Contradiction

A *contradiction* is always false. What is proven is always true and since a contradiction is always false it cannot be proven.
Central to mathematics is the concept of a proof, i.e. given a set of premises, how can one infer, derive or deduce the conclusion. A conclusion is deducible from a set of premises if it is not possible for all the premises to be true and the conclusion false.
For example, the conclusion, Q, can be deduced from the set of premises $\{P, P \rightarrow Q\}$ i.e. it is not possible for both P and $P \rightarrow Q$ to be true and the sentence Q to be false.
If P and $P \rightarrow Q$ are both true then Q must also be true.
From the Truth Table of the operator, $\rightarrow$, the only situation where $P \rightarrow Q$ is false is when P is true and Q is false. Therefore, if we suppose Q to be false and we have assumed that P is true, then $P \rightarrow Q$ is false contradicting our assumption that $P \rightarrow Q$ is true.

Tautology

The sentences in Propositonal Logic that are always true are called **Tautologies**, e.g. $P \vee \neg P$ is a Tautology.
If a propositional sentence P is a contradiction i.e. always false (e.g. $P \wedge \neg P$) then its negation is a Tautology i.e. always true.
Since $P \wedge \neg P$ is a contradiction, then $\neg(P \wedge \neg P)$ is a Tautology.
By De Morgan's Law, $\neg(P \wedge \neg P) = \neg P \vee P$.
In Propositional Logic, Tautologies correspond to Logical Truths.

Tautology (Cont'd)

Definition

Tautology
If a propositional sentence, P, evaluates to T in all states (i.e. P is always true) then P is a **Tautology**. In particular, the constant, *True* i.e. $\top$ is a Tautology.

Notation $\models$

"P is a Tautology" is abbreviated as "$\models P$". The symbol $\models$ is pronounced 'double turnstyle' as there is also a 'single turnstyle' symbol $\vdash$.

Read "$\models P$" as "P is a Tautology".

Tautology (Cont'd)

If P is a Tautology then it is always true, i.e. P is true no matter what values are assigned to the variables in P. We can check whether a sentence, P, is a Tautology by using a Truth Table.
Example: $\models p \vee \neg p$ as

p	$p \vee \neg p$
F	T
T	T

Exercise: Check $\models P \equiv \neg\neg P$.

$\models p \rightarrow q \equiv \neg q \rightarrow \neg p$

Example: $\models p \rightarrow q \equiv \neg q \rightarrow \neg p$ as

p	q	$p \rightarrow q$	$\equiv$	$\neg$	q	$\rightarrow$	$\neg$	p
F	F	**T**	**T**	T		**T**	T	
F	T	**T**	**T**	F		**T**	T	
T	F	**F**	**T**	T		**F**	F	
T	T	**T**	**T**	F		**T**	F	

$\models p \wedge q \equiv p \equiv q \equiv p \vee q$

Example: Show $\models p \wedge q \equiv p \equiv q \equiv p \vee q$
Restoring brackets we get: $((((p \wedge q) \equiv p) \equiv q) \equiv (p \vee q))$

p	q	$(((p \wedge q)$	$\equiv$	$p)$	$\equiv$	$q)$	$\equiv$	$(p \vee q)$
F	F	F	T		**F**		**T**	**F**
F	T	F	T		**T**		**T**	**T**
T	F	F	F		**T**		**T**	**T**
T	T	T	T		**T**		**T**	**T**

It can be shown that the operator, $\equiv$, is associative, i.e.

$$(p \equiv q) \equiv r \; = \; p \equiv (q \equiv r)$$

As a consequence we can rewrite $p \wedge q \equiv p \equiv q \equiv p \vee q$ for example, as $(p \wedge q \equiv p) \equiv (q \equiv p \vee q)$ or as $p \wedge q \equiv (p \equiv q \equiv p \vee q)$.

Logical Implication

Logical Implication vs Conditional

There is a close connection between the conditional operator, $\rightarrow$, and mathematical/logical implication.
In mathematics, 'implies' normally means 'logically implies'
For propositional sentences, P and Q,

P **Logically Implies** Q iff $P \rightarrow Q$ is a Tautology.

We use "$P \models Q$ " for "P Logically Implies Q", $\therefore$

$$P \models Q \quad \mathit{iff} \quad \models P \rightarrow Q$$

If P **Logically Implies** Q then whenever P is True then so is Q, i.e. it is not possible for P to be True and Q False. If $P \rightarrow Q$ is a Tautology (always True) then there is no state in which P is True and Q is False as if there were such at state then $P \rightarrow Q$ would be False in that state.

Logical Implication (Cont'd)

More generally, if Ps is a set of sentences and P and Q are sentences then

$$Ps \cup \{P\} \models Q \quad \textit{iff} \quad Ps \models P \rightarrow Q$$

$Ps \cup \{P\}$ can be abbreviated to Ps, P so that

$$Ps,\ P \models Q \quad \textit{iff} \quad Ps \models P \rightarrow Q$$

Consider $Ps \models Q$: if Ps is empty, i.e. $\{\} \models Q$ then

$$\{\} \models Q \quad \textit{iff} \quad \models Q$$

i.e. if Q is logically implied from the empty set of premises then Q is a Tautology.

Premises and Conclusion = Argument

Let the set of propositional sentences $Ps = \{P_1, P, \ldots, P_n\}$ then $Ps \models Q$ can be written as

$$P_1,\ P_2, \ldots,\ P_n \models Q$$

$P_1,\ P_2, \ldots,\ P_n$ are the Premises and Q is the Conclusion.
An argument with premises, $P_1,\ P_2, \ldots,\ P_n$ and a conclusion, Q, is valid iff $P_1,\ P_2, \ldots,\ P_n \models Q$
i.e. an argument is valid if the premises logically imply the conclusion.
An argument is valid whenever the premises are True then so is the conclusion, i.e. it is not possible for the all the premises to be True and the conclusion False.

Premises and Conclusion (Cont'd)

Since

$$P_1, P_2, \ldots, P_n \models Q$$

is the same as

$$P_1, P_2, \ldots, P_{n-1} \models P_n \to Q$$

we get by continuing,

$$P_1, P_2, \ldots, P_n \models Q \text{ iff } \models P_1 \to P_2 \to \cdots \to P_n \to Q$$

Recall that the operator $\to$ associates to the right.

Alternative Notation: $P_1, P_2, \ldots, P_n \models Q$ can be written as

$$\frac{P_1, P_2, \ldots, P_n}{Q}$$

Proof by Contradiction

Negation

If $P \models \bot$ then $\models P \to \bot$ i.e. $\models \neg P$.
$\therefore$ $\neg P$ is a Tautology, i.e. P is a contradiction.
i.e. if $P \models \bot$ then P is a contradiction.

Also, if $\neg P \models \bot$ then $\models \neg P \to \bot$ i.e. $\models \neg\neg P$ i.e. $\models P$
$\therefore$ if $\neg P$ implies a contradiction then P is a Tautology.

Transitivity of $\rightarrow$

Transitivity of $\rightarrow$

For any propositional sentences, P, Q and R,
the argument with premises, $P \rightarrow Q,\ Q \rightarrow R$ and conclusion
$P \rightarrow R$
is valid i.e.
$P \rightarrow Q,\ Q \rightarrow R \models P \rightarrow R$ i.e.

$$\frac{P \rightarrow Q,\ Q \rightarrow R}{P \rightarrow R}$$

as it is not possible for both $P \rightarrow Q$ and $Q \rightarrow R$ to be true and $P \rightarrow R$ to be false.

Transitivity of $\rightarrow$(Cont'd)

Show $P \rightarrow Q,\ Q \rightarrow R \models\ P \rightarrow R$
i.e. show $P \rightarrow Q\ \models\ (Q \rightarrow R) \rightarrow P \rightarrow R$
i.e. show $\models (P \rightarrow Q) \rightarrow (Q \rightarrow R) \rightarrow P \rightarrow R$
Restoring brackets, show $\models (P \rightarrow Q) \rightarrow ((Q \rightarrow R) \rightarrow (P \rightarrow R))$

P	Q	R	$(P \rightarrow Q)$	$\rightarrow$	$((Q \rightarrow R)$	$\rightarrow$	$(P \rightarrow R))$
F	F	F	**T**	**T**	T	**T**	T
F	F	T	**T**	**T**	T	**T**	T
F	T	F	**T**	**T**	F	**T**	T
F	T	T	**T**	**T**	T	**T**	T
T	F	F	**F**	**T**	T	**F**	F
T	F	T	**F**	**T**	T	**T**	T
T	T	F	**T**	**T**	F	**T**	F
T	T	T	**T**	**T**	T	**T**	T

Logically Equivalent

Logically Equivalent

P is **logically equivalent** to Q iff $\models P \equiv Q$

Use "$P = Q$" for "P is logically equivalent to Q" $\therefore$

$$P = Q \quad iff \quad \models P \equiv Q$$

In particular,

$$P = Q \quad iff \quad P \models Q \; and \; Q \models P$$

Double Negation

$P = \neg\neg P$ as $\models P \equiv \neg\neg P$

De Morgan's Laws

De Morgan's Laws

Show $\neg(P \wedge Q) = \neg P \vee \neg Q$ i.e. show $\models \neg(P \wedge Q) \equiv \neg P \vee \neg Q$

P	Q	$\neg$	$(P$	$\wedge$	$Q)$	$\equiv$	$\neg P$	$\vee$	$\neg Q$
F	F	**T**		F		**T**	T	**T**	T
F	T	**T**		F		**T**	T	**T**	F
T	F	**T**		F		**T**	F	**T**	T
T	T	**F**		T		**T**	F	**F**	F

Also,

- $\neg(P \vee Q) = \neg P \wedge \neg Q$

Set Theory Application

Recall from Set Theory: $\overline{X \cap Y} = \overline{X} \cup \overline{Y}$.
Taking advantage of the Tautology $\neg(P \wedge Q) \equiv \neg P \vee \neg Q$ an alternative proof of $\overline{X \cap Y} = \overline{X} \cup \overline{Y}$ can be given.
Show $\overline{X \cap Y} = \overline{X} \cup \overline{Y}$

Proof.

$z \in \overline{X \cap Y}$
$\equiv \neg(z \in X \cap Y)$
$\equiv \neg(z \in X \wedge z \in Y)$
$\{$Tautology: $\neg(P \wedge Q) \equiv \neg P \vee \neg Q$ $\}$
$\equiv \neg(z \in X) \vee \neg(z \in Y)$
$\equiv z \in \overline{X} \vee z \in \overline{Y}$
$\equiv z \in \overline{X} \cup \overline{Y}$ $\square$

$P \to Q \to R = P \land Q \to R$

Show $P \to Q \to R = P \land Q \to R$
i.e. show $\models P \to (Q \to R) \equiv P \land Q \to R$

P	Q	R	P	$\to$	$(Q \to R)$	$\equiv$	$P \land Q$	$\to$	R
F	F	F	F	*T*	T	**T**	F	*T*	F
F	F	T	F	*T*	T	**T**	F	*T*	T
F	T	F	F	*T*	F	**T**	F	*T*	F
F	T	T	F	*T*	T	**T**	F	*T*	T
T	F	F	T	*T*	T	**T**	F	*T*	F
T	F	T	T	*T*	T	**T**	F	*T*	T
T	T	F	T	*F*	F	**T**	T	*F*	F
T	T	T	T	*T*	T	**T**	T	*T*	T

Cont'd

Since $P_1, P_2 \models Q$ is the same as $\models P_1 \to P_2 \to Q$ and $P_1 \to P_2 \to Q = P_1 \wedge P_2 \to Q$ then $P_1, P_2 \models Q$ is the same as $\models P_1 \wedge P_2 \to Q$.
But, $\models P_1 \wedge P_2 \to Q$ is the same as $P_1 \wedge P_2 \models Q$ $\therefore$

$$P_1, P_2 \models Q \text{ is the same as } P_1 \wedge P_2 \models Q \;.$$

More generally,

$$P_1,\ P_2, \ldots,\ P_n \models Q \text{ is the same as } P_1 \wedge P_2, \cdots \wedge P_n \models Q$$

Proving an Argument

Is the following argument valid:

Either the vicar or the butler shot the earl.
If the butler shot the earl then the butler was not drunk at 9pm.
Either the vicar is a liar or the butler was drunk at 9pm

therefore

either the vicar is a liar or he shot the earl

Abbreviate the propositions in the above argument.

- s: the vicar shot the earl
- b: the butler shot the earl
- d: the butler was drunk at 9pm
- c: the vicar is a liar

Argument

Need to show the following:

$$\frac{s \vee b,\;\; b \to \neg d,\;\; c \vee d}{c \vee s}$$

i.e. show

$$s \vee b,\;\; b \to \neg d,\;\; c \vee d \models c \vee s$$

To show this, show

$$\models s \vee b \to (b \to \neg d) \to c \vee d \to c \vee s$$

Show by Truth Table

To show $\models s \vee b \to (b \to \neg d) \to c \vee d \to c \vee s$, use a truth table and show that in all states it is true. Since the sentence has 4 variables, 16 rows are needed in the Truth Table.
(Restore brackets for clarity; recall $\to$ is right associative)

s	b	d	c	$(s \vee b)$	$\to$	$((b \to \neg d) \to ((c \vee d) \to (c \vee s)))$
F	F	F	F		T	
					T	
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	
					T	
T	T	T	T		T	

From the truth table, conclude that

$$\models s \vee b \to (b \to \neg d) \to c \vee d \to c \vee s$$

Therefore the original argument is valid.

Alternative

From above, since $P \to Q \to R = P \wedge Q \to R$ rewrite

$$\models s \vee b \to (b \to \neg d) \to c \vee d \to c \vee s$$

as

$$\models (s \vee b) \wedge (b \to \neg d) \wedge (c \vee d) \to c \vee s$$

Using Short Cuts to create Truth Table

Using Short Cuts for Evaluating sentences

Consider the Truth Table for $(s \vee b) \wedge (b \rightarrow \neg d) \wedge (c \vee d) \rightarrow c \vee s$.

If any of $(s \vee b), (b \rightarrow \neg d), (c \vee d)$ is F, then $(s \vee b) \wedge (b \rightarrow \neg d) \wedge (c \vee d)$ is F.

From the Truth Table for the operator, $\rightarrow$, if P is F then $P \rightarrow Q$ is T and also if Q is T then $P \rightarrow Q$ is T.

From the Truth Table for the operator, $\vee$, if either of P or Q is T then so is $P \vee Q$.

Truth Table with Short Cuts

First 8 rows in Truth Table

s	b	d	c	$((s \vee b)$	$\wedge$	$(b \to \neg d)$	$\wedge$	$(c \vee d))$	$\to$	$(c \vee s)$
F	F	F	F	F					**T**	
F	F	F	T	F					**T**	
F	F	T	F	F					**T**	
F	F	T	T	F					**T**	
F	T	F	F					F	**T**	
F	T	F	T						**T**	T
F	T	T	F			F			**T**	
F	T	T	T			F			**T**	

Cont'd

Next 8 rows in Truth Table

Since in the following 8 rows, the variable, s, is T the so is $c \vee s$.

s	b	d	c	$((s \vee b)$	$\wedge$	$(b \to \neg d)$	$\wedge$	$(c \vee d))$	$\to$	$(c \vee s)$
T	F	F	F						**T**	T
T	F	F	T						**T**	T
T	F	T	F						**T**	T
T	F	T	T						**T**	T
T	T	F	F						**T**	T
T	T	F	T						**T**	T
T	T	T	F						**T**	T
T	T	T	T						**T**	T

Cont'd

In general,

$$P, P_2, \ldots, P_n \models Q$$

may be rewitten as

$$P_1 \wedge P_2, \cdots \wedge P_n \models Q$$

i.e.

$$\models P_1 \wedge P_2, \cdots \wedge P_n \rightarrow Q$$

To show:

$$P, P_2, \ldots, P_n \text{ logically implies } Q$$

we show

$$P_1 \wedge P_2, \cdots \wedge P_n \rightarrow Q \text{ is a Tautology}$$

Refutation Approach

Refutation Approach

The Rufutation Approach may be more efficient than using Truth Tables.
To show
$\models (s \vee b) \wedge (b \rightarrow \neg d) \wedge (c \vee d) \rightarrow c \vee s$
show that there is no state in which
$((s \vee b) \wedge (b \rightarrow \neg d) \wedge (c \vee d)) \rightarrow c \vee s$ is F.
That is, attempt to refute $(s \vee b) \wedge (b \rightarrow \neg d) \wedge (c \vee d) \rightarrow c \vee s$
i.e. attempt to make it false.
Attempted Refutation:
Suppose $(s \vee b) \wedge (b \rightarrow \neg d) \wedge (c \vee d) \rightarrow c \vee s$ is F
then
$(s \vee b) \wedge (b \rightarrow \neg d) \wedge (c \vee d)$ is T and
$c \vee s$ is F.

Refutation (Cont'd)

i.e.

$$\begin{array}{cc} ((s \vee b) \wedge (b \rightarrow \neg d) \wedge (c \vee d)) & c \vee s \\ T & F \end{array}$$

$\therefore$ from the properties of $\wedge$ and $\vee$,

$$\begin{array}{ccccc} (s \vee b) & (b \rightarrow \neg d) & (c \vee d) & c & s \\ T & T & T & F & F \end{array}$$

1. From $c = F$ then $d = T$ since $(c \vee d) = T$.
2. From $s = F$ and $(s \vee b) = T$, then $b = T$
3. Since $b = T$ and $(b \rightarrow \neg d) = T$, then $\neg d = T$, i.e. $d = F$.
4. From 1. it is deduced that $d = T$ and from 3. that $d = F$ and therefore have a conflict i.e. a contradiction.
5. $\therefore$ the attempt at refutation failed.

Cont'd

From the above, it is not possible to make $((s \vee b) \wedge (b \rightarrow \neg d) \wedge (c \vee d)) \rightarrow c \vee s$ false.

The argument

Either the vicar or the butler shot the earl.
If the butler shot the earl then the butler was not drunk at 9pm.
Either the vicar is a liar or the butler was drunk at 9pm

therefore

either the vicar is a liar or he shot the earl

is valid as it is not possible to make all the premises $(s \vee b), (b \rightarrow \neg d), (c \vee d)$ true and the conclusion, $c \vee s$, false.