MA2C03: TUTORIAL 13 PROBLEMS
GRAPH THEORY

1) Let (V, E) be the graph with vertices a, b, c, d, and e and edges ab, bd, be, ac, cd, and ae. Does this graph have a Hamiltonian circuit? Justify your answer.
2) For what type of n does the complete graph K_n have an Eulerian circuit? Justify your answer.
3) For what type of n does the complete graph K_n have an Eulerian trail? Justify your answer.
4) For what type of n does the complete graph K_n have a Hamiltonian circuit? Justify your answer.
5) For what type of p and q does the complete bipartite graph $K_{p,q}$ have an Eulerian circuit? Justify your answer.
6) For what type of p and q does the complete bipartite graph $K_{p,q}$ have an Eulerian trail? Justify your answer.
7) For what type of p and q does the complete bipartite graph $K_{p,q}$ have a Hamiltonian circuit? Justify your answer.

Solution: 1) Yes, it has a Hamiltonian circuit. One example is aebdca.

2) In a complete graph K_n every vertex is connected to every other vertex, so the degree of every vertex is $n-1$. By the theorem we proved in class, we need $n-1$ to be even, so n must be odd. Note that we need $n \geq 3$ to have a circuit in the first place, so for $n \geq 3$, n odd K_n has an Eulerian circuit.

3) Since all vertices have the same degree in the complete graph K_n, we cannot be in the case where all but two of the vertices have odd degree and the rest have even degree unless $n = 2$. Therefore, K_n has an Eulerian trail only for $n = 2$.

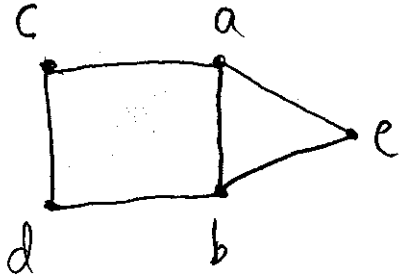
4) Since every vertex is connected to every other vertex in K_n, not only does K_n have a Hamiltonian circuit for every $n \geq 3$ but furthermore we can get a Hamiltonian circuit for every possible listing of the vertices where the first and the last coincide.

5) Recall that a bipartite graph satisfies that its vertices are partitioned into two sets V_1 and V_2 such that $V_1 \cap V_2 = \emptyset$ and $V_1 \cup V_2 = V$, the set of all vertices. In the case of the complete bipartite graph $K_{p,q}$, the number of elements in V_1 is p, and the number of elements in V_2 is q. Therefore, $\forall v \in V_1$, deg v $= q$, and $\forall v \in V_2$, deg v $= p$ as the graph is a complete bipartite graph. For the degrees of all vertices to be even,

we must have that both p and q are even to guarantee the existence of an Eulerian circuit. Furthermore, the total number of vertices should be at least 3 for a circuit to exist, so $p \geq 2$, $q \geq 2$ and both are even.

6) Either $p \geq 1$ is odd and $q = 2$ or vice versa $p = 2$ and $q \geq 1$ is odd as we need two vertices to have odd degree and the rest to have even degrees and the degrees of vertices in the same set of the partition, V_1 and V_2, is the same.

7) We must have $p = q \geq 2$ for a Hamiltonian circuit to exist as we hop from a vertex in V_1 to a vertex in V_2 and back.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