Petit Samuel

CS 1003 - Math Homework III

Q1 - $A = \begin{pmatrix} -5 & -2 & -2 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$

I am going to calculate the determinant first, that will also check if A has an inverse (if $\det(A) \neq 0$).

$$\det(A) = 1 \times \begin{vmatrix} -2 & -2 \\ 1 & 0 \end{vmatrix} - 0 \times \begin{vmatrix} -5 & -2 \\ 2 & 0 \end{vmatrix} + 1 \times \begin{vmatrix} -5 & -2 \\ 2 & 1 \end{vmatrix}$$

$$= 1 \times (0 - (-2)) - 0 + 1 \times (-5 - (-4))$$
$$= 2 - 0 - 1 = 1.$$

I am now going to calculate the matrix of cofactors

$$\tilde{A} = \begin{pmatrix} \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} & (-1) \times \begin{vmatrix} 2 & 0 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix} \\ (-1) \times \begin{vmatrix} -2 & -2 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} -5 & -2 \\ 1 & 1 \end{vmatrix} & (-1) \times \begin{vmatrix} -5 & -2 \\ 1 & 0 \end{vmatrix} \\ \begin{vmatrix} -2 & -2 \\ 1 & 0 \end{vmatrix} & (-1) \times \begin{vmatrix} -5 & -2 \\ 2 & 0 \end{vmatrix} & \begin{vmatrix} -5 & -2 \\ 2 & 1 \end{vmatrix} \end{pmatrix}$$

$$\tilde{A} = \begin{pmatrix} 1 & -2 & -1 \\ 2 & -3 & -2 \\ 2 & -4 & -1 \end{pmatrix}.$$

$$A^{-1} = \frac{1}{1} \begin{pmatrix} 1 & -2 & -1 \\ 2 & -3 & -2 \\ 2 & -4 & -1 \end{pmatrix}^T = \begin{pmatrix} 1 & 2 & 2 \\ -2 & -3 & -4 \\ -1 & -2 & -1 \end{pmatrix}.$$

Q2. $B = \begin{pmatrix} 4 & 2 & -1 \\ 2 & 4 & 1 \\ -1 & 1 & 3 \end{pmatrix}$

With $\vec{v}$ the eigenvectors of B, λ the eigenvalues of B.

Let's first find the characteristic equation for B:

$$B\vec{v} = \lambda\vec{v}$$
$$\Leftrightarrow B\vec{v} - \lambda I\vec{v} = 0$$
$$\Leftrightarrow (B - \lambda I)\vec{v} = 0.$$

$$B - \lambda I = \begin{pmatrix} 4 & 2 & -1 \\ 2 & 4 & 1 \\ -1 & 1 & 3 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 2 & -1 \\ 2 & 4 & 1 \\ -1 & 1 & 3 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} = \begin{pmatrix} 4-\lambda & 2 & -1 \\ 2 & 4-\lambda & 1 \\ -1 & 1 & 3-\lambda \end{pmatrix}.$$

$$\det(B - \lambda I) = \begin{vmatrix} 4-\lambda & 2 & -1 \\ 2 & 4-\lambda & 1 \\ -1 & 1 & 3-\lambda \end{vmatrix}$$

$$= (-1) \times \begin{vmatrix} 2 & -1 \\ 4-\lambda & 1 \end{vmatrix} - 1 \times \begin{vmatrix} 4-\lambda & -1 \\ 2 & 1 \end{vmatrix} + (3-\lambda) \times \begin{vmatrix} 4-\lambda & 2 \\ 2 & 4-\lambda \end{vmatrix}$$

$$= (-1)\left[2 - (4-\lambda)(-1)\right] - \left[4-\lambda+2\right] + (3-\lambda)\left[(4-\lambda)^2 - 4\right]$$

$$= (-1)(2 - (\lambda - 4)) - 6 + \lambda + (3-\lambda)(12 - 8\lambda + \lambda^2)$$

$$= (-1)(6-\lambda) - 6 + \lambda + \left[36 - 24\lambda + 3\lambda^2 - 12\lambda + 8\lambda^2 - \lambda^3\right]$$

$$= (-6) + \lambda - 6 + \lambda + 36 - 36\lambda + 11\lambda^2 - \lambda^3$$

$$= -\lambda^3 + 11\lambda^2 - 34\lambda + 24.$$

The characteristic equation is then:

$$-\lambda^3 + 11\lambda^2 - 34\lambda + 24 = 0$$

$$\Leftrightarrow \boxed{\lambda^3 - 11\lambda^2 + 34\lambda - 24 = 0.}$$

Let's now check if 1 is a solution of that equation

$1^3 - 11 \cdot (1)^2 + 34 \cdot 1 - 24 = 1 - 11 + 34 - 24$
$= 0.$

I am looking to writte the expression in the form: $(\lambda - 1)(a\lambda^2 + b\lambda + c)$
with a, b and c real numbers.

$(\lambda - 1)(a\lambda^2 + b\lambda + c) = a\lambda^3 + b\lambda^2 + c\lambda - a\lambda^2 - b\lambda - c$
$= a\lambda^3 + (b-a)\lambda^2 + (c-b)\lambda - c.$

We are looking for the values of a, b and c as:

$a\lambda^3 + (b-a)\lambda^2 + (c-b)\lambda - c = \lambda^3 - 11\lambda^2 + 34\lambda - 2$

$$\begin{cases} a = 1 \\ b - a = -11 \\ c - b = 34 \\ -c = -24 \end{cases} \Leftrightarrow \begin{cases} a = 1 \\ b - 1 = -11 \\ c - b = 34 \\ c = -24 \end{cases} \Leftrightarrow \begin{cases} a = 1 \\ b = -10 \\ c - (-10) = 34 \\ c = -24 \end{cases}$$

$$\Leftrightarrow \begin{cases} a = 1 \\ b = -10 \\ c = 24 \\ c = 24. \end{cases}$$

By identification we then know:
$(\lambda - 1)(\lambda^2 - 10\lambda + 24) = \lambda^3 - 11\lambda^2 + 34\lambda - 24.$

I am now going to solve the equation:

$\lambda - 1 = 0$
$\Leftrightarrow \lambda = 1$

$\lambda^2 - 10 + 24 = 0$
$\Delta = (-10)^2 - 4 \cdot 1 \cdot 24 = 4$

we then know:
$\lambda_1 = \frac{-(-10) - \sqrt{4}}{2 \cdot 1} = 4$
$\lambda_2 = \frac{10 + \sqrt{4}}{2} = 6.$

The eigenvalues of B are 1, 6 and 4.

I am now going to find the eigen vectors associated with B: using Gaussian Elimination.

- Case 1: $\lambda = 1$, we are looking for $\vec{v}$ where $(B - 1.I) = 0$.

$$B - I = \begin{pmatrix} 3 & 2 & -1 \\ 2 & 3 & 1 \\ -1 & 1 & 2 \end{pmatrix} \begin{matrix} R1 \\ R2 \\ R3 \end{matrix}$$

; I am now going to reduce the augmented Matrix $(B - I : \vec{0})$ to reduced row echelon form.

$R3 \leftrightarrow -1.R3$
$R3 \leftrightarrow R1$

$$\longrightarrow \begin{pmatrix} 1 & -1 & 2 & 0 \\ 2 & 3 & 1 & 0 \\ 3 & 2 & -1 & 0 \end{pmatrix} \begin{matrix} R1 \\ R2 \\ R3 \end{matrix}$$

$R2 \to R2 + (-2)R1$
$R3 \to R3 + (-3)R1$

$$\begin{pmatrix} 1 & -1 & -2 & 0 \\ 0 & 5 & 5 & 0 \\ 0 & 5 & 5 & 0 \end{pmatrix} \begin{matrix} R1 \\ R2 \\ R3 \end{matrix} \longleftarrow$$

$$R2 \to \frac{1}{5} R2 \longrightarrow \begin{pmatrix} 1 & -1 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 5 & 5 & 0 \end{pmatrix} \begin{matrix} R1 \\ R2 \\ R3 \end{matrix}$$

$R1 \to R1 + R2$
$R3 \to R3 - 5R$

$$\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{matrix} R1 \\ R2 \\ R3 \end{matrix} \longleftarrow$$

$$\begin{cases} x_1 - x_3 = 0 \\ x_2 + x_3 = 0 \end{cases} \quad \text{with } x_3 = t$$

$$\begin{cases} x_1 - t = 0 \\ x_2 + t = 0 \end{cases} \iff \begin{cases} x_1 = t \\ x_2 = -t \end{cases}$$

So: $\vec{v_1} = \begin{pmatrix} t \\ -t \\ t \end{pmatrix} = t \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$. with $t \in \mathbb{R}$.

$\vec{v_1}$ are the eigenvectors of B associated with the value $\lambda = 1$.

I am now going to use the same method for the 2 other eigenvalues.

· Case 2 : $\lambda = 4$.

$$(B - 4I) = \begin{pmatrix} 0 & 2 & -1 \\ 2 & 0 & 1 \\ -1 & 1 & -1 \end{pmatrix} ; (B - 4I : 0) = \begin{pmatrix} 0 & 2 & -1 & 0 \\ 2 & 0 & 1 & 0 \\ -1 & 1 & -1 & 0 \end{pmatrix}$$

$R3 \to (-1) \cdot R3$
$R3 \leftrightarrow R1$

$$\longrightarrow \begin{array}{c} \begin{pmatrix} 1 & -1 & 1 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 \end{pmatrix} \end{array} \begin{array}{l} R1 \\ R2 \\ R3 \end{array}$$

$$\begin{array}{l} R1 \\ R2 \\ R3 \end{array} \begin{pmatrix} 1 & -1 & 1 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 2 & -1 & 0 \end{pmatrix} \qquad \begin{array}{l} R2 \to R2 - 2R1 \\ R2 \leftrightarrow R3 \\ \longleftarrow \end{array}$$

$$R2 \to \frac{1}{2} R2 \quad \longrightarrow \begin{pmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 2 & -1 & 0 \end{pmatrix} \begin{array}{l} R1 \\ R2 \\ R3 \end{array}$$

$$\begin{array}{l} R1 \\ R2 \\ R3 \end{array} \begin{pmatrix} 1 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \begin{array}{l} R1 \to R1 + R2 \\ R3 \to R3 - 2R2 \\ \longleftarrow \end{array}$$

$$\begin{cases} x_1 + \frac{1}{2} x_3 = 0 \\ x_2 - \frac{1}{2} x_3 = 0 \end{cases} \qquad \text{with } x_3 = t$$

$$\begin{cases} x_1 + \frac{1}{2} t = 0 \\ x_2 - \frac{1}{2} t = 0 \end{cases} \iff \begin{cases} x_1 = -\frac{1}{2} t \\ x_2 = \frac{1}{2} t \end{cases}$$

So: $\vec{v_2} = \begin{pmatrix} -\frac{1}{2} t \\ \frac{1}{2} t \\ t \end{pmatrix} = t \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 1 \end{pmatrix}$.

with $t \in \mathbb{R}$. $\vec{v_2}$ are the eigenvectors of B associated with the value $\lambda = 4$

Case 3: $\lambda = 6$

$$(B - 6I) = \begin{pmatrix} -2 & 2 & -1 \\ 2 & -2 & 1 \\ -1 & 1 & -3 \end{pmatrix}. \quad (B-6I \,|\, 0) = \left(\begin{array}{ccc|c} -2 & 2 & -1 & 0 \\ 2 & -2 & 1 & 0 \\ -1 & 1 & -3 & 0 \end{array}\right)$$

$R3 \to (-1) \cdot R3$
$R3 \leftrightarrow R1$

$$\longrightarrow \left(\begin{array}{cccc} 1 & -1 & 3 & 0 \\ 2 & -2 & 1 & 0 \\ -2 & 2 & -1 & 0 \end{array}\right) \begin{array}{l} R1 \\ R2 \\ R3 \end{array}$$

$R2 \to R2 - 2R1$
$R3 \to R3 + 2R1$

$$\longleftarrow \left(\begin{array}{cccc} 1 & -1 & 3 & 0 \\ 0 & 0 & -5 & 0 \\ 0 & 0 & 5 & 0 \end{array}\right) \begin{array}{l} R1 \\ R2 \\ R3 \end{array}$$

$R2 \to -\frac{1}{5} R2$

$$\longrightarrow \left(\begin{array}{cccc} 1 & -1 & 3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 5 & 0 \end{array}\right) \begin{array}{l} R1 \\ R2 \\ R3 \end{array}$$

$R3 \to R3 - 5R2$
$R1 \to R1 - 3R2$

$$\longleftarrow \begin{array}{l} R1 \\ R2 \\ R3 \end{array} \left(\begin{array}{cccc} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\begin{cases} x_1 - x_2 = 0 \\ x_3 = 0 \end{cases} \iff \begin{cases} x_1 = x_2 \\ x_3 = 0 \end{cases}$$

if $x_1 = t$:

$$\begin{cases} x_1 = t \\ x_2 = t \\ x_3 = 0. \end{cases}$$

$$\vec{v_3} = \begin{pmatrix} t \\ t \\ 0 \end{pmatrix} = t \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

with $t \in \mathbb{R}$. $\vec{v_3}$ are the eigenvectors of B associated with the value $\lambda = 6$.