

Notes on Lie Groups

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1 Lie groups and Lie algebras

Remark 1.1. The space of all vector fields is an ∞ -dimension Lie algebra, which is the Lie algebra of ∞ -dimension Lie group $\text{Diff}(M) : \{f : M \rightarrow M \mid f \text{ is a diffeomorphism}\}$.

Let X be a left-invariant vector field on G , integral lines/flow lines for X always exist (First locally exists, then use the property of left-invariance to extend the local solution). Then $\psi : (\mathbb{R}, +) \rightarrow G$ via $t \mapsto \psi_t(e)$ is a group homomorphism.

Example 1.1. Let $G = SO(3)$, then $\mathfrak{g} = \mathfrak{so}(3)$, define

$$\varphi_t(e) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(t) & \sin(t) \\ 0 & -\sin(t) & \cos(t) \end{pmatrix} \quad \text{then } X = \frac{d}{dt} \Big|_{t=0} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(t) & \sin(t) \\ 0 & -\sin(t) & \cos(t) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \in \mathfrak{g}$$

Example 1.2. Let $G = GL(n, \mathbb{R}) \subseteq M_{n \times n}(\mathbb{R})$, then $T_I GL(n, \mathbb{R}) = M_{n \times n}(\mathbb{R})$. The 1-parameter group of $X \in T_I GL(n, \mathbb{R})$ is $\varphi^X(t) = e^{tX} = \sum_{n=0}^{\infty} \frac{(tX)^n}{n!}$.

Remark 1.2. $e^X = \sum_{n=0}^{\infty} \frac{X^n}{n!}$ convergent absolutely since $\|A^n\| \leq \|A\|^n$. And $\det(e^A) = e^{\text{tr} A}$.

Definition 1.1. $X \in \mathfrak{g} = T_e G$, let $\varphi^X(t)$ to be the 1-parameter group of X , define $e^X := \varphi^X(1)$, easy to see that $e^{sX} = \varphi^X(s)$.

Now consider $\exp : \mathfrak{g} \rightarrow G$, $d(\exp)_0 : \mathfrak{g} \rightarrow \mathfrak{g}$, then $d(\exp)_0(w) = \frac{\partial}{\partial s} \Big|_{s=0} \exp(0 + sw) = w$. Therefore $d(\exp)_0 = Id_{\mathfrak{g}}$. The inverse function theorem implies the following important consequence.

Proposition 1.1. \exp is a diffeomorphism of a neighborhood of 0 to a neighborhood of $e \in G$.

Remark 1.3. There exists a local inverse for \exp , denoted as “log” on a neighborhood of $e \in G$. There exists a (semi)-explicit formular for log: “Bakes-Campbell-Hausdorff” theorem

$$\log(e^X \cdot e^Y) = X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}[X, [X, Y]] + \frac{1}{12}[Y, [X, Y]] + \text{higher brackets.}$$

In particular, if \mathfrak{g} is abelian, then $\log(e^X \cdot e^Y) = X + Y$.

Remark 1.4. If \mathfrak{g} is abelian, G is connected, then G is abelian, vice versa. (\exp is a diffeomorphism at 0, there is a neighborhood U of e such that U is abelian, then G is generated by U given that G connected).

If \mathfrak{g} is abelian, G connected, then $\exp : \mathfrak{g} \rightarrow G$ is onto and group homomorphism. Therefore, $G \cong \mathfrak{g}/\text{Ker}(\exp)$. $\text{Ker}(\exp)$ is 0-dimensional and intersects with some neighborhood of 0 is 0. Therefore $\text{Ker}(\exp)$ is a discrete subgroup of $(\mathfrak{g}, +)$, i.e. a lattice $\cong \mathbb{Z}^r$, where $r \leq \dim \mathfrak{g}$.

As a result, if G is abelian, then $G \cong \mathbb{R}^r \times \mathbb{R}^{n-r}/\mathbb{Z}^r \cong T^r \times \mathbb{R}^{n-r}$ for $0 \leq r \leq n$. In particular, if G is compact, then G is torus.

In general, $\exp : \mathfrak{g} \rightarrow G$ is not a global diffeomorphism.

Example 1.3. $U(1) = \{z \in \mathbb{C} \mid z\bar{z} = 1\} \cong S^1$ and $\mathfrak{u}(1) = \{z \in \mathbb{C} \mid z + \bar{z} = 0\} \cong i\mathbb{R}$. Hence not a global diffeomorphism.

Surjectivity fails in general.

Example 1.4. $\exp : \mathfrak{sl}(2, \mathbb{R}) \rightarrow SL(2, \mathbb{R})$ is not surjective. But \exp is surjective if G is compact and connected.

Let $\psi : G \rightarrow H$ be a smooth Lie group homomorphism, the 1-parameter groups are homomorphisms $(\mathbb{R}, +) \rightarrow G$, the image of a 1-parameter group in G is also a 1-parameter group in H . And the following diagram commutes

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{d\psi_e} & \mathfrak{h} \\ \exp \downarrow & & \downarrow \exp \\ G & \xrightarrow{\psi} & H \end{array}$$

There is a “natural” homomorphism associated to a Lie group G . $\forall g \in G$, define $c_g : G \rightarrow G$ via $x \mapsto g \cdot g^{-1}$. Now c_g fixes e , so $dc_g : \mathfrak{g} \rightarrow \mathfrak{g}$. Note that $c_g \circ c_h = c_{g \cdot h}$ so $c_g \circ c_{g^{-1}} = Id_G$ hence dc_g is invertible.

$dc_g \circ dc_h = dc_{g \cdot h}$ gives a group homomorphism of Lie groups $\text{Ad} : G \rightarrow \text{Aut}(\mathfrak{g})$ called adjoint map

Example 1.5. This is an example of representation of G . $\text{Ad} : G \rightarrow \text{Aut}(\mathfrak{g})$