5.6 The Sine Law

YOU WILL NEED

- dynamic geometry software (optional)

Communication *Tip*

To perform a calculation to a high degree of accuracy, save intermediate answers by using the memory keys of your calculator. Round only after the very last calculation.

GOAL

Solve two-dimensional problems by using the sine law.

LEARN ABOUT the Math

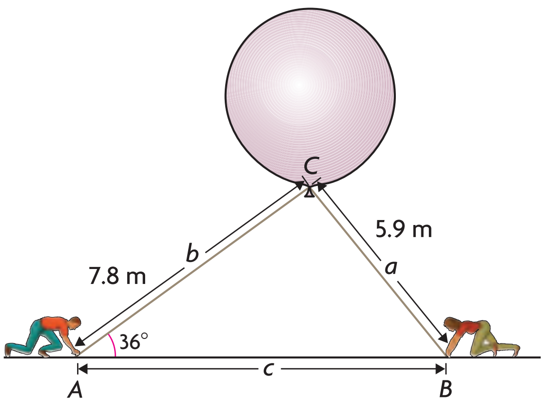
Albert and Belle are part of a scientific team studying thunderclouds. The team is about to launch a weather balloon into an active part of a cloud. Albert's rope is 7.8 m long and makes an angle of 36° with the ground. Belle's rope is 5.9 m long.

? How far, to the nearest tenth of a metre, is Albert from Belle?

EXAMPLE 1 Using the sine law to calculate an unknown length

Determine the distance between Albert and Belle.

Adila's Solution: Assuming that Albert and Belle are on Opposite Sides of the Balloon



From the problem, it is not clear how Albert, Belle, and the balloon are positioned relative to each other. I assumed that Albert and Belle are on opposite sides of the balloon. I drew a sketch of this situation.

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

In $\triangle ABC$, I knew one angle and two sides. To determine AB (side c), I needed to know $\angle C$. So I first calculated $\angle B$ using the **sine law**.

$$\frac{5.9}{\sin 36^\circ} = \frac{7.8}{\sin B}$$

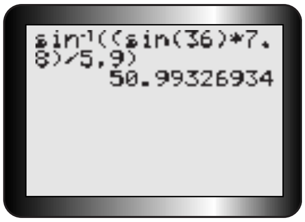
$$(\overset{1}{\cancel{\sin 36^\circ}} \times \sin B)\left(\frac{5.9}{\underset{1}{\cancel{\sin 36^\circ}}}\right) = (\sin 36^\circ \times \overset{1}{\cancel{\sin B}})\left(\frac{7.8}{\underset{1}{\cancel{\sin B}}}\right)$$

To solve for $\angle B$, I multiplied both sides of the equation by the lowest common denominator $(\sin 36^\circ \times \sin B)$ to eliminate the fractions.

$$\frac{(\sin B)(\overset{1}{\cancel{5.9}})}{\underset{1}{\cancel{5.9}}} = \frac{(\sin 36^\circ)(7.8)}{5.9}$$

Then I divided both sides by 5.9 to isolate sin B.

$$\angle B = \sin^{-1}\left(\frac{(\sin 36^\circ)(7.8)}{5.9}\right)$$



I used my calculator to evaluate $\angle B$.

$$\angle B \doteq 51°$$

I rounded to the nearest degree.

$$\angle A + \angle B + \angle C = 180°$$
$$36° + 51° + \angle C = 180°$$
$$\angle C = 180° - (36° + 51°)$$
$$= 93°$$

I calculated $\angle C$ by using the fact that all three interior angles add up to 180°.

$$\frac{a}{\sin A} = \frac{c}{\sin C}$$

Then I used the sine law again to determine c.

$$\frac{5.9}{\sin 36°} = \frac{c}{\sin 93°}$$

$$\frac{5.9}{\sin 36°} \times \sin 93° = \frac{c}{\cancel{\sin 93°}_{1}} \times \cancel{\sin 93°}^{1}$$

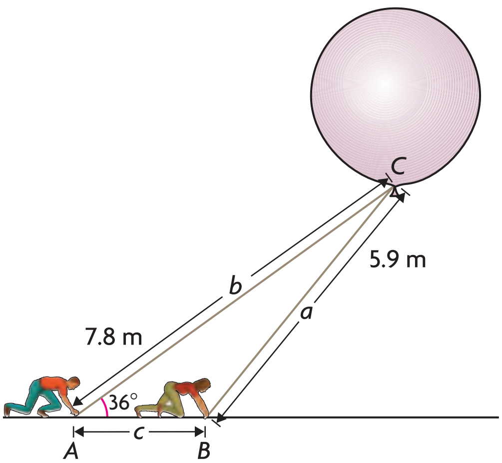
To solve for c, I multiplied both sides of the equation by sin 93°.

$$10.0 \text{ m} \doteq c$$

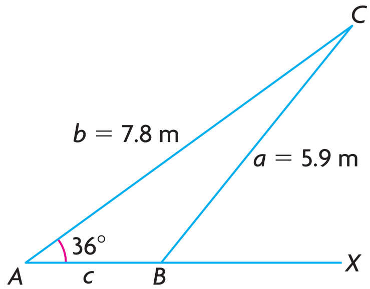
I rounded to the nearest tenth.

If Albert and Belle are on opposite sides of the balloon, they are about 10.0 m apart.

Reuben's Solution: Assuming that Albert and Belle are on the Same Side of the Balloon



The problem did not state how Albert, Belle, and the balloon are positioned relative to each other. I assumed that Albert and Belle are on the same side of the balloon. I drew a sketch of this situation.



In $\triangle ABC$, I knew one angle and two sides. If I knew $\angle C$, I could determine c using the sine law. First I had to determine $\angle B$ in order to get $\angle C$.

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

I used the sine law to calculate $\angle B$.

$$\frac{5.9}{\sin 36°} = \frac{7.8}{\sin B}$$

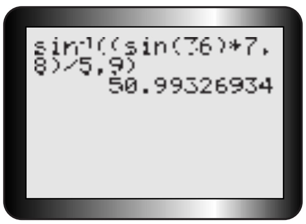
$$(\cancel{\sin 36°} \times \sin B)\left(\frac{5.9}{\cancel{\sin 36°}}\right) = (\sin 36° \times \cancel{\sin B})\left(\frac{7.8}{\cancel{\sin B}}\right)$$

To solve for $\angle B$, I first multiplied both sides of the equation by the lowest common denominator ($\sin 36° \times \sin B$) to eliminate the fractions.

$$\frac{(\sin B)(\cancel{5.9})}{\cancel{5.9}} = \frac{(\sin 36°)(7.8)}{5.9}$$

I divided both sides by 5.9 to isolate $\sin B$.

$$\angle B = \sin^{-1}\left(\frac{(\sin 36°)(7.8)}{5.9}\right)$$



I used my calculator to evaluate $\angle B$. I rounded to the nearest degree.

$$\angle CBX \doteq 51°$$

51° is the value of the related acute angle $\angle CBX$, but I wanted the obtuse angle $\angle CBA$ in the triangle.

$$\angle CBA = 180° - 51°$$
$$= 129°$$

$$\angle C = 180° - (\angle A + \angle CBA)$$
$$= 180° - (36° + 129°)$$
$$= 15°$$

I calculated $\angle C$ by using the fact that all three interior angles add up to 180°.

$$\frac{a}{\sin A} = \frac{c}{\sin C}$$

I used the sine law again to calculate side c.

$$\frac{5.9}{\sin 36°} = \frac{c}{\sin 15°}$$

$$\sin 15° \times \frac{5.9}{\sin 36°} = \cancel{\sin 15°}^{1} \times \frac{c}{\cancel{\sin 15°}_{1}}$$

To solve for c, I multiplied both sides of the equation by sin 15°.

$$2.6 \text{ m} \doteq c$$

I rounded to the nearest tenth.

If Albert and Belle are on the same side of the balloon, they are about 2.6 m apart.

Reflecting

A. Why is the situation in Example 1 called **the ambiguous case of the sine law**?

B. What initial information was given in this problem?

C. What is the relationship between sin B in Adila's solution and sin B in Reuben's solution? Explain why both values of sine are related.

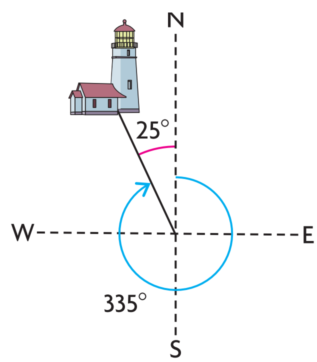
D. Calculate the height of $\triangle ABC$ in both solutions. What do you notice? Compare this value with the length of a and b.

the ambiguous case of the sine law

a situation in which 0, 1, or 2 triangles can be drawn given the information in a problem. This occurs when you know two side lengths and an angle *opposite* one of the sides rather than *between* them (an SSA triangle). If the given angle is acute, 0, 1, or 2 triangles are possible. If the given angle is obtuse, 0 or 1 triangle is possible (see the In Summary box for this lesson).

bearing

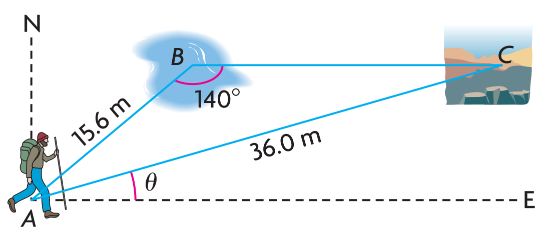
the direction in which you have to move in order to reach an object. A bearing is a clockwise angle from magnetic north. For example, the bearing of the lighthouse shown is 335°.



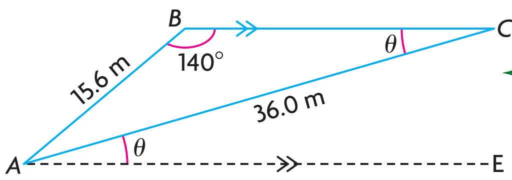
APPLY the Math

EXAMPLE 2 Using the sine law in the ambiguous case to calculate the only possible angle

Karl's campsite is 15.6 m from a lake and 36.0 m from a scenic lookout as shown. From the lake, the angle formed between the campsite and the lookout is 140°. Karl starts hiking from his campsite to go to the lookout. What is the **bearing** of the lookout from Karl's position ($\angle NAC$)?



Sara's Solution



Based on the given information, this is an SSA triangle. But since the given angle is obtuse, only one situation had to be considered. In $\triangle ABC$, I knew that AE is parallel to BC. Since $\angle C$ and θ are alternate angles between parallel lines, they are equal.

$$\frac{\sin C}{c} = \frac{\sin B}{b}$$

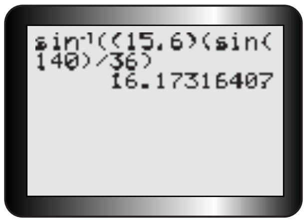
I needed to calculate an angle, so I set up the sine law equation with the angles in the numerators.

$$\frac{\sin \theta}{15.6} = \frac{\sin 140^\circ}{36.0}$$

$$\cancel{15.6}^{1} \times \frac{\sin \theta}{\cancel{15.6}_{1}} = 15.6 \times \frac{\sin 140^\circ}{36.0}$$

To solve for $\sin \theta$, I multiplied both sides of the equation by 15.6.

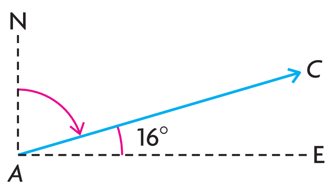
$$\theta = \sin^{-1}\left(\frac{(15.6)(\sin 140^\circ)}{36.0}\right)$$



I used the inverse sine function on my calculator to determine $\angle C$ (angle θ).

$$\theta \doteq 16^\circ$$

I rounded to the nearest degree.



In order to state the bearing of the lookout, I needed to know the complementary angle of 16°.

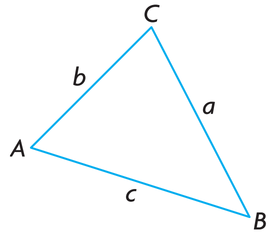
$$\angle NAC = 90^\circ - 16^\circ$$
$$= 74^\circ$$

So I subtracted 16° from 90°.

From Karl's campsite, the lookout has a bearing of about 74°.

In Summary

Key Ideas



- The sine law states that in any $\triangle ABC$, the ratios of each side to the sine of its opposite angle are equal.

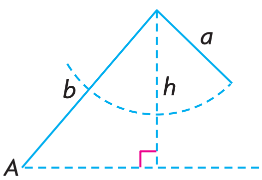
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \quad \text{or} \quad \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

- Given any triangle, the sine law can be used if you know
  - two sides and one angle opposite a given side (SSA) or
  - two angles and any side (AAS or ASA)
- The ambiguous case arises in a SSA (side, side, angle) triangle. In this situation, depending on the size of the given angle and the lengths of the given sides, the sine law calculation may lead to 0, 1, or 2 solutions.

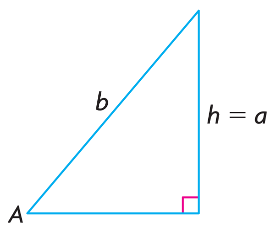
Need to Know

- In the ambiguous case, if $\angle A$, a, and b are given and $\angle A$ is acute, there are four cases to consider. In each case, the height of the triangle is $h = b \sin A$.

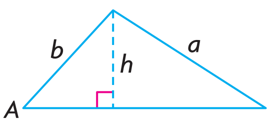
If $\angle A$ is acute and $a < h$, no triangle exists.



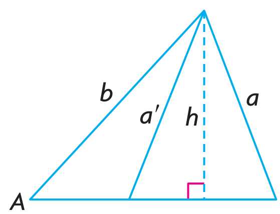
If $\angle A$ is acute and $a = h$, one right triangle exists.



If $\angle A$ is acute and $a > b$, one triangle exists.

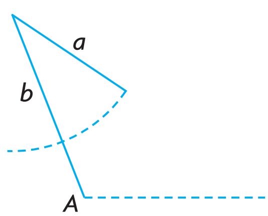


If $\angle A$ is acute and $h < a < b$, two triangles exist.

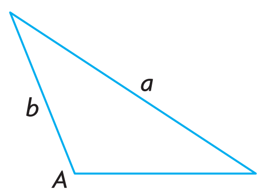


If $\angle A$, a, and b are given and $\angle A$ is obtuse, there are two cases to consider.

If $\angle A$ is obtuse and $a < b$ or $a = b$, no triangle exists.



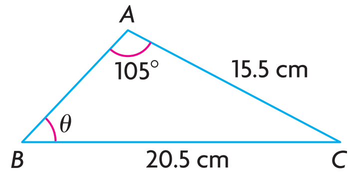
If $\angle A$ is obtuse and $a > b$, one triangle exists.



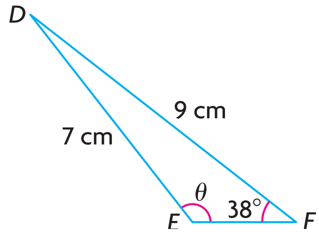
CHECK Your Understanding

1. Determine the measure of angle θ to the nearest degree.

a)



b)



2. A triangular plot of land is enclosed by a fence. Two sides of the fence are 9.8 m and 6.6 m long, respectively. The other side forms an angle of 40° with the 9.8 m side.

a) Draw a sketch of the situation.

b) Calculate the height of the triangle to the nearest tenth. Compare it to the given sides.

c) How many lengths are possible for the third side? Explain.

3. Determine whether it is possible to draw a triangle, given each set of information. Sketch all possible triangles where appropriate. Label all side lengths to the nearest tenth of a centimetre and all angles to the nearest degree.

a) $a = 5.2$ cm, $b = 2.8$ cm, $\angle B = 65°$

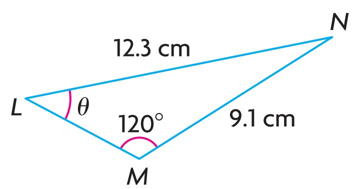
b) $b = 6.7$ cm, $c = 2.1$ cm, $\angle C = 63°$

c) $a = 5.0$ cm, $c = 8.5$ cm, $\angle A = 36°$

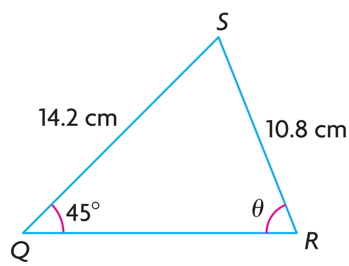
PRACTISING

4. Determine the measure of angle θ to the nearest degree.

a)



b)



5. K Where appropriate, sketch all possible triangles, given each set of information. Label all side lengths to the nearest tenth of a centimetre and all angles to the nearest degree.

a) $a = 7.2$ mm, $b = 9.3$ mm, $\angle A = 35°$

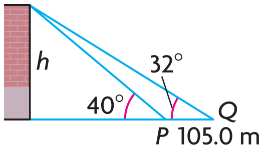
b) $a = 7.3$ m, $b = 14.6$ m, $\angle A = 30°$

c) $a = 1.3$ cm, $b = 2.8$ cm, $\angle A = 33°$

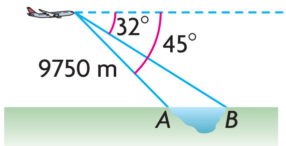
d) $c = 22.2$ cm, $\angle A = 75°$, $\angle B = 43°$

6. The trunk of a leaning tree makes an angle of 12° with the vertical. To prevent the tree from falling over, a 35.0 m rope is attached to the top of the tree and is pegged into level ground some distance away. If the tree is 20.0 m from its base to its top, calculate the angle the rope makes with the ground to the nearest degree.

7. **A** A building of height h is observed from two points, P and Q, that are 105.0 m apart as shown. The angles of elevation at P and Q are 40° and 32°, respectively. Calculate the height, h, to the nearest tenth of a metre.



8. A surveyor in an airplane observes that the angle of depression to two points on the opposite shores of a lake are 32° and 45°, respectively, as shown. What is the width of the lake, to the nearest metre, at those two points?



9. The Pont du Gard near Nîmes, France, is a Roman aqueduct. An observer in a hot-air balloon some distance away from the aqueduct determines that the angle of depression to each end is 54° and 71°, respectively. The closest end of the aqueduct is 270.0 m from the balloon. Calculate the length of the aqueduct to the nearest tenth of a metre.

10. **T** A wind tower at the top of a hill casts a shadow 30 m long along the side of the hill. An observer at the farthest edge of the shadow from the tower estimates the angle of elevation to the top of the tower to be 34°. If the slope of the hill is 13° from the horizontal, how high is the tower to the nearest metre?

11. Carol is flying a kite on level ground, and the string forms an angle of 50° with the ground. Two friends standing some distance from Carol see the kite at angles of elevation of 66° and 35°, respectively. One friend is 11 m from Carol. For each question below, state all possible answers to the nearest metre.
    a) How high is the kite above the ground?
    b) How long is the string?
    c) How far is the other friend from Carol?

12. The Huqiu Tower in China was built in 961 CE. When the tower was first built, its height was 47 m. Since then it has tilted 2.8°, so it is called China's Leaning Tower. There is a specific point on the ground where you can be equidistant from both the top and the bottom of the tower. How far is this point from the base of the tower? Round your answer to the nearest metre.

13. Your neighbour claims that his lot is triangular, with one side 430 m long and the adjacent side 110 m long. The angle opposite one of these sides is 35°. Determine the other side length of this lot to the nearest metre and the interior angles to the nearest degree.

14. **C** In $\triangle LMN$, $\angle L$ is acute. Using a sketch, explain the relationship between $\angle L$, sides l and m, and the height of the triangle for each situation.
    a) Only one triangle is possible.
    b) Two triangles are possible.
    c) No triangle is possible.

Extending

15. A sailor out in a lake sees two lighthouses 11 km apart along the shore and gets bearings of 285° from his present position for lighthouse A and 237° for lighthouse B. From lighthouse B, lighthouse A has a bearing of 45°.
    a) How far, to the nearest kilometre, is the sailor from both lighthouses?
    b) What is the shortest distance, to the nearest kilometre, from the sailor to the shore?

16. The *Algomarine* is a cargo ship that is 222.5 m long. On the water, small watercraft have the right of way. However, bulk carriers cruise at nearly 30 km/h, so it is best to stay out of their way: If you pass a cargo ship within 40 m, your boat could get swamped! Suppose you spot the *Algomarine* on your starboard (right) side headed your way. The bow and stern of the carrier appear separated by 12°. The captain of the *Algomarine* calls you from the bridge, located at the stern, and says that you are 8° off his bow.
    a) How far, to the nearest metre, are you from the stern?
    b) Are you in danger of being swamped?

17. The Gerbrandy Tower in the Netherlands is an 80 m high concrete tower, on which a 273.5 m guyed mast is mounted. The lower guy wires form an angle of 36° with the ground and attach to the tower 155 m above ground. The upper guy wires form an angle of 59° with the ground and attach to the mast 350 m above ground. How long are the upper and lower guy wires? Round your answers to the nearest metre.