Q: Can we represent $X = \frac{1}{2^{p_1}} + \frac{1}{2^{p_2}} + \dots + \frac{1}{2^{p_k}}$ in an easier to understand form?

A: Yes, we bring the fractions to the same denominator:

$$x = \frac{1}{2^{p_1}} + \frac{1}{2^{p_2}} + \dots + \frac{1}{2^{p_k}} = \frac{2^{p_k-p_1}}{2^{p_k-p_1}\cdot 2^{p_1}} + \frac{2^{p_k-p_2}}{2^{p_k-p_2}\cdot 2^{p_2}} + \dots + \frac{2^{p_k-p_{k-1}}}{2^{p_k-p_{k-1}}\cdot 2^{p_{k-1}}} +$$

$$+ \frac{1}{2^{p_k}} = \frac{2^{p_k-p_1} + 2^{p_k-p_2} + \dots + 2^{p_k-p_{k-1}} + 1}{2^{p_k}} = \frac{\text{odd natural number}}{\text{power of } 2}$$

$= \frac{m}{2^n}$ for $m \in \mathbb{N}$ odd and $n \in \mathbb{N}^*$ as $p_1 < p_2 < \dots < p_k$ so the differences $p_k - p_1, p_k - p_2, \dots, p_k - p_{k-1}$ are all positive integers.

So the sequence in $(0,1)$ that has two decimal binary expansions is $\{\frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}, \dots\} = B$. Note that B is countably infinite as each set $B_n = \{0 < \frac{odd}{2^n} < 1\}$ is finite, $B = \bigcup_{n=1}^{\infty} B_n$ is countable by our corollary, and the countably infinite sequence $\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots\} \subseteq B$, which means the countable set B must be countably infinite. Now let us examine the binary expansions of the elements $y \in B$. $\forall y \in B$, $y = \frac{1}{2^{p_1}} + \frac{1}{2^{p_2}} + \dots + \frac{1}{2^{p_k}}$ for $p_1, \dots, p_k \in \mathbb{N}^*$, $p_1 < p_2 < \dots < p_k$. The two binary expansions corresponding to y, $b_{y,1}$ and $b_{y,2}$ are of the form $0.x_1 x_2 \dots x_{p_k-1} x_{p_k} x_{p_k+1} \dots$, where $x_1, \dots, x_{p_k-1}$ are common to $b_{y,1}$ and $b_{y,2}$, whereas $x_{p_k}, x_{p_k+1}, \dots$ differ. Now $x_j = \begin{cases} 1 & \text{if } j = p_1, p_2, \dots, p_{k-1} \\ 0 & \text{otherwise} \end{cases}$ for $1 \le j \le p_k$ is the common part corresponding to $\frac{1}{2^{p_1}} + \frac{1}{2^{p_2}} + \dots + \frac{1}{2^{p_{k-1}}}$, whereas the difference comes from the two possible ways of representing the last term in the sum $\frac{1}{2^{p_k}}$, namely $10000\dots$ or $011111\dots$. Therefore, $b_{y,1}$ has $x_{p_k} = 1$ and $x_j = 0$ $\forall j > p_k$ (corresponding to $1000\dots$), whereas $b_{y,2}$ has $x_{p_k} = 0$ and $x_j = 1$ $\forall j > p_k$ (corresponding to $0111\dots$). Let $s_{y,1} \in A$ be the sequence corresponding to $b_{y,1}$ in A, the set of all sequences of 0's and 1's, i.e. if $b_{y,1} = 0.x_1 x_2 x_3 \dots$ $s_{y,1} = \{x_1, x_2, x_3, \dots\}$. Let $s_{y,2} \in A$ be the sequence corresponding to $b_{y,2}$. We now define $B_1 = \{b_{y,1} \mid y \in B\}$, $B_2 = \{b_{y,2} \mid y \in B\}$, $A_1 = \{s_{y,1} \mid y \in B\}$, $A_2 = \{s_{y,2} \mid y \in B\}$. B is in one-to-one correspondence to B_1, B_2, A_1, A_2 by construction, so $B \sim B_1$, $B \sim B_2$, $B \sim A_1$, $B \sim A_2$, but B is countably infinite $\Rightarrow A_1, A_2, B_1, B_2$ are all countably infinite.

We have just one more observation to make regarding the correspondence between sequences of 0's and 1's in A and elements of $(0,1)$, namely that the ~~two~~ zero sequence $\{0, 0, \ldots\}$ corresponds to the binary expansion $0.000\cdots = 0 \notin (0,1)$ since $(0,1) = \{x \in \mathbb{R} \mid 0 < x < 1\}$ and the one sequence $\{1, 1, 1, \ldots\}$ corresponds to the binary expansion $0.11111 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = \sum_{n=1}^{\infty} \frac{1}{2^n} = 1 \notin (0,1)$. Now we can finally prove that $(0,1)$ is uncountably infinite:

<u>Proposition</u> $(0,1)$ is uncountably infinite.

<u>Proof</u> We define a map $f : (0,1) \to \{0.x_1x_2x_3\ldots \mid x_j \in \{0,1\}\ \forall j \geq 1\}$ as follows

$$f(y) = \begin{cases} b_{y,1} & \text{if } y \in B \quad \leftarrow \text{the first of the two possible binary expansions} \\ 0.x_1x_2\cdots & \text{if } y \notin B \quad \leftarrow \text{the unique binary expansion.} \end{cases}$$

By our previous discussion, f is a bijection as defined $\Rightarrow$ $(0,1) \sim \{0.x_1x_2x_3\ldots \mid x_j \in \{0,1\}\ \forall j \geq 1\}$. Also by our previous discussion $\{0.x_1x_2x_3\ldots \mid x_j \in \{0,1\}\ \forall j \geq 1\} \sim A \setminus (A_2 \cup \{0,0,\ldots\} \cup \{1,1,\ldots\})$

set of all sequences of 0's and 1's — the constant zero sequence — the constant one sequence

Therefore, $(0,1) \sim A \setminus (A_2 \cup \{0,0,\ldots\} \cup \{1,1,\ldots\})$ since $\sim$ is transitive (it is an equivalence relation).