MA2C03: TUTORIAL 17 PROBLEMS
COUNTABILITY OF SETS

For each of the following sets, determine whether it is finite, countably infinite, or uncountably infinite. Justify your answer.

1) $\{a^p \mid p \in \mathbb{N} \text{ and } a = e^{q\pi i} \text{ for } q \in \mathbb{Q}\}$

2) $\{a^p \mid p \in \mathbb{N} \text{ and } a = e^{q\pi i} \text{ for } q \in \mathbb{R} \setminus \mathbb{Q}\}$

3) $\{(x, y, z) \in \mathbb{N}^3 \mid x^2 + y^2 = z^2 \text{ and } x, y, z \in \mathbb{N}^*\}$, the Pythagorean triplets that give the lengths of the legs and the hypothenuse of a right triangle.

4) $\displaystyle\bigcup_{q \in \mathbb{Q}} L_q$ where $L_q = \{(x, y) \in \mathbb{R}^2 \mid x = q\}$.

5) $\{(x, y) \in \mathbb{R}^2 \mid y = x^2\}$

6) $\mathcal{P}(J_n) \times \mathcal{P}(\mathbb{N})$, where $J_n = \{1, \ldots, n\}$ and $\mathcal{P}(A)$ is the power set of a set A.

7) $\mathbb{R}^n$ for $n \geq 1$.

8) Prove that the language generated by a regular expression is countable. Give an example of a regular expression that generates a finite language and another example of a regular expression that generates a countably infinite language. Justify your answers.

Solution: 1) $\{a^p \mid p \in \mathbb{N} \text{ and } a = e^{q\pi i} \text{ for } q \in \mathbb{Q}\}$ is a finite set. Let $q = \dfrac{r}{s}$ for $r, s \in \mathbb{Z}$, $s \neq 0$, $(r, s) = 1$. Therefore, $a^p = e^{\frac{pr\pi i}{s}}$, which assumes one of s values $e^{\frac{\pi i}{s}}, e^{\frac{2\pi i}{s}}, \ldots, e^{\frac{(s-1)\pi i}{s}}, e^{\frac{s\pi i}{s}}$ depending upon the value of p. We conclude that our set is finite

$$\{a^p \mid p \in \mathbb{N} \text{ and } a = e^{q\pi i} \text{ for } q \in \mathbb{Q}\} = \left\{e^{\frac{\pi i}{s}}, e^{\frac{2\pi i}{s}}, \ldots, e^{\frac{(s-1)\pi i}{s}}, e^{\frac{s\pi i}{s}}\right\}.$$

2) $A = \{a^p \mid p \in \mathbb{N} \text{ and } a = e^{q\pi i} \text{ for } q \in \mathbb{R} \setminus \mathbb{Q}\}$ is countably infinite. Since $q \in \mathbb{R} \setminus \mathbb{Q}$, $a^{p_1} \neq a^{p_2}$ if $p_1 \neq p_2$, so the map $f : \mathbb{N} \to A$ given by $f(p) = a^p$ is a bijection. Therefore,

$$A = \{a^p \mid p \in \mathbb{N} \text{ and } a = e^{q\pi i} \text{ for } q \in \mathbb{R} \setminus \mathbb{Q}\} \sim \mathbb{N}.$$

3) $\{(x, y, z) \in \mathbb{N}^3 \mid x^2 + y^2 = z^2 \text{ and } x, y, z \in \mathbb{N}^*\} \subset \mathbb{N}^3$, and we know from class that $\mathbb{N}^3$ is countably infinite. Therefore, our set can be finite or countably infinite. We remark that

$$(3, 4, 5) \in \{(x, y, z) \in \mathbb{N}^3 \mid x^2 + y^2 = z^2 \text{ and } x, y, z \in \mathbb{N}^*\}$$

as $3^2+4^2=9+16=25=5^2$. Furthermore,

$$(3p,4p,5p)\in\{(x,y,z)\in\mathbb{N}^3 \;\big|\; x^2+y^2=z^2 \text{ and } x,y,z\in\mathbb{N}^*\}$$

for every $p\in N^*$ as $3^2p^2+4^2p^2=9p^2+16p^2=5^2p^2$. Since $\mathbb{N}^*\sim\mathbb{N}$ is countably infinite, our set must likewise be countably infinite.

4) $L_q=\{(x,y)\in\mathbb{R}^2 \;\big|\; x=q\}=\{q\}\times\mathbb{R}\sim\mathbb{R}$. Therefore, $\displaystyle\bigcup_{q\in\mathbb{Q}}L_q$ is a union of disjoint uncountably infinite sets, so it must itself be uncountably infinite.

5) Consider the subset A of $\{(x,y)\in\mathbb{R}^2 \;\big|\; y=x^2\}$ given by

$$A=\{(x,y)\in\mathbb{R}^2 \;\big|\; y=x^2\}\cap[(0,1)\times\mathbb{R}].$$

The function $f(x)=x^2=y$ is a bijection on $(0,1)$ (easy to check). Therefore, $\{(x,y)\in\mathbb{R}^2 \;\big|\; y=x^2\}\cap[(0,1)\times\mathbb{R}]\sim(0,1)$, so the set A is uncountably infinite as we proved in class that $(0,1)$ was uncountably infinite. Since $A\subset\{(x,y)\in\mathbb{R}^2 \,\big|\, y=x^2\}$, the set $\{(x,y)\in\mathbb{R}^2 \,\big|\, y=x^2\}$ must itself be uncountably infinite. Note that we have employed here a very standard technique for showing a set is uncountably infinite. It suffices to show it has an uncountably infinite subset.

6) We proved in Michaelmas term that the number of elements of a set with n elements is 2^n, so $\mathcal{P}(J_n)$ is a finite set with 2^n elements. By contrast, we proved in class that $\mathcal{P}(\mathbb{N})$ is uncountably infinite. The set consisting of the empty set belongs to the power set of any set, so $\{\emptyset\}\in\mathcal{P}(J_n)$. Therefore, $\{\emptyset\}\times\mathcal{P}(\mathbb{N})\subset\mathcal{P}(J_n)\times\mathcal{P}(\mathbb{N})$, but $\{\emptyset\}\times\mathcal{P}(\mathbb{N})\sim\mathcal{P}(\mathbb{N})$. We conclude that $\mathcal{P}(J_n)\times\mathcal{P}(\mathbb{N})$ has an uncountably infinite subset, so it itself must be uncountably infinite.

7) For $n=1$, we have already shown in class that $\mathbb{R}^1=\mathbb{R}$ was uncountably infinite. Now for $n\geq 2$ consider

$$\mathbb{R}^n=\{(x_1,\ldots,x_n) \;\big|\; x_i\in\mathbb{R}\;\forall i\}.$$

The set

$$\mathbb{R}\times\{0\}\cdots\{0\}=\{(x_1,0\ldots,0) \;\big|\; x_1\in\mathbb{R}\}\subset\mathbb{R}^n,$$

but $\mathbb{R}\times\{0\}\cdots\{0\}\sim\mathbb{R}$, which is uncountably infinite. Therefore, $\mathbb{R}^n$ has an uncountably infinite subset, which means it must itself be uncountably infinite.

8) By definition, a set is countable, if it is finite or countably infinite. A regular expression is built up from $\emptyset$, ϵ, and the letters of the alphabet A via the Kleene star $*$, concatenation, and union. The Kleene star makes a countably infinite set out of a finite one. Concatenation gives a set whose size matches the size of the biggest set in the concatenation. In other words, the concatenation of strings from two finite sets

will yield a finite set. The concatenation of strings from a finite set with a countably infinite set will yield a countably infinite set, whereas the concatenation of strings from two countably infinite sets yields a countably infinite set. Union behaves just like concatenation. Therefore, from a finite set via the Kleene star, union, and concatenation, we can only obtain a finite set or a countably infinite set. This concludes our proof. To give the required examples, let us consider the binary alphabet $A = \{0, 1\}$. The regular expression $\{01\} \cup \{11\}$ yields a regular language with two elements, whereas the regular expression $0^* \cup 1^*$ gives the regular language consisting of all strings of just 0's and all strings of just 1's, which is countably infinite as the sequence of strings ϵ, 0, 00, 000, etc. is inside this language.