Assume G is a compact and connected Lie group, let $\mathfrak{g}$ be the Lie algebra G, then* $\mathrm{H}^*\Omega^*(G) \cong \mathrm{H}^*(\Lambda^*\mathfrak{g}^*)$.

Corollary 3.1. *Two locally isomorphic compact connected Lie groups have isomorphic cohomology rings.*

3.2 Semi-simple Lie Algebras

Definition 3.2. The derived series of a Lie algebra $\mathfrak{L}$ is a sequence of ideals of $\mathfrak{L}$ by $\mathfrak{L}^{(0)} = \mathfrak{L}, \mathfrak{L}^{(1)} = [\mathfrak{L}, \mathfrak{L}], \mathfrak{L}^{(2)} = [\mathfrak{L}^{(1)}, \mathfrak{L}^{(1)}], \cdots, \mathfrak{L}^{(i)} = [\mathfrak{L}^{(i)}, \mathfrak{L}^{(i)}]$. Call $\mathfrak{L}$ *solvable* if $\mathfrak{L}^{(n)} = 0$ for some n. It is clear that if $\mathfrak{I}, \mathfrak{J}$ are solvable ideals of $\mathfrak{L}$, then so is $\mathfrak{I} + \mathfrak{J}$. This proves the existence of a unique maximal solvable ideal, called the *radical* of $\mathfrak{L}$ and denoted $\mathrm{Rad}\mathfrak{L}$. In case $\mathrm{Rad}\mathfrak{L} = 0$, L is called *semi-simple*.

A Lie group G is called semi-simple, if its Lie algebra $\mathfrak{g}$ is semi-simple.

Definition 3.3. A *representation* of a Lie algebra $\mathfrak{L}$ is a homomorphism $\phi : \mathfrak{L} \to \mathfrak{gl}(V)$. The adjoint representation is defined as $\mathrm{ad} : \mathfrak{L} \to \mathfrak{gl}(\mathfrak{L})$ and $\mathrm{ad} : x \mapsto \mathrm{ad}_x$, where $\mathrm{ad}_x(y) = [x, y]$.

Proposition 3.2. *If Lie algebra $\mathfrak{L}$ is semi-simple, then*

(a) $\mathfrak{L} = [\mathfrak{L}, \mathfrak{L}]$, *and all ideals and homomorphic images of $\mathfrak{L}$ are semi-simple. Moreover, each ideal of $\mathfrak{L}$ is a sum of certain simple ideals of $\mathfrak{L}$.*

(b) *The adjoint representation of $\mathfrak{L}$ is faithful.*

(c) *Every finite dimentional representation of $\mathfrak{L}$ is completely reducible.*

Proof. See Humphreys [5, ChapterII]. □

Remark 3.2. In fact, property (c) is equivalent to the definition of semi-simple Lie algebra $\mathfrak{L}$.

Corollary 3.2. *If $\mathfrak{L}$ is semi-simple then* $\mathrm{H}^1(\Lambda^*\mathfrak{L}^*) = \{0\}$.

Theorem 3.2. *A compact connected Lie group G is semi-simple if and only if its fundamental group is finite.*

Proof. Assume G is semi-simple, thus its Lie algebra $\mathfrak{g}$ is semi-simple, then $\mathrm{H}^1(\Lambda^*\mathfrak{g}^*) = \{0\}$, it follows from Theorem 3.1 that $\mathrm{H}^1(\Omega^*(M)) = \{0\}$. It follows from universal coefficient theorems that the 1-dim homology group of G is finite, which is the abelianization of fundamental group. Another fact is that fundamental group of topological group is always abelian thus they coincides with each other, and thus finite.

Conversely, if the fundamental group is finite. Then the universal covering group $\widetilde{G}$ of G is compact. Since $\widetilde{G}$ is simply connected, every representation of the Lie algebra $\mathfrak{g}$ is induced by some representation of $\widetilde{G}$ [2, p.113]. $\widetilde{G}$ being compact, every representation of $\widetilde{G}$ is completely reducible. Hence the same follows for every representation of $\mathfrak{g}$. Hence $\mathfrak{g}$ is semi-simple and so is G. □

Definition 3.4. A semi-simple Lie algebra (over $\mathbb{R}$) will be called compact if it is the Lie algebra of some compact connected Lie group.

Guided by the results of equation (4) we define a cochain $\omega \in \Lambda^n\mathfrak{L}^*$ to be invariant if

$$\sum_{i=1}^{n}(-1)^i\omega\big([x, x_i], x_1, \cdots, \widehat{x}_i, \cdots, x_n\big) \qquad \forall\, x, x_i \in \mathfrak{L}. \tag{7}$$

The same computations as in Theorem 2.2 shows that

Proposition 3.3. *Every invariant cochain is cocycle.*

Here comes an analogue of Theorem 2.3. The proof is purely algebraic, which will not be shown here.

Theorem 3.3. *Let $\mathfrak{L}$ be a semi-simple Lie algebra over a field of characteristic 0. Every cohomology class of $\mathrm{H}^n(\Lambda^*\mathfrak{L})$ contains exactly one invariant cocycle. The invariant cocycles constitute a ring isomorphic with the cohomology ring $\mathrm{H}^*(\Lambda^*\mathfrak{L})$.*

3.3 The Bilinear Form of A Representation

Definition 3.5. Let (ρ, V) be a representation of a Lie algebra $\mathfrak{L}$. For $x, y \in \mathfrak{L}$, define

$$B(x, y) = \mathrm{Trace}\,(\rho_x, \rho_y). \tag{8}$$

Then B is a bilinear symmetric form on $\mathfrak{L}$ and $B([x, z], y) = B(x, [z, y])$, since

$$\begin{aligned} B([x, z], y) &= \mathrm{Trace}\,(\rho_{[x,z]}, \rho_y) = \mathrm{Trace}\,(\rho_z\rho_x\rho_y - \rho_x\rho_z\rho_y) \\ &= \mathrm{Trace}\,\big(\rho_x(\rho_z\rho_y - \rho_y\rho_z)\big) = B\big(x, [z, y]\big). \end{aligned}$$

Lemma 3.1. If $\mathfrak{L}$ is semi-simple and the representation ρ is faithful, the bilinear form associated with ρ is nondegenerate.

Notice that the above lemma is a generation of the following theorem

Theorem 3.4. *A Lie algebra $\mathfrak{L}$ is semi-simple if and only if it Killing form is nondegenerate.*

Proof. See Humphreys [5, p.22]. □

Remark 3.3. The lemma 3.1 is due to a well known theorem of Cartan. The matrix which represents the bilinear form of a representation has been considered by Casimir and is sometimes called Casimir's matrix. The fact that it is regular was essential in the algebraic proof given by van der Waerden of the full reducility of representations of semi-simple Lie algebras.

Using Theorem 3.3 we shall now prove the following theorem.

Theorem 3.5. *If $\mathfrak{L}$ is a semi-simple Lie algebra over a field of characteristic 0, then* $\mathrm{H}^1(\Lambda^*\mathfrak{L}^*) = \{0\}, \mathrm{H}^2(\Lambda^*\mathfrak{L}^*) = \{0\}$ *and* $\mathrm{H}^3(\Lambda^*\mathfrak{L}^*) \neq \{0\}$.

Proof. The first part of the theorem was proved earlier. In order to prove $\mathrm{H}^2(\Lambda^*\mathfrak{L}^*) = \{0\}$, it suffices to show that every invariant 2-cocycle $\omega \in \mathfrak{L}$ is zero. Since ω is cocycle we have

$$\delta\omega(a, b, c) = \omega\big([a, b], c\big) - \omega\big([a, c], b\big) + \omega\big([b, c], a\big).$$

The first two terms cancel out since ω is invariant. Hence $\omega([b, c], a) = 0$. But $[\mathfrak{L}, \mathfrak{L}] = \mathfrak{L}$, whence $\omega = 0$.

In ordder to prove $\mathrm{H}^3(\Lambda^*\mathfrak{L}^*) \neq \{0\}$ it suffices to exhibit a nonvanishing invariant 3-cochain in L. Consider the Killing form κ: $\kappa(x, y) = \mathrm{Trace}\,(\mathrm{ad}_x\mathrm{ad}_y)$. Then κ is a bilinear symmetric form on $\mathfrak{L}$. Furthermore we have $\kappa\big([x, z], y\big) = \kappa\big(x, [y, z]\big)$. Now define ω to be

$$\omega(a, b, c) = \kappa\big([a, b], c\big), \qquad \text{for all } a, b, c \in \mathfrak{L}.$$

It is clear that ω is alternating thus an element of $\Lambda^3\mathfrak{L}^*$. Further

$$\begin{aligned} \omega\big([a, x], b, c\big) + \omega\big(a, [b, x], c\big) + \omega\big(a, b, [c, x]\big) &= \kappa\big(\big[[a, x], b\big], c\big) + \kappa\big(\big[a, [b, x]\big], c\big) + \kappa\big([a, b], [c, x]\big) \\ &= \kappa\big(\big[[a, b], x\big], c\big) - \kappa\big([a, b], [x, c]\big) = 0. \end{aligned}$$

Thus ω is invariant. If ω is identically zero then $\kappa\big([x, y], z\big)$ is identicallly zero. Since $\mathfrak{L} = [\mathfrak{L}, \mathfrak{L}]$, this implies that $\kappa(x, y) = 0$ for all $x, y \in \mathfrak{L}$, which is rediculous. □

Corollary 3.3. *The n-dimensional sphere S^n is a group manifold for the values $n = 1, 3$ only.*

Proof. Suppose that S^n is a group manifold and $n > 1$. Since S^n is compact it can be represented as a Lie group G. Since $n > 1$, S^n is simply-connected and therefore, by Theorem 3.2, G is semi-simple. The preceding theorem then implies that $n = 3$. □

4 Cohomology Groups Associated with A Representation

Definition 4.1. Let $\mathfrak{L}$ be a Lie algebra over a field K of characteristic 0, and let (ρ, V) be a finite dimensional representation of $\mathfrak{L}$ over K. A k-linear alternating mapping of $\mathfrak{L}$ into V will be called a k-dim V-cochain or shorter k-V-cochain. The k-V-cochains form a space $\Lambda^k(\mathfrak{L}, V)$. By definition $\Lambda^0(\mathfrak{L}, V) = V$.

We define a linear mapping $\omega \mapsto \delta\omega$ of $\Lambda^k(\mathfrak{L}, V)$ into $\Lambda^{k+1}(\mathfrak{L}, V)$ by the formula

$$\begin{aligned}(\delta\omega)(v_1, \cdots, v_{k+1}) &= \sum_{i=1}^{k+1}(-1)^{i+1}\rho(v_i)\omega\big(v_1, \cdots, \widehat{v}_i, \cdots, v_{n+1}\big) \\ &\quad + \sum_{i<j}(-1)^{i+j}\omega\big([v_i, v_j], v_1, \cdots, \widehat{v}_i, \cdots, \widehat{v}_j, \cdots, v_{n+1}\big).\end{aligned}$$

If $k = 0$, then $\omega \in V$, and $\delta\omega$ is defined by $(\delta\omega)(V) = \rho(V)\omega$. After some tedious and routine calculation, we claim that $\delta^2\omega = 0$ for any k-V-cochain ω.

The cohomology group obtained using k-V-cochain is denoted as $\mathrm{H}^k(\mathfrak{L}, V)$. By definition, we have that $\mathrm{H}^0(\mathfrak{L}, V)$ is the subspace of the invariant elements of V under representation ρ. For $k > \dim\mathfrak{L}$ we have $\mathrm{H}^k(\mathfrak{L}, V) = 0$.

Remark 4.1. In the process, two notations are necessary to go with. For each $x \in \mathfrak{L}$, we define a linear mapping $\mathfrak{L}_x : \Lambda^k(\mathfrak{L}, V) \to \Lambda^k(\mathfrak{L}, V)$ by setting

$$(\mathfrak{L}_x\omega)(v_1, \cdots, v_k) = \rho_x\big(\omega(v_1, \cdots, v_k)\big) - \sum_{i=1}^{k}(-1)^{i+1}\omega\big([v_i, x], v_1, \cdots, \widehat{v}_i \cdots, v_k\big)$$

if $k = 0$, $\mathfrak{L}_x$ is defined by $\mathfrak{L}_x f = \rho_x f$.

For $x \in \mathfrak{L}$ we also define a linear mapping $\iota_x : \Lambda^{k+1}(\mathfrak{L}, V) \to \Lambda^k(\mathfrak{L}, V)$ by setting

$$\iota_x\omega(v_1, \cdots, v_k) = \omega(x, v_1, \cdots, v_k)$$

A direct substitution into the definitions yields the following three formula, for $x, y \in \mathfrak{L}$

$$\begin{aligned}\mathfrak{L}_x &= \iota_x\delta + \delta\iota_x \\ \iota_{[x,y]} &= \mathfrak{L}_y\iota_x - \iota_x\mathfrak{L}_y = [\iota_x, \mathfrak{L}_y] = \big[\mathfrak{L}_x, \iota_y\big] \\ \mathfrak{L}_{[x,y]} &= \big[\mathfrak{L}_x, \mathfrak{L}_y\big]\end{aligned} \tag{9}$$

Theorem 4.1. *If $\mathfrak{L}$ is a semi-simpleLie algebra over a field of characteristic 0, and (ρ, V) is an irreducible and nontrivial representation of $\mathfrak{L}$ then $\mathrm{H}^k(\mathfrak{L}, V) = 0$ for all dimensions k.*

Proof. the representation ρ is not trivial, the kernel $\mathfrak{L}_0$ of ρ is not all of $\mathfrak{L}$, since $\mathfrak{L}$ is semi-simple there is an ideal $\mathfrak{L}_1$ in $\mathfrak{L}$ such that $\mathfrak{L}$ is the direct sum of $\mathfrak{L}_0$ and $\mathfrak{L}_1$. Let $y_1, \cdots, y_n$ be a base in $\mathfrak{L}_1$. Since the bilinearform $B(y, z)$ associated with the representation ρ of $\mathfrak{L}_1$ is nondegenerate (by Lemma 3.1) we can select a dual base $z_1, \cdots, z_n$ for $\mathfrak{L}_1$ such that

$$B(y_i, z_j) = \delta^i_j.$$

For each $x \in \mathfrak{L}$ we have $[y_i, x] \in \mathfrak{L}_1$ and therefore $[y_i, x] = \sum_{j=1}^n c_{ij}y_j$. Similarly $[x, z_i] = \sum_{j=1}^n c_{ij}y_j$. We shall prove that $cij = d_{ij}$. Indeed we have $B\big([y_i, x], z_j\big) = B\big(\sum_k c_{ik}y_k, z_j\big) = c_{ij}$ and similarly $B\big(y_i, [x, z_j]\big) = d_{ij}$. Hence $c_{ij} = d_{ij}$ by formula (8). This implies the following two propositions for any $x \in \mathfrak{L}_1$ and any linear function f of $\mathfrak{L}$ to V.

$$\sum_{i=1}^n \rho_{[y_i,x]}f(z_i) = \sum_{i=1}^n \rho_{y_i}f\big([x, z_i]\big) \tag{10}$$

$$\sum_{i=1}^n \big(\rho_{y_i}\rho_{[x,z_i]} - \rho_{[y_i,x]}\rho_{z_i}\big) = 0. \tag{11}$$

Consider the linear transformation of V into itself

$$\Gamma = \sum_{i=1}^{n} \rho_{y_i}\rho_{z_i}.$$

Since the trace is n it follows that $\Gamma \neq 0$. Further

$$\begin{aligned} \Gamma\rho_x &= \sum_{i=1}^{n} \rho_{y_i}\rho_{z_i}\rho_x = \sum_{i=1}^{n} \rho_{y_i}\big(\rho_{[x,z_i]} - \rho_x\rho_{z_i}\big) \\ &= \sum_{i=1}^{n} \big(\rho_{y_i}\rho_{[x,z_i]} - \rho_{[y_i,x]}\rho_{z_i} + \rho_x\rho_{y_i}\rho_{z_i}\big) = \rho_x\Gamma. \end{aligned}$$

Since Γ commutes with each ρ_x the space $\Gamma(V)$ is an invariant subspace of V. But the representation ρ is irreducible hence $\Gamma(V) = V$ and Γ has an inverse Γ^{-1}. Clearly $\Gamma^{-1}\rho_x = \rho_x\Gamma^{-1}$.

Let $\omega \in \Lambda^k(\mathfrak{L}, V)$ and let λ be a linear transformation $V \to V$. Then $\lambda\omega \in \Lambda^k(\mathfrak{L}, V)$. A direct computation shows that

$$(\delta\lambda\omega - \lambda\delta\omega)(v_1, \cdots, v_{k+1}) = \sum_{i=1}^{k+1}(-1)^{i+1}\big(\rho_{v_i}\lambda - \lambda\rho_{v_i}\big)\omega\big(v_1, \cdots, \widehat{v}_i, v_{k+1}\big). \tag{12}$$

Suppose now that ω is a cocycle. Consider the cochains $\iota_{z_i}\omega \in \Lambda^{k-1}(\mathfrak{L}, V)$ and let $g = \sum_{i=1}^{n} \rho_{y_i}\iota_{z_i}\omega$. Then by equation (9)

$$\begin{aligned} \delta g &= \sum_{i=1}^{n} \big(\delta\rho_{y_i}\iota_{z_i}\omega - \rho_{y_i}\delta\iota_{z_i}\omega\big) + \sum_{i=1}^{n} \rho_{y_i}\delta\iota_{z_i}\omega \\ &= \sum_{i=1}^{n} \big(\delta\rho_{y_i}\iota_{z_i}\omega - \rho_{y_i}\delta\iota_{z_i}\omega\big) + \sum_{i=1}^{n} \rho_{y_i}\mathfrak{L}_{z_i}\omega - \sum_{i=1}^{n} \rho_{y_i}\iota_{z_i}\delta\omega \\ &= \sum_{i=1}^{n} \big(\delta\rho_{y_i}\iota_{z_i}\omega - \rho_{y_i}\delta\iota_{z_i}\omega\big) + \sum_{i=1}^{n} \rho_{y_i}\mathfrak{L}_{z_i}\omega \end{aligned}$$

Hence by equation (12)

$$\begin{aligned} (\delta g)(v_1, \cdots, v_k) &= \sum_{i=1}^{n}\sum_{j=1}^{k}(-1)^{j+1}\big(\rho_{v_j}\rho_{y_i} - \rho_{y_i}\rho_{v_j}\big)\iota_{z_i}\omega(v_1, \cdots, \widehat{v}_i, v_{k+1}) + \sum_{i=1}^{n} \rho_{y_i}\mathfrak{L}_{z_i}\omega(v_1, \cdots, v_k) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{k}(-1)^{j+1}\rho_{[y_i,v_j]}\iota_{z_i}\omega(v_1, \cdots, \widehat{v}_i, v_{k+1}) + \sum_{i=1}^{n} \rho_{y_i}\rho_{z_i}\omega(v_1, \cdots, v_k) \\ &\quad - \sum_{i=1}^{n}\sum_{j=1}^{k}(-1)^{j+1}\rho_{y_i}\omega\big([v_j, z_i], x_1, \cdots, \widehat{v}_j, \cdots, v_k\big). \end{aligned}$$

The two double sums cancel in view of equation (10) and therefore $\delta g = \iota_{z_i}\omega$. Since Γ^{-1} and ρ_x commute, equation (12) implies that $\delta\Gamma^{-1}g = \Gamma^{-1}\delta g = \Gamma^{-1}\Gamma\omega = \omega$. Hence ω is a coboundary, this completes the proof. □

Remark 4.2. This theorem shows that in the semi-simple case nothing is gained by studying cohomology groups over representations.

Afterwards, we pay more attention on the group $\mathrm{H}^1(\mathfrak{L}, V)$ and $\mathrm{H}^2(\mathfrak{L}, V)$. In fact any representation ρ decomposes into irreducible representations and this carries with it a direct decomposition of the cohomology groups. Hence $\mathrm{H}^k(\mathfrak{L}, V)$ is isomorphic with the direct sum of several copies of $\mathrm{H}^k(\Lambda^*\mathfrak{L}^*)$. we have the following result

Theorem 4.2. *A Lie algebra $\mathfrak{L}$ over a field of characteristic 0 is semi-simple if and only if* $\mathrm{H}^1(\mathfrak{L}, V) = \{0\}$ *for every representation* (ρ, V) *of* $\mathfrak{L}$.

4.1 Extensions of Lie Algebras

Let $\widetilde{\mathfrak{L}}$ and $\mathfrak{L}$ be two Lie algebras and an onto homomorphism $\phi : \widetilde{\mathfrak{L}} \twoheadrightarrow \mathfrak{L}$. The pair $(\widetilde{\mathfrak{L}}, \phi)$ is called an *extension*. The kernel V of ϕ is called the kernel of the extension. If $[V, V] = 0$ the extension is said to have an abelian kernel. An extension is called *inessential* if there exists a subalgebra $\mathfrak{L}'$ of $\widetilde{\mathfrak{L}}$, which is mapped by ϕ isomorphically onto $\mathfrak{L}$. As a vector space $\widetilde{\mathfrak{L}}$ is then the direct sum $\mathfrak{L}' \oplus V$.

Theorem 4.3. *If every extension of $\mathfrak{L}$ with an abelian kernel is inessential, then every extension of $\mathfrak{L}$ is inessential.*

The extensions of $\mathfrak{L}$ with an abelian kernel will be now studied in greater detail. Let $(\widetilde{\mathfrak{L}}, \phi)$ be such an extension with kernel V. We select a linear mapping $u : \mathfrak{L} \to \widetilde{\mathfrak{L}}$ such that $\phi u(x) = x$ for each $x \in \mathfrak{L}$. For each $x \in \mathfrak{L}$ and $w \in V$ we then have a mapping $\rho_x w = [w, u(x)]$ of V into itself. Since $[V, V] = 0$ it can be easily verified that ρ_x is independent of the choice of u. Further $\rho_y \rho_x w - \rho_x \rho_y w = \big[[w, u(x)], u(y)\big] - \big[[w, u(y)], u(x)\big] = \big[w, [u(x), u(y)]\big] = \big[w, u([x, y])\big] = \rho_{[x,y]} w$, and therefore ρ is a representationof $\mathfrak{L}$ with V a representation space.

Suppose now that the space V and the representation ρ of $\mathfrak{L}$ in V is given. Any pair $(\widetilde{\mathfrak{L}}, \phi)$ where ϕ is a homomorphism of $\widetilde{\mathfrak{L}}$ onto $\mathfrak{L}$ with kernel V, such that $[V, V] = 0$ and which leads to the given representation ρ, will be called an extension of $\mathfrak{L}$ by ρ. Two such extensions $(\widetilde{\mathfrak{L}}_1, \phi_1)$ and $(\widetilde{\mathfrak{L}}_2, \phi_2)$ are isomorphic if there is an isomorphism $\theta : \widetilde{\mathfrak{L}}_1 \to \widetilde{\mathfrak{L}}_2$ such that $\theta(w) = w$ for $w \in V$ and that $\phi_1 = \phi_2 \theta$.

Given an extension $(\widetilde{\mathfrak{L}}, \phi)$ of $\mathfrak{L}$ by ρ we select a linear mapping $u : \mathfrak{L} \to \widetilde{\mathfrak{L}}$ with $\phi u(x) = x$. Since $\phi\big([u(x), u(y)]\big) = [x, y] = \phi u([x, y])$ therefore there is an element $f(x, y) \in V$ such that

$$\big[u(x), u(y)\big] = f(x, y) + u\big([x, y]\big).$$

The function f is called the factor set corresponding to the function u. Clearly f is a 2-V-cochain. For $x, y, z \in \mathfrak{L}$ we have

$$\begin{aligned} \big[[u(x), u(y)], u(z)\big] &= \big[f(x, y), u(z)\big] + \big[u([x, y]), u(z)\big] \\ &= \rho_z f(x, y) + f\big([x, y], z\big) + u\big(\big[[x, y], z\big]\big). \end{aligned}$$

Therefore Jacobi's identity gives

$$\rho_z f(x, y) + \rho_y f(z, x) + \rho_x f(y, z) + f\big([x, y], z\big) + f\big([z, x], y\big) + f\big([y, z], x\big) = 0$$

or $(\delta f)(x, y, z) = 0$. It follows that the factor set f is a 2-V-cocycle.

If $\bar{u}$ is a different linear mapping $\bar{u} : \mathfrak{L} \to \widetilde{\mathfrak{L}}$ such that $\phi \bar{u}(x) = x$, then $h(x) = \bar{u}(x) - u(x) \in V$ and h is a 1-V-cochain. If $\bar{f}$ is the factor set corresponding to $\bar{u}$ then

$$\begin{aligned} \bar{f}(x, y) &= \big[u(x), \bar{u}(y)\big] - u\big([x, y]\big) = \big[u(x), u(y)\big] + \big[h(x), u(y)\big] \\ &\quad + \big[u(x), h(y)\big] + \big[h(x), h(y)\big] - u\big([x, y]\big) - h\big([x, y]\big) \\ &= f(x, y) - \big[\rho_x h(y) - \rho_y h(x) + h([x, y])\big] \end{aligned}$$

so that $\bar{f} = f - \delta h$. This proves that the cohomology class of f is independent of the choice of u. Thus to each extension $(\widetilde{\mathfrak{L}}, \phi)$ corresponds a definite element of $\mathrm{H}^2(\mathfrak{L}, \rho)$. Clearly two isomorphic extensions determine the same element of $\mathrm{H}^2(\mathfrak{L}, \rho)$. Given an element of $\mathrm{H}^2(\mathfrak{L}, \rho)$ a corresponding extension can be constructed as follows. Let f be a cocycle belonging to the given element of $\mathrm{H}^2(\mathfrak{L}, \rho)$. As a vectorspace $(\widetilde{\mathfrak{L}}$ is the direct sum $\mathfrak{L}' \oplus V$, with $\phi(x, w) = x$ and $u(x) = (x, 0)$. Commutationin $\mathfrak{L}$ is definedby the formula

$$\big[(x_1, w_1), (x_2, w_2)\big] - \big([x_l, x_2], \rho_{x_2} w_1 - \rho_{x_2} V_2 + f(x_l, x_2)\big).$$

It is further easy to see that an extension is inessential if and only if the corresponding cohomology class is zero.

This implies the following theorem

Theorem 4.4. *The elements of $\mathrm{H}^2(\mathfrak{L}, \rho)$ are in a 1-1 correspondence with isomorphism classes of extensions of $\mathfrak{L}$ by ρ.*

Corollary 4.1. *In order that every extension of* $\mathfrak{L}$ *over* ρ *be inessential it is necessary and sufficient that* $\mathrm{H}^2(\mathfrak{L}, \rho) = 0$.

Combining this with Theorem 4.3, we find

Theorem 4.5. *In order that every extension of a Lie algebra* $\mathfrak{L}$ *be inessential it is necessary and sufficient that* $\mathrm{H}^2(\mathfrak{L}, \rho) = 0$ *for every representation* ρ.

In view of Theorems 3.5 and 4.1 this condition is satisfied when $\mathfrak{L}$ is a semi-simple Lie algebra over a field of characteristic 0. Thus we obtain Levi's theorem

Theorem 4.6. *If* $\mathfrak{L}$ *is a semi-simpleLie algebra over a field of characteristic 0, then every extension of* $\mathfrak{L}$ *is inessential.*

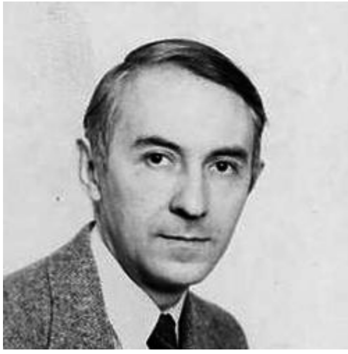

Figure 1: Claude Chevalley 1909-1984

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