Theorem: Let (V, E) be a tree, then $\#(E) = \#(V) - 1$, where $\#(E)$ is the number of edges of the tree and $\#(V)$ is the number of vertices.

Proof: Use strong induction of $\#(V)$.

Base Case: $\#(V) = 1$. The graph is trivial $\Rightarrow \#(E) = 0$, so $0 = 1 - 1$ as needed.

Inductive Step: Suppose that every tree with m vertices ($\#(V) = m$) has $m - 1 = \#(v) - 1 = \#(E)$ edges. We seek to prove that if (V, E) is a tree with $m + 1$ vertices, then it has m edges.
By the previous theorem, (V, E) has one pendent vertex. Let that vertex be v. Since deg $v = 1$, then there is only one edge incident to v. Let vw be that edge. w is then the only vertex of (V, E) adjacent to v. We wish to reduce to the inductive hypothesis, the most natural way is to delete v from V and vw from E. Let $V' = V \backslash \{v\}$ and $E' = E \backslash \{vw\}$. (V', E') is a subgraph of (V, E) such that $\#(V') = \#(V) - 1$ and $\#(E') = \#(E) - 1$. To use the inductive hypothesis, we must show (V', E') is a tree, **i.e.** (V', E') is connected and $(V'E')$ contains no circuits. $\forall v_1, v_2 \in V'$, since (V, E) is a tree hence connected, $\exists$ path from v_1 to v_2 in (V, E). This path cannot pass through v because deg $v = 1 \Rightarrow$ it would have to pass through w twice contradicting the fact that it is a path (all vertices are distinct) $\Rightarrow$ this path is in $(V', E') \Rightarrow (V'E')$ is connected.
(V', E') is a subgraph of (V, E), which is a tree, hence does not contain any circuits, so (V', E') contains no circuits.
(V', E') is thus a tree, $\Rightarrow$ by the inductive hypothesis, $\#(V') = \#(V) - 1 = \#(E') - 1 = \#(E) - 1 - 1 = \#(E) - 2 \Rightarrow \#(V) - 1 = \#(E) - 2 \Leftrightarrow \#(V) = \#(E) - 1$ as needed.

qed

(L.i.d.)

<u>Theorem</u> Let (V, E) be a tree, $\forall v, w \in V$ w/ $v \neq w$, $\exists!$ path in (V, E) from v to w.

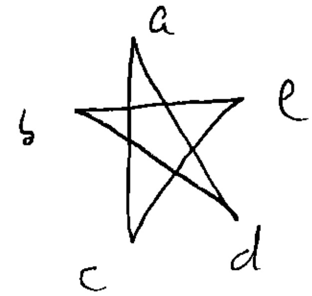
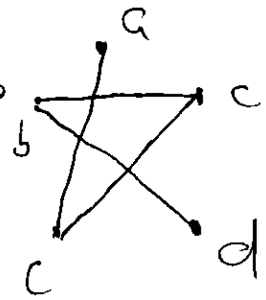
<u>Proof</u> (V, E) is a tree $\Rightarrow$ (V, E) is connected $\Rightarrow$ $\exists$ path from v to w. Assume $\exists$ 2 distinct paths from v to w. By a previous Theorem, we deduce (V, E) contains a circuit (recall that one criterion for having a circuit in a graph was the existence of two distinct paths between two vertices) $\Rightarrow\Leftarrow$ (V, E) is a tree hence it contains no circuits $\Rightarrow$ the path between v and w in (V, E) is unique.

(Q.e.d.)

Spanning Trees

Task For any graph, construct a subgraph containing all the vertices of the original graph such that this subgraph is a tree.

Def A spanning tree in a graph (V, E) is a subgraph of the graph (V, E), which is a tree and includes every vertex in V.

Example  the pentagram has 

as a spanning tree (we delete the edge ad from the pentagram so that there is no circuit).

Remark A graph (V, E) may have more than one spanning tree, i.e. spanning trees are not unique.

Theorem Every connected graph contains a spanning tree.

Proof Let (V, E) be a connected graph. Let $\mathcal{C}$ be the collection of all connected subgraphs (V', E') of the graph (V, E) with $V' = V$ (i.e. containing all vertices of the original graph). The original graph $(V, E) \in \mathcal{C}$, so $\mathcal{C}$ is not empty. Choose (V, E') in $\mathcal{C}$ such that the number of edges $\#(E')$ is minimal, i.e. (V, E') is such that $\forall (V, E'') \in \mathcal{C}$, $\#(E') \leq \#(E'')$.

Claim (V, E') is the required spanning tree.

Proof of claim: (V, E') is connected and has the same vertices as (V, E) since it belongs to $\mathcal{C}$. We just need to show that (V,E') is a tree, i.e. that it contains no circuits. We prove so indirectly, i.e. by contradiction. Assume (V,E') contains a circuit. Let vw be one of the edges traversed by a circuit in (V,E'). Let $E'' = E' - \{vw\}$ (we take out that edge). There still exists a walk from vertex v to vertex w via the remaining edges of the circuit. Note that since (V,E') is connected there exists a walk from every vertex in V to v via edges in E' and therefore to either v or w via edges in E''. Since there exists a walk from v to w via edges in E'', every vertex in V is connected to v via a walk whose edges belong to $E'' \Rightarrow (V,E'')$ is connected $\Rightarrow (V,E'') \in \mathcal{C}$, but $\#(E'') = \#(E') - 1 \Rightarrow\Leftarrow$ as (V,E') was selected to be the graph in $\mathcal{C}$ w/ the least number of edges $\Rightarrow (V,E')$ cannot contain a circuit $\Rightarrow (V,E')$ is the required spanning tree.

(q.e.d.)

<u>Corollary</u> Let (V,E) be a connected graph w/ $\#(V)$ vertices and $\#(E)$ edges. If $\#(E) = \#(V) - 1$, then (V,E) is a tree.

<u>Proof</u> By the previous theorem, every connected graph contains a spanning tree, and by a previous theorem proven during the section on trees, that tree has $\#(V)-1$ edges $\Rightarrow$ the spanning tree has the same number of edges as (V,E) and is its subgraph by definition $\Rightarrow (V,E)$ is its own spanning tree $\Rightarrow (V,E)$ is a tree.

(q.e.d.)

Constructing spanning trees

<u>Task</u> Given a connected undirected graph, investigate two ways of constructing a spanning tree for it.

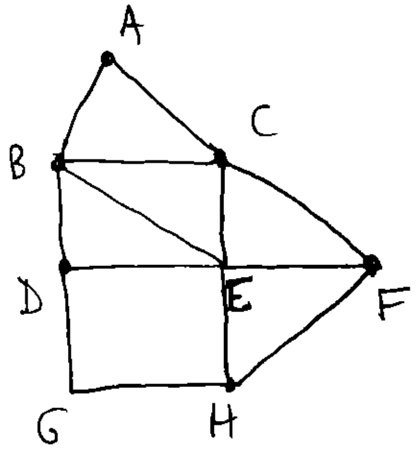
Let (V, E) be a connected undirected graph. We can proceed in one of two ways to construct a spanning tree for it:

(1) Start w/ (V, E) itself. Break up all of its circuits by deleting one edge per circuit.

(2) Start w/ an edge in E. Let this edge be vw. Add back all remaining vertices in $V - \{v, w\}$ by adding in one edge in E per vertex such that at each step the subgraph of (V, E) that we have is both connected <u>AND</u> a tree.

Remark Note that algorithm 1 is akin to the proof of the Theorem that every connected graph has a spanning tree.

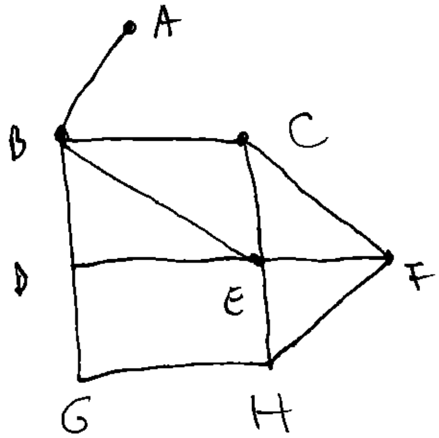
Example Consider



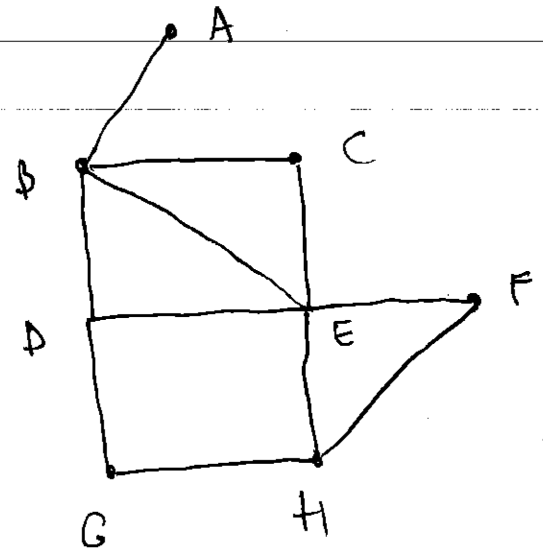
We shall illustrate both 1 and 2 on this graph.

First procedure 1:

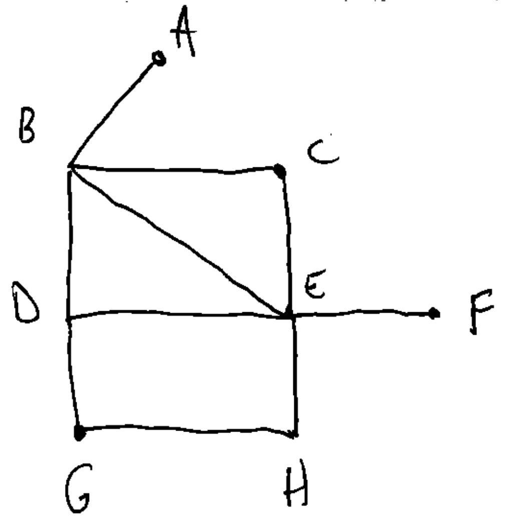
Note ABCA is a circuit. We have a choice which edge to delete. Let us choose to delete AC.



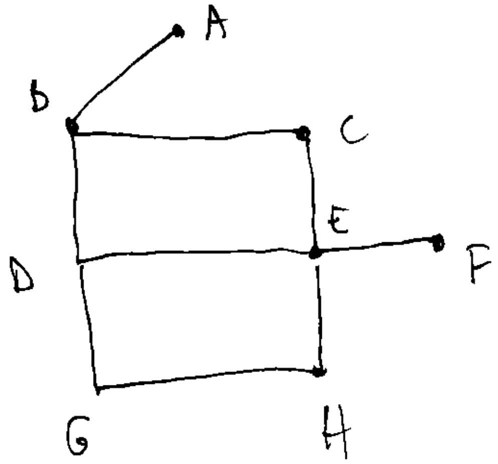
CEFC is a circuit, We choose to delete CF.



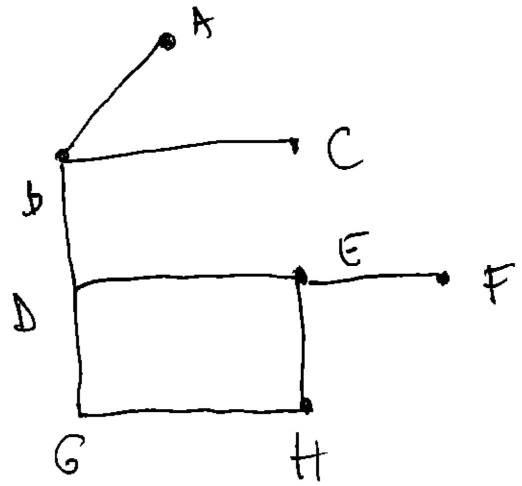
HFHE is a circuit. We choose to delete FH.



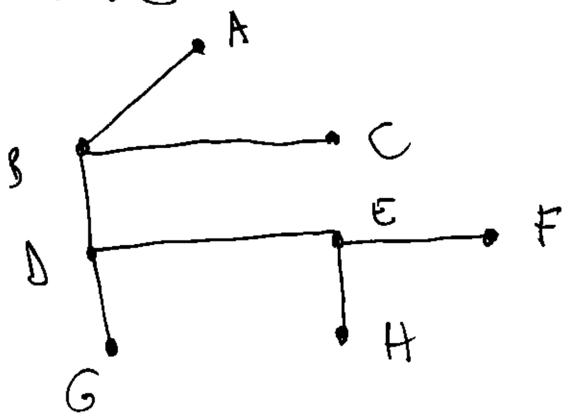
BDEB is a circuit, We choose to delete BE.



BCEDB is a circuit. We choose to delete CE.



DEHGD is a circuit. We choose to delete GH.



The graph that is left doesn't seem to have any circuits. We check that it is a tree using the formula we proved earlier in the course that for a tree $\#(E) = \#(V) - 1$. (46)

$$V = \{A, B, C, D, E, F, G, H\} \Rightarrow \#(V) = 8$$

$$E' = \{AB, BC, BD, DE, EF, DG, EH\} = 7 = \#(V) - 1$$

So (V, E') that we have constructed is a tree and hence the spanning tree of the original (V, E).