Set Theory

Set Theory

Set Theory

Set Theory

1. Preliminaries
2. Russell's Paradox
3. Set Operations
4. Set Properties
5. Cardinality of Sets
6. Power Set
7. Cantor's Theorem on Infinite Sets.

Set Theory

Set Theory

Set Theory

1. Preliminaries
2. Russell's Paradox
3. Set Operations
4. Set Properties
5. Cardinality of Sets
6. Power Set
7. Cantor's Theorem on Infinite Sets.

Defining Sets

Sets can be defined by extension i.e. listing the elements or by a property.

- By Extension i.e. listing the elements:
  e.g. $A = \{2, 3, 5, 7, 11, 13, 17, 19\}$
- By a Property (or a Predicate):
  e.g. $B = \{n \mid n < 20 \;\; and \;\; n \; is \; prime\}$
  - $\{\}$ is the empty set; it has no elements.
    e.g. $\{\} = \{x \mid x \neq x\}$.

Notation: $\in$
Read $x \in S$ as "x is an element of the set S" or "x is in S".
Query:
Does the set $E = \{x \mid x = x\}$, the set of everything, exist.
i.e. $x \in E \equiv x \notin \{\}$

Subset $\subseteq$

$X \subseteq Y$ iff every element of X is an element of Y.
i.e. there is no element, z, such that $z \in X$ and $z \notin Y$.

Theorem

$\{\} \subseteq X$, *for all sets* X.

Since there are no elements in the empty set, $\{\}$, there is no element, z, in the empty set $\{\}$ such $z \in \{\}$ and $z \notin X$ hence $\{\} \subseteq X$.

Equality of Sets

Equality of Sets

Two sets X and Y are equal if they have the same elements, i.e. all the elements in X are in Y and all the elements in Y are in X.

$$X = Y \text{ iff } X \subseteq Y \text{ and } Y \subseteq X$$

Example

Sets
$A = \{2, 3, 5, 7, 11, 13, 17, 19\}$ and
$B = \{n | n < 20 \ \ and \ \ n \, is \, prime\}$
are equal.

Example: Equal Sets

Example

Let $S = \{x \in \mathbb{R} \mid x^2 < x\}$ and $T = \{x \in \mathbb{R} \mid 0 < x < 1\}$
Show $S = T$.
Show i) $S \subseteq T$ and ii) $T \subseteq S$

Proof.

i) $S \subseteq T$
Let $x \in S$ then
$x^2 < x$
$\equiv x^2 - x < 0$
$\equiv x(x-1) < 0$

x and $x - 1$ have opposite signs or parity;

Example: Equal Sets

Example

Let $S = \{x \in \mathbb{R} \mid x^2 < x\}$ and $T = \{x \in \mathbb{R} \mid 0 < x < 1\}$
Show $S = T$.
Show i) $S \subseteq T$ and ii) $T \subseteq S$

Proof.

i) $S \subseteq T$
Let $x \in S$ then
$x^2 < x$
$\equiv x^2 - x < 0$
$\equiv x(x-1) < 0$

x and $x - 1$ have opposite signs or parity; □

Proof cont'd

Proof.

Case: $x > 0$ and $x - 1 < 0$

If $x > 0$ and $x - 1 < 0$ i.e. $x < 1$ then $0 < x < 1$.

$\therefore$ (therefore) $x \in T$.

Case: $x < 0$ and $x - 1 > 0$

Impossible, as $x - 1 < x$. $\square$

Proof Cont'd

Proof.

ii) $T \subseteq S$

Let $x \in T$ then $0 < x < 1$

$\therefore x < 1$,

$\{$multiply both sides by x and since $0 < x\}$

$\therefore x^2 < x$,

$\therefore x \in S$. $\square$

Number Sets

- $\mathbb{N}$, Natural Numbers {0,1,2,...}
- $\mathbb{Z}$, Integers {...-2, -1, 0, 1, 2, ...}
- $\mathbb{Q}$, Rationals (Fractions) $\{\frac{a}{b} \mid a, b \in \mathbb{Z}, b \neq 0\}$
- $\mathbb{R}$, Real numbers e.g. all the points on the number line.

Number Sets

- $\mathbb{N}$, Natural Numbers {0,1,2,...}
- $\mathbb{Z}$, Integers {...-2, -1, 0, 1, 2, ...}
- $\mathbb{Q}$, Rationals (Fractions) $\{\frac{a}{b} \mid a, b \in \mathbb{Z}, b \neq 0\}$
- $\mathbb{R}$, Real numbers e.g. all the points on the number line.

Number Sets

- $\mathbb{N}$, Natural Numbers {0,1,2,...}
- $\mathbb{Z}$, Integers {...-2, -1, 0, 1, 2, ...}
- $\mathbb{Q}$, Rationals (Fractions) $\{\frac{a}{b} \mid a, b \in \mathbb{Z}, b \neq 0\}$
- $\mathbb{R}$, Real numbers e.g. all the points on the number line.

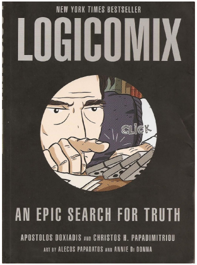
Number Sets

- $\mathbb{N}$, Natural Numbers {0,1,2,...}
- $\mathbb{Z}$, Integers {...-2, -1, 0, 1, 2, ...}
- $\mathbb{Q}$, Rationals (Fractions) $\{\frac{a}{b} \mid a, b \in \mathbb{Z}, b \neq 0\}$
- $\mathbb{R}$, Real numbers e.g. all the points on the number line.

Bertrand Russell

Logicomix Book:

https://www.logicomix.com/en/index.html
https://www.youtube.com/watch?v=XebglmXrgEc



Christos Papadimitriou Talk

To Microsoft Research

https://www.youtube.com/watch?v=rM8xw2EDBYc

Bertrand Russell

Bertrand Russell (1872–1970) was a British philosopher, logician, essayist and social critic best known for his work in mathematical logic and analytic philosophy.
Over the course of a long career, Russell also made significant contributions to a broad range of other subjects, including ethics, politics, educational theory, the history of ideas and religious studies, cheerfully ignoring Hooke's admonition to the Royal Society against "meddling with Divinity, Metaphysics, Moralls, Politicks, Grammar, Rhetorick, or Logick" (Kreisel 1973, 24). In addition, generations of general readers have benefited from his many popular writings on a wide variety of topics in both the humanities and the natural sciences. Like Voltaire, to whom he has been compared (Times of London 1970, 12)), he wrote with style and wit and had enormous influence.

(https://plato.stanford.edu/entries/russell/)

Russell's Paradox

Example

Library Reference Books
A library has reference books, some of which refer to themselves and some which do not. The Librarian creates a new reference book, R, with the entries of all those (and only those) reference books that do not reference themselves. The book, R, is a reference book and the librarian now considers whether they should include a reference to R in the book R.

Russell's Paradox Cont'd

Should R include a reference to itself

- Case: Include a reference to R in R:
  By definition of R, it should contain references to books that do not reference themselves. Since by assumption a reference to R is included in R then, by definition, R should not include a reference to R.
- Case: Exclude a reference to R in R:
  By assumption, R does not contain a reference to R hence by definition of R, the book R should include a reference to R.

In either case, we get a contradiction.

Russell's Paradox Cont'd

Should R include a reference to itself

- Case: Include a reference to R in R:
  By definition of R, it should contain references to books that do not reference themselves. Since by assumption a reference to R is included in R then, by definition, R should not include a reference to R.
- Case: Exclude a reference to R in R:
  By assumption, R does not contain a reference to R hence by definition of R, the book R should include a reference to R.

In either case, we get a contradiction.

Russell's Paradox Cont'd

Should R include a reference to itself

- Case: Include a reference to R in R:
  By definition of R, it should contain references to books that do not reference themselves. Since by assumption a reference to R is included in R then, by definition, R should not include a reference to R.
- Case: Exclude a reference to R in R:
  By assumption, R does not contain a reference to R hence by definition of R, the book R should include a reference to R.

In either case, we get a contradiction.

Russell's Paradox Cont'd

Should R include a reference to itself

- Case: Include a reference to R in R:
  By definition of R, it should contain references to books that do not reference themselves. Since by assumption a reference to R is included in R then, by definition, R should not include a reference to R.
- Case: Exclude a reference to R in R:
  By assumption, R does not contain a reference to R hence by definition of R, the book R should include a reference to R.

In either case, we get a contradiction.

Simpler Version

Consider a library with 25 reference books, labeled A to Y. Some of these reference themselves, some don't. Consider adding a new reference book, Z, which is made of just the entries of the books in the library which do not reference themselves. If book A does not list itself then book Z includes an entry for A. Should Z reference itself or not?

More precision is needed. For example, include a time limit. Book Z is to include all books that do not reference themselves by 12noon. Therefore, since book Z is not in the library at 12noon, it will not be mentioned in his own entries.

Simpler Version Cont'd

Continuing with the time limit, books A to Y exist at 12 noon but book Z does not. Book Z is created by 1pm containing entries to all books that do not reference themselves by 12noon and so book Z does not contain a reference to itself. Time continues and consider all reference books in library at 1pm. Since book Z is in the library, update Z to include a reference to Z. At 2pm, update book Z to exclude a reference to book Z and continue on forever. Book Z is never completed and so is more of a project than a book in the library.

Normal Sets

A set is called *normal* (or ordinary) if it does not contain itself. The set of numbers $s = \{1, 2, 3\}$ is normal as $s \notin \{1, 2, 3\}$, i.e. $s \notin s$. It is not easy to think of a set that is not normal but a possibility would be the set of everything i.e. E above.
Consider a set, N, the set of all normal sets:
i.e. For any x, $x \in N$ iff x is a normal set.
$N = \{x \mid x \notin x\}$
$x \in N \equiv x \notin x$
Substituting N for x we get
$N \in N \equiv N \notin N$
This is a contradiction. So there is a problem in defining a set using the property, $x \notin x$.

(Local) Universal Set

Also there is a problem in defining a set $E = \{x \mid x = x\}$ and so in standard set theory, the set of all sets is not regarded as a set.
If one is to use set thery, one begins by first determining the set of elements of interest. That is, one defines a local universal set, U.
For example, in mathematics, the set $\mathbb{R}$ (real numbers) is frequently used as a local universal set, U.

Set Operations

Union, intersection, difference, complement.

- **Union**: $X \cup Y = \{z \mid z \in X \text{ or } z \in Y\}$
  $z \in X \cup Y$ iff either $z \in X$ or $z \in Y$ or z is in both X and Y
  Note: $X \subseteq X \cup Y$ and $Y \subseteq X \cup Y$
- **Intersection**: $X \cap Y = \{z \mid z \in X \text{ and } z \in Y\}$
  $z \in X \cap Y$ iff z is in both X and Y
  Note: $X \cap Y \subseteq X$ and $X \cap Y \subseteq Y$
- **Difference**: $X - Y = \{z \mid z \in X \text{ and } z \notin Y\}$
  $z \in X - Y$ iff ($z \in X$ but z is not in Y).
  Alternative Notation: $X \setminus Y$
- **Complement**: $\overline{X} = U - X$ where U is a universal set of elements of interest.
  In particular, $\overline{\{\}} = U$ and $\overline{U} = \{\}$
  Alternative Notation: X'

Set Operations

Union, intersection, difference, complement.

- **Union**: $X \cup Y = \{z \mid z \in X \text{ or } z \in Y\}$
  $z \in X \cup Y$ iff either $z \in X$ or $z \in Y$ or z is in both X and Y
  Note: $X \subseteq X \cup Y$ and $Y \subseteq X \cup Y$
- **Intersection**: $X \cap Y = \{z \mid z \in X \text{ and } z \in Y\}$
  $z \in X \cap Y$ iff z is in both X and Y
  Note: $X \cap Y \subseteq X$ and $X \cap Y \subseteq Y$
- **Difference**: $X - Y = \{z \mid z \in X \text{ and } z \notin Y\}$
  $z \in X - Y$ iff ($z \in X$ but z is not in Y).
  Alternative Notation: $X \setminus Y$
- **Complement**: $\overline{X} = U - X$ where U is a universal set of elements of interest.
  In particular, $\overline{\{\}} = U$ and $\overline{U} = \{\}$
  Alternative Notation: X'

Set Operations

Union, intersection, difference, complement.

- **Union**: $X \cup Y = \{z \mid z \in X \text{ or } z \in Y\}$
  $z \in X \cup Y$ iff either $z \in X$ or $z \in Y$ or z is in both X and Y
  Note: $X \subseteq X \cup Y$ and $Y \subseteq X \cup Y$
- **Intersection**: $X \cap Y = \{z \mid z \in X \text{ and } z \in Y\}$
  $z \in X \cap Y$ iff z is in both X and Y
  Note: $X \cap Y \subseteq X$ and $X \cap Y \subseteq Y$
- **Difference**: $X - Y = \{z \mid z \in X \text{ and } z \notin Y\}$
  $z \in X - Y$ iff ($z \in X$ but z is not in Y).
  Alternative Notation: $X \backslash Y$
- **Complement**: $\overline{X} = U - X$ where U is a universal set of elements of interest.
  In particular, $\overline{\{\}} = U$ and $\overline{U} = \{\}$
  Alternative Notation: X'

Set Operations

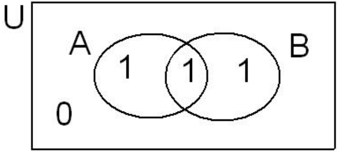
Union, intersection, difference, complement.

- **Union**: $X \cup Y = \{z \mid z \in X \text{ or } z \in Y\}$
  $z \in X \cup Y$ iff either $z \in X$ or $z \in Y$ or z is in both X and Y
  Note: $X \subseteq X \cup Y$ and $Y \subseteq X \cup Y$
- **Intersection**: $X \cap Y = \{z \mid z \in X \text{ and } z \in Y\}$
  $z \in X \cap Y$ iff z is in both X and Y
  Note: $X \cap Y \subseteq X$ and $X \cap Y \subseteq Y$
- **Difference**: $X - Y = \{z \mid z \in X \text{ and } z \notin Y\}$
  $z \in X - Y$ iff ($z \in X$ but z is not in Y).
  Alternative Notation: $X \setminus Y$
- **Complement**: $\overline{X} = U - X$ where U is a universal set of elements of interest.
  In particular, $\overline{\{\}} = U$ and $\overline{U} = \{\}$
  Alternative Notation: X'

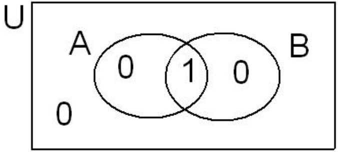
Venn Diagrams

Venn Diagrams

$A \cup B$

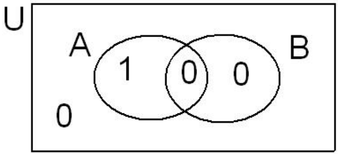


$A \cap B$

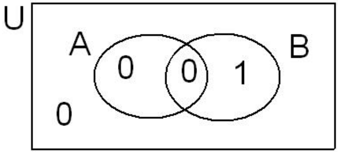


Venn Diagrams (Cont'd)

$A - B$

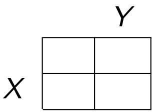


$B - A$



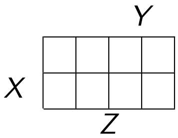
Karnaugh Map

Instead of Circles or Ellipses representing subsets of U, one can use Squares or Rectangles. The various subsets are represented as equal area divisions of U. If we are concerned with 2 subsets X and Y of U then in a Karnaugh Map, the subset X is respresented as the bottom half division and the subset Y as the right half division.



Karnaugh Map (Cont'd)

For 3 subsets X, Y and Z of U, we have the Karnaugh Map



X is the 'bottom half'. Y the 'right half' and Z the 'middle half' of the Universal set, U.

Veitch .v. Karnaugh

An ancestor of the Karnaugh Map was proposed by Marquand in 1881. Veitch rediscovered the Marquand diagram in his 1952 article "A Chart Method for Simplifying Truth Functions",
The 'Veitch/Marquand Chart' was modified by Karnaugh in 1953 and it is more convenient to use what will be referred to as Karnaugh Maps.

Karnaugh: Union and Intersection

Karnaugh_Map: Union and Intersection

e.g. $X \cup Y$

$$X \cup Y = \begin{array}{c|c|c|} & \multicolumn{2}{c}{Y} \\ \cline{2-3} & 0 & 1 \\ \cline{2-3} X & 1 & 1 \\ \cline{2-3} \end{array}$$

e.g. $X \cap Y$

$$X \cap Y = \begin{array}{c|c|c|} & \multicolumn{2}{c}{Y} \\ \cline{2-3} & 0 & 0 \\ \cline{2-3} X & 0 & 1 \\ \cline{2-3} \end{array}$$

Karnaugh: Union and Intersection

Karnaugh_Map: Union and Intersection

e.g. $X \cup Y$

$$X \cup Y = \begin{array}{c|c|c|} & \multicolumn{2}{c}{Y} \\ \cline{2-3} & 0 & 1 \\ \cline{2-3} X & 1 & 1 \\ \cline{2-3} \end{array}$$

e.g. $X \cap Y$

$$X \cap Y = \begin{array}{c|c|c|} & \multicolumn{2}{c}{Y} \\ \cline{2-3} & 0 & 0 \\ \cline{2-3} X & 0 & 1 \\ \cline{2-3} \end{array}$$

Karnaugh Map

Each of the 'cells' in the Karnaugh Map represent a subset of the whole 'box' which represents the Universal set, U.

Universal Set

	$\overline{Y}$	Y
$\overline{X}$	$\overline{X} \cap \overline{Y}$	$\overline{X} \cap Y$
X	$X \cap \overline{Y}$	$X \cap Y$

Disjoint_Sets

Sets A and B are **disjoint** iff $A \cap B = \{\}$. For example, $A \cap \overline{A} = \{\}$.

From above, $X \cup Y$ is the union of disjoint subsets of U.

$$X \cup Y = (X \cap \overline{Y}) \cup (X \cap Y) \cup (\overline{X} \cap Y)$$

One Variable Sets

Set expression with one variable:

e.g. set X is represented as

$$X = \quad \begin{array}{c|c|} \cline{2-2} & 0 \\ \cline{2-2} X & 1 \\ \cline{2-2} \end{array}$$

e.g. $\overline{X}$, the complement of X

$$\overline{X} = \quad \begin{array}{c|c|} \cline{2-2} & 1 \\ \cline{2-2} X & 0 \\ \cline{2-2} \end{array}$$

Note: $\overline{\overline{X}} = X$.

Set Difference

e.g. $X - Y$

$$X - Y = \begin{array}{c|c|c|} & \multicolumn{2}{c}{Y} \\ \cline{2-3} & 0 & 0 \\ \cline{2-3} X & 1 & 0 \\ \cline{2-3} \end{array}$$

Note: $X - Y = X \cap \overline{Y}$

e.g. $Y - X$

$$Y - X = \begin{array}{c|c|c|} & \multicolumn{2}{c}{Y} \\ \cline{2-3} & 0 & 1 \\ \cline{2-3} X & 0 & 0 \\ \cline{2-3} \end{array}$$

Note: $Y - X = Y \cap \overline{X}$

Set Difference

e.g. $X - Y$

$$X - Y = \begin{array}{c|c|c|} & \multicolumn{2}{c}{Y} \\ \cline{2-3} & 0 & 0 \\ \cline{2-3} X & 1 & 0 \\ \cline{2-3} \end{array}$$

Note: $X - Y = X \cap \overline{Y}$

e.g. $Y - X$

$$Y - X = \begin{array}{c|c|c|} & \multicolumn{2}{c}{Y} \\ \cline{2-3} & 0 & 1 \\ \cline{2-3} X & 0 & 0 \\ \cline{2-3} \end{array}$$

Note: $Y - X = Y \cap \overline{X}$

Defining Set Operations

Example

Define a set operator, $\oplus$, such that $X \oplus Y = (X \cap \overline{Y}) \cup (Y \cap \overline{X})$

$$
\begin{array}{cc} X \cap \overline{Y} & Y \\ X & \begin{array}{|c|c|} \hline 0 & 0 \\ \hline 1 & 0 \\ \hline \end{array} \end{array}
\quad \cup \quad
\begin{array}{cc} Y \cap \overline{X} & Y \\ X & \begin{array}{|c|c|} \hline 0 & 1 \\ \hline 0 & 0 \\ \hline \end{array} \end{array}
$$

'Union' these to get

$$
X \oplus Y \quad = \quad \begin{array}{cc} & Y \\ X & \begin{array}{|c|c|} \hline 0 & 1 \\ \hline 1 & 0 \\ \hline \end{array} \end{array}
$$

Defining Set Operations

Example

Define a set operator, $\oplus$, such that $X \oplus Y = (X \cap \overline{Y}) \cup (Y \cap \overline{X})$

$$X \cap \overline{Y} \quad \begin{array}{c|c|c} & \multicolumn{2}{c}{Y} \\ & 0 & 0 \\ X & 1 & 0 \end{array} \quad \cup \quad Y \cap \overline{X} \quad \begin{array}{c|c|c} & \multicolumn{2}{c}{Y} \\ & 0 & 1 \\ X & 0 & 0 \end{array}$$

'Union' these to get

$$X \oplus Y \quad = \quad \begin{array}{c|c|c} & \multicolumn{2}{c}{Y} \\ & 0 & 1 \\ X & 1 & 0 \end{array}$$

Example Exercise

Exercise: Determine whether

$$(X - Y) \cup (Y - X) = (X \cup Y) - (X \cap Y)$$

LHS $(X - Y) \cup (Y - X)$

$$X - Y \quad \begin{array}{c|c|c|} & \multicolumn{2}{c}{Y} \\ \cline{2-3} & 0 & 0 \\ \cline{2-3} X & 1 & 0 \\ \cline{2-3} \end{array} \quad \cup \quad Y - X \quad \begin{array}{c|c|c|} & \multicolumn{2}{c}{Y} \\ \cline{2-3} & 0 & 1 \\ \cline{2-3} X & 0 & 0 \\ \cline{2-3} \end{array}$$

'Union' these to get

$$(X - Y) \cup (Y - X) \quad = \quad \begin{array}{c|c|c|} & \multicolumn{2}{c}{Y} \\ \cline{2-3} & 0 & 1 \\ \cline{2-3} X & 1 & 0 \\ \cline{2-3} \end{array}$$

Exercise Cont'd

RHS $(X \cup Y) - (X \cap Y)$

$$\begin{array}{cc} & Y \\ X \cup Y & \\ & \begin{array}{|c|c|} \hline 0 & 1 \\ \hline 1 & 1 \\ \hline \end{array} \\ X & \end{array} \quad - \quad \begin{array}{cc} & Y \\ X \cap Y & \\ & \begin{array}{|c|c|} \hline 0 & 0 \\ \hline 0 & 1 \\ \hline \end{array} \\ X & \end{array}$$

'Difference' these to get

$$(X \cup Y) - (X \cap Y) \quad = \quad \begin{array}{cc} & Y \\ X & \begin{array}{|c|c|} \hline 0 & 1 \\ \hline 1 & 0 \\ \hline \end{array} \end{array}$$

Therefore

$X \oplus Y$
$= (X \cap \overline{Y}) \cup (Y \cap \overline{X})$
$= (X - Y) \cup (Y - X)$
$= (X \cup Y) - (X \cap Y)$

Exercise Cont'd

RHS $(X \cup Y) - (X \cap Y)$

$$X \cup Y \quad \begin{array}{c|c|c} & \multicolumn{2}{c}{Y} \\ \hline & 0 & 1 \\ \hline X & 1 & 1 \end{array} \quad - \quad X \cap Y \quad \begin{array}{c|c|c} & \multicolumn{2}{c}{Y} \\ \hline & 0 & 0 \\ \hline X & 0 & 1 \end{array}$$

'Difference' these to get

$$(X \cup Y) - (X \cap Y) \quad = \quad \begin{array}{c|c|c} & \multicolumn{2}{c}{Y} \\ \hline & 0 & 1 \\ \hline X & 1 & 0 \end{array}$$

Therefore

$X \oplus Y$

$= (X \cap \overline{Y}) \cup (Y \cap \overline{X})$

$= (X - Y) \cup (Y - X)$

$= (X \cup Y) - (X \cap Y)$

3-Variable Set Expressions

Set expression with 3 variables

Example

$X \cup \overline{Y} \cup Z$

			Y	
	1	1	1	0
X	1	1	1	1
		Z		

Cont'd

Example

Determine $X \cup (X \cap Z) \cup (Y \cap Z)$

$$\begin{array}{cc} & \qquad\qquad Y \\ & \begin{array}{|c|c|c|c|} \hline 0 & 0 & 0 & 0 \\ \hline 1 & 1 & 1 & 1 \\ \hline \end{array} \\ X & \\ & Z \end{array} \cup \quad X \cap Z \quad \begin{array}{cc} & \qquad\qquad Y \\ & \begin{array}{|c|c|c|c|} \hline 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 1 & 0 \\ \hline \end{array} \\ X & \\ & Z \end{array}$$

$$\cup \quad Y \cap Z \quad \begin{array}{cc} & \qquad\qquad Y \\ & \begin{array}{|c|c|c|c|} \hline 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 1 & 0 \\ \hline \end{array} \\ X & \\ & Z \end{array}$$

Cont'd

'Union' these together to get:

$$\begin{array}{c|c|c|c|c|}
\multicolumn{1}{c}{} & \multicolumn{2}{c}{} & \multicolumn{2}{c}{Y} \\ \cline{2-5}
 & 0 & 0 & 1 & 0 \\ \cline{2-5}
X & 1 & 1 & 1 & 1 \\ \cline{2-5}
\multicolumn{1}{c}{} & \multicolumn{4}{c}{Z}
\end{array}$$

$(X \oplus Y) \oplus Z$

Example

Example_A: Determine $(X \oplus Y) \oplus Z$

Let $A = X \oplus Y$ i.e. $A = (X \cap \overline{Y}) \cup (Y \cap \overline{X}))$ $\therefore$

$$A = \begin{array}{c|c|c|c|c|} & & & \multicolumn{2}{c}{Y} \\ \cline{2-5} & 0 & 0 & 1 & 1 \\ \cline{2-5} X & 1 & 1 & 0 & 0 \\ \cline{2-5} & & \multicolumn{2}{c}{Z} & \end{array}$$

$(X \oplus Y) \oplus Z$ Cont'd

$$A \oplus Z = (A \cap \overline{Z}) \cup (Z \cap \overline{A})$$

$$
\begin{array}{c|c|c|c|c|}
\multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{2}{c}{Y} \\
\cline{2-5}
 & 0 & 0 & 1 & 1 \\
\cline{2-5}
X & 1 & 1 & 0 & 0 \\
\cline{2-5}
\multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{2}{c}{Z} & \multicolumn{1}{c}{}
\end{array}
\quad \cap \quad
\begin{array}{c|c|c|c|c|}
\multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{2}{c}{Y} \\
\cline{2-5}
 & 1 & 0 & 0 & 1 \\
\cline{2-5}
X & 1 & 0 & 0 & 1 \\
\cline{2-5}
\multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{2}{c}{Z} & \multicolumn{1}{c}{}
\end{array}
$$

$$A \quad \cap \quad \overline{Z}$$

$$\cup$$

$$Z \quad \cap \quad \overline{A}$$

$$
\begin{array}{c|c|c|c|c|}
\multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{2}{c}{Y} \\
\cline{2-5}
 & 0 & 1 & 1 & 0 \\
\cline{2-5}
X & 0 & 1 & 1 & 0 \\
\cline{2-5}
\multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{2}{c}{Z} & \multicolumn{1}{c}{}
\end{array}
\quad \cap \quad
\begin{array}{c|c|c|c|c|}
\multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{2}{c}{Y} \\
\cline{2-5}
 & 1 & 1 & 0 & 0 \\
\cline{2-5}
X & 0 & 0 & 1 & 1 \\
\cline{2-5}
\multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{2}{c}{Z} & \multicolumn{1}{c}{}
\end{array}
$$

$(X \oplus Y) \oplus Z$ Cont'd

i.e.

$$\begin{array}{c|c|c|c|c|} \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{2}{c}{Y} \\ \cline{2-5} & 0 & 0 & 0 & 1 \\ \cline{2-5} X & 1 & 0 & 0 & 0 \\ \cline{2-5} \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{2}{c}{Z} & \multicolumn{1}{c}{} \end{array}$$

$$\cup$$

$$\begin{array}{c|c|c|c|c|} \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{2}{c}{Y} \\ \cline{2-5} & 0 & 1 & 0 & 0 \\ \cline{2-5} X & 0 & 0 & 1 & 0 \\ \cline{2-5} \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{2}{c}{Z} & \multicolumn{1}{c}{} \end{array}$$

i.e.

$$(X \oplus Y) \oplus Z \quad = \quad \begin{array}{c|c|c|c|c|} \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{2}{c}{Y} \\ \cline{2-5} & 0 & 1 & 0 & 1 \\ \cline{2-5} X & 1 & 0 & 1 & 0 \\ \cline{2-5} \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{2}{c}{Z} & \multicolumn{1}{c}{} \end{array}$$

$X \oplus (Y \oplus Z)$

Example

Example_B Determine $X \oplus (Y \oplus Z)$
Consider $\oplus$ as 'addition mod 2'

Let $B = Y \oplus Z$ $\therefore$ by adding individual cells 'mod 2' we get:

$$
\begin{aligned}
B &= \begin{array}{c|c|c|c|c|} \multicolumn{1}{c}{} & \multicolumn{2}{c}{} & \multicolumn{2}{c}{Y} \\ \cline{2-5} & 0 & 0 & 1 & 1 \\ \cline{2-5} X & 0 & 0 & 1 & 1 \\ \cline{2-5} \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{2}{c}{Z} & \multicolumn{1}{c}{} \end{array} \oplus \begin{array}{c|c|c|c|c|} \multicolumn{1}{c}{} & \multicolumn{2}{c}{} & \multicolumn{2}{c}{Y} \\ \cline{2-5} & 0 & 1 & 1 & 0 \\ \cline{2-5} X & 0 & 1 & 1 & 0 \\ \cline{2-5} \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{2}{c}{Z} & \multicolumn{1}{c}{} \end{array} \\
&= \begin{array}{c|c|c|c|c|} \multicolumn{1}{c}{} & \multicolumn{2}{c}{} & \multicolumn{2}{c}{Y} \\ \cline{2-5} & 0 & 1 & 0 & 1 \\ \cline{2-5} X & 0 & 1 & 0 & 1 \\ \cline{2-5} \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{2}{c}{Z} & \multicolumn{1}{c}{} \end{array} \\
&= Y \oplus Z
\end{aligned}
$$

$X \oplus (Y \oplus Z)$ Cont'd

$$\therefore X \oplus (Y \oplus Z) = X \oplus B$$

$$\begin{array}{cc} & Y \\ X & \begin{array}{|c|c|c|c|}\hline 0 & 0 & 0 & 0 \\ \hline 1 & 1 & 1 & 1 \\ \hline \end{array} \\ & Z \end{array} \quad \oplus \quad \begin{array}{cc} & Y \\ X & \begin{array}{|c|c|c|c|}\hline 0 & 1 & 0 & 1 \\ \hline 0 & 1 & 0 & 1 \\ \hline \end{array} \\ & Z \end{array}$$

$$X \quad \oplus \quad B$$

$$= \quad \begin{array}{cc} & Y \\ X & \begin{array}{|c|c|c|c|}\hline 0 & 1 & 0 & 1 \\ \hline 1 & 0 & 1 & 0 \\ \hline \end{array} \\ & Z \end{array}$$

$$= \quad X \oplus (Y \oplus Z)$$

Disjoint Union, Symmetric Difference

From **Example_A** and **Example_B**

$$(X \oplus Y) \oplus Z = X \oplus (Y \oplus Z)$$

Since $\oplus$ is associative, brackets can be included to suit.
From above, $X \oplus Y \oplus Z$ consists of 'cells' i.e.

$$X \oplus Y \oplus Z = (X \cap \overline{Y} \cap \overline{Z}) \cup (\overline{X} \cap \overline{Y} \cap Z) \cup (X \cap Y \cap Z) \cup (\overline{X} \cap Y \cap \overline{Z})$$

The set operator, $\oplus$, is referred to as 'Disjoint Union' or 'Symmetric Difference'. From above:

$$\begin{aligned} X \oplus Y &= (X \cap \overline{Y}) \cup (Y \cap \overline{X}) \\ &= (X - Y) \cup (Y - X) \\ &= (X \cup Y) - (X \cap Y) \end{aligned}$$

Labelling Cells in Karnaugh Map

The 'cells' in a Karnaugh Map can be labelled using binary numbers, xyz.

		yz 00	01	11	10
	0	000	001	011	010
x	1	100	101	111	110

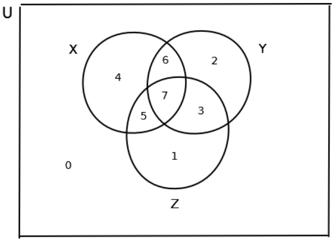
Note: Moving from one cell to a neighbour is a one 'bit' change.
Using Decimal numbers, the cells are labelled as:

		yz 00	01	11	10
	0	0	1	3	2
x	1	4	5	7	6

Labelling Venn Diagrams

While not as systematic as labelling Karnaugh Maps, Venn diagrams may also be labelled.

XYZ	Decimal
000	0
001	1
010	2
011	3
100	4
101	5
110	6
111	7



Binary representation of the sets $\{\}$ and U

An alternative notation for the empty set, $\{\}$, is $\emptyset$.
In the current context, use $X \backslash Y$ for $X - Y$ (Set Difference).
Consider the set $\{\emptyset,\ U\}$ then

$X \oplus Y$	$\emptyset$	U
$\emptyset$	$\emptyset$	U
U	U	$\emptyset$

$X \cup Y$	$\emptyset$	U
$\emptyset$	$\emptyset$	U
U	U	U

$X \cap Y$	$\emptyset$	U
$\emptyset$	$\emptyset$	$\emptyset$
U	$\emptyset$	U

$X \backslash Y$	$\emptyset$	U
$\emptyset$	$\emptyset$	$\emptyset$
U	U	$\emptyset$

Also, $\overline{\emptyset} = U$ and $\overline{U} = \emptyset$.

Binary Representation

Consider the set $B = \{0, 1\}$. Let $x, y \in B$.
The operation, $\oplus$, is binary addition and the operations $\cup$, $\cap$ and $\setminus$ can be defined by a table.

$x \oplus y$	0	1
0	**0**	**1**
1	**1**	**0**

$x \cup y$	0	1
0	**0**	**1**
1	**1**	**1**

$x \cap y$	0	1
0	**0**	**0**
1	**0**	**1**

$x \setminus y$	0	1
0	**0**	**0**
1	**1**	**0**

Also, $\overline{0} = 1$ and $\overline{1} = 0$.
In some textbooks, the symbol, 0, is used for the empty set, $\{\}$, and the symbol, 1, is used for the Universal set, U.

$\cap$, $\cup$, $\oplus$ in terms of $(+ \text{ mod } 2)$ and $(* \text{ mod } 2)$

Let $x, y \in B$ where $B = \{0, 1\}$.
$x \cap y$, $x \cup y$ and $x \oplus y$ can be defined in terms of the Binary Arithmetic operators $(+ \text{ mod } 2)$ and $(* \text{ mod } 2)$.

$$\begin{aligned} x \cap y &= (x * y)\, mod\, 2 \\ x \cup y &= (x + y + x * y)\, mod\, 2 \\ x \oplus y &= (x + y)\, mod\, 2 \end{aligned}$$

Set Difference and Set Complement

Note: The Set Difference operator, $\backslash$, is not the same as the Binary Arithmetic subtraction operator and Set Complement is not the same as Binary Arithmetic negation.
Recall $X \backslash Y = \{z \mid z \in X \text{ and } z \notin Y\}$ and $\overline{Y} = U \backslash Y$.
In terms of Binary Arithmetic

$$x \backslash y = (x * (y+1)) \, mod \, 2 \; \therefore$$
$$\overline{y} = 1 \backslash y = (1 * (y+1) \, mod \, 2 = (y+1) \, mod \, 2$$

$\therefore$ In Binary Arithmetic, $0 \backslash 1 = 0 = \overline{1}$ and $0 \backslash 0 = 0$.
Also, $1 \backslash 0 = 1$ and $1 \backslash 1 = 0$.

Set Difference and Subtraction

In Arithmetic, negation is defined in terms of, 0, the Identity for $+$, i.e.

$$x + (-x) = 0$$

where 0 is the Identity for $+$ i.e. $x + 0 = 0 + x = x$.
In Binary Arithmetic, $-0 = 0$ and $-1 = 1$.
Subtraction is defined so that $x - y = x + (-y)$
$\therefore 0 - x = -x$:
$0 - 1 = -1 = 1$ and $0 - 0 = -0 = 0$.
Also, $1 - 0 = 1$ and $1 - 1 = 0$.
Note: In Binary Arithmetic, $1 + 1 = 0$.