MA2C03: ASSIGNMENT 2 SOLUTIONS

1) (20 points)

(a) Describe the formal language over the alphabet $\{a, b, c\}$ generated by the context-free grammar whose non-terminals are $\langle S\rangle$ and $\langle A\rangle$, whose start symbol is $\langle S\rangle$, and whose production rules are the following:
(1) $\langle S\rangle \to a\langle S\rangle$
(2) $\langle S\rangle \to b\langle A\rangle$
(3) $\langle A\rangle \to b\langle A\rangle$
(4) $\langle A\rangle \to c\langle A\rangle$
(5) $\langle A\rangle \to c$
(6) $\langle S\rangle \to a$
In other words, describe the structure of the strings generated by this grammar.

(b) Is this grammar regular? Justify your answer.

(c) Is this grammar in normal form? If it is not in normal form, then modify it to make it be in normal form. Explain why it generates the same language after your modifications.

(d) Write down a regular expression that gives the language from part (a) and justify your answer.

Solution: (5 points total: 2 for realizing the form of words in this language and 3 points for the justification) (a) $L = \{a^i \mid i \geq 2\} \cup \{a^i bwc \mid i \geq 0,\ w \in \{b, c\}^*\}$. Rules (1) and (6) produce $\{a^i \mid i \geq 2\}$. Rule (1) may or may not be applied. If it is, it produces a word a^i for $i \geq 0$ at the beginning. Rules (2) and (5) produce b and c respectively, while rules (3) and (4) give some word in terms of b's and c's between the b and the c produced by rules (2) and (5).

(b) (5 points total: 2 for the answer and 3 for the justification) Yes, all six production rules are of type (i), (ii), or (iii) from the definition of a regular grammar.

(c) (5 points total: 2 for the answer, 2 for the modification, and 1 point for the justification) No, rules (5) and (6) are of type ii. Add two non-terminals to split each of the rules (5) and (6) into two rules, one of type (i) and one of type (iii) as we did in lecture. For example, we can add non-terminals $\langle D\rangle$ and $\langle E\rangle$ and replace rule (5) by
(5) $\langle A\rangle \to c\langle D\rangle$

(7) $\langle D\rangle \to \epsilon$

and rule (6) by

(6) $\langle S\rangle \to a\langle E\rangle$

(8) $\langle E\rangle \to \epsilon$.

The newly added non-terminals do not appear in any other production rule of the grammar. Rules (5) and (7) altogether produce the same strings as the initial rule (5) did. Likewise, rules (6) and (8) together produce the same strings as the initial rule (6) did. As a result, the new production rules yield the same language as the initial ones.

(d) (5 points total: 3 for the answer and 2 for the justification) $(a \circ a \circ a^*) \cup (a^* \circ b \circ \{b, c\}^* \circ c)$.

$(a \circ a \circ a^*)$ corresponds to $\{a^i \mid i \geq 2\}$, while $(a^* \circ b \circ \{b, c\}^* \circ c)$ corresponds to $\{a^i bwc \mid i \geq 0,\ w \in \{b, c\}^*\}$.

2) (10 points) Let L be the language consisting of all binary numbers divisible by 4. Note that any binary number starting with 0 and containing more than one symbol is considered improper and should be rejected.

(a) Draw a deterministic finite state acceptor that accepts the language L. Carefully label all the states including the starting state and the finishing states as well as all the transitions. Make sure you justify it accepts all strings in the language L and no others.

(b) Devise a regular grammar in normal form that generates the language L. Be sure to specify the start symbol, the non-terminals, and all the production rules. Make sure you justify it generates all strings in the language L and no others.

Solution: (a) (5 points) A binary number divisible by 4 is either zero itself or ends in two zeroes. The DFA needs to have two branches out of the start node, one to handle a word beginning in zero, which unless is zero itself, is not proper, and one for words starting in 1. Only those starting in 1 that end in two 0's can be accepted. See drawing of the DFA at the end of the solution set.

(b) (5 points) We write down the production rules of the regular grammar corresponding to the DFA drawn in part (a) following the algorithm described in lecture that gives the one-to-one correspondence between a DFA and the production rules of a regular grammar:

(1) $\langle S\rangle \to 0\langle A\rangle$

(2) $\langle A\rangle \to \epsilon$

(3) $\langle S\rangle \to 1\langle B\rangle$

(4) $\langle B\rangle \to 1\langle B\rangle$

(5) $\langle B\rangle \to 0\langle C\rangle$
(6) $\langle C\rangle \to 1\langle B\rangle$
(7) $\langle C\rangle \to 0\langle D\rangle$
(8) $\langle D\rangle \to 0\langle D\rangle$
(9) $\langle D\rangle \to 1\langle B\rangle$
(10) $\langle D\rangle \to \epsilon$

Since the DFA in part (a) accepts L, this regular grammar generates exactly L.

3) (10 points) Let M be the language over the alphabet $\{a, b, c\}$ given by $M = \{a^i b^j c^k \mid i, j, k \geq 0 \quad j = i + k\}$.

(a) Use the Pumping Lemma to show this language is not regular.
(b) Write down the production rules of a context-free grammar that generates exactly M.

Solution: (a) (5 points) Assume the language M is regular, so it must have some pumping length p. In order to reduce the proof to just one case, consider the word $a^p b^{p+k} c^k \in M$ for $k \geq 0$. By the Pumping Lemma, the word $w = a^p b^{p+k} c^k$ can be decomposed as xuy, where the length $|xu| \leq p$. Therefore, $x = a^i$ and $u = a^j$ for $i, j \in \mathbb{N}$, $i + j = p$, and $j \geq 1$. Consider $xu^2y = a^{p+j} b^{p+k} c^k$. $p + j + k > p + k$ because $j \geq 1$, so $xu^2y \notin M$. We have derived the required contradiction. M cannot be a regular language.

(b) (5 points)

(1) $\langle S\rangle \to \langle A\rangle\langle B\rangle$
(2) $\langle A\rangle \to a\langle A\rangle b$
(3) $\langle B\rangle \to b\langle B\rangle c$
(4) $\langle A\rangle \to \epsilon$
(5) $\langle B\rangle \to \epsilon$

4) (20 points) Let (V, E) be the graph with vertices a, b, c, d, e, f, g, and h, and edges ab, bc, cd, de, ae, ac, ad, af, cg, and dh.

(a) Draw this graph.
(b) Write down this graph's incidence table and its incidence matrix.
(c) Write down this graph's adjacency table and its adjacency matrix.
(d) Is this graph complete? Justify your answer.
(e) Is this graph bipartite? Justify your answer.
(f) Is this graph regular? Justify your answer.
(g) Does this graph have any regular subgraph? Justify your answer.
(h) Give an example of an isomorphism φ from the graph (V, E) to itself satisfying that $\varphi(c) = d$.

(i) Is the isomorphism from part (h) unique or can you find another isomorphism ψ that is distinct from φ but also satisfies that $\psi(c) = d$? Justify your answer.

Solution: (a) (2 points) See graph drawn at the end of the solution set.

(b) (2 points: 1 point each the table and the matrix) The incidence table is

	ab	bc	cd	de	ae	ac	ad	af	cg	dh
a	1	0	0	0	1	1	1	1	0	0
b	1	1	0	0	0	0	0	0	0	0
c	0	1	1	0	0	1	0	0	1	0
d	0	0	1	1	0	0	1	0	0	1
e	0	0	0	1	1	0	0	0	0	0
f	0	0	0	0	0	0	0	1	0	0
g	0	0	0	0	0	0	0	0	1	0
h	0	0	0	0	0	0	0	0	0	1

and the incidence matrix is

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

(c) (2 points: 1 point each the table and the matrix) The adjacency table is

	a	b	c	d	e	f	g	h
a	0	1	0	0	1	1	0	0
b	1	0	1	0	0	0	0	0
c	0	1	0	1	0	0	1	0
d	0	0	1	0	1	0	0	1
e	1	0	0	1	0	0	0	0
f	1	0	0	0	0	0	0	0
g	0	0	1	0	0	0	0	0
h	0	0	0	1	0	0	0	0

and the adjacency table is

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

(d) (2 points: 1 for the answer and 1 for the justification) No, edge bf is not part of the graph.

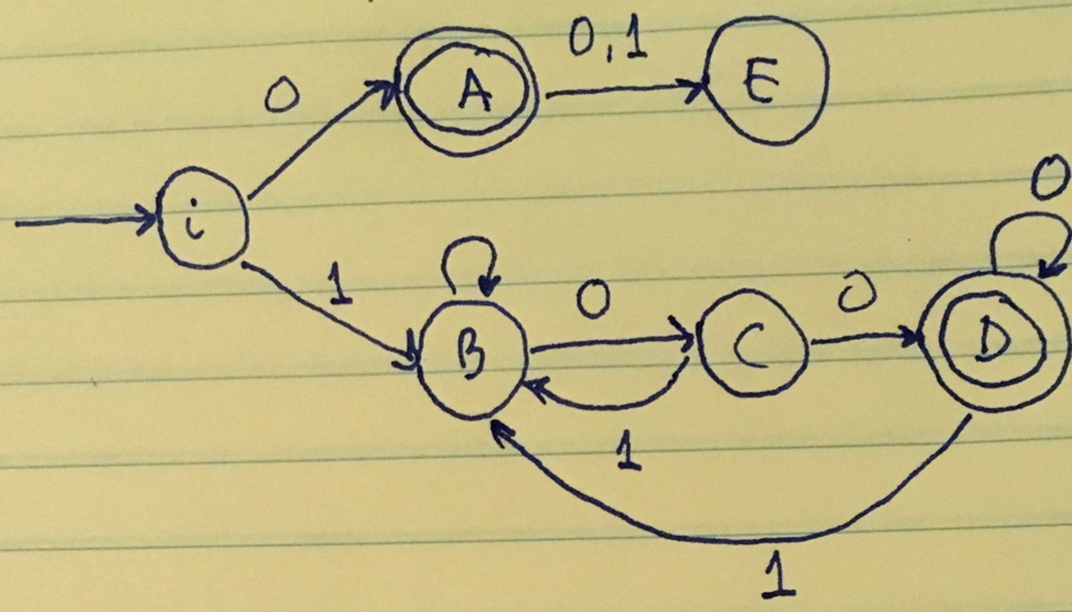
(e) (2 points: 1 for the answer and 1 for the justification) Yes, the partitioning of the set of vertices is $\{a, c, d\}$ and $\{b, e, f, g, h\}$.

(f) (2 points: 1 for the answer and 1 for the justification) No, $deg(a) = 5$, while $deg(b) = 2$.

(h) (4 points: points decked if not all vertex assignments in the map lead to an isomorphism) $\varphi(a) = a$, $\varphi(f) = f$, $\varphi(b) = e$, $\varphi(e) = b$, $\varphi(c) = d$, $\varphi(d) = c$, $\varphi(g) = h$, and $\varphi(h) = g$.

(i) (2 points: 1 for the answer and 1 for the justification) Yes, as only vertices c and d have degree 3, and our isomorphism φ has as its requirement that it exchanges them. Since g is connected to c and h is connected to d, φ must exchange g and h and has no degree of freedom in the process. Likewise, since b is connected to c and e is connected to d, φ must exchange b and e and cannot do anything different than that. As both b and e are connected to a, which is the only vertex of the graph of degree 5, φ must send a to itself. As f is a pendant vertex connected to a, φ must likewise send f to itself. φ is therefore unique.

2(a)



4(a)