MA2C03 - DISCRETE MATHEMATICS - TUTORIAL NOTES
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1 Set theory

Example 1.1. Prove $A \setminus (A \setminus B) \subseteq B$.

Proof.
This is done by examining where the elements lie in the set to the left of $\subseteq$ and proving they also lie in B. To this end, take $x \in A \setminus (A \setminus B)$. By the definition of $X \setminus Y = X \cap Y^c$, we have

$$x \in A \setminus (A \setminus B) \Rightarrow x \in A \cap (A \setminus B)^c \Rightarrow x \in A \text{ AND } x \in (A \setminus B)^c$$

Applying the definition of $\setminus$ again we conclude $x \in A$ AND $x \in (A \cap B^c)^c$. Using De Morgan's laws for the later, we get $x \in A$ AND $x \in A^c \cup (B^c)^c$. Let's focus more on the later, with the knowledge that $x \in A$.

$$x \in A^c \cup (B^c)^c \Rightarrow x \in A^c \text{ OR } x \in (B^c)^c \Rightarrow x \in A^c \text{ OR } x \in B$$

Since x cannot be in both A and A^c at the same time, we conclude $x \in B$ (now ignoring A). What we have shown:

$$\forall x (x \in A \setminus (A \setminus B) \Rightarrow x \in B)$$

So $A \setminus (A \setminus B) \subseteq B$ as required. ■

Example 1.2. $A = B \Leftrightarrow A \subseteq B \wedge B \subseteq A$

Proof.
I want to emphasise that this is how two sets should be proved equivalent - show A is a subset of B, and vice versa. As Prof. Nicoara writes in the online notes, this comes from the tautology $(P \leftrightarrow Q) \leftrightarrow (P \rightarrow Q \wedge Q \rightarrow P)$. The contrapositive of this statement can also be used to show two sets are not equal - the contrapositive can be deduced from the above tautology. ■

2 Logic

Example 2.1. Show via a truth table $P \to Q \equiv \neg Q \to \neg P$ ("$P \to Q$ is equivalent to $\neg Q \to \neg P$").

Proof.
We want to show the two statements are equivalent; if one is true, so must the other be. The truth table will show this, as the columns for $P \to Q$ and $\neg Q \to \neg P$ will be identical.

P	Q	$\neg P$	$\neg Q$	$P \to Q$	$\neg Q \to \neg P$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

Example 2.2. Explain *why* induction works.

Proof.
When we prove a proposition $P(n)$ by induction, we are proving $\forall n(P(n))$ - we do this by proving two statements;

(1) $P(1)$

(2) $\forall n(P(n) \to P(n+1))$

Suppose we were unsuccessful in our proof; it's not true that $\forall n(P(n))$ - namely $\exists m \neg P(m)$. Note that we only use induction on finite numbers, meaning $m < \infty$.

- By (1), we know $P(1)$.
- By (2), with $n = 1$ we know $P(1) \to P(2)$.
- By Modus Ponens (MP), we know $P(2)$. Modus Ponens means "from X, $X \to Y$, infer Y".

- Continuing this pattern, we eventually know $P(m-1)$ and $P(m-1) \to P(m)$.
- By MP we know $P(m)$; a contradiction as m was a finite number such that $\neg P(m)$ was true.

Thus, by a proof by contradiction, we conclude induction 'works'. ■

Example 2.3. For an example of a proof by induction, e.g. $\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$, see the class/online blackboard notes.

3 Relations

Example 3.1. The binary relation $\leq$ is *not* an equivalence relation - why is this? What is $\leq$?

Proof.
For the why, we'll leave that as an exercise for the reader (HINT: to prove this, run through the definition of an *equivalence relation* and see if $\leq$ satisfies the three properties).
What is $\leq$? $\leq$ is a relation between real numbers and is a *total order*. That is, a partial order that is *total*, i.e. every two elements of $\mathbb{R}$ can be compared under $\leq$. Logically, this is written $\forall x, y(x \leq y \vee y \leq x)$.
Orders come in two flavours; *total* and *partial* - the difference being if all the elements can be compared or not.

- On $\mathbb{R}$, $\leq$ is a *total* order - it satisfies the definition of a *partial order* **and** every two elements can be compared. To clear a point of possible confusion; yes, it's a type of partial order, but there's more to it than that.
  $\leq$ is a total order on any subset of $\mathbb{R}$ as well; like $\mathbb{Q}$ or $\mathbb{Z}$.
- Consider set inclusion; $\subseteq$. This is a partial order on sets. However this order is not total; consider the sets $A = \{1, 2, 3\}$ and $B = \{2, 3, 4\}$. Then $A \not\subseteq B$ and $B \not\subseteq A$ - these sets can't be compared using $\subseteq$.
- I gave another example during the tutorial - binary trees. When we discuss binary trees later in the course, you'll see the relation

$$A \leq B \Leftrightarrow A \text{ is an ancestor of } B$$

forms a partial order.

■

Example 3.2. Prove $\equiv_3$ is an equivalence relation.

Proof.
This is covered in the online notes. We also spoke briefly about how there wasn't anything special about the "3", i.e. equivalence modulo n is also an equivalence relation, for all n. ■