MA2C03: ASSIGNMENT 1
DUE BY FRIDAY, NOVEMBER 18
SOLUTION SET PART II

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4) (a) Give an example of a relation that is reflexive but **NOT** symmetric or transitive. Justify your answer.
(b) Give an example of a relation that is symmetric but **NOT** reflexive or transitive. Justify your answer.
(c) Give an example of a relation that is transitive but **NOT** reflexive or symmetric. Justify your answer.

(Hint: As your examples, you may either use relations that you have encountered before in predicate logic or other contexts in mathematics **OR** construct your own relations. If you construct your own, use the simplest scenario that fits what you are trying to do.)

Solution.

(a) Let $A = \{1, 2, 3\}$ and define a relation R_A on A by;
$$R_A = \{(1,1),(2,2),(3,3),(1,2),(2,3)\}$$
Then say $xR_Ay \Leftrightarrow (x, y) \in R_A$.
R_A is reflexive by definition, but is not symmetric (as $1R_A2$ but $\neg(2R_A1)$) and is not transitive (as $1R_A2$ and $2R_A3$ but not $1R_A3$).

(b) "Not equal to" ($\neq$) on $\mathbb{R}$ is by definition not reflexive, but is symmetric ($1 \neq 2 \Rightarrow 2 \neq 1$). It is also not transitive - if it was, $1 \neq 2 \wedge 2 \neq 1 \Rightarrow 1 \neq 1$, a contradiction.
Also, students could reference $Q3$ for this problem.

(c) $<$ on $\mathbb{R}$ fits these requirements exactly. It's not reflexive ($5 \not< 5$), not symmetric ($3 < 5 \not\Rightarrow 5 < 3$) but it is transitive ($a < b < c \Rightarrow a < c$).

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Grading rubric: 10 points total - 5 points for (a), 3 points for (b) and 2 points for (c).
Common mistakes: For (a), the relation one defines doesn't have to already exist, or have a name. The main problem was students not

being clear on the definitions of reflexive, symmetric and transitive, or forgetting to prove one of the properties.

5) Use mathematical induction to prove the geometric series formula, which states that for any $a, r \in \mathbb{R}$ with $r \neq 1$ and any $n \in \mathbb{N}^*$,

$$a + ar + ar^2 + \cdots + ar^{n-1} = a\frac{(1-r^n)}{(1-r)}.$$

Solution.
Fix $a, r \neq 1 \in \mathbb{N}$.
Base case: $n = 1$.
Then

$$a\frac{(1-r^1)}{(1-r)} = a(1) = a$$

as required.
Induction step: Assume true for $n = k$.
Prove true for $n = k + 1$.

$$a + ar + ar^2 + \cdots + ar^{k-1} + ar^k = a\frac{(1-r^k)}{(1-r)} + ar^k$$
$$= a\left(\frac{(1-r^k)}{(1-r)} + \frac{(1-r)r^k}{(1-r)}\right) = a\left(\frac{(1-r^k) + (r^k - r^{k+1})}{(1-r)}\right)$$
$$= a\frac{(1-r^{k+1})}{(1-r)}$$

as required.
Therefore by induction the proposition holds, namely

$$a + ar + ar^2 + \cdots + ar^{n-1} = a\frac{(1-r^n)}{(1-r)}.$$

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Grading rubric: 10 points total - 3 points for proving the base case, 2 points for stating the assumption and 5 points for proving the inductive step.
Common mistakes: The main problem was in the "Prove true for $n = k + 1$" step, students were taking

$$a + ar + ar^2 + \cdots + ar^k = a\frac{(1-r^{k+1})}{(1-r)}$$

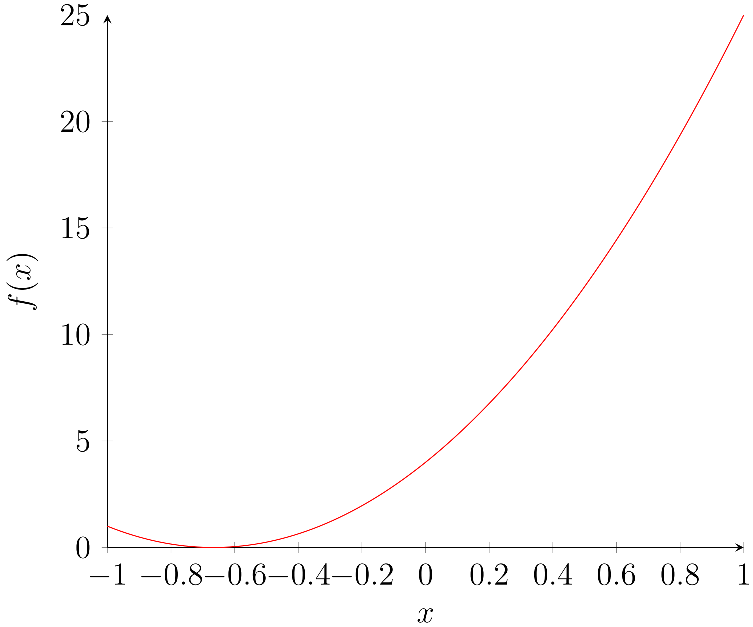
then showing "$a = a$" or some other tautology. This is logically incorrect; showing

$$P(k+1) \Rightarrow \text{Tautology}$$

is true doesn't show $P(k+1)$ is true.
Although it didn't lose one marks, students should still write a small blurb at the end of their induction as a conclusion so the proof doesn't end abruptly.

6) Let $f : [-1,1] \to [0,25]$ be the function defined by $f(x) = 9x^2 + 12x + 4$ for all $x \in [-1,1]$. Determine whether or not this function is injective and whether or not it is surjective. Justify your answers. Recall that $[-1,1]$ is the set of all real numbers between -1 and 1 with the endpoints of -1 and 1 included in the set.

Solution.
Plotting the function we get:



By the horizontal line test this function is not injective;
Note $9x^2 + 12x + 4 = (3x+2)^2$ so it has a double root at $x = -\frac{2}{3}$. By taking the derivative we see this is a minimum; thus on either side of $x = -\frac{2}{3}$ the function is increasing. It then fails the horizontal line test because of this.
It's also possible to disprove injectivity by giving an example; the most popular one was noting $f(-1) = f(-\frac{1}{3}) = 1$. Thus

$$\exists x, y((x \neq y) \wedge (f(x) = f(y)))$$

disproving injectivity, which states

$$\forall x, y((x \neq y) \Rightarrow (f(x) \neq f(y)))$$

This function is surjective; as $f(-\frac{2}{3}) = 0$ and $f(1) = 25$ by the intermediate value theorem f takes every value in between. (The intermediate value theorem relies on the continuity of f.) ■

Grading rubric: 10 points total - 5 points for disproving injectivity with 2 for the answer and 3 for the justification and 5 points for proving surjectivity with 2 for the answer and 3 for the justification.
Common mistakes: There were a number of answers saying something along the line of "the domain and the range share elements therefore the function can't be injective" or "$0, 1 \in [-1, 1]$ and $0, 1 \in [0, 25]$ therefore not injective". In general what is in the domain and range is irrelevant until we take the function into consideration, thus these answers received no marks. As a counter example to these answers: Consider the function

$$f : [-1, 1] \to [-1, 1] \qquad \text{given by} \qquad f(x) = x$$

This is an injective function, and the range and the domain are equal.

7) In class we introduced $\mathbb{Z}_n$ as the set of equivalence classes determined by division mod n for $n \geq 1$, $n \in \mathbb{N}$. Let $n = 11$. Consider the binary operation $*$ on $\mathbb{Z}_{11}$ defined by

$$x * y = x \oplus_{11} y \oplus_{11} 2$$

for all $x, y \in \mathbb{Z}_{11}$. Recall that by definition $x \oplus_{11} y = x + y \bmod 11$. What is the identity element for $(\mathbb{Z}_{11}, *)$? Prove that $(\mathbb{Z}_{11}, *)$ is a monoid. Which elements of $(\mathbb{Z}_{11}, *)$ are invertible? Is $(\mathbb{Z}_{11}, *)$ a group? Justify your answers.

Solution.

(1) 9 is the identity element for $\mathbb{Z}_{11}$; this follows as for all $x \in \mathbb{Z}_{11}$

$$x * 9 = x \oplus_{11} 9 \oplus_{11} 2 = x \oplus_{11} (9 \oplus_{11} 2) = x \oplus_{11} 0 = x$$

Similarly $9 * x = x$;

$$9 * x = 9 \oplus_{11} x \oplus_{11} 2 = x \oplus_{11} 9 \oplus_{11} 2 = x$$

- here we use the commutativity of $\oplus_{11}$, which follows from the commutativity of $+$. Note that we are also using the associativity of $\oplus_{11}$, which follows from the associativity of $+$ on $\mathbb{Z}$. The Cayley table is as follows:

Element		0	1	2	3	4	5	6	7	8	9	10
0		2	3	4	5	6	7	8	**9**	10	0	1
1		3	4	5	6	7	8	**9**	10	0	1	2
2		4	5	6	7	8	**9**	10	0	1	2	3
3		5	6	7	8	**9**	10	0	1	2	3	4
4		6	7	8	**9**	10	0	1	2	3	4	5
5		7	8	**9**	10	0	1	2	3	4	5	6
6		8	**9**	10	0	1	2	3	4	5	6	7
7		**9**	10	0	1	2	3	4	5	6	7	8
8		10	0	1	2	3	4	5	6	7	8	**9**
9		0	1	2	3	4	5	6	7	8	**9**	10
10		1	2	3	4	5	6	7	8	**9**	10	0

(2) $(\mathbb{Z}_{11}, *)$ is a monoid as $\oplus_{11}$ is a binary operation (closed) and by (1) the element 9 is the identity. We also note $*$ is associative; this follows from the associativity of $\oplus_{11}$.

(3) The following elements are invertible:

Element	0	1	2	3	4	5	6	7	8	9	10
Inverse	7	6	5	4	3	2	1	0	10	9	8

(by reading off of the table). Point of possible confusion: 0 is not the identity to this monoid; 9 is. Thus an element and its inverse should $*$ to 9, not 0.

(4) $(\mathbb{Z}_{11}, *)$ is a group as it's a monoid where every element is invertible.

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Grading rubric: 10 points total - 1 point for naming the identity element, 5 points for proving it's a monoid, 3 points for identifying all of the invertible elements, 1 point for saying it's a group.
Common mistakes: Nearly everyone lost a mark in proving that 9 was the identity; to prove e is an identity one needs to show

$$\forall x(x * e = e * x = x)$$

Just showing $x * e = x$ doesn't cut it. A number or people also lost marks proving things about commutativity - the operation of a monoid or group doesn't *need* to be commutative; by proving (or attempting to prove) it is commutative makes it seem one doesn't know the definitions.