Q1 $\begin{pmatrix} 3 & 2 & 1 & 11 \\ 1 & 1 & 1 & 4 \\ 2 & 1 & 2 & 5 \end{pmatrix}$

$$\xrightarrow{R1 \leftrightarrow R2} \begin{pmatrix} 1 & 1 & 1 & 4 \\ 3 & 2 & 1 & 11 \\ 2 & 1 & 2 & 5 \end{pmatrix}$$

$$\begin{array}{rrrrr} R2: & 3 & 2 & 1 & 11 \\ -3R1: & -3 & -3 & -3 & -12 \\ \hline & 0 & -1 & -2 & -1 \end{array}$$

$$\begin{array}{rrrrr} R3: & 2 & 1 & 2 & 5 \\ -2R1: & -2 & -2 & -2 & -8 \\ \hline & 0 & -1 & 0 & -3 \end{array}$$

$$\xrightarrow[R3-2R1]{R2-3R1} \begin{pmatrix} 1 & 1 & 1 & 4 \\ 0 & -1 & -2 & -1 \\ 0 & -1 & 0 & -3 \end{pmatrix}$$

$$\xrightarrow{(-1)R3 \leftrightarrow (-1)R2} \begin{pmatrix} 1 & 1 & 1 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 1 & 2 & 1 \end{pmatrix}$$

$$\begin{array}{rrrrr} R1: & 1 & 1 & 1 & 4 \\ -R2: & 0 & -1 & 0 & -3 \\ \hline & 1 & 0 & 1 & 1 \end{array}$$

$$\begin{array}{rrrrr} R3: & 0 & 1 & 2 & 1 \\ -R2: & 0 & -1 & 0 & -3 \\ \hline & 0 & 0 & 2 & -2 \end{array}$$

$$\xrightarrow[R3-R2]{R1-R2} \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 2 & -2 \end{pmatrix}$$

$$\xrightarrow{\frac{1}{2}R3} \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

$$\begin{array}{rrrrr} R1: & 1 & 0 & 1 & 1 \\ -R3: & 0 & 0 & -1 & 1 \\ \hline & 1 & 0 & 0 & 2 \end{array}$$

$$\xrightarrow{R1-R3} \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

Hence $x=2$, $y=3$, $z=-1$ is the unique solution of the linear system

Q2 $\begin{vmatrix} 3 & 1 & 5 \\ 2 & 2 & 4 \\ 4 & 1 & 5 \end{vmatrix} = 3 \begin{vmatrix} 2 & 4 \\ 1 & 5 \end{vmatrix} - 1 \begin{vmatrix} 2 & 4 \\ 4 & 5 \end{vmatrix} + 5 \begin{vmatrix} 2 & 2 \\ 4 & 1 \end{vmatrix}$

$$= 3(10-4) - 1(10-16) + 5(2-8)$$

$$= 3(6) - 1(-6) + 5(-6)$$

$$= 18 + 6 - 30$$

$$= -6$$

Sample Inclass Test Solutions.

Q3 The characteristic equation is got from

$$\det\begin{pmatrix} 5-\lambda & 4 & 2 \\ 4 & 5-\lambda & 2 \\ 2 & 2 & 2-\lambda \end{pmatrix} = (5-\lambda)[(5-\lambda)(2-\lambda) - 2(2)] - 4[4(2-\lambda) - 2(2)] + 2[4(2) - 2(5-\lambda)]$$

$$= (5-\lambda)[10 - 5\lambda - 2\lambda + \lambda^2 - 4] - 4(8 - 4\lambda - 4) + 2(8 - 10 + 2\lambda)$$

$$= -\lambda^3 + 12\lambda^2 - 21\lambda + 10$$

So the chacteristic equation is $\lambda^3 - 12\lambda^2 + 21\lambda - 10 = 0$

Possible rational roots are $\pm 1, \pm 2, \pm 5, \pm 10$

$\lambda = 1$ gives $1 - 12 + 21 - 10 = 0$ so $\lambda = 1$ is one of the eigenvalues.

$$\lambda^3 - 12\lambda^2 + 21\lambda - 10 = 0$$
$$(\lambda - 1)(\lambda^2 - 11\lambda + 10) = 0$$
$$(\lambda - 1)(\lambda - 1)(\lambda - 10) = 0$$

So the eigenvalues are $\lambda = 1$, $\lambda = 1$ and $\lambda = 10$

(Note $\lambda = 1$ is a repeated eigenvalue).

Q4 For $f(x) = 2\cos(3x)$ we have

$$\begin{aligned} f(x) &= 2\cos(3x) & f(0) &= 2 \\ f'(x) &= -6\sin(3x) & f(0) &= 0 \\ f''(x) &= -18\cos(3x) & f(0) &= -18 \\ f^{(3)}(x) &= 54\sin(3x) & f(0) &= 0 \\ f^{(4)}(x) &= 162\cos(3x) & f(0) &= 162 \end{aligned}$$

The quartic Taylor Polynomial about 0 for $f(x) = 2\cos(3x)$ is

$$\begin{aligned} p(x) &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 \\ &= 2 + 0(x) + \frac{-18}{2!}x^2 + \frac{0}{3!}x^3 + \frac{162}{4!}x^4 \\ &= 2 - 9x^2 + \frac{27}{4}x^4 \end{aligned}$$

Q5

$$\begin{aligned}
f(x) &= x(1+x)^{-\frac{1}{2}} - \ln(1+x) \\
&= x\left(1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3 + \dots\right) - \left(x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots\right) \\
&= x - \frac{1}{2}x^2 + \frac{3}{8}x^3 - \frac{5}{16}x^4 - x + \frac{1}{2}x^2 - \frac{1}{3}x^3 + \frac{1}{4}x^4 + \dots \\
&= \left(\frac{3}{8} - \frac{1}{3}\right)x^3 + \left(-\frac{5}{16} + \frac{1}{4}\right)x^4 + \dots \\
&= \frac{1}{24}x^3 - \frac{1}{16}x^4 + \dots
\end{aligned}$$

So the first two terms requested are

$$\frac{1}{24}x^3 - \frac{1}{16}x^4$$

$$\begin{aligned}
f(0.02) &= \frac{1}{24}(0.02)^3 - \frac{1}{16}(0.02)^4 \\
&= \frac{1}{24}(0.000008) - \frac{1}{16}(0.00000016) \\
&= 3.2 \times 10^{-7} \quad \text{(to 2 significant figures)}
\end{aligned}$$