Notes on Complex Geometry

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1 The Frobenius theorem

Definition 1.1. Let X be a n−dimensional manifold, and $E \subseteq T_X$ be a $\mathcal{C}^1$ vector sub-bundle of rank k. Such an E is called a **distribution** of X. A distribution E is called **integrable** if locally there exists a $\mathcal{C}^1$ map $\phi(U) : U \to \mathbb{R}^{n-k}$ such that every fiber E_x is identified with $\operatorname{Ker} \mathrm{d}\phi_x$.

Theorem 1.1 (Frobenius)**.** *A distribution E is integrable if and only if for all vector fields $\chi, \psi \in \Gamma(X, E)$, the bracket $[\chi, \psi] \in \Gamma(X, E)$.*

Proof. "⇒": if integrable, then locall $U \subseteq X, \Gamma(U, E)$ consists of vector fields χ that annihilate the functions $f_i = x_i \circ \phi$, where ϕ is defined above. If χ, ψ annihilate f_i, then the bracket $[\chi, \psi](f_i) = 0$.
"⇐": We prove this by induction on k. If $k = 1$, the result is implied by the Flow–Bow theorem which states that a nontrivial $\mathcal{C}^1$ vector field is locally diffeomorphic to a nontrivial constant vector field. Let $U \subseteq X$ open, and admits a non-trivial section χ of E over U, and a sub-mersion $\phi : U \to \mathbb{R}^{n-1}$ whose fibers are the trajectories of χ. By the Flow–Box theorem, we may assume that U is diffeomorphic to $(0, 1) \times V$, V is an open set of $\mathbb{R}^{n-1}$, that ϕ is the second projection and $\chi = \frac{\partial}{\partial t}$. The following lemma is essential.

Lemma 1.1. The integrability condition implies that there exists a distribution F of rank $k - 1$ on V such that $E = (\phi_*)^{-1}(F)$. Moreover, E satisfies the integrability condition iff F does.

Here $\phi_* : T_{U,x} \to T_{V,\phi(x)}$ is the differential of ϕ. Admitting the lemma, this gives locally $\psi : V \to \mathbb{R}^{n-k}$ whose fibers are integral manifolds of the distribution F. Then the fibers of $\psi \circ \phi : U \to \mathbb{R}^{n-k}$ are integral manifolds of the distribution E. □

Proof of Lemma 1.1. At each point $x \in U$, the differential $\phi_* : T_{U,x} \to T_{V,\phi(x)}$ defines a vector subspace $F_x = \phi_*(E_x) \subseteq T_{V,\phi(x)}$ of rank $k - 1$, since $\langle \chi \rangle = \operatorname{Ker}\phi_*$ is contained in E. Obviously, $E_x = (\phi_*)^{-1}(F_x)$. To show define a sub-bundle F, We want to show that F_x depends on the point $\phi(x)$, not on the choice of the point x in the fibre $\phi^{-1}(\phi(x))$, i.e. $K_t := \phi_* E_{t,v} \subseteq T_{V,v}$ is independent of t. For this, we need the following lemmata.

Lemma 1.2. Let $K \subseteq (0, 1) \times \mathbb{R}^m$ be a differentiable vector sub-bundle of rank k of the trivial bundle of rank m over $(0, 1)$. If $\forall \sigma \in \Gamma(K)$, $\dfrac{\mathrm{d}\sigma}{\mathrm{d}t} \in K$, then K is trivial bundle.

To apply this lemma, consider the sub-bundle $K = \phi_* E \subseteq (0, 1) \times T_{V,v}$ over the fibre $(\phi)^{-1}(v) = (0, 1) \times \{v\}$. We now check the following lemma

Lemma 1.3. Let $\sigma \in \Gamma(E)$ over $(0, 1) \times \{v\}$, then $\phi_*(\sigma) \in \Gamma(K)$ and we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=t_0} (\phi_*(\sigma)) = \phi_{*,(t_0,v)}([\chi, \tilde{\sigma}])$$

in $T_{V,v}$, where $\tilde{\sigma} \in \Gamma(E)$ over $(0, 1) \times V$ extending σ.

When the integrability condition is satisfied, we can define a sub-bundle F of T_V by $F_v := \phi_* E_{t,v} \subseteq T_{V,v}$ for arbitrary $t \in (0, 1)$. It suffices to see the sub-bundle F constructed in this way satisfied the integrability condition. Now for every $\sigma, \rho \in \Gamma(V, F)$, there exist $\tilde{\sigma}, \tilde{\rho} \in \Gamma((0, 1) \times V, E)$ extending σ and ρ. Therefore $[\sigma, \rho]_v = \phi_{*,(t,v)}([\tilde{\sigma}, \tilde{\rho}]_{t,v})$, hence F satisfies the integrability condition. □

2 The Newlander–Nirenberg theorem

Theorem 2.1 (Newlander–Nirenberg)**.** *The almost complex structure $\mathcal{J}$ is integrable if and only if we have $[T_X^{0,1}, T_X^{0,1}] \subseteq T_X^{0,1}$.*

Remark 2.1. By passing to the conjugate, this is equivalent to the condition that the bracket of two vector fields of type $(1,0)$ is of type $(1,0)$.

Following Weil (1957), we assume $(X, \mathcal{J})$ are real analytic, then this theorem follows from the following analytic version of the Frobenius theorem 1.1.

Theorem 2.2. *Let X be a complex manifold of dimension n, and let E be a holomorphic distribution of rank k over X. Then E is integrable in the holomorphic sense if and only if we have the integrability condition $[E, E] \subseteq E$.*

Here the integrability in the holomorphic sense means that X is covered by open sets U such that there exists a holomorphic submersive map $\phi_U : U \to \mathbb{C}^{n-k}$ satisfying $E_u = \mathrm{Ker}(\phi_* : T_{U,u} \to T_{\mathbb{C}^{n-k},\phi(u)})$ for every $u \in U$.

Proof of Theorem 2.2. First reduce to the real Frobenius theorem, then the real distribution $\Re E \subseteq T_{X,\mathbb{R}}$ also satisfies the Frobenius integrability condition, hence is integrable. Assume X is covered by open sets U such that there exists a submersion $\phi_U, : U \to V$, where V is open in $\mathbb{R}^{2(n-k)}$, satisfying $(\Re E)_u = \mathrm{Ker}(\phi_{*,u} : T_{U,u,\mathbb{R}} \to T_{\mathbb{R}^{2(n-k)},\phi(u)})$, $\forall u \in U$.

We want to show there exists a complex structure on the image of ϕ for which ϕ is holomorphic. Then we may $\mathbb{C}-$ linearly extend $\mathrm{Ker}(\phi_{*,u}) = (\Re E)_u$ to $E_u \subseteq T_{U,u}$ the holomorphic tangent space. If $v = \phi(u), T_{V,v} = T_{U,u}/(\Re E)_u$ and as $\Re E$ is stable under the endomorphism $\mathfrak{J}$ corresponding to the almost complex structure on T_U, there is an induced complex structure on $T_{V,v}$. We want to show this complex structure does not depend on the point in the fibre of ϕ above v. Hence there exists an almost complex structure on T_V for which $\mathrm{d}\phi$ is $\mathbb{C}-$linear at every point. Finally, we take a complex submanifold of U transverse to the fibres of ϕ_U, which exists up to restricting U. Via ϕ_U, this submanifold becomes locally isomorphic to V, and this isomorphism is compatible with the almost complex structures. Thus the alomost complex structure on $\mathcal{J}\phi_U$ is integrable, it makes ϕ_U into a holomorphic map. ☐

Proof of Theorem 2.1 in the real analytic case. Since everything is local, we may assume X is an open set $U \subseteq \mathbb{R}^{2n}$ and that $\mathcal{J}$ is a real analytic map with values in $\mathrm{End}(\mathbb{R}^{2n})$, satisfying $\mathcal{J}^2 = -Id$. Up to restricting U, we may assume that $\mathcal{J}$ is given by a convergent power series and gives a holomorphic map $\mathcal{J}$ from an open set $U_{\mathbb{C}} \subseteq \mathbb{C}^{2n}$ to $\mathrm{End}(\mathbb{C}^{2n})$, which satisfies $\mathcal{J}^2 = -Id$. Now we define $E_{\mathbb{C},u} = T_{U_{\mathbb{C}},u}^{0,1} \cong \mathbb{C}^{2n}$ to be the eigenspace associated to the eigenvalue $-i$ of $\mathcal{J}$, and $E_{\mathbb{C}}$ is a holomorphic distribution of $\mathrm{rank}_{\mathbb{C}} = n$.

Similarly, $\Gamma(U_{\mathbb{C}}, T_{U_{\mathbb{C}}}^{0,1})$ are generated over $\mathbb{C}$ by the $\chi + i\mathcal{J}\chi$, where χ is a real or complex vector field on $U_{\mathbb{C}}$. If the integrability condition for $T_X^{0,1}$ holds in the Theorem 2.1, then the holomorphic distribution $E_{\mathbb{C}}$ satisfies the integrability condition of Theorem 2.2. The distributioin $E_{\mathbb{C}}$ is thus integrable, which locally gives a holomorphic submersion $\phi_{\mathbb{C}} : U_{\mathbb{C}} \to \mathbb{C}^n$, whose fibres are the integral holomorphic submanifolds of the distribution $E_{\mathbb{C}}$.

We want to show the restriction of ϕ to U is a local diffeomorphism. Indeed, along U, $T_{U,u} \subseteq T_{U_{\mathbb{C}},u} \cong \mathbb{C}^{2n}$ can be identified with $\mathbb{R}^{2n}$, while $\Re E_{\mathbb{C}}$ can be identified with $T_{U,u}^{0,1}$. These two spaces are thus transvere. Hence $\phi_{\mathbb{C},*}|_{T_U}$ is an isomorphism, so that $\phi|_U$ is a diffeomorphism in the neighborhood of u.

Second, we show that the complex structure induced on U has an associated almost complex structure given precisely by $\mathcal{J}$. We need to check that the isomorphism $\phi_{\mathbb{C},*} : T_{U,u} \to T_{\mathbb{C}^n,\phi(u)}$ identifies $\mathfrak{J}$ with the complex structure operator on $\mathbb{C}^n$. Check that the isomorphism comes from the following composition:

$$T_{U,u} \hookrightarrow T_{U_{\mathbb{C}},u} \to T_{U_{\mathbb{C}},u}/(\Re E_{\mathbb{C}})_u$$

and $\Re E_{\mathbb{C},u} \subseteq T_{U_{\mathbb{C}},u}^{0,1}$ is generated by $\chi + i\mathfrak{J}\chi$, where $\chi \in \Gamma(T_U)$. Therefore, we have $\chi = -i\mathfrak{J}\chi$ for $\chi \in T_{U,u}$, i.e. $i\chi = \mathfrak{J}\chi$.

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References

[1] Claire Voisin, *Hodge Theory and Complex Algebraic Geometry I*.