CS2010: ALGORITHMS AND DATA STRUCTURES

Lecture 10: Recursion vs Iteration

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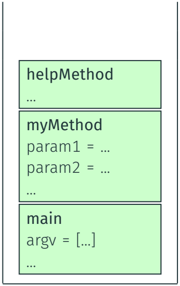
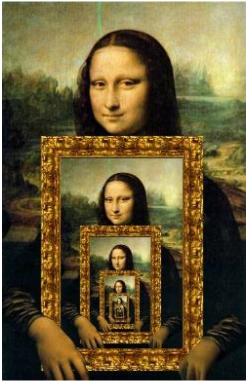
HOW METHODS EXECUTE

- Call stack: is a stack maintained by the Java runtime system
- One call stack frame (aka activation record) for each running instance of a method: contains all information necessary to execute the method
  - **references** to parameter values and local objects, return address etc.
- Objects themselves are stored in another part of memory: the heap
- every time a method is called, a new stack frame is pushed on the call stack.
- every time a method returns, the top-most stack frame is popped.



Recursion: when something is defined in terms of itself.

Infinite Recusion



Well-founded Recursion

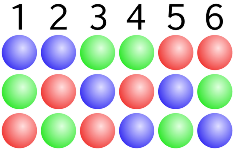
Principle: A method is recursive when its definition **calls the method itself**.

A correct recursive method should be well-founded: it should terminate/must end up at a base case.

Classic example: factorial – in math written as: $n!$

Math definition:

$$0! = 1$$
$$n! = n \cdot (n-1)! \quad \text{when } n > 0$$



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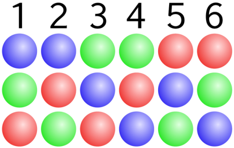

Classic example: Fibonacci numbers

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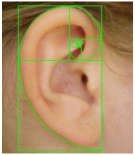

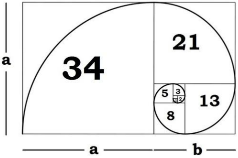

RECURSION

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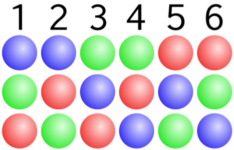

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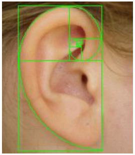

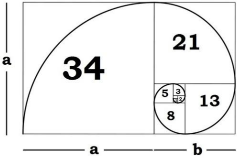

It is convenient to implement recursive math definitions using recursive methods.

Recursive implementation:

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int fac(int n) {



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Recursive implementation:

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int fac(int n) {
  if (n == 0)
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IMPLEMENTATION OF FACTORIAL

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→ Closely matches the mathematical definition.

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  - → specify how smaller solutions **compose** into the solutions of recursive cases
  - → it is usually a **top-down calculation**

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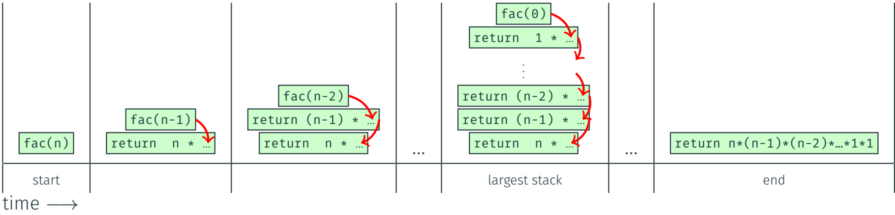
→ Q: memory space for call stack frames? (max # frames on call stack)
A: $\Theta(n)$ space
All this call stack space is needed because of the `return n *` …



Recursive implementation using accumulator: (H/W: can you implement the accumulator version bottom-up?)

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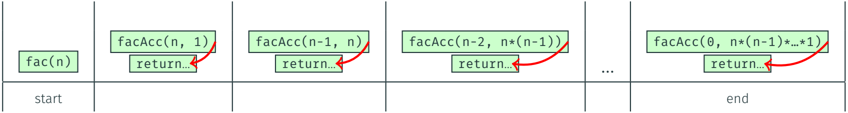

From **tail recursive implementation** ⟶ iterative implementation:

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- Running time of iterative implementation: $\Theta(n)$
- Stack space of iterative implementation: $\Theta(1)$
  In functional languages this simple translation is done by the compiler!

Math definition:

$$\begin{aligned} fib(0) &= 1 \\ fib(1) &= 1 \\ fin(n) &= fib(n-1) + fib(n-2) \quad \text{when } n > 1 \end{aligned}$$

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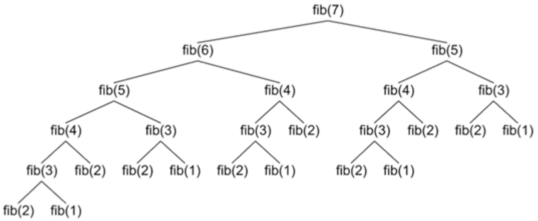
- → Running time: non-tight upper bound: $O(2^n)$ tight bound: $\Theta(fib(n))$
  - → # of recursive calls of fib(n) is the size of the binary tree of recursive calls with n levels ($\leq 2^n$); however this is not a complete tree; thus $O(2^n)$.



FIBONACCI NUMBERS

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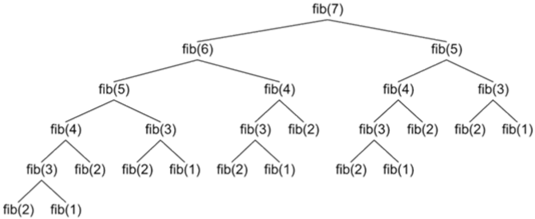

- Call stack space: $\Theta(n)$

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Recursive implementation with accumulator (tail recursion):

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int fibAcc(int n, int last, int secondToLast) {
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→ This is much trickier! Read it again off-line and understand why it works.

→ It is a bottom-up calculation using two accumulators.

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→ Running time: $\Theta(n)$

→ Call stack space: $\Theta(n)$ in Java and $\Theta(1)$ in other languages.

Tail-recursive implementation ⟶ iterative implementation

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```
int fib(int n) {
  int last = 1; int secondToLast = 1;
  for ( ; !(n <= 1); n--) {
    int tmpLast = last;
    last = last + secondToLast;
    secondToLast = tmpLast;
  }
  return last;
}
```

→ Worst Case Asymptotic Running time: $\Theta(n)$

→ Call stack space: $\Theta(1)$

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Attempt 1: try all numbers n from **+inf** down to 1 until you find one that has the above property.

Attempt 2: try all numbers n from **x** down to 1 until you find one that has the above property (because gcd will necessarily be $\leq min(x, y)$).

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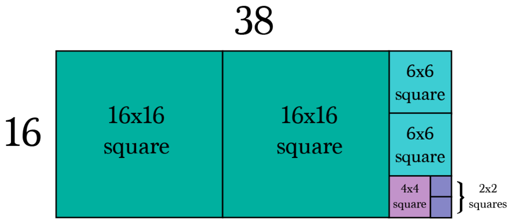
Attempt 3: Euclid's algorithm a Divide & Conquer approach:

Euclid's Theorem: gcd(x,y) = gcd(y, x % y).



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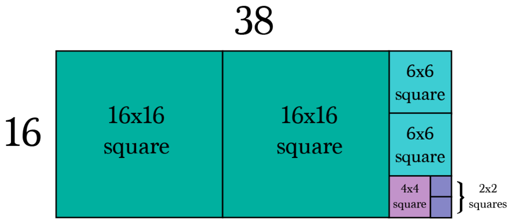

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- Q: How many iterations in the worst case?
- **Gabriel Lame's Theorem (1844):** #iterations $< 5 \cdot h$
  Where $h =$ digits of $min(x, y)$ (here this is x) in base 10.
- A: $O(lg(x))$

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RECURSION VS ITERATION: WHICH ONE TO CHOOSE?

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- Iterative algorithms usually have easier proofs of memory usage (and in Java use less memory)
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  - because the compiler maintains a stack of previous calls for us.
  - We will use recursive implementations for most operations over **trees**.

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- Recursive algorithms are sometimes easier to implement when we have complex data structures.
  - because the compiler maintains a stack of previous calls for us.
  - We will use recursive implementations for most operations over **trees**.
- We *could* have used recursive implementations for operations over lists & arrays but:
  - they are not simpler than the iterative implementations
  - Java will need $O(n)$ call stack space to execute them.

Homework 1: Implement Euclid's algorithm using

→ A recursive method.
→ An iterative method.

Homework 2: give recursive implementations for:

→ binary search over an array
→ linear search over a linked list

Homework 3: Implement **search** on a Binary Search Tree (next lecture) using recursion and compare with the iterative version in the book.

Homework 4**: give an iterative implementation for method **put** on binary search tree.