Mathematics CS1003

Dr. Meriel Huggard

Room 1.09 Lloyd Institute
School of Computer Science and Statistics
Trinity College Dublin, IRELAND
http://mymodule.tcd.ie
Meriel.Huggard@tcd.ie

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Eigenvalues and Eigenvectors I

Considering a matrix A, we want to calculate

- its eigenvalues
- its eigenvectors

Definition:

We want to find real numbers, λ, and non-zero vectors, $\mathbf{v}$; where they exist; such that $\mathbf{v}$ and $A\mathbf{v}$ are scalar multiples of each other:

$$A\mathbf{v} = \lambda\mathbf{v}$$

- λ is called an eigenvalue of A.
- $\mathbf{v}$ is the eigenvector of A corresponding to λ.

Eigenvalues and Eigenvectors II

Our equation

$$A\mathbf{v} = \lambda\mathbf{v}$$

may be rewritten as:

$$A\mathbf{v} - \lambda\mathbf{v} = 0$$

or, using the identity matrix I:

$$(A - \lambda I)\mathbf{v} = 0$$

To find when this has a non-trivial solution we need to find when

$$\det(A - \lambda I) = 0$$

This is called the characteristic equation of A.

When we expand this we obtain the characteristic polynomial of A.

EXAMPLE : Find the eigenvalues of the matrix

$$A = \begin{pmatrix} 5 & 6 & 2 \\ 0 & -1 & -8 \\ 1 & 0 & -2 \end{pmatrix}.$$

Eigenvalues and Eigenvectors IV

SOLUTION: We found that

$$\det(A - \lambda I) = -\lambda^3 + 2\lambda^2 + 15\lambda - 36.$$

To find solutions to $\det(A - \lambda I) = 0$ i.e., to solve

$$\lambda^3 - 2\lambda^2 - 15\lambda + 36 = 0.$$

- Find integer valued solutions. Such solutions divide the constant term (36). Possibilities: $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 9, \pm 12, \pm 18, \pm 36$.
- $\lambda = 3$: $3^3 - 2.3^2 - 15.3 + 36 = 0$.
- Now factor out $\lambda - 3$:

$$(\lambda - 3)(\lambda^2 + \lambda - 12) = \lambda^3 - 2\lambda^2 - 15\lambda + 36.$$

Eigenvalues and Eigenvectors V

- Solve $\lambda^2 + \lambda - 12 = 0$ by formula:

$$\lambda = \frac{-1 \pm \sqrt{1^2 - 4.1. - 12}}{2} = \frac{-1 \pm 7}{2}$$

  Thus $\lambda = -4$ or 3.

- So

$$\begin{aligned} \det(A - \lambda I) &= -\lambda^3 + 2\lambda^2 + 15\lambda - 36 \\ &= (\lambda - 3)(\lambda - 3)(\lambda + 4) \end{aligned}$$

The eigenvalues of A are $\lambda = -4, 3$. Note that $\lambda = 3$ is a repeated root of the characteristic equation.

Eigenvalues and Eigenvectors VI

Once the eigenvalues of a matrix A have been found, we can find the eigenvectors by Gaussian Elimination.

1. For each eigenvalue λ, we have

$$(A - \lambda I)\mathbf{x} = \mathbf{0},$$

   where $\mathbf{x}$ is the eigenvector associated with eigenvalue λ.

2. Find $\mathbf{x}$ by Gaussian elimination. That is, convert the augmented matrix

$$\left(A - \lambda I \,\vdots\, \mathbf{0} \right)$$

   to reduced row echelon form, and solve the resulting linear system.

Eigenvalues and Eigenvectors VII

<u>EXAMPLE</u>: Find the eigenvectors of

$$A = \begin{pmatrix} 5 & 6 & 2 \\ 0 & -1 & -8 \\ 1 & 0 & -2 \end{pmatrix}.$$

we know that the eigenvalues of A are $\lambda = -4, 3$ with 3 being a repeated root (twice).

<u>SOLUTION</u>:

- **Case 1:** $\lambda = -4$

  We must find vectors $\mathbf{x}$ which satisfy $(A - \lambda I)\mathbf{x} = \mathbf{0}$:

$$\lambda = -4 \text{ gives us } A - \lambda I = \begin{pmatrix} 9 & 6 & 2 \\ 0 & 3 & -8 \\ 1 & 0 & 2 \end{pmatrix}.$$

Eigenvalues and Eigenvectors IX

- Construct the augmented matrix $\left(A - \lambda I \,\vdots\, \mathbf{0}\right)$ and convert it to row echelon form

$$\begin{array}{l}\left(\begin{array}{cccc} 9 & 6 & 2 & 0 \\ 0 & 3 & -8 & 0 \\ 1 & 0 & 2 & 0 \end{array}\right)\end{array}\begin{array}{l}\text{R1}\\\text{R2}\\\text{R3}\end{array} \xrightarrow{R1 \leftrightarrow R3} \left(\begin{array}{cccc} 1 & 0 & 2 & 0 \\ 0 & 3 & -8 & 0 \\ 9 & 6 & 2 & 0 \end{array}\right)\begin{array}{l}\text{R1}\\\text{R2}\\\text{R3}\end{array}$$

$$\xrightarrow{R3 \to R3 - 9 \times R1} \left(\begin{array}{cccc} 1 & 0 & 2 & 0 \\ 0 & 3 & -8 & 0 \\ 0 & 6 & -16 & 0 \end{array}\right)\begin{array}{l}\text{R1}\\\text{R2}\\\text{R3}\end{array}$$

$$\xrightarrow{R2 \to 1/3 \times R2} \left(\begin{array}{cccc} 1 & 0 & 2 & 0 \\ 0 & 1 & -8/3 & 0 \\ 0 & 6 & -16 & 0 \end{array}\right)\begin{array}{l}\text{R1}\\\text{R2}\\\text{R3}\end{array}$$

$$\xrightarrow{R3 \to R3 - 6 \times R2} \left(\begin{array}{cccc} 1 & 0 & 2 & 0 \\ 0 & 1 & -8/3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)\begin{array}{l}\text{R1}\\\text{R2}\\\text{R3}\end{array}$$

Eigenvalues and Eigenvectors X

- Rewrite as a linear system

$$\begin{aligned} x_1 \quad + 2x_3 &= 0 \\ x_2 - 8/3x_3 &= 0 \end{aligned}$$

  or, introducing parameters

$$\begin{aligned} x_1 &= -2t \\ x_2 &= 8/3t \\ x_3 &= t \end{aligned}$$

- Thus

$$\mathbf{x} = \begin{pmatrix} -2t \\ 8/3t \\ t \end{pmatrix} = t \begin{pmatrix} -2 \\ 8/3 \\ 1 \end{pmatrix} \quad \text{for any } t \in \mathbb{R}$$

  are eigenvectors of A associated with the eigenvalue $\lambda = -4$.

Eigenvalues and Eigenvectors XI

- **Case 2**: $\lambda = 3$

  We seek vectors x for which $(A - \lambda I)\mathbf{x} = \mathbf{0}$.

$$\lambda = 3 \Rightarrow A - \lambda I = \begin{pmatrix} 2 & 6 & 2 \\ 0 & -4 & -8 \\ 1 & 0 & -5 \end{pmatrix}.$$

Eigenvalues and Eigenvectors XII

- Construct the augmented matrix $\left(A - \lambda I \,\vdots\, \mathbf{0}\right)$ and reduce it to row echelon form.

$$\begin{array}{cc} \begin{pmatrix} 2 & 6 & 2 & 0 \\ 0 & -4 & -8 & 0 \\ 1 & 0 & -5 & 0 \end{pmatrix} & \begin{array}{l} \text{R1} \\ \text{R2} \\ \text{R3} \end{array} \end{array}$$

$$\overset{R1 \leftrightarrow R3}{\longrightarrow} \quad \begin{array}{cc} \begin{pmatrix} 1 & 0 & -5 & 0 \\ 0 & -4 & -8 & 0 \\ 2 & 6 & 2 & 0 \end{pmatrix} & \begin{array}{l} \text{R1} \\ \text{R2} \\ \text{R3} \end{array} \end{array}$$

$$\overset{R3 \to R3 - 2 \times R1}{\longrightarrow} \quad \begin{array}{cc} \begin{pmatrix} 1 & 0 & -5 & 0 \\ 0 & -4 & -8 & 0 \\ 0 & 6 & 12 & 0 \end{pmatrix} & \begin{array}{l} \text{R1} \\ \text{R2} \\ \text{R3} \end{array} \end{array}$$

$$\overset{R2 \to -1/4 \times R2}{\longrightarrow} \quad \begin{array}{cc} \begin{pmatrix} 1 & 0 & -5 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 6 & 12 & 0 \end{pmatrix} & \begin{array}{l} \text{R1} \\ \text{R2} \\ \text{R3} \end{array} \end{array}$$

$$\overset{R3 \to R3 - 6 \times R2}{\longrightarrow} \quad \begin{array}{cc} \begin{pmatrix} 1 & 0 & -5 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & \begin{array}{l} \text{R1} \\ \text{R2} \\ \text{R3} \end{array} \end{array}$$

Eigenvalues and Eigenvectors XIII

- Rewrite this as a linear system:

$$\begin{aligned} x_1 \qquad - 5x_3 &= 0 \\ x_2 + 2x_3 &= 0. \end{aligned}$$

  or, introducing parameter t,

$$\begin{aligned} x_1 &= 5t \\ x_2 &= -2t \\ x_3 &= t \end{aligned}$$

- Thus

$$\mathbf{x} = \begin{pmatrix} 5t \\ -2t \\ t \end{pmatrix} = t \begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix} \quad \text{for any } t \in \mathbb{R}$$

  are eigenvectors associated with eigenvalue $\lambda = 3$.

Eigenvalues and Eigenvectors XIV

Conclusions:

- The solution of the linear system will involve at least one parameter, so there are infinitely many eigenvectors.
- However, all eigenvectors have a very specific form.