Notes on Complex Geometry

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February 18, 2019

1 Frobenius integrability theorem

Definition 1.1. Let X be a $n-$dimensional manifold, and $E \subseteq T_X$ be a $\mathcal{C}^1$ vector sub-bundle of rank k. Such an E is called a **distribution** of X. A distribution E is called **integrable** if locally there exists a $\mathcal{C}^1$ map $\phi(U) : U \to \mathbb{R}^{n-k}$ such that every fiber E_x is identified with $\operatorname{Ker} \mathrm{d}\phi_x$.

Theorem 1.1 (Frobenius)**.** *A distribution E is integrable if and only if for all vector fields $\chi, \psi \in \Gamma(X, E)$, the bracket $[\chi, \psi] \in \Gamma(X, E)$.*

Proof. "$\Rightarrow$": if integrable, then locall $U \subseteq X, \Gamma(U, E)$ consists of vector fields χ that annihilate the functions $f_i = x_i \circ \phi$, where ϕ is defined above. If χ, ψ annihilate f_i, then the bracket $[\chi, \psi](f_i) = 0$.
"$\Leftarrow$": We prove this by induction on k. If $k = 1$, the result is implied by the Flow–Bow theorem which states that a nontrivial $\mathcal{C}^1$ vector field is locally diffeomorphic to a nontrivial constant vector field. Let $U \subseteq X$ open, and admits a non-trivial section χ of E over U, and a sub-mersion $\phi : U \to \mathbb{R}^{n-1}$ whose fibers are the trajectories of χ. By the Flow–Box theorem, we may assume that U is diffeomorphic to $V \times (0, 1)$, V is an open set of $\mathbb{R}^{n-1}$, that ϕ is the first projection and $\chi = \frac{\partial}{\partial t}$. The following lemma is essential.

Lemma 1.1. The integrability condition implies that there exists a distribution F of rank $k - 1$ on V such that $E = (\phi_*)^{-1}(F)$. Moreover, E satisfies the integrability condition iff F does.

Here $\phi_* : T_{U,x} \to T_{V,\phi(x)}$ is the differential of ϕ. Admitting the lemma, this gives locally $\psi : V \to \mathbb{R}^{n-k}$ whose fibers are integral manifolds of the distribution F. Then the fibers of $\psi \circ \phi : U \to \mathbb{R}^{n-k}$ are integral manifolds of the distribution E.

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References

[1] Claire Voisin, *Hodge Theory and Complex Algebraic Geometry I.*