

# Notes on Lie Groups

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## 1 Lie groups and Lie algebras

**Remark 1.1.** The space of all vector fields is an  $\infty$ -dimension Lie algebra, which is the Lie algebra of  $\infty$ -dimension Lie group  $\text{Diff}(M) : \{f : M \rightarrow M \mid f \text{ is a diffeomorphism}\}$ .

Let  $X$  be a left-invariant vector field on  $G$ , integral lines/flow lines for  $X$  always exist (First locally exists, then use the property of left-invariance to extend the local solution). Then  $\psi : (\mathbb{R}, +) \rightarrow G$  via  $t \rightarrow \psi_t(e)$  is a group homomorphism.

**Example 1.1.** Let  $G = SO(3)$ , then  $\mathfrak{g} = \mathfrak{so}(3)$ , define

$$\varphi_t(e) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(t) & \sin(t) \\ 0 & -\sin(t) & \cos(t) \end{pmatrix} \quad \text{then } X = \left. \frac{d}{dt} \right|_{t=0} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(t) & \sin(t) \\ 0 & -\sin(t) & \cos(t) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \in \mathfrak{g}$$

**Example 1.2.** Let  $G = GL(n, \mathbb{R}) \subseteq M_{n \times n}(\mathbb{R})$ , then  $T_I GL(n, \mathbb{R}) = M_{n \times n}(\mathbb{R})$ . The 1-parameter group of  $X \in T_I GL(n, \mathbb{R})$  is  $\varphi^X(t) = e^{tX} = \sum_{n=0}^{\infty} \frac{(tX)^n}{n!}$ .

**Remark 1.2.**  $e^X = \sum_{n=0}^{\infty} \frac{X^n}{n!}$  convergent absolutely since  $\|A^n\| \leq \|A\|^n$ . And  $\det(e^A) = e^{\text{tr} A}$ .

**Definition 1.1.**  $X \in \mathfrak{g} = T_e G$ , let  $\varphi^X(t)$  to be the 1-parameter group of  $X$ , define  $e^X := \varphi^X(1)$ , easy to see that  $e^{sX} = \varphi^X(s)$ .

Now consider  $\exp : \mathfrak{g} \rightarrow G$ ,  $d(\exp)_0 : \mathfrak{g} \rightarrow \mathfrak{g}$ , then  $d(\exp)_0(w) = \left. \frac{\partial}{\partial s} \right|_{s=0} \exp(0 + sw) = w$ . Therefore  $d(\exp)_0 = Id_{\mathfrak{g}}$ . The inverse function theorem implies the following important consequence.

**Proposition 1.1.**  $\exp$  is a diffeomorphism of a neighborhood of 0 to a neighborhood of  $e \in G$ .

**Remark 1.3.** There exists a local inverse for  $\exp$ , denoted as “log” on a neighborhood of  $e \in G$ . There exists a (semi)-explicit formula for log: “Baker-Campbell-Hausdorff” theorem

$$\log(e^X \cdot e^Y) = X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}[X, [X, Y]] + \frac{1}{12}[Y, [X, Y]] + \text{higher brackets}.$$

In particular, if  $\mathfrak{g}$  is abelian, then  $\log(e^X \cdot e^Y) = X + Y$ .

**Remark 1.4.** If  $\mathfrak{g}$  is abelian,  $G$  is connected, then  $G$  is abelian, vice versa. ( $\exp$  is a diffeomorphism at 0, there is a neighborhood  $U$  of  $e$  such that  $U$  is abelian, then  $G$  is generated by  $U$  given that  $G$  is connected).

If  $\mathfrak{g}$  is abelian,  $G$  connected, then  $\exp : \mathfrak{g} \rightarrow G$  is onto and group homomorphism. Therefore,  $G \cong \mathfrak{g}/\text{Ker}(\exp)$ .  $\text{Ker}(\exp)$  is 0-dimensional and intersects with some neighborhood of 0 is 0. Therefore  $\text{Ker}(\exp)$  is a discrete subgroup of  $(\mathfrak{g}, +)$ , i.e. a lattice  $\cong \mathbb{Z}^r$ , where  $r \leq \dim \mathfrak{g}$ .

As a result, if  $G$  is abelian, then  $G \cong \mathbb{R}^r \times \mathbb{R}^{n-r}/\mathbb{Z}^r \cong T^r \times \mathbb{R}^{n-r}$  for  $0 \leq r \leq n$ . In particular, if  $G$  is compact, then  $G$  is torus.

In general,  $\exp : \mathfrak{g} \rightarrow G$  is not a global diffeomorphism.

**Example 1.3.**  $U(1) = \{z \in \mathbb{C} \mid z\bar{z} = 1\} \cong S^1$  and  $\mathfrak{u}(1) = \{z \in \mathbb{C} \mid z + \bar{z} = 0\} \cong i\mathbb{R}$ . Hence not a global diffeomorphism.

Surjectivity fails in general.

**Example 1.4.**  $\exp : \mathfrak{sl}(2, \mathbb{R}) \rightarrow SL(2, \mathbb{R})$  is not surjective. But  $\exp$  is surjective if  $G$  is compact and connected.

Let  $\psi : G \rightarrow H$  be a smooth Lie group homomorphism, the 1-parameter groups are homomorphisms  $(\mathbb{R}, +) \rightarrow G$ , the image of a 1-parameter group in  $G$  is also a 1-parameter group in  $H$ . And the following diagram commutes

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{d\psi_e} & \mathfrak{h} \\ \exp \downarrow & & \exp \downarrow \\ G & \xrightarrow{\psi} & H \end{array}$$

There is a “natural” homomorphism associated to a Lie group  $G$ .  $\forall g \in G$ , define  $c_g : G \rightarrow G$  via  $x \mapsto g \cdot x \cdot g^{-1}$ . Now  $c_g$  fixes  $e$ , so  $dc_g : \mathfrak{g} \rightarrow \mathfrak{g}$ . Note that  $c_g \circ c_h = c_{g \cdot h}$  so  $c_g \circ c_{g^{-1}} = Id_G$  hence  $dc_g$  is invertible.

$dc_g \circ dc_h = dc_{g \cdot h}$  gives a group homomorphism of Lie groups  $\text{Ad} : G \rightarrow \text{Aut}(\mathfrak{g})$  called adjoint map

**Example 1.5.** This is an example of representation of  $G$ .  $\text{Ad} : G \rightarrow \text{Aut}(\mathfrak{g})$