Notes on Algebraic Topology

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Abstract

This notes collect what I think are interesting and important.

1 Category

Example 1.1. We define a cotravariant functor

$$\mathrm{Fun} : \underline{\mathrm{Top}} \to \underline{\mathrm{Ring}}, \quad X \mapsto F(X) = \mathrm{Hom}(X, \mathbb{R})$$

$F(X)$ are continuous real functions on X. A classical theorem of Gelfand-Kolmogoroff says that two compact Hausdorff spaces X, Y are homeomorphic if and only if $\mathrm{Fun}(X), \mathrm{Fun}(Y)$ are ring isomorphic.

Proposition 1.1. *For a compact smooth manifold M, $M \cong \mathrm{Hom}_{\mathbb{R}}(\Omega^0(M), \mathbb{R})$*

Proof. Define map $\alpha : M \to \mathrm{Hom}_{\mathbb{R}}(\Omega^0(M), \mathbb{R})$ via $p \mapsto \alpha_p : f \mapsto f(p)$ is clearly injective.

For $T \in \mathrm{Hom}_{\mathbb{R}}(\Omega^0(M), \mathbb{R})$, we want to show there exists $p \in M$ such that $T(f)$ implies $f(p) = 0$. If so, $T(f - T(f)) = T(f) - T(f)T(1) = 0$, then $[f - T(f)](p) = 0$, therefore, $T(f) = f(p) = \alpha_p(f)$. Assume the hypothesis fails, then for every $p \in M$, there exists $f_p \in \Omega^0(M)$ such that $f_p(p) > 0$ and $T(f_p) = 0$. We may take a bump function ρ_p such that $p \in \mathrm{supp}(\rho_p) \subseteq \{f_p > 0\}$, then $T(f_p\rho_p) = T(f_p)T(\rho_p) = 0$. Therefore without loss of generality we may assume $f_p \geqslant 0$ on M. When M is compact, there exists a finite set $\{p_i\}_{i=1}^n$ such that the function $f := \sum_{i=1}^n f_{p_i} > 0$, which means f is invertible. However $T(f) = \sum_{i=1}^n T(f_{p_i}) = 0$ and $T(1) = T(f)T(f^{-1}) = 0$, absurd. This completes the proof. □

Proposition 1.2. *The contravariant functor $\Omega^0 : \underline{\mathrm{Mfd}}_{\mathbb{R}}^{cpt} \to \underline{\mathrm{CA}}_{\mathbb{R}}$ is fully faithful. Here we denote by $\underline{\mathrm{CA}}_{\mathbb{R}}$ the category of commutative algebra ober $\mathbb{R}$.*

Proof. For compact manifolds M, N. $\Omega^0 : \mathrm{Hom}(M, N) \to \mathrm{Hom}_{\mathbb{R}}(\Omega^0(N), \Omega^0(M))$ is clearly injective. For any $T \in \mathrm{Hom}_{\mathbb{R}}(\Omega^0(N), \Omega^0(M))$ induces a pullback:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbb{R}}(\Omega^0(M), \mathbb{R}) \arrow[r, "T^*"] \arrow[d, "\cong"'] & \mathrm{Hom}_{\mathbb{R}}(\Omega^0(N), \mathbb{R}) \arrow[d, "\cong"'] \\
M \arrow[r, "F"] & N
\end{tikzcd}$$

We use the fact that a continuous map $\varphi : M \to N$ is smooth $\Leftrightarrow \forall f \in \Omega^0(N), f \circ \varphi \in \Omega^0(M)$ and show that $T = F^*$(it is not that obvious), hence F is smooth. □

Theorem 1.1. *Let $X \in \underline{\mathrm{Top}}$. We define a category $\Pi_1(X)$ as follows:*

- $\mathrm{Obj}(\Pi_1(X)) = X$.
- $\mathrm{Hom}_{\Pi_1(X)}(x_0, x_1)$ = *pass classes from x_0 to x_1.*
- $1_{x_0} = i_{x_0}$, *the constant map.*

*Then $\Pi_1(X)$ defines a category which is in fact a groupoid. The inverse of $[\gamma]$ is given by $[\gamma]^{-1}$. $\Pi_1(X)$ is called the **fundamental groupoid** of X.*

Let $\mathcal{C}$ be a groupoid. Let $A \in \text{Obj}(\mathcal{C})$, then $\text{Aut}_{\mathcal{C}}(A) := \text{Hom}_{\mathcal{C}}(A, A)$ forms a group. For any $f : A \to B$, it induces a group isomorphism $Ad_f : \text{Aut}_{\mathcal{C}}(A) \to \text{Aut}_{\mathcal{C}}(B)$ via $g \mapsto f \circ g \circ f^{-1}$.

This naturally defines a functor $\mathcal{C} \to \underline{\text{Group}} : A \to \text{Aut}_{\mathcal{C}}(A) \quad f \to Ad_f$.

Specialize to the topological space, we find a functor $\Pi_1(X) \to \underline{\text{Group}}$. Let $x_0 \in X$, the group $\pi_1(X, x_0) := \text{Aut}_{\Pi_1}(X)(x_0)$ is called the fundamental group of the pointed space (X, x_0). If X is path connected, then for $x_0, x_1 \in X$, we have the group isomorphism: $\pi_1(X, x_0) \cong \pi_1(X, x_1)$.

Let $f : X \to Y$ be a continuous map. It defines a functor $\Pi_1(f) : \Pi_1(X) \to \Pi_1(Y)$ via $x \mapsto y, [\gamma] \mapsto [f \circ \gamma]$. Then Π_1 defines a functor $\Pi_1 : \underline{\text{Top}} \to \underline{\text{Groupoid}}$ via $X \mapsto \Pi_1(X)$. Here morphism in $\underline{\text{Groupoid}}$ are given by natural transformations.

Proposition 1.3. *Let $f, g : X \to Y$ be continuous maps that are homotopic by $F : X \times I \to Y$. Define path class*

$$\tau_{x_0} := [F|_{x_0 \times I}] = \text{Hom}_{\Pi_1(Y)}(f(x_0), g(x_0))$$

Then τ defines a natural transformation $\tau : \Pi_1(f) \to \Pi_1(g)$

This propsition can be pictured by the following diagram:

$$\begin{tikzcd} X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow F"] & Y \end{tikzcd} \Rightarrow \begin{tikzcd} \Pi_1(X) \arrow[r, bend left, "\Pi_1(f)"] \arrow[r, bend right, "\Pi_1(g)"'] \arrow[r, phantom, "\Downarrow \tau"] & \Pi_1(Y) \end{tikzcd}$$

Theorem 1.2. *Let $f : X \to Y$ be a homotopy equivalence. Then $\Pi_1(f) : \Pi_1(X) \to \Pi_1(Y)$ is an equivalence of categories. In particular, it induces group isomorphism $\pi_1(X, x_0) \cong \pi_1(Y, f(x_0))$*

2 Covering and Fibration

Definition 2.1. Let $p : E \to B$ be continuous. A trivilazation of p over an open set $U \subseteq B$ is a homeomorphism $\varphi : p^{-1}(U) \to U \times F$ over U, i.e. the following diagram commutes

$$\begin{tikzcd} p^{-1}(U) \arrow[rr, "\varphi"] \arrow[dr] & & U \times F \arrow[dl] \\ & U & \end{tikzcd}$$

p is called **locally trivial** if there exists an open over $\mathcal{U}$ of B such that p has a trivialization over each open $U \in \mathcal{U}$. Such p is also called a **fiber bundle** and F is called the fiber.

Definition 2.2. A **covering** is a locally trivial fiber bundle with discrete fiber F.

Definition 2.3. Let $p : E \to B, f : X \to B$. A **lifting** of f along p is a map $F : X \to E$ such that $p \circ F = f$

$$\begin{tikzcd} & E \arrow[d, "p"] \\ X \arrow[ur, dashed, "F"] \arrow[r, "f"] & B \end{tikzcd}$$

Lemma 2.1. Let $p : E \to B$ be a covering. Let

$$D = \{(x, x) \in E \times E \,|\, x \in E\} \quad Z = \{(x, y) \in E \times E \,|\, p(x) = p(y)\}$$

Then $D \subseteq Z$ is open and closed.

Theorem 2.1 (Uniqueness of lifting)**.** *Let $p : E \to B$ be a covering. Let $F_0, F_1 : X \to E$ be two liftings of f. Suppose X is conneceted and F_0, F_1 agree somewhere. Then $F_0 = F_1$.*

Proof. Consider the map $\hat{F} := (F_0, F_1) : X \to Z$. $\hat{F}(X) \cap D \neq \varnothing$. The above lemma implies $\hat{F} \subseteq D$. $\square$

Definition 2.4. A map $p : E \to B$ is said to have the **homotopy lifting property**(HLP) with repect to X if for any map $\tilde{f} : X \to E$ and $F : X \times I \to B$ such that $p \circ \tilde{f} = F|_{X\times 0}$, then there exists a lifting $\tilde{F}$ of F along p such that $\tilde{F}|_{X\times 0} = \tilde{f}$, i.e. the following diagram commutes

$$\begin{tikzcd} X \times 0 \arrow[r, "\tilde{f}"] \arrow[d, hook] & E \arrow[d, "p"] \\ X \times I \arrow[r, "F"'] \arrow[ur, dashed, "\exists \tilde{F}"] & B \end{tikzcd}$$

A map $p : E \to B$ is called a **fibration**(of **Hurewicz fibration**) if p has HLP for any space.

Example 2.1. A projection is a fibration.

Theorem 2.2. *A covering is a fibration.*

Corollary 2.1. *Let $p : E \to B$ be a covering. Then $\Pi_1(p) : \Pi_1(E) \to \Pi_1(B)$ is a faithful functor. In particular, the induced map $\pi_1(E, e) \to \pi_1(B, p(e))$ is injective.*

Let $p : E \to B$ be a covering. Let $\gamma : I \to B$ be a path in B from b_1 to b_2. Then it defines a map $T_\gamma : p^{-1}(b_1) \to p^{-1}(b_2)$ via $e_1 \mapsto \tilde{\gamma}(1)$, where $\tilde{\gamma}$ is a lift of γ with initial condition $\tilde{\gamma}(0) = e_1$.

Assume $[\gamma_1] = [\gamma_2]$ in B, HLP implies that $T_{\gamma_1} = T_{\gamma_2}$. We define a well-defined map

$$T : \mathrm{Hom}_{\Pi_1(B)}(b_1, b_2) \to \mathrm{Hom}_{\underline{\mathrm{Set}}}(p^{-1}(b_1), p^{-1}(b_2)) \quad \text{via } [\gamma] \mapsto T_{[\gamma]}.$$

Proposition 2.1. *The following data*

$$T : \Pi_1(B) \to \underline{\mathrm{Set}} \quad \textit{via } b \mapsto p^{-1}(b),\ [\gamma] \mapsto T_{[\gamma]}$$

defines a functor, called the ***transport functor****. In particular, we have a well-defined map $\pi_1(B, b) \to \mathrm{Aut}(p^{-1}(b))$.*

Proposition 2.2. *Let $p : E \to B$ be a covering, E path-connected. Let $e \in E, b = p(e) \in B$. Then the action of $\pi_1(B, b)$ on $p^{-1}(b)$ is transitive, whose stabilizer at e is $\pi_1(E, e)$(or more acurately $p_*(\pi_1(E, e))$). In other words,*

$$p^{-1}(b) \cong \pi_1(B, b)/\pi_1(E, e)$$

as a coset space.

Theorem 2.3 (Lifting Criterion)**.** *Let $p : E \to B$ be a covering. $f : X \to B$ for X path-connected and locally path connected. Let $e \in E, x_0 \in X$ such that $f(x_0) = p(e)$. Then there exists a lift F of f with $F(x_0) = e$ if and only if*

$$f_*(\pi_1(X, x_0)) \subseteq p_*(\pi_1(E, e)).$$

Definition 2.5. A left (right) $G-$**principal covering** is a covering $p : E \to B$ with a left (right) properly discontinuous $G-$action on E over B

$$\begin{tikzcd} E \arrow[rr, "g"] \arrow[dr, "p"'] & & E, \arrow[dl, "p"] & \forall g \in G \\ & B & \end{tikzcd}$$

such that the induced map $E/G \to B$ is a homeomorphism.

Example 2.2. $ex : \mathbb{R} \to S^1$ is a $\mathbb{Z}-$principal covering for the action $n : t \mapsto t + n$, $\forall n \in \mathbb{Z}$.

Example 2.3. $S^n \to \mathbb{R}P^n \cong S^n / \mathbb{Z}_2$ is a $\mathbb{Z}_2-$principal covering.

Proposition 2.3. *Let $p : E \to B$ be a $G-$principal covering. Then transportation commutes with $G-$action, i.e.*

$$T_{[\gamma]} \circ g = g \circ T_{[\gamma]}, \quad \forall g \in G, \gamma \textit{ a path in } B.$$

Theorem 2.4. *Let $p : E \to B$ be a G–principal covering, E path-connected, $e \in E, b = p(e)$. Then we have an exact sequence of groups*

$$1 \to \pi_1(E,e) \to \pi_1(B,b) \to G \to 1.$$

In other words, $\pi_1(E,e)$ is a normal subgroup of $\pi_1(B,b)$ and $G = \pi_1(B,b)/\pi_1(E,e)$.

Proof. Let $F = p^{-1}(b)$. The previous proposition implies that $\pi_1(B,b)$– action and G– action on F commute. It induces a $\pi_1(B,b) \times G$– action on F. Consider its stabilizer at e and two projections

$$\begin{tikzcd} & \mathrm{Stab}_e(\pi_1(B,b) \times G) \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ \pi_1(B,b) & & G \end{tikzcd}$$

p_1 is an isomorphism since G acts transitively on F, p_2 is an epimorphism with $\ker(p_2) = \mathrm{Stab}_e(\pi_1(B,b)) = \pi_1(E,e)$, since G acts faithfully on F. □

Example 2.4. Apply this theorem to the covering $ex : \mathbb{R} \to S^1$, we find a group isomorphism

$$\deg : \pi_1(S^1) = \mathbb{Z}$$

which is called the **degree map**.

Proposition 2.4. *If $i : A \subseteq X$ is a retract, then $i_* : \pi_1(A) \to \pi_1(X)$ is injective.*

Corollary 2.2. *Let D^2 be the unit disk in $\mathbb{R}^2$. Then its boundary S^1 is not a retract of D^2.*

Theorem 2.5 (Brouwer fixed point Theorem)**.** *Let $f : D^2 \to D^2$. Then there exists $x \in D^2$ such that $f(x) = x$.*

Proof. Assume f has no fixed point. Let l_x be the ray starting from $f(x)$ pointing toward x. Then we have define a retraction $r : D^2 \to S^1, \quad x \to l_x \bigcap \partial D^2$. Contradiction. □

Theorem 2.6 (Fundamental Theorem of Algebra)**.** *Let $f(x) = x^n + c_1 x^{n-1} + \cdots + c_n$ be a polynomial with $c_i \in \mathbb{C}, n > 0$. Then there exists $a \in \mathbb{C}$ such that $f(a) = 0$.*

Proof. Assume f as no root in $\mathbb{C}$. Define a homotopy

$$F : S^1 \times I \to S^1, \quad F(e^{2\pi i\theta}, t) = \frac{f(\tan(\pi t/2)e^{2\pi i\theta})}{|f(\tan(\pi t/2)e^{2\pi i\theta})|}$$

Then $\deg(F|_{S^1\times 0}) = 0$ and $\deg(F|_{S^1\times 1}) = n$. Contradiction. □

References

[1] Gary Sivek, *The Analytic Class Number Formula.* gsivek@mit.edu, May 19,2005.