Notes on Lie Groups

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1 Lie groups and Lie algebras

Remark 1.1. The space of all vector fields is an $\infty-$dimension Lie algebra, which is te Lie algebra of $\infty-$dimension Lie group $\mathrm{Diff}(M) := \{f : M \to M \mid f \text{ is a diffeomorphism}\}$.

Let X be a left-invariant vector field on G, integral lines/flow lines for X always exist (First locally exists, then use the property of left-invariance to extend the local solution). Then $\psi : (\mathbb{R}, +) \to G$ via $t \to \psi_t(e)$ is a group homomorphism.

Example 1.1. Let $G = SO(3)$, then $\mathfrak{g} = \mathfrak{so}(3)$, define

$$\varphi_t(e) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(t) & \sin(t) \\ 0 & -\sin(t) & \cos(t) \end{pmatrix} \quad \text{then } X = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(t) & \sin(t) \\ 0 & -\sin(t) & \cos(t) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \in \mathfrak{g}$$

Example 1.2. Let $G = GL(n, \mathbb{R}) \subseteq M_{n\times n}(\mathbb{R})$, then $T_I GL(n, \mathbb{R}) = M_{n\times n}(\mathbb{R})$. The 1-parameter group of $X \in T_I GL(n, \mathbb{R})$ is $\varphi^X(t) = e^{tX} = \sum\limits_{n=0}^{\infty} \dfrac{(tX)^n}{n!}$.

Remark 1.2. $e^X = \sum\limits_{n=0}^{\infty} \dfrac{X^n}{n!}$ convergent absolutely since $\|A^n\| \leqslant \|A\|^n$. And $\det(e^A) = e^{trA}$.

Definition 1.1. $X \in \mathfrak{g} = T_e G$, let $\varphi^X(t)$ to be the 1-parameter group of X, define $e^X := \varphi^X(1)$, easy to see that $e^{sX} = \varphi^X(s)$.

Now consider $\exp : \mathfrak{g} \to G$, $\mathrm{d}(\exp)_0 : \mathfrak{g} \to \mathfrak{g}$, then $\mathrm{d}(\exp)_0(w) = \frac{\partial}{\partial s}\Big|_{s=0} \exp(0 + sw) = w$. Therefore $\mathrm{d}(\exp)_0 = Id_{\mathfrak{g}}$. The inverse function theorem implies the following important consequence.

Proposition 1.1. $\exp$ *is a diffeomorphism of a neighborhood of* 0 *to a neighborhood of* $e \in G$.

Remark 1.3. There exists a local inverse for exp, denoted as "log " on a neighborhood of $e \in G$. There exists a (semi)-explicit formular for log: "Bakes-Campbell-Hausdorff" theorem

$$\log(e^X \cdot e^Y) = X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}[X, [X, Y]] + \frac{1}{12}[Y, [X, Y]] + \text{ higer brackets.}$$

In particular, if $\mathfrak{g}$ is abelian, then $\log(e^X \cdot e^Y) = X + Y$.

Remark 1.4. If $\mathfrak{g}$ is abelian, G is connected, then G is abelian, vice versa. (exp is a diffeomorphism at 0, there is a neighborhood U of e such that U is abelian, then G is generated by U given that G connected).

If $\mathfrak{g}$ is abelian, G connected, then $\exp : \mathfrak{g} \to G$ is onto and group homomorphism. Therefore, $G \cong \mathfrak{g}/\mathrm{Ker}(\exp)$. $\mathrm{Ker}(\exp)$ is 0-dimensional and intersects with some neighborhood of 0 is 0. Therefore $\mathrm{Ker}(\exp)$ is a discrete subgroup of $(\mathfrak{g}, +)$, i.e. a lattice $\cong \mathbb{Z}^r$, where $r \leqslant \dim\mathfrak{g}$.

As a result, if G is abelian, then $G \cong \mathbb{R}^r \times \mathbb{R}^{n-r}/\mathbb{Z}^r \cong T^r \times \mathbb{R}^{n-r}$ for $0 \leqslant r \leqslant n$. In particular, if G is compact, then G is torus.

In general, $\exp : \mathfrak{g} \to G$ is not a global diffeomorphism.

Example 1.3. $U(1) = \{z \in \mathbb{C} \,|\, z\bar{z} = 1\} \cong S^1$ and $\mathfrak{u}(1) = \{z \in \mathbb{C} \,|\, z + \bar{z} = 0\} \cong i\mathbb{R}$. Hence not a global diffeomorphism.

Surjectiveity fails in general.

Example 1.4. $\exp : \mathfrak{sl}(2,\mathbb{R}) \to SL(2,\mathbb{R})$ is not surjective. But exp is surjective if G is compact and connected.

Let $\psi : G \to H$ be a smooth Lie group homomorphism, the 1-parameter groups are homomorphisms $(\mathbb{R},+) \to G$, the image of a 1-parameter group in G is also a 1-parameter group in H. And the following diagram commutes

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "d\psi_e"] \arrow[d, "\exp"'] & \mathfrak{h} \arrow[d, "exp"] \\ G \arrow[r, "\psi"] & H \end{tikzcd}$$

There is a "natural" homomorphism associated to a Lie group G. $\forall g \in G$, define $c_g : G \to G$ via $x \mapsto g \cdot g^{-1}$. Now c_g fixes e, so $\mathrm{d}c_g : \mathfrak{g} \to \mathfrak{g}$. Note that $c_g \circ c_h = c_{g \cdot h}$ so $c_g \circ c_{g^{-1}} = Id_G$ hence $\mathrm{d}c_g$ is invertible.

$\mathrm{d}c_g \circ \mathrm{d}c_h = dc_{g \cdot h}$ gives a group homomorphism of Lie groups $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$ called adjoint map.

Remark 1.5. This is an example of representation of G. $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$.

Example 1.5. If G is a matrix group, Ad is just the conjugation of matrices in $\mathfrak{g}$.

$$g(I + h + \cdots)g^{-1} = I + ghg^{-1} + \cdots$$

- $SO(n) := \{A \in M_{n\times n}(\mathbb{R}) \,|\, A^T A = I\} \quad \mathfrak{so}(n) := \{h \in M_{n\times n}(\mathbb{R}) \,|\, h^T + h = 0\}$. Note that

$$(AhA^{-1})^T = (AhA^T)^T = Ah^T A^T = -AhA^T = -AhA^{-1} \in \mathfrak{so}(n).$$

Differentiate $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g}) = GL(\mathfrak{g})$, then we have $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})(:=$ all linear maps $\mathfrak{g} \to \mathfrak{g})$. Calculation: $\mathrm{Ad}(g) = \mathrm{d}(c_g)_e$. Let $g = (1 + h + \cdots)$. $\forall x \in \mathfrak{g}$,

$$g \cdot x \cdot g^{-1} = (1 + h + \cdots) \cdot x \cdot (1 + h + \cdots)^{-1} = (1 + h + \cdots) \cdot x \cdot (1 - h + \cdots) = 1 + [h, x] + \cdots$$

Therefore $\mathrm{ad}(h) : x \mapsto [h, x]$. It implies the following commutative diagram

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\mathrm{ad}"] \arrow[d, "\exp"'] & \mathrm{End}(\mathfrak{g}) \arrow[d, "\exp"] \\ G \arrow[r, "\mathrm{Ad}"] & \mathrm{Aut}(\mathfrak{g}) \end{tikzcd}$$

Remark 1.6. Let $\varphi : G \to H$ be a Lie group homomorphism, so $\varphi \circ c_g = c_{\varphi(g)}$, and $X\mathfrak{g}, \exp(X) = g \in G$

$$(\mathrm{d}\varphi)_e \circ (\mathrm{d}c_g)_e = (\mathrm{d}c_{\varphi(g)})_e \circ (\mathrm{d}\varphi)_e, \quad (\mathrm{d}\varphi)_e \circ \mathrm{Ad}(g) = \mathrm{Ad}(\varphi(g)) \circ (\mathrm{d}\varphi)_e.$$

Differentiate at g, then we have

$$(\mathrm{d}\varphi)_e \circ \mathrm{ad}(X) = \mathrm{ad}((\mathrm{d}\varphi)_e X) \circ (\mathrm{d}\varphi)_e.$$

Apply to $Y \in \mathfrak{g}$ and $(\mathrm{d}\varphi)_e[X, Y] = [(\mathrm{d}\varphi)_e(X), (\mathrm{d}\varphi)_e(Y)]$.

Remark 1.7. If G is abelian, then c_g is trivial. therefore $\mathrm{Ad}(g) = Id \in \mathrm{Aut}(\mathfrak{g})$, i.e. $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$. This implies $\mathrm{ad} \equiv 0$, i.e. $\mathfrak{g}$ is abelian. Recall that if $\mathfrak{g}$ is abelian, then G_0 (the identity component) is abelian.