Notes on Lie Groups

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1 Lie groups and Lie algebras

Remark 1.1. The space of all vector fields is an $\infty-$dimension Lie algebra, which is te Lie algebra of $\infty-$dimension Lie group $\mathrm{Diff}(M) := \{f : M \to M \,|\, f \text{ is a diffeomorphism}\}$.

Let X be a left-invariant vector field on G, integral lines/flow lines for X always exist (First locally exists, then use the property of left-invariance to extend the local solution). Then $\psi : (\mathbb{R}, +) \to G$ via $t \to \psi_t(e)$ is a group homomorphism.

Example 1.1. Let $G = SO(3)$, then $\mathfrak{g} = \mathfrak{so}(3)$, define

$$\varphi_t(e) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(t) & \sin(t) \\ 0 & -\sin(t) & \cos(t) \end{pmatrix} \quad \text{then } X = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(t) & \sin(t) \\ 0 & -\sin(t) & \cos(t) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \in \mathfrak{g}$$

Example 1.2. Let $G = GL(n, \mathbb{R}) \subseteq M_{n\times n}(\mathbb{R})$, then $T_I GL(n, \mathbb{R}) = M_{n\times n}(\mathbb{R})$. The 1-parameter group of $X \in T_I GL(n, \mathbb{R})$ is $\varphi^X(t) = e^{tX} = \sum_{n=0}^{\infty} \dfrac{(tX)^n}{n!}$.

Remark 1.2. $e^X = \sum_{n=0}^{\infty} \dfrac{X^n}{n!}$ convergent absolutely since $||A^n|| \leqslant ||A||^n$. And $\det(e^A) = e^{trA}$.

Definition 1.1. $X \in \mathfrak{g} = T_e G$, let $\varphi^X(t)$ to be the 1-parameter group of X, define $e^X := \varphi^X(1)$, easy to see that $e^{sX} = \varphi^X(s)$.

Now consider $\exp : \mathfrak{g} \to G$, $\mathrm{d}(\exp)_0 : \mathfrak{g} \to \mathfrak{g}$, then $\mathrm{d}(\exp)_0(w) = \frac{\partial}{\partial s}\Big|_{s=0} \exp(0 + sw) = w$. Therefore $\mathrm{d}(\exp)_0 = Id_{\mathfrak{g}}$. The inverse function theorem implies the following important consequence.

Proposition 1.1. $\exp$ *is a diffeomorphism of a neighborhood of* 0 *to a neighborhood of* $e \in G$.

Remark 1.3. There exists a local inverse for exp, denoted as "log " on a neighborhood of $e \in G$. There exists a (semi)-explicit formular for log: "Bakes-Campbell-Hausdorff" theorem

$$\log(e^X \cdot e^Y) = X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}[X, [X, Y]] + \frac{1}{12}[Y, [X, Y]] + \text{ higer brackets.}$$

In particular, if $\mathfrak{g}$ is abelian, then $\log(e^X \cdot e^Y) = X + Y$.

Remark 1.4. If $\mathfrak{g}$ is abelian, G is connected, then G is abelian, vice versa. (exp is a diffeomorphism at 0, there is a neighborhood U of e such that U is abelian, then G is generated by U given that G connected).

If $\mathfrak{g}$ is abelian, G connected, then $\exp : \mathfrak{g} \to G$ is onto and group homomorphism. Therefore, $G \cong \mathfrak{g}/\mathrm{Ker}(\exp)$. Ker(exp) is 0-dimensional and intersects with some neighborhood of 0 is 0. Therefore Ker(exp) is a discrete subgroup of $(\mathfrak{g}, +)$, i.e. a lattice $\cong \mathbb{Z}^r$, where $r \leqslant \dim\mathfrak{g}$.

As a result, if G is abelian, then $G \cong \mathbb{R}^r \times \mathbb{R}^{n-r}/\mathbb{Z}^r \cong T^r \times \mathbb{R}^{n-r}$ for $0 \leqslant r \leqslant n$. In particular, if G is compact, then G is torus.

In general, $\exp : \mathfrak{g} \to G$ is not a global diffeomorphism.

Example 1.3. $U(1) = \{z \in \mathbb{C} \,|\, z\bar{z} = 1\} \cong S^1$ and $\mathfrak{u}(1) = \{z \in \mathbb{C} \,|\, z + \bar{z} = 0\} \cong i\mathbb{R}$. Hence not a global diffeomorphism.

Surjectiveity fails in general.

Example 1.4. $\exp : \mathfrak{sl}(2, \mathbb{R}) \to SL(2, \mathbb{R})$ is not surjective. But exp is surjective if G is compact and connected.

Let $\psi : G \to H$ be a smooth Lie group homomorphism, the 1-parameter groups are homomorphisms $(\mathbb{R}, +) \to G$, the image of a 1-parameter group in G is also a 1-parameter group in H. And the following diagram commutes

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\mathrm{d}\psi_e} & \mathfrak{h} \\ {\scriptstyle \exp}\downarrow & & \downarrow{\scriptstyle exp} \\ G & \xrightarrow{\psi} & H \end{array}$$

There is a "natural" homomorphism associated to a Lie group G. $\forall g \in G$, define $c_g : G \to G$ via $x \mapsto g{\cdot}g^{-1}$. Now c_g fixes e, so $\mathrm{d}c_g : \mathfrak{g} \to \mathfrak{g}$. Note that $c_g \circ c_h = c_{g \cdot h}$ so $c_g \circ c_{g^{-1}} = Id_G$ hence $\mathrm{d}c_g$ is invertible.

$\mathrm{d}c_g \circ \mathrm{d}c_h = dc_{g \cdot h}$ gives a group homomorphism of Lie groups $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$ called adjoint map.

Remark 1.5. This is an example of representation of G. $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$.

Example 1.5. If G is a matrix group, Ad is just the conjugation of matrices in $\mathfrak{g}$.

$$g(I + h + \cdots)g^{-1} = I + ghg^{-1} + \cdots$$

- $SO(n) := \{A \in M_{n \times n}(\mathbb{R}) \,|\, A^T A = I\}$ $\quad \mathfrak{so}(n) := \{h \in M_{n \times n}(\mathbb{R}) \,|\, h^T + h = 0\}$. Note that

$$(AhA^{-1})^T = (AhA^T)^T = Ah^T A^T = -AhA^T = -AhA^{-1} \in \mathfrak{so}(n).$$

Differentiate $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g}) = GL(\mathfrak{g})$, then we have $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$(:= all linear maps $\mathfrak{g} \to \mathfrak{g}$). Calculation: $\mathrm{Ad}(g) = \mathrm{d}(c_g)_e$. Let $g = (1 + h + \cdots)$. $\forall x \in \mathfrak{g}$,

$$g \cdot x \cdot g^{-1} = (1 + h + \cdots) \cdot x \cdot (1 + h + \cdots)^{-1} = (1 + h + \cdots) \cdot x \cdot (1 - h + \cdots) = 1 + [h, x] + \cdots$$

Therefore $\mathrm{ad}(h) : x \mapsto [h, x]$. It implies the following commutative diagram

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\mathrm{ad}} & \mathrm{End}(\mathfrak{g}) \\ {\scriptstyle \exp}\downarrow & & \downarrow{\scriptstyle \exp} \\ G & \xrightarrow{\mathrm{Ad}} & \mathrm{Aut}(\mathfrak{g}) \end{array}$$

Remark 1.6. Let $\varphi : G \to H$ be a Lie group homomorphism, so $\varphi \circ c_g = c_{\varphi(g)}$, and $X\mathfrak{g}, \exp(X) = g \in G$

$$(\mathrm{d}\varphi)_e \circ (\mathrm{d}c_g)_e = (\mathrm{d}c_{\varphi(g)})_e \circ (\mathrm{d}\varphi)_e, \quad (\mathrm{d}\varphi)_e \circ \mathrm{Ad}(g) = \mathrm{Ad}(\varphi(g)) \circ (\mathrm{d}\varphi)_e.$$

Differentiate at g, then we have

$$(\mathrm{d}\varphi)_e \circ \mathrm{ad}(X) = \mathrm{ad}((\mathrm{d}\varphi)_e X) \circ (\mathrm{d}\varphi)_e.$$

Apply to $Y \in \mathfrak{g}$ and $(\mathrm{d}\varphi)_e[X, Y] = [(\mathrm{d}\varphi)_e(X), (\mathrm{d}\varphi)_e(Y)]$.

Remark 1.7. If G is abelian, then c_g is trivial. therefore $\mathrm{Ad}(g) = Id \in \mathrm{Aut}(\mathfrak{g})$, i.e. $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$. This implies $\mathrm{ad} \equiv 0$, i.e. $\mathfrak{g}$ is abelian. Recall that if $\mathfrak{g}$ is abelian, then G_0 (the identity component) is abelian.

Example 1.6. Let V_n be the space of homogeneous polynomials with degree n in z_1, z_2. $SU(2)$ has a standard representaion on V_n. Since the conjugate of the torus Π cover $SU(2)$, we calculate the character of the element in the torus.

$$\chi\Big|_{\pi = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}} = \sum_{k=0}^{n} e^{i(n-2k)\theta} = e^{in\theta} + e^{i(n-2)\theta} + \cdots + e^{-in\theta}.$$

Remark 1.8 (Clebsch–Gordan formular)**.**

$$V_k \otimes V_l = V_{k+l} \oplus V_{k+l-2} \oplus \cdots \oplus V_{k-l}, \quad k \geqslant l.$$

For proof, via character.

Example 1.7 (Motivating idea from finite groups)**.** For a finite group G, let V_i be its irreducible representations

$$\mathbb{C}G = \bigoplus_{i=1}^{r} (\mathrm{dim} V_i) V_i \left(= \bigoplus_{i=1}^{r} V_i \otimes V_i^* \text{ as a } G \times G\text{– module} \right)$$

Theorem 1.1 (Peter–Weyl theorem)**.** *Let G be a compact Lie group*

$$L^2(G) \cong \bigoplus_{V_i:\ irred} (\mathrm{dim} V_i) V_i \left(= V_i \otimes V_i^* \ \textit{as a } G \times G - \textit{reprentation} \right)$$

Remark 1.9. The representation $g \mapsto \langle g \cdot v_i, v_j \rangle$ arise on the RHS as matrix entries of the irreducible representation $\varphi : G \to \mathrm{End}(V_i)$. These generate a dense space of $L^2(G)$. In fact, the proof of Peter–Weyl tells these generate a dense subspace of

$$\mathcal{C}(G) = \{\text{continuous functions: } G \to \mathbb{C}\}$$

in the supreme norm.

Corollary 1.1. *Every compact Lie group is a matrix group, i.e. $G \hookrightarrow GL(n, \mathbb{C})$ for some n, it means that G has a finite faithful representation.*

Proof. We use the density of the representation functions in $\mathcal{C}(G)$. Let $g \neq 1$: this acn be separated from 1 by a function in $\mathcal{C}(G)$, i.e. $\exists f \in \mathcal{C}(G)$, such that $f(g) \neq f(1)$. By the density, we can actually finid a representation function f doing this. Let K_1 be the kernel of the associated representation $\varphi : G \to \mathrm{End}(V_1)$. If $K_1 = \{1\}$, we are done, since V_1 is our faithful representation. If not, pick $g_2 \neq 1 \in K_1$ and repeat argument, get new representation K_2. Now $V_1 \oplus V_2$ has kernel $K_1 \cap K_2 \subsetneq K_1$. Repeat this process and we get a strictly descending chain $K_1 \supseteq K_1 \cap K_2 \supseteq \cdots$ of embedded Lie subgroups of G. So each is compact thus has finitely many components. At each stage, either the dimension or the number of components drops. This process must terminate as $\mathrm{dim} G < \infty$ and $\#\{\text{components}\} < \infty$. Therefore, we get a faithful representation. $\square$

Remark 1.10. In fact, the statement of the corollary is equivalent to the density of the representation functions in $\mathcal{C}(G)$, via Stone–Weierstrass theorem.

Back to finite groups. The character χ_{reg} of the regular representation is supported at identity.

$$\chi_{\mathrm{reg}}(e) = |G|, \quad \chi_{\mathrm{reg}}(g) = 0 \text{ if } g \neq e.$$

Therefore, $\langle \chi_{V_i}, \chi_{\mathrm{reg}} \rangle = \dfrac{1}{|G|} \sum\limits_{g \in G} \chi_{V_i}(g) \chi_{\mathrm{reg}}(g) = \chi_{V_i}(e) = \mathrm{dim} V_i$. Notice that it give a decomposition of $\mathbb{C}G$. In a sense, this too generalizes to compact Lie groups.

Example 1.8. Let $G = S^1$, by Peter–Weyl theorem, we have a decompostion:

$$L^2(S^1) \cong \bigoplus_{n \in \mathbb{Z}} U_n, \quad S^1 \circlearrowright U_n \text{ by } e^{in\theta}.$$

It is a classical result by Weierstrass that the triangular polynomials are dense in $L^2(S^1)$. Then we take $\chi_{reg}(e^{i\theta}) = \sum\limits_{n \in \mathbb{Z}} e^{in\theta}$, which is not convergent in the general sense. But we can view this as a distribution, i.e. gadget acting on funcitons to scalars.

$$\begin{aligned} \chi_{\mathrm{reg}} : f(\theta) \mapsto & \int_{S^1} f(\theta) \textstyle\sum_{n=-\infty}^{\infty} e^{in\theta} \mathrm{d}\theta \\ = & \textstyle\sum_{n \in \mathbb{Z}} n\text{'th Fourier coefficients of } f \\ = & f(0) \end{aligned}$$

So χ_{reg} is Dirac delta function at $\theta = 0$, i.e. $e^{i0} = 1 \in S^1$.