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# When Von Neumann meets Grothendieck: On Algebraic Cellular Automata

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**Abstract**

We investigate some general properties of algebraic cellular automata, i.e., cellular automata over groups whose alphabets are affine algebraic sets and which are locally defined by regular maps. When the ground field is assumed to be uncountable and algebraically closed, we prove that such cellular automata always have a closed image with respect to the prodiscrete topology on the space of configurations and that they are reversible as soon as they are bijective.

## Contents



## 1 Introduction

### Background

A cellular automaton is a discrete model studied in computer science, mathematics, physics, complexity science, theoretical biology and microstructure modeling. The concept was originally discovered in the 1940s by Stanislaw Ulam and John von Neumann [1]. Von Neumann used cellular automata to serve as theoretical models for self-reproducing machines. Deep connections with complexity theory and logic emerged from the discovery that some cellular automata are universal Turing machines.

In this lecture, our main interest is to discuss the surjective properties of algebraic cellular autamata, namely cellular automata over groups whose alphabets are affine algebraic sets and whose local defining maps are regular. Thus, the Ax-Grothendieck theorem 2.3 plays a big role in the process. This present article grew out from numerous readings of [3].

Some basic definitions and notaions are provided as follows.

Let $G$ be a group and let $A$ be an affine algebraic set over $K$. The set $A^G = \{x : G \to A\}$ is called the set of *configurations* over a group $G$ and the *alphabet* $A$. We equip $A^G = \prod_{g \in G} A$ with its *prodiscrete* topology (i.e., the product topology obtained by taking the discrete topology on each factor $A$ of $A^G$). The action of $G$ on $A^G$ defined by

$$gx(h) = x(g^{-1}h) \qquad \forall g, h \in G,\, x \in A^G \tag{1}$$

is called the *G-shift* on $A^G$. Given a configuration $x \in A^G$ and a subset $\Omega \subseteq G$. the element $x|_\Omega \in A^\Omega$ defined by $x|_\Omega(g) = x(g)$ for all $g \in \Omega$ is called the *restriction* of $x$ to $\Omega$ or the *pattern* of $x$ supported by $\Omega$.

**Definition 1.1.** An *cellular automaton* over the group $G$ and the alphabet $A$ is a map $\tau : A^G \to A^G$ satisfying the following condition: there exist a finite subset $M \subseteq G$ and a map $\mu : A^M \to A$ such that

$$\tau(x)(g) = \mu((g^{-1}x)|_M) \qquad \forall x \in A^G, g \in G \tag{2}$$

Such a set $M$ is then called a *memory* set of $\tau$ and $\mu$ is called the *local defining map* for $\tau$ associated with $M$.

**Definition 1.2.** A cellular automaton $\tau : A^G \to A^G$ is said to be an *algebraic cellular automaton* over the field $K$ if the alphabet $A$ is an affine algebraic set over K and if for some (or, equivalently, any) memory set $M \subseteq G$ of $\tau$, the associated local defining map $\mu : A^M \to A$ is regular.

**Remark 1.** This definition suggests that whether the local defining map is a regular map does not depend on the choice of the memory set. The statement shall be proved in section 3.

**Example 1.** Every cellular automaton with finite alphabet $A$ may be regarded as an algebraic cellular automaton. Indeed, it suffices to embed $A$ as a subset of some field $K$ and then observe that, if $M$ is a finite set, any map $\mu : A^M \to A$ is the restriction of some polynomial map $P : K^M \to K$ (which can be made explicit by using Lagrange interpolation formula).

**Example 2.** Let $K$ be a field, $A$ an affine algebraic set over $K$, and $f : A \to A$ a regular map. Let $G$ be a group and fix an element $g_0 \in G$. Then the map $\tau : A^G \to A^G$, defined by $\tau(x) = f(x(gg_0))$ for all $x \in A^G$ and $g \in G$, is an algebraic cellular automaton with memory set $\{g_0\}$ and local defining map $f$. Note that $\tau$ is injective (resp. surjective) if and only if $f$ is injective (resp. surjective).

## Main Results

We recall that a map $f : X \to Y$ between topological spaces $X$ and $Y$ has the *closed image property* if its image $f(X)$ is closed in $Y$. When the alphabet $A$ is a finite set, it immediately follows from the compactness of $A^G$ and the fact that $A^G$ is Hausdorff that every cellular automaton $\tau : A^G \to A^G$ has the closed image property.

Our first result on algebraic cellular automata is a particular case of [2, Section 4.D].

**Theorem 1.1.** *Let $G$ be a group, $K$ an uncountable algebraically closed field, and $A$ an affine algebraic set over $K$. Then every algebraic cellular automaton $\tau : A^G \to A^G$ has the closed image property with respect to the prodiscrete topology on $A^G$.*

Recall that a group is called *residually finite* if the intersection of all its finite index subgroups is trivial. For example, by a theorem of Mal'cev, every finitely generated linear group is residually finite. From Theorem 1.1 and the Ax-Grothendieck theorem 2.3,[8] on injective endomorphisms of algebraic varieties, we shall deduce the following [2, Section 4.E']

**Corollary 1.1.** *Let $G$ be a residually finite group (e.g., $G = \mathbb{Z}^d$), $K$ an uncountable algebraically closed field, and $A$ an affine algebraic set over $K$. Then every injective algebraic cellular automaton $\tau : A^G \to A^G$ is surjective and hence bijective.*

Given a group $G$ and a set $A$, a cellular automaton $\tau : A^G \to A^G$ is called *reversible* if $\tau$ is bijective and its inverse map $\tau^{-1} : A^G \to A^G$ is also a cellular automaton. When the alphabet $A$ is finite, it easily follows from the compactness of $A^G$ and the Curtis-Hedlund theorem that every bijective cellular automaton $\tau : A^G \to A^G$ is reversible (see [4, Theorem 1.10.2]).

In this paper, for algebraic cellular automata, we shall prove the following result:

**Theorem 1.2.** *Let $G$ be a group, $K$ an uncountable algebraically closed field, and $A$ an affine algebraic set over $K$. Then every bijective algebraic cellular automaton $\tau : A^G \to A^G$ is reversible.*

The article is organized as follows. Our familiar results on affine algebraic sets will be given without proof unless necessary. In section 2, we recall some preliminary properties of cellular autamata in section 3. We prove that the projective limit of a projective sequence of nonempty constructible sets over an uncountable algebraically closed field is never empty (Theorem 3.1). This last result, a Mittag-Lefflertype statement, is a key ingredient in the proof of Theorem 1.1 and Theorem 1.2. In section 3, we establish Theorem 1.1 and Corollary 1.1. The proof of Theorem 1.2 is given in section 4.

# 2 Basic results from algebraic geometry

## 2.1 Constructible Sets And Chevalley's Theorem

Let $A$ be an affine algebraic set over a field $K$. One says that a subset $L \subseteq A$ is *locally closed* in $A$ if there exists an open subset $U \subseteq A$ and a closed subset $V \subseteq A$ such that $L = U \cap V$. This is equivalent to $L$ being open in its closure $\overline{L} \subseteq A$.

One says that a subset $C \subseteq A$ is *constructible* if $C$ is finite union of locally closed subsets of $A$. The set of constructible subsets of $A$ is closed under finite unions, finite intersections, and taking complements in $A$. We shall use the following elementary result (see for example [10, AG Section 1.3])

**Proposition 2.1.** *Let $A$ be an affine algebraic set over a field $K$ and suppose that $C$ is a constructible subset of $A$. Then there is an open dense subset $U$ of $C$ such that $\overline{U} \subseteq C$.*

We shall also use the following theorem due to C. Chevalley (see for example [10, AG Section 10.2])

**Theorem 2.1** (Chevalley)**.** *Let $K$ be an algebraically closed field. Let $A$ and $B$ be affine algebraic sets over $K$, and let $f : A \to B$ be a regular map. Then every constructible subset $C \subseteq A$ has a constructible image $f(C) \subseteq B$. In particular, $f(A)$ is a constructible subset of $B$.*

## 2.2 Dimension

The dimension of an affine algebraic set $A$ is equal to the Krull dimension of its coordinate ring $K[A]$. If in addition $A$ is irreducible, then $\dim(A)$ is also equal to the transcendence degree of its function field $K(A)$ over $K$. Let $A$ and $B$ be irreducible affine algebraic sets over $K$. Let $f : A \to B$ be a regular map and let $f^* : K[B] \to K[A]$ denote the induced ring homomorphism. One says that $f$ is a finite morphism if $K[A]$ is finitely generated as a $f^*(K[B])$-module. Every finite morphism $f : A \to B$ between irreducible affine algebraic sets is closed (see [12, Proposition 8.7]).

We shall use the following result (see [12, Theorem 8.12]) which can be deduced from Emmy Noether's normalization lemma:

**Theorem 2.2.** *Let $K$ be an algebraically closed field and let $A$ be an irreducible affine algebraic set over $K$ such that $\dim(A) = d$. Then there exists a surjective finite morphism $f : A \to K^d$.*

## 2.3 The Ax-Grothendieck Theorem

We recall the following result without proof:

**Theorem 2.3** (Ax-Grothendieck)**.** *Let $K$ be an algebraically closed field and let $A$ be an affine algebraic set over $K$. Then every injective regular map $f : A \to A$ is surjective and hence bijective.*

*Proof.* A cohomological proof of the theorem was given by A. Borel in [11]. ☐

**Remark 2.** (a) The theorem fails if the hypothesis that $K$ is algebraically closed is removed. In characteristic 0, it suffices to consider the injective polynomial map $f : \mathbb{Q} \to \mathbb{Q}$ defined by $f(x) = x^3$, which is not surjective since $2 \in f(\mathbb{Q})$.
(b) When $A$ is an affine algebraic set over an algebraically closed field $K$ of characteristic 0, it is known that the inverse map of any bijective regular map $f : A \to A$ is also regular (see [9, Proposition 17.9.6]).
(c) When $K$ is an algebraically closed field of characteristic $p > 0$ and $A$ is an affine algebraic set over $K$, the inverse map of a bijective map $f : A \to A$ need not to be regular. For example, the inverse map of the Frobenius automorphism $f : K \to K$ defined by $f(x) = x^p$ is not regular since there is no polynomial $P \in K[t]$ such that $P(x)^p = x$ for all $x \in K$.
(d) It is known [13] that every injective regular map $f : A \to A$, where $A$ is a real affine algebraic set, is bijective. However, its inverse need not to be regular. For example, the inverse of the bijective polynomial map $f : R \to R$ defined by $f(x) = x^3$ is the map $x \mapsto \sqrt[3]{x}$ which is not polynomial.
(e) The inverse map of a bijective regular map between distinct algebraic subsets may fail to be regular even if the ground field is algebraically closed and of characteristic 0. For example, the map $f : \mathbb{C} \to Z(t_1^2 - t_2^3) \subseteq \mathbb{C}^2$ given by $f(t) = (t^3, t^2)$ is bijective and regular but its inverse map is not regular. Otherwise, this would imply the existence of a polynomial $P \in \mathbb{C}[t_1, t_2]$ such that $P(z^3, z^2) = z$ for all $z \in \mathbb{C}$. This is impossible since, for any $P \in \mathbb{C}[t_1, t_2]$, the expression $P(z^3, z^2)$ is polynomial in $z$ with each non-constant monomial of degree at least 2.

# 3 Algebraic Cellular automata

## 3.1 Prodiscrete Topology

Let $G$ be a group and $A$ a set. We equip $A^G$ with the prodiscrete topology. This is the smallest topology on $A^G$ for which the projection map $\pi_g : A^G \to A$, given by $\pi_g(x) = x(g)$, is continuous for every $g \in G$. The elementary cylinders

$$C(g, a) = \pi_g^{-1}(a) = \{x \in A^G : x(g) = a\} \quad (g \in G, a \in A)$$

are both open and closed in $A^G$. A subset $U \subseteq A^G$ is open if and only if $U$ can be expressed as a (finite or infinite) union of finite intersections of elementary cylinders.

If $x \in A^G$, a neighborhood base of $x$ is given by the sets

$$V(x, \Omega) = \{y \in A^G : x|_\Omega = y|_\Omega\} = \bigcap_{g \in \Omega} C(g, x(g)) \tag{3}$$

where $\Omega$ runs over all finite subsets of $G$.

**Remark 3.** (a) The space $A^G$ is Hausdorff with prodiscrete topology (see [4, Proposition 1.2.1]).
(b) The action of $G$ on $A^G$ is continuous (see [4, Proposition 1.2.2]).

## 3.2 Periodic Configurations

Let $G$ be a group, $A$ be a set and let $H$ be a subgroup of $G$. A configuration $x \in A^G$ is called $H$-periodic if $x$ is fixed by $H$ (i.e., $hx = x \ \forall h \in H$). Let $\mathrm{Fix}(H)$ denote the subset of $A^G$ consisting of all $H$-periodic configurations.

**Proposition 3.1.** *Let $H$ be a subgroup of $G$. Then the set $\mathrm{Fix}(H)$ is closed in $A^G$ for the prodiscrete topology.*

*Proof.* We have $\mathrm{Fix}(H) = \bigcap_{h \in H} \{x \in A^G : hx = x\}$. It follows from Remark 3 that it is closed. □

**Proposition 3.2.** *Let $H$ be a subgroup of $G$ and let denote by $\rho : G \to H\backslash G$ the canonical surjection. Then the map $\rho^* : A^{H\backslash G} \to \mathrm{Fix}(H)$ defined by $\rho^*(y) = y \circ \rho$ for all $y \in A^{H\backslash G}$ is bijective.*

*Proof.* If $y_1, y_2 \in A^{H\backslash G}$ satisfy $y_1 \circ \rho = y_2 \circ \rho$, then $y_1 = y_2$ since $\rho$ is surjective. Thus $\rho^*$ is injective. If $x \in \mathrm{Fix}(H)$, then $hx = x$ for all $h \in H$. Thus, the configuration $x$ is constant on each right coset of $G$ modulo $H$, that is, $x$ is in the image of $\rho^*$. This shows that $\rho^*$ is surjective. □

## 3.3 Induction And Restriction of Cellular Automata

Let $G$ be a group and let $A$ be a set. Let $H$ be a subgroup of $G$. Let $CA(G,H;A)$ denote the set consisting of all cellular automata $\tau : A^G \to A^G$ admitting a memory set $M$ such that $M \subseteq H$. Thus, $CA(G,H;A)$ is the subset of $CA(G;A)$ consisting of the cellular automata whose minimal memory set is contained in $H$. It is easy to check that $CA(G,H;A)$ is a submonoid of $CA(G;A)$.

Let $\tau \in CA(G,H;A)$. Let $M$ be a memory set for $\tau$ such that $M \subseteq H$ and let $\mu : A^M \to A$ denote the associated local defining map. Then, the map $\tau_H : A^H \to A^H$ defined by

$$\tau_H(x)(h) = \mu((h^{-1}x)|_M) \qquad \forall x \in A^H, h \in H$$

is a cellular automaton over the group $H$ with memory set $M$ and local defining map $\mu$. One says that $\tau_H$ is the *restriction* of the cellular automaton $\tau$ to $H$.

Conversely, let $\sigma : A^H \to A^H$ be a cellular automaton with memory set $M$ and local defining map $\mu : A^M \to A$. Then the map $\sigma^G : A^G \to A^G$ defined by

$$\sigma^G(\tilde{x})(g) = \mu((g^{-1}\tilde{x})|_M) \qquad \forall \tilde{x} \in A^G, g \in G$$

is a cellular automaton over $G$ with memory set $M$ and local defining map $\mu$. One says that $\sigma^G$ is the cellular automaton *induced* by $\sigma$. It is easy to show that:

**Proposition 3.3.** *The map $\tau \mapsto \tau_H$ is a monoid isomorphism from $CA(G,H;A)$ onto $CA(H;A)$ whose inverse is the map $\sigma \mapsto \sigma^G$ .*

*Proof.* See [4, Propositioin 1.7.2] □

Let $\tau \in CA(G,H;A)$, notice that

$$A^G = \prod_{c \in G\setminus H} A^c$$

We have $\tilde{x} = (\tilde{x}|_c)_{c \in G\setminus H}$ for each $\tilde{x} \in A^G$, where $\tilde{x}|_c \in A^c$ denotes the restriction of $\tilde{x}$ to $c$. If $c \in G \setminus H$, and $g \in c$ then $\tau(\tilde{x})(g)$ depends only on $\tilde{x}|_c$ ($M \subseteq H \Rightarrow gM \subseteq c$). It deduces that

$$\tau = \prod_{c \in G\setminus H} \tau_c \tag{4}$$

$\tau_c : A^c \to A^c$ is the unique map which satisfies $\tau_c(\tilde{x}|_c) = (\tau(\tilde{x}))|_c$. When $c = H$, $\tau_c = \tau_H : A^H \to A^H$ is the cellular automaton obtained by restriction of $\tau$ to $H$. Given $c \in G \setminus H$ and $g \in c$, define $\phi_g : H \to c$ taking $h \mapsto gh$ for all $h \in H$. Then $\phi_g$ induces a bijective map $\phi_g^* : A^c \to A^H$ given by $\phi_g^*(x) = x \circ \phi_g$. It turns out that the maps $\tau_c$ and $\tau_H$ are conjugate by $\phi_g^*$ (see [4, Proposition 1.7.3])

$$\tau_c = (\phi_g^*)^{-1} \circ \tau_H \circ \phi_g^* \tag{5}$$

We shall use the following result:

**Proposition 3.4.** *Let $G$ be a group, $A$ a set, and $H$ a subgroup of $G$. Suppose that $\tau : A^G \to A^G$ is a cellular automaton over $G$ admitting a memory set contained in $H$ and let $\tau_H : A^H \to A^H$ denote the cellular automaton over $H$ obtained by restriction. Then the following holds:*
*(i) $\tau$ is bijective if and only if $\tau_H$ is bijective;*
*(ii) $\tau$ is reversible if and only if $\tau_H$ is reversible;*
*(iii) $\tau(A^G)$ is closed in $A^G$ for the prodiscrete topology if and only if $\tau_H(A^H)$ is closed in $A^H$ for the prodiscrete topology;*
*(iv) when $A$ is an affine algebraic set over a field $K$, then $\tau$ is algebraic if and only if $\tau_H$ is algebraic.*

*Proof.* For (i), it immediately follows from (4) and (5). For (ii), see [4, Proposition1.10.4]. Assertion (iii) is established in [5, Theorem 1.2]. Assertion (iv) immediately follows from the definition of an algebraic cellular automaton since $\tau$ and $\tau_H$ admits a commmon local defining map. □

## 3.4 Minimal Memory

Let $G$ be a group and let $A$ be a set. Given a cellular automaton $\tau : A^G \to A^G$ and a memory set $M \subseteq G$ for $\tau$, we denote by $\mu_M : A^M \to A$ the local defining map for $\tau$ associated with $M$. Observe that $\mu_M$ is entirely determined by $\tau$ and $M$ since we have

$$\mu_M(y) = \tau(x)(1_G) \qquad \forall y \in A^M$$

where $x \in A^G$ is any configuration satisfying $x|_M = y$. Note that this formula deduces that any two memory sets of $\tau$, say $M_1$ and $M_2$, $M_1 \cap M_2 \neq \varnothing$.

If $M$ is a memory set, $M \subseteq M'$ finite, $\mu_{M'} = \mu_M \circ p$ is the associated defining map, where $p : A^{M'} \to A^M$ is the canonical map, thus memory set is not unique. However, we shall see that every automaton admits a unique memory set of minimal cardinality.

**Lemma 3.1.** Let $\tau : A^G \to A^G$ be a cellular automaton. Let $S_1$ and $S_2$ be memory sets for $\tau$. Then $S_1 \cap S_2$ is also a memory set for $\tau$.

*Proof.* If $x, y \in A^G$, s.t., $x|_{S_1 \cap S_2} = y|_{S_1 \cap S_2}$. Choose $z \in A^G$ s.t. $z|_{S_1} = x|_{S_1}$ and $z|_{S_2} = y|_{S_2}$. Then $\mu(x|_{S_1 \cap S_2}) = \tau(x)(1_G) = \tau(z)(1_G) = \tau(y)(1_G) = \mu(y|_{S_1 \cap S_2})$ ☐

Let $\overline{S}_\tau$ be the set of all memory sets of $\tau$. Then $S_0 = \bigcap_{S \in \overline{S}_\tau} S$ is finite and the minimal memory set of $\tau$.

The following proposition shows that the regularity of the local defining map does not depend on the choice of the memory set:

**Proposition 3.5.** *Let $G$ be a group and let $A$ be an affine algebraic set over a field $K$. Let $\tau : A^G \to A^G$ be a cellular automaton. Then the following conditions are equivalent:*
*$(i)$ there exists a memory set $M$ of $\tau$ such that the associated local defining map $\tau_M : A^M \to A$ is regular;*
*$(ii)$ for any memory set $M$ of $\tau$, the associated local defining map $\tau_M : A^M \to A$ regular.*

*Proof.* Consider the minimal memory set $M_0$ of $\tau$ and fix an arbitrary poing $a_0 \in A$. We have $M_0 \subseteq M$ and $\mu_{M_0} = \mu_M \circ \iota$ ,where $\iota : A^{M_0} \to A^M$ is the embedding defined by

$$\iota(y)(g) = \begin{cases} y(g) & \text{if } g \in M_0 \\ a_0 & \text{if } g \in M \backslash M_0 \end{cases}$$

for all $y \in A^{M_0}$. It follows that the map $\mu_{M_0}$ is regular. On the other hand, if $M'$ is another memory set, $\mu_{M'} = \mu_{M_0} \circ \pi$, where $\pi : A^{M'} \to A^{M_0}$ is the projection map. We deduce that $\mu_{M'}$ is a regular map. ☐

## 3.5 Dynamical Characterization of Residual Finiteness

Recall that the set $A^G$ is equipped with the prodiscrete topology and that $G$ acts on $A^G$ by the left shift defined by (1).

**Proposition 3.6.** *If Group $G$ is residually finite, then for every set $A$, the set of points of $A^G$ which have a finite $G$-orbit is dense in $A^G$.*

**Remark 4.** The other direction of the statement is also correct (see [4, Theorem 2.7.2]).

**Lemma 3.2.** Let $G$ be a group. Let $H_1$ and $H_2$ be subgroups of finite index of $G$. Then the subgroup $H = H_1 \cap H_2$ is of finite index in $G$.

*Proof.* Since "$xH = yH \in G/H$" $\Leftrightarrow$ "$xH_1 = yH_1 \in G/H_1$ and $xH_2 = yH_2 \in G/H_2$", there is an injective map $f : G/H \to G/H_1 \times G/H_2$ giving $gH \mapsto (gH_1, gH_2)$. As the sets $G/H_1$ and $G/H_2$ are finite by hypothesis, we deduce that $G/H$ is finite. ☐

**Lemma 3.3.** Let $G$ be a residually finite group and let $\Omega$ be a finite subset of $G$. Then there exists a normal subgroup of finite index $K$ of $G$ such that the restriction of the canonical homomorphism $\rho : G \to G/K$ to $\Omega$ is injective.

*Proof.* Consider the finite subset $S = \{g^{-1}h : g, h \in \Omega \text{ and } g \neq h\} \subseteq G$. Since $G$ is residually finite, we can find, for every $s \in S$, a normal subgroup of finite index $N_s \subseteq G$ such that $s \notin N_s$. The set $K = \bigcap_{s \in S} N_s$ is a normal subgroup of finite index in $G$ by Lemma 3.2. Let $\rho : G \to G/K$ be the canonical homomorphism. If $g$ and $h$ are distinct elements in $\Omega$, then $g^{-1}h \notin K$ and hence $\rho(g) \neq \rho(h)$. $\square$

*Proof of Propsition 3.6.* Suppose that $G$ is residually finite. Let $A$ be a set and let $W$ be a neighborhood of a point $x$ in $A^G$. Let us show that $W$ contains a configuration with finite $G$-orbit. Consider a finite subset $\Omega \subseteq G$ such that

$$V(x, \Omega) = \{y \in A^G : y|_\Omega = x|_\Omega\} \subseteq W$$

By Lemma 3.3, we can find a normal subgroup of finite index $K \subseteq G$ such that the restriction to $\Omega$ of the canonical homomorphism $\rho : G \to G/K$ is injective. This implies that the map $\phi : A^{G/K} \to A^\Omega$ defined by $\phi(z) = (z \circ \rho)|_\Omega$ is surjective. Thus we can find an element $z_0 \in A^{G/K}$ such that the configurations $z_0 \circ \rho$ and $x$ coincide on $\Omega$, that is, such that $z_0 \circ \rho \in V(x, \Omega)$. On the other hand, the configuration $z_0 \circ \rho$ is $K$-periodic. As $K$ is of finite index in $G$, we deduce that the $G$-orbit of $z_0 \circ \rho$ is finite. Thus $W$ contains a configuration whose $G$-orbit is finite. $\square$

### 3.6 Projective Sequences of Constructible Sets

**Definition 3.1.** A *projective sequence* of sets is a sequence $(X_n)_{n\in\mathbb{N}}$ of sets equipped with maps $f_{nm} : X_m \to X_n$, defined for all $n, m \in \mathbb{N}$ with $m \geq n$, satisfying the following conditions:
(PS-1): $f_{nn}$ is the identity map on $X_n \qquad \forall n \in \mathbb{N}$;
(PS-2): $f_{nk} = f_{nm} \circ f_{mk} \qquad \forall n, m, k \in \mathbb{N}$ with $k \geq m \geq n$.
Observe that the projective sequence $(X_n, f_{nm})$ is entirely determined by the maps $g_n = f_{n,n+1}$ for all $n \in \mathbb{N}$.

**Definition 3.2.** Let $(X_n, f_{nm})$ be a projective sequence of sets. The *projective limit* $X = \varprojlim X_n$ of the projective sequence $(X_n)$ in the subset $X \subseteq \prod_{n\in\mathbb{N}} X_n$ consisting of the sequences $x = (x_n)_{n\in\mathbb{N}}$ satisfying $x_n = f_{nm}(x_m)$ for all $n, m \in \mathbb{N}$ such that $m \geq n$. Note that there is a canonical map $\pi_n : X \to X_n$ sending $x$ to $x_n$ and that one has $\pi_n = f_{nm} \circ \pi_m$ for all $m, n \in \mathbb{N}$ with $m \geq n$.

Property (PS-2) implies that, for each $n \in \mathbb{N}$, the sequence of sets $f_{nm}(X_m), m \geq n$, is non-increasing. Let us set, for each $n \in \mathbb{N}$,

$$X'_n = \bigcap_{m \geq n} f_{nm}(X_m) \qquad \forall n \in \mathbb{N}.$$

The set $X'_n$ is called the set of *universal elements* in $X_n$. Observe that $f_{nm}(X'_m) \subseteq X'_n$ for all $m \geq n$. Thus, the map $f_{nm}$ induces by restriction a map $f'_{nm} : X'_m \to X'_n$ for all $m \geq n$. Then $(X_n, f_{nm})$ is a projective sequence which is called the *universal projective sequence* associated with the projective sequence $(X_n, f_{nm})$. It is clear that the projective sequences $(X_n, f_{nm})$ and $(X_n, f_{nm})$ have the same projective limit. The following result belongs to the prosperous family of Mittag-Leffler type statements. (see e.g. [6, TG II. Section 5], [7, Section I.3])

**Proposition 3.7.** *Let $(X_n, f_{nm})$ be a projective sequence of sets and let $(X'_n, f'_{nm})$ denote the associated universal projective sequence of sets. Let $X = \varprojlim X_n = \varprojlim X'_n$ denote their common projective limit. Suppose that all maps $f'_{nm} : X'_m \to X'_n, m, n \in \mathbb{N}$ and $m \geq n$ are surjective. Then all canonical maps $\pi'_m : X \to X'_n, m \in \mathbb{N}$, are surjective. In particular, if $X'_m \neq \varnothing$ for all $m \in \mathbb{N}$, then one has $X \neq \varnothing$.*

*Proof.* Let $x'_m \in X'_m$. As the maps $f'_{k,k+1}, k \geq m$, are surjective, we can construct by induction a sequence $(x'_k)_{k\geq m}$ such that $x'_k = f'_{k,k+1}(x'_{k+1})$ for all $k \geq m$. Then the sequence $(x_n)_{n\in\mathbb{N}}$ where $x_n = x'_n$ if $n \geq m$ and $x_n = f_{nm}(x'_m)$ if $m \geq n$, is in $X$ and satisfies $x'_m = \pi'_m(x)$. This shows that $\pi'_m$ is surjective. $\square$

**Remark 5.** Observe that, for the maps $f'_{nm}, m \geq n$, to be surjective, it suffices that the maps $f'_{n,n+1}$ are surjective. Also, for the sets $X'_n$ to be nonempty, $n \in \mathbb{N}$, it suffices that the set $X'_0$ is nonempty.

**Definition 3.3.** Let $K$ be a field. We say that a projective sequence $(X_n, f_{nm})$ is a *projective sequence of constructible sets* over $K$ if there is a projective sequence $(A_n, F_{nm})$ consisting of affine algebraic sets $A_n$ over $K$ and regular maps $F_{nm} : A_m \to A_n$ satisfying the following conditions:
(PSC-1) $X_n$ is a constructible subset of $A_n$ for every $n \in \mathbb{N}$;
(PSC-2) $F_{nm}(X_m) \subseteq X_n$ and $f_{nm}$ is the restriction of $F_{nm}$ to $X_m$ for all $m, n \in \mathbb{N}$ such that $m \geq n$.

The following result is an essential ingredient in the proofs of Theorem 1.1 and Theorem 1.2.

**Theorem 3.1.** *Let $K$ be an uncountable algebraically closed field and let $(X_n, f_{nm})$ be a projective sequence of nonempty constructible sets over $K$. Then one has $X = \varprojlim X_n \neq \varnothing$.*

Let us first prove Theorem 3.1 in the particular case where the projective sequence is given by inclusion maps. (cf [2, (CIP) p.127])

**Proposition 3.8.** *Let $K$ be an uncountable algebraically closed field and let $A$ be an affine algebraic set over $K$. Suppose that $(C_n)_{n\in\mathbb{N}}$ is a sequence of nonempty constructible subsets of $A$ such that $C_{n+1} \subseteq C_n$ for all $n \in \mathbb{N}$. Then one has $\bigcap_{n\in\mathbb{N}} C_n \neq \varnothing$.*

We start by establishing two auxiliary results which are valid over any uncountable ground field.

**Lemma 3.4.** Let $K$ be an uncountable (not necessarily algebraically closed) field and let $(Q_n)_{n\in\mathbb{N}}$ be a sequence of nonzero polynomials in $K[t_1, \dots, t_m]$. Then there exists a point $a \in K^m$ such that $Q_n(a) \neq 0$ for all $n \in \mathbb{N}$.

*Proof.* Induction on m. For $m = 1$, Obvious. Suppose now that $m \geq 2$ and that the result is true for polynomials in $m-1$ indeterminates. Let $S$ denote the set of $n \in \mathbb{N}$ such that the indeterminate $t_m$ occurs in $Q_n$. Thus, we have $Q_n \in K[t_1, \dots, t_{m-1}]$ for all $n \in \mathbb{N}\backslash S$. For $n \in S$, let $R_n \in K[t_1, \dots, t_{m-1}]$ denote the coefficient of the highest degree power of $t_m$ occurring in $Q_n$. By our induction hypothesis, we can find $b \in K^{m-1}$ such that $R_n(b) \neq 0$ for all $n \in S$ and $Q_n(b) \neq 0$ for all $n \in \mathbb{N}\backslash S$. As $Q_n(b, t_m)$ is a nonzero polynomial in $t_m$ for all $n \in S$, it follows from the case $m = 1$ that we can find $t \in K$ such that $Q_n(b, t) \neq 0$ for all $n \in S$. Then the point $a = (b, t) \in K^m$ satisfies $Q_n(a) \neq 0$ for all $n \in \mathbb{N}$. □

**Lemma 3.5.** Let $K$ be an uncountable (not necessarily algebraically closed) field and let $(\Omega_n)_{n\in\mathbb{N}}$ be a sequence of nonempty open subsets of $K^m$. Then one has $\bigcap_{n\in\mathbb{N}} \Omega_n \neq \varnothing$.

*Proof.* As the special open subsets form a basis for the Zariski topology on $K^m$, we can find, for each $n \in \mathbb{N}$, a nonzero polynomial $Q_n \in K[t_1, \dots, t_m]$ such that $V_n = K^m\backslash Z(Q_n) = \{a \in K^m : Q_n(a) \neq 0\}$ satisfies $V_n \subseteq \Omega_n$. By Lemma 3.4, we have $\bigcap_{n\in\mathbb{N}} V_n \neq \varnothing$. Consequently, we also have $\bigcap_{n\in\mathbb{N}} \Omega_n \neq \varnothing$. □

*Proof of Proposition 3.8.* we can find, for each $n \in \mathbb{N}$, a nonempty open subset $U_n$ of $A$ such that $U_n \subseteq C_n$. If $A = A_1 \cup A_2 \cup \cdots \cup A_s$ is the decomposition of $A$ into irreducible components, then we have

$$U_n = (U_n \cap A_1) \cup (U_n \cap A_2) \cup \cdots \cup (U_n \cap A_s) \neq \varnothing$$

It follows that we can find an index $1 \leq i \leq s$ and an increasing map $\varphi : \mathbb{N} \to \mathbb{N}$ such that $U_{\varphi(n)} \cap A_i \neq \varnothing$ for all $n \in \mathbb{N}$.

Let $d = \dim(A_i)$. Since $A_i$ is irreducible and the closed subset $F_n = A_i\backslash U_{\varphi(n)}$ is strictly contained in $A_i$, we have $\dim(F_n) < d$. On the other hand, it follows from Theorem 2.2 that we can find a surjective finite morphism $f : A_i \to K^d$. As every finite morphism is closed, the set $L_n = f(F_n)$ is closed in $K^d$. We have $\dim(L_n) \leq \dim(F_n) < d$ and therefore $L_n \neq K^d$ for all $n \in \mathbb{N}$. By Lemma 3.5, the nonempty open subsets $\Omega_n = K^d\backslash L_n \subseteq K^d$ satisfy $\bigcap_{n\in\mathbb{N}} \Omega_n \neq \varnothing$. As

$$f\Big(\bigcup_{n\in\mathbb{N}} F_n\Big) = \bigcup_{n\in\mathbb{N}} f(F_n) = \bigcup_{n\in\mathbb{N}} L_n = K^d\backslash \bigcap_{n\in\mathbb{N}} \Omega_n$$

it follows that $f(\bigcup_{n\in\mathbb{N}} F_n) \neq K^d$. As $f$ is surjective, this implies that $\bigcup_{n\in\mathbb{N}} F_n \neq A_i$ and hence $\bigcap_{n\in\mathbb{N}} U_{\varphi(n)} \neq \varnothing$. Since $U_{\varphi(n)} \subseteq C_{\varphi(n)} \subseteq C_n$ for all $n \in \mathbb{N}$, we conclude that $\bigcap_{n\in\mathbb{N}} C_n \neq \varnothing$. □

**Remark 6.** Proposition 3.8 becomes false when the ground field $K$ is countable even if $K$ is algebraically closed (e.g., when K is the algebraic closure of either $\mathbb{Q}$, or of the field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ of cardinality $p$ where $p$ is a prime number). Indeed, if $K$ is countable, say $K = \{a_n : n \in \mathbb{N}\}$, then the sequence of constructible subsets $C_n = K\backslash\{a_0, a_1, \dots, a_n\} \subseteq K$ $(n \in \mathbb{N})$ has an empty intersection.

*Proof of Theorem 3.1.* Let $(A_n, F_{nm})$ be a projective sequence of affine algebraic sets and regular maps satisfying conditions (PSC-1) and (PSC-2) above. Let $(X'_n, f'_{nm})$ denote the universal projective sequence associated with the projective sequence $(X_n, f_{nm})$. For all $m \neq n$, the image set $f_{nm}(X_m) = F_{nm}(X_m)$ is a constructible subset of $A_n$ by Chevalley's theorem (Theorem 2.1). As the sequence $f_{nm}(X_m), m =$

$n, n+1, \ldots$ is a non-increasing sequence of nonempty constructible subsets of the affine algebraic set $A_n$, we deduce from Proposition 3.8 that

$$X'_n = \bigcap_{m \geq n} f_{nm}(X_m) \neq \varnothing. \qquad \forall n \in \mathbb{N}$$

Thus, by Proposition 3.7, it suffices to show that all maps $f'_{nm}, m \geq n$, are surjective. To see this, let $m, n \in \mathbb{N}$ with $m \geq n$ and suppose that $x'_n \in X'_n$. Then, for all $k \geq n$, we have $x'_n \in f_{nk}(X_k)$ so that we can find $y_k \in X_k$ such that $f_{nk}(y_k) = x'_n$. For $k \geq m$, the element $z_k = f_{mk}(y_k)$ satisfies $f_{nm}(z_k) = f_{nm} \circ f_{mk}(y_k) = f_{nk}(y_k) = x'_n$. We deduce that $f_{nm}^{-1}(x'_n) \cap f_{mk}(X_k) \neq \varnothing$ for all $k \geq m$. Now observe that $f_{nm}^{-1}(x'_n) \cap f_{mk}(X_k)$ is constructible in $A_m$. Indeed, $f_{nm}^{-1}(x'_n) = F_{nm}^{-1}(x'_n) \cap X_m$, is constructible in $A_m$ since it is the intersection of a closed subset with a constructible subset of $A_m$, and $f_{mk}(X_k) = F_{mk}(X_k)$ is constructible in $A_m$ by Chevalley's theorem (Theorem 2.1). By applying again Proposition 3.8, we deduce that

$$f_{nm}^{-1}(x'_n) \supseteq \bigcap_{k \geq m} \left( f_{nm}^{-1}(x'_n) \cap f_{mk}(X_k) \right) \neq \varnothing.$$

Consequently, the map $f'_{nm} : X'_m \to X'_n$ is surjective. $\square$

# 4 The Last Few Steps to Our Theorems

We shall use the abundant results above to prove Theorem 1.1 and Corollary 1.1.

*Proof of Theorem 1.1.* Let $\tau : A^G \to A^G$ be an algebraic cellular automaton. Let $M \subseteq G$ be a memory set for $\tau$ and let $\mu : A^M \to A$ denote the associated local defining map. We discuss in two cases.
(i) if group $G$ is countable. Then we can find a sequence $(E_n)_{n \in \mathbb{N}}$ of finite subsets of $G$ such that $G = \bigcup_{n \in \mathbb{N}} E_n, M \subseteq E_0$, and $E_n \subseteq E_{n+1}$ for all $n \in \mathbb{N}$. Consider, for each $n \in \mathbb{N}$, the finite subset $F_n \subseteq G$ defined by $F_n = \{g \in G : gM \subseteq E_n\} = \bigcap_{m \in M} E_n m^{-1} \supseteq E_n$. Note that $G = \bigcup_{n \in \mathbb{N}} F_n, 1_G \in F_0$, and $F_n \subseteq F_{n+1}$ for all $n \in \mathbb{N}$.

It follows from formula (2) that if $x, x' \in A^G$, such that $x|_{E_n} = x'|_{E_n}$ then $\tau(x)|_{F_n} = \tau(x')|_{F_n}$. Therefore, we can define a map $\tau_n : A^{E_n} \to A^{F_n}$ by setting $\tau_n(u) = (\tau(x))|_{F_n}$ for all $u \in A^{E_n}$, where $x \in A^G$ denotes an arbitrary configuration extending $u$. Observe that both $A^{E_n}$ and $A^{F_n}$ are affine algebraic sets as they are finite Cartesian powers of the affine algebraic set $A$. Moreover, the map $\tau_n$ is regular from formula (2) and the fact that $\mu : A^M \to A$ is regular.

Let $y \in A^G$ and suppose that $y \in \overline{\tau(A^G)}$. Then it follows from the expression (3) that for all $n \in \mathbb{N}$, we can find $z_n \in A^G$ such that $y|_{F_n} = (\tau(z_n))|_{F_n}$. Consider, for each $n \in \mathbb{N}$, $X_n = \tau_n^{-1}(y|_{F_n}) \subseteq A^{E_n}$ is a nonempty algebraic set. Observe that, for all $m \geq n$, the restriction map $A^{E_m} \to A^{E_n}$ induces a regular map $f_{nm} : X_m \to X_n$. Conditions (PS-1) and (PS-2) are trivially satisfied so that $(X_n, f_{nm})$ is a projective sequence of nonempty constructible (in fact, algebraic) sets. By Theorem 3.1, we have $\varprojlim X_n \neq \varnothing$. Choose an element $(x_n)_{n \in \mathbb{N}} \in \varprojlim X_n$. Thus $x_n \in A^{E_n}$ and $x_{n+1}|_{E_n} = x_n|_{E_n}$ for all $n \in \mathbb{N}$. As $G = \bigcup_{n \in \mathbb{N}} E_n$, we deduce that there exists a (unique) configuration $x \in A^G$ such that $x|_{E_n} = x_n$ for all $n \in \mathbb{N}$. Moreover, we have $\tau(x)|_{F_n} = \tau_n(x_n) = y_n = y|_{F_n}$ for all $n \in \mathbb{N}$. As $G = \bigcup_{n \in \mathbb{N}} F_n$, this shows that $\tau(x) = y$. This completes the proof that $\tau$ has the closed image property in the case when $G$ is countable.
(ii)In general, for arbitrary (possibly uncountable) group $G$. Let $H$ denote the subgroup of $G$ generated by $M$. Observe that $H$ is countable since $M$ is finite. The restriction cellular automaton $\tau_H : A^H \to A^H$ is algebraic by Propostion 3.4 (iv). Thus, by the first part of the proof, $\tau_H$ has the closed image property, that is $\tau_H(A^H)$ is closed in $A^H$ for the prodiscrete topology. By applying Proposition 3.4 (iii), we deduce that $\tau(A^G)$ is also closed in $A^G$ for the prodiscrete topology. Thus $\tau$ has the closed image property. $\square$

Let $A, M, N$ be sets. We recall that, given a map $\rho : M \to N$, it induces a map $\rho^* : A^N \to A^M$ defined by $\rho^*(y) = y \circ \rho$ for all $y \in A^N$.

**Lemma 4.1.** Let $K$ be a field and let $A$ be an affine algebraic set over $K$. Suppose that we are given a map $\rho : M \to N$, where $M$ and $N$ are finite sets. Then the induced map $\rho^* : A^N \to A^M$ is regular.

*Proof.* We have $\rho^*(y)(m) = y(\rho(m))$ for all $m \in M$ and $y \in A^N$. It follows that each coordinate map of $\rho^*$ is one of the projection maps $A^N \to A$ and is therefore regular. Consequently, $\rho^*$ is regular. $\square$

Let $G$ be a group and let $A$ be a set. Suppose that $H$ is a subgroup of $G$. Consider the set $H\backslash G$ and the canonical surjection $\rho_H : G \to H\backslash G$ taking $g \mapsto Hg$. We have that the map $\rho_H^* : A^{H\backslash G} \to \mathrm{Fix}(H)$ is bijective (see Proposition 3.2). Observe now that if $\tau : A^G \to A^G$ is a cellular automaton, then one has $\tau(\mathrm{Fix}(H)) \subseteq \mathrm{Fix}(H)$ since $\tau$ is $G$-equivariant. We denote by $\tau_H : \mathrm{Fix}(H) \to \mathrm{Fix}(H)$ restricted by $\tau$, and by $\widetilde{\tau_H} : A^{H\backslash G} \to A^{H\backslash G}$ the conjugate of $\tau$ by $\rho_H^*$ (see formula (5)).

*Proof of Corollary 1.1.* Suppose that $\tau : A^G \to A^G$ is an injective algebraic cellular automaton. Denote by $\mathcal{F}$ the set of all finite index subgroups of $G$.

Let $H \in \mathcal{F}$ and let $\rho_H : G \to H\backslash G$ taking $g \mapsto Hg$. Then $H\backslash G$ is finite. We claim that the map $\widetilde{\tau_H} : A^{H\backslash G} \to A^{H\backslash G}$ is regular. To see this, it suffices to prove that, for each $g \in G$, the map $\pi_g : A^{H\backslash G} \to A$ defined by $\pi_g(y) = \widetilde{\tau_H}(Hg)$ is regular. Choose a memory set $M$ for $\tau$ and let $\mu : A^M \to A$ denote the associated local defining map. Consider the map $\psi : M \to H\backslash G$ defined by $\psi(m) = \rho_H(gm)$ for all $m \in M$ and the induced map $\psi^* : A^{H\backslash G} \to A^M$. Then we have $\pi_g = \mu \circ \psi^*$. The map $\mu$ is regular since $\tau$ is algebraic. On the other hand, $\psi^*$ is regular by Lemma 4.1. It follows that $\pi_g$ is regular, which proves our claim. Now observe that $\tau_H : \mathrm{Fix}(H) \to \mathrm{Fix}(H)$ is injective since it is the restriction of $\tau$. As $\widetilde{\tau_H}$ is conjugate to $\tau_H$, we deduce that $\widetilde{\tau_H}$ is injective as well. It follows that $\widetilde{\tau_H}$ is surjective by the Ax-Grothendieck theorem (Theorem 2.3). Thus, $\tau_H$ is also surjective and hence $\mathrm{Fix}(H)=\tau_H(\mathrm{Fix} H) \subseteq \tau(A^G)$.

Let $E \subseteq A^G$ denote the set of configurations whose orbit under the $G$-shift is finite. Then we have

$$E = \bigcup_{H \in \mathcal{F}} \mathrm{Fix}(H) \subseteq \tau(A^G).$$

On the other hand, it follows from Proposition 3.6 that the residual finiteness of $G$ implies that $E$ is dense in $A^G$. As $\tau(A^G)$ is closed in $A^G$ by Theorem 1.1, we conclude that $\tau(A^G) = A^G$. $\square$

*Proof of Theorem 1.2.* Let $\tau : A^G \to A^G$ be a bijective algebraic cellular automaton. We have to show that the inverse map $\tau^{-1} : A^G \to A^G$ is a cellular automaton.
(i) If group $G$ is countable. Let us show that the following local property is satisfied by $\tau^{-1}$ :

- $(*)$ there exists a finite subset $N \subseteq G$ such that, for any $y \in A^G$, the element $\tau^{-1}(y)(1_G)$ only depends on the restriction of $y$ to $N$.

This will show that $\tau$ is reversible. Indeed, if $(*)$ holds for some finite subset $N \subseteq G$, then there exists a (uniqe) map $\nu : A^N \to A$ such that $\tau^{-1}(y)(1_G) = \nu(y|_N)$ for all $y \in A^G$. Now, the $G$-equivariance of $\tau$ implies the $G$-equivaiance of $\tau^{-1}$ (since $\tau(gx) = g\tau(x) \Rightarrow g\tau^{-1}(y) = \tau^{-1}(gy)$, for $y = \tau(x)$). Consequently, we get

$$\tau^{-1}(y)(g) = g^{-1}\tau^{-1}(y)(1_G) = \tau^{-1}(g^{-1}y)(1_G) = \nu((g^{-1}y)|_N) \qquad \forall y \in A^G, g \in G$$

which implies that $\tau^{-1}$ is the cellular automaton with memory set $N$ and local defining map $\nu$.

If the condition $(*)$ is not satisfied. Let $M$ be a memory set $\tau$ such that $1_G \in M$. Since $G$ is countable, it follows the same construction process of proof in the theorem 1.1 that $G = \bigcup_{n\in\mathbb{N}} E_n = \bigcup_{n\in\mathbb{N}} F_n$ with the associated relation between $E_n$ and $F_n$ such that $M \subseteq E_0$ and $1_G \in F_0$.

Since $(*)$ is not satisfied, we can find, for each $n \in \mathbb{N}$, two configurations $y'_n, y''_n \in A^G$ such that $y'_n|_{F_n} = y''_n|_{F_n}$ and $\tau^{-1}(y'_n)(1_G) \neq \tau^{-1}(y''_n)(1_G)$. Recall from the proof of Theorem 1.1, that $\tau$ induces, for each $n \in \mathbb{N}$, a regular map $\tau_n : A^{E_n} \to A^{F_n}$ given by $\tau_n(u) = (\tau(x))|_{F_n}$ for every $u \in A^{E_n}$, where $x \in A^G$ is any configuration extending $u$.

Consider, for each $n \in \mathbb{N}$, the subset $X_n \subseteq A^{E_n} \times A^{E_n}$ consisting of all pairs $(u, v) \in A^{E_n} \times A^{E_n}$ such that $\tau_n(u) = \tau_n(v)$ and $u(1_G) \neq v(1_G)$. Note that $X_n$ is locally closed and hence constructible in the affine algebraic set $A^{E_n} \times A^{E_n}$ for the Zariski topology since it is the intersection of a closed subset with an open set. Note also that $X_n$ is not empty, since $((\tau^{-1}(y'_n))|_{E_n}, (\tau^{-1}(y''_n))|_{E_n}) \in X_n$. Observe that , for $m \geq n$, the restriction map $\rho_{nm} : A^{E_m} \to A^{E_n}$ gives us a regular map $\pi_{nm} = \rho_{nm} \times \rho_{nm} : A^{E_m} \times A^{E_m} \to A^{E_n} \times A^{E_n}$, which induces by restriction a map $f_{nm} : X^m \to X^n$. Conditions (PS-1) and (PS-2) are trivially satisfied, so that $(X_n, f_{nm})$ is a projective sequence of nonempty constructible sets. Thus, we have $\varprojlim X_n \neq \varnothing$ by Theorem 3.1. Choose an element $(p_n)_{n\in\mathbb{N}} \in \varprojlim X_n$. Thus $p_n = (u_n, v_n) \in A^{E_n} \times A^{E_n}$

an $u_{n+1}|_{E_n} = u_n|_{E_n}$ (resp. $v_{n+1}$ and $v_n$) for all $n \in \mathbb{N}$. As $G = \bigcup_{n\in\mathbb{N}} E_n$, there exists a unique configuration $x' \in A^G$ (resp. $x'' \in A^G$) such that $x'_n|_{E_n} = u_n$ (resp. $x''|_{E_n} = v_n$) for all $n \in \mathbb{N}$. Moreover, we have

$$(\tau(x'))|_{F_n} = \tau_n(u_n) = \tau_n(v_n) = (\tau(x''))|_{F_n} \qquad \forall n \in \mathbb{N}$$

As $G = \bigcup_{n\in\mathbb{N}} F_n$, this shows that $\tau(x') = \tau(x'')$. On the other hand, we have $x'(1_G) = u_0(1_G) \neq v_0(1_G) = x''(1_G)$ and hence $x' \neq x''$. This contradicts the injectivity of $\tau$ and therefore completes the proof that $\tau$ is reversible in the case when $G$ is countable.

(ii)In general, for arbitrary (possibly uncountable) group $G$. Choose a memory $M \subseteq G$ for $\tau$ and denote by $H$ the subgroup of $G$ generated by $M$. Thus $H$ is countable. Observe that $H$ is countable since $M$ is finite. By assertions (iv) and (i) of Proposition 3.4, the restriction cellular automaton $\tau_H : A^H \to A^H$ is algebraic and bijective. It then follows from the first part of the proof that $\tau_H$ is reversible. This implies that $\tau$ is reversible as well by assertion (ii) of Propostion 3.4. $\square$

**Remark 7.** Under the hypothesis of the proceding theorem, it may happen that the inverse cellular autamaton is not algebraic. For example, Let $K$ be an uncountable algebraically closed field of characteristic $p > 0$ and consider the Frobenius automorphism $f : K \to K$ given by $\lambda \mapsto \lambda^p$. Then the map $\tau : K^G \to K^G$, defined by $\tau(x)(g) = f(x(g)) \quad \forall x \in K^G, \forall g \in G$, is a bijective algebraic cellular automaton with memory set $1_G$, Therefore $\tau^{-1}$ is not algebraic since $f^{-1}$ is not polynomial.

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