<u>Theorem</u> Let $\{A_n\}_{n=1,2,\dots}$ be a sequence of countably infinite sets. Let $S = \bigcup_{n=1}^{\infty} A_n$. Then S is countably infinite.

<u>Proof</u> Each A_n is countably infinite $\Leftrightarrow A_n \sim J \quad \forall n \geq 1$

$\Leftrightarrow A_n = \{x_{nk}\}_{k=1,2,\dots} = \{x_{n1}, x_{n2}, x_{n3}, \dots\}$

We use the indices like for the entries of a matrix. The first index tells us which A_m set the element belongs to, while the second index tells us where that element is in the enumeration (the counting) of A_m.

Write
$$\begin{array}{cccc} x_{11} & x_{12} & x_{13} & x_{14} \cdots \\ x_{21} & x_{22} & x_{23} & x_{24} \cdots \\ x_{31} & x_{32} & x_{33} & x_{34} \cdots \\ x_{41} & & & \end{array}$$

$$\{x_{11}, x_{12}, x_{21}, x_{31}, x_{22}, x_{13}, x_{14}, x_{23}, x_{32}, x_{41}, \dots\}$$

$= \bigcup_{m=1}^{\infty} A_m = S$ is countably infinite because even if some x_{ij}'s are the same $A_m \subseteq S$ $\forall m \geq 1$ and $A_m \sim J$. (q.e.d.)

Corollary 1 Suppose an indexing set I is countable, and $\forall i \in I$, A_i is countable, Then $T = \bigcup_{i \in I} A_i$ is countable.

Proof: The biggest set we can obtain here is when I is countably infinite and each A_i is countably infinite. By the previous theorem, T is countably infinite in that case. Therefore, T is at most countably infinite (may be finite if I is finite and each A_i is finite), so T is countable.

(q.e.d.)

Corollary 2 Let A be a countably infinite set, and let $A^m = \underbrace{A \times \dots \times A}_{m \text{ times}} = \{(a_1, a_2, \dots, a_m) \mid a_1, a_2, \dots, a_m \in A\}$.

Then A^m is countably infinite.

Proof We use induction:

Base case $m = 1$ $A^1 = A \sim J \Rightarrow A^1$ is countably infinite.

Inductive step Assume A^{m-1} is countably infinite.

$A^m = A^{m-1} \times A = \{(b, a) \mid b \in A^{m-1}, a \in A\}$

$\forall b \in A^{m-1}$ $S_b = \{(b, a) \in A^m \mid a \in A\} \sim J \sim A \Rightarrow S_b$ is countably infinite. $A^n = \bigcup_{b \in A^{n-1}} S_b \sim J$ by Corollary 1, so A^n is indeed countably infinite as claimed. (q.e.d.)

<u>Corollary 3</u> $\mathbb{N}^n$ is countably infinite $\forall n \geq 1$.

<u>Proof</u> $\mathbb{N} \sim J$, so the result follows from Corollary 2. (q.e.d.)

<u>Corollary 4</u> $\mathbb{Z}^n$ is countably infinite $\forall n \geq 1$.

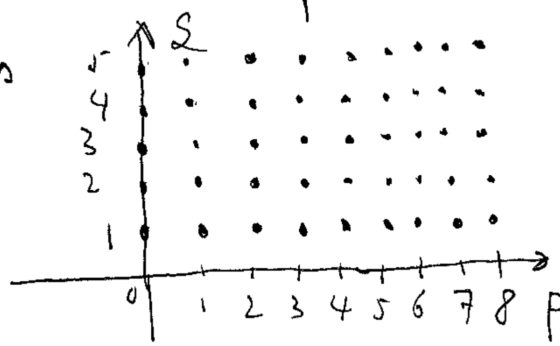
<u>Proof</u> We showed $\mathbb{Z} \sim J$, so the result follows from Corollary 2. (q.e.d.)

<u>Corollary 5</u> $\mathbb{Q}$ is countably infinite.

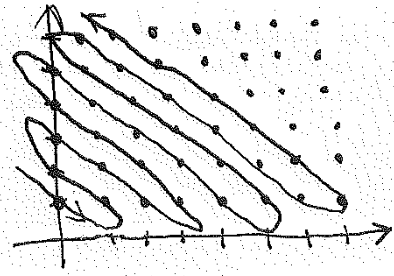
<u>Proof</u> $\mathbb{Q} = \{ \frac{p}{q} \mid q \neq 0,\ p,q \in \mathbb{Z},\ \underbrace{(p,q)=1}_{\text{no common factors}} \}$, but we can represent $\mathbb{Q}$ as $\{(p,q) \mid q \neq 0\ \ p,q \in \mathbb{Z}\}/\sim\ \subseteq \mathbb{Z}^2$, where $(p_1,q_1) \sim (p_2,q_2)$ $\iff \frac{p_1}{q_1} = \frac{p_2}{q_2} \iff p_1 q_2 = p_2 q_1$ by cross multiplication.

We also know $\mathbb{Z} \subseteq \mathbb{Q}$ (let $q = 1$). Therefore, $\mathbb{Q}$ is sandwiched between $\mathbb{Z} = \mathbb{Z}^1$ and $\mathbb{Z}^2$, both of which are countably infinite $\Rightarrow \mathbb{Q}$ is countably infinite. (q.e.d.)

<u>Remark</u> We can give a visual representation of the previous argument as follows



The dots are pairs (p,q) w/ $q \neq 0$ $p,q \in \mathbb{Z}$, which form a lattice. We can use the snake trick from the Theorem to show that the positive rationals $\mathbb{Q}^+ = \{ \frac{p}{q} \in \mathbb{Q} \mid \frac{p}{q} > 0 \}$ are countably infinite.



Similarly, we can show $\mathbb{Q}^- = \{ \frac{p}{q} \in \mathbb{Q} \mid \frac{p}{q} < 0 \}$ is countably infinite.

Then $\mathbb{Q} = \mathbb{Q}^- \cup \{0\} \cup \{\mathbb{Q}^+\}$ is countably infinite by Corollary 1. $\llcorner$

Next, show the set of sequences of 0's and 1's is uncountably infinite. We will use this result to show other sets are uncountably infinite.

<u>Theorem</u> Let A be the set of all sequences $s = \{x_1, x_2, \ldots\} = \{x_n\}_{n=1,2,\ldots}$ such that $x_n \in \{0,1\}$ $\forall n \geq 1$. Then A is uncountably infinite.

<u>Remark</u> This result is proven via a clever construction, which is due to Georg Cantor (1845-1918), a very famous German mathematician who invented set theory. Cantor also came up w/ the diagonal argument (snake trick) we used to prove a countably infinite union of countably infinite sets is countably infinite, the idea that sizes of sets should be understood via bijections ($A \sim B$ for A, B sets), as well as the notions of countably infinite and uncountably infinite.

<u>Proof</u> Assume A is countable $\Leftrightarrow A = \{s_1, s_2, \ldots\}$, where $s_j = \{x_n^j\}_{n=1,2,\ldots}$ for $x_n^j = 0$ or $x_n^j = 1$. We will now construct a sequence s_0 of 0's and 1's that cannot be in the enumeration $\{s_1, s_2, \ldots\}$. Let s_0 be such that $x_j^0 = \begin{cases} 1 & \text{if } x_j^j = 0 \\ 0 & \text{if } x_j^j = 1 \end{cases}$ In other words, s_0 differs from each s_j in the j^{th} element $\Rightarrow s_0 \notin \{s_1, s_2, \ldots\}$, but s_0 is a sequence of 0's and 1's $\Rightarrow s_0 \in A \Rightarrow\Leftarrow$

(q.e.d.)

<u>Corollary</u> The power set $\mathcal{P}(\mathbb{N})$ of $\mathbb{N}$ is uncountably infinite.

<u>Remark</u> Recall our proof that if B is a set w/ n elements, $\#(B) = n$, then its power set $\mathcal{P}(B)$ has 2^n elements based on the "on/off"

idea. For each element of B, we have the choice to include it in our subset ("on") or not to include it ("off"). Therefore, we have 2 choices for each element and $\#(B) = n$, so $\# \mathcal{P}(B) = 2^n$.

Proof $\mathbb{N} \sim J$, so we can write $\mathbb{N} = \{x_1, x_2, \ldots\}$. When we form a subset of $\mathbb{N}$, for each i, we can include x_i or leave it out. Say we represent including x_i by 1 and leaving x_i out by 0. Then each subset of $\mathbb{N}$ can be represented uniquely as a sequence of 0's and 1's. In fact, there is a one-to-one correspondence between the subsets of $\mathbb{N}$ and the sequences of 0's and 1's. Therefore $\mathcal{P}(\mathbb{N}) \sim A$, where A is the set of all sequences of 0's and 1's, but we showed in the previous Theorem that A is uncountably infinite $\Rightarrow \mathcal{P}(\mathbb{N})$ is uncountably infinite.

(q.e.d.)

We shall also use the one-to-one correspondence with the set of sequences of 0's and 1's in order to prove $\mathbb{R}$ is uncountably infinite. The argument proceeds in two steps:

1. We show $\mathbb{R} \sim (0,1)$ via a cleverly chosen bijection.
2. We set up a correspondence between $(0,1)$ and the set A of all sequences of 0's and 1's via a binary expansion.

Step 1 is the following proposition:

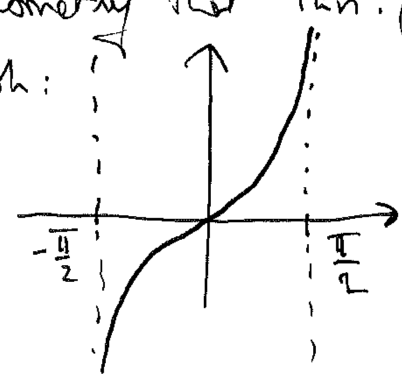
Proposition $\mathbb{R}$ is in bijective correspondence with the interval $(0,1)$.

Remark $(0,1) \subsetneq \mathbb{R}$, but we saw infinite sets can be in one-to-one correspondence with one of their proper subsets.

Proof Recall from trigonometry that $\tan : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}$ is a bijection. Here is the graph:

$$\tan x = \frac{\sin x}{\cos x}$$

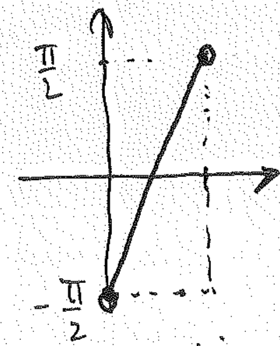
$$\cos\left(-\frac{\pi}{2}\right) = \cos\left(\frac{\pi}{2}\right) = 0$$



$x = -\frac{\pi}{2}$, $x = \frac{\pi}{2}$ are asymptotes of the graph.

We now use a linear function, a bijection, to show $(0,1) \sim \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$g(x) = \pi x - \frac{\pi}{2}$. Here is the graph:



The composition of two bijections is itself a bijection $\Rightarrow \tan(g(x)) =$
$= \tan\left(\pi x - \frac{\pi}{2}\right)$ is a bijection from $(0,1)$ to $\mathbb{R}$. The map we want
$f : \mathbb{R} \to (0,1)$ is its inverse $f(x) = \left(\tan\left(\pi x - \frac{\pi}{2}\right)\right)^{-1}$ as the inverse of a bijection is itself a bijection.

(q.e.d.)

Step (2) is a bit more complicated: To each $x \in (0,1)$, we want to associate $0.x_1x_2x_3\ldots\ldots$, where after the decimal $\{x_1, x_2, x_3, \ldots\}$ is a sequence of 0's and 1's. In other words, we are giving a binary expansion of every $x \in (0,1)$ as $0.x_1x_2x_3\ldots = 0 + \frac{1}{2}\cdot x_1 + \frac{1}{4}\cdot x_2 + \frac{1}{8}x_3 + \ldots$

$$= 0 + \frac{1}{2^1}\cdot x_1 + \frac{1}{2^2}\cdot x_2 + \frac{1}{2^3}x_3 + \ldots = 0 + \sum_{n=1}^{\infty} \frac{1}{2^n}\cdot x_n = \sum_{n=1}^{\infty} \frac{1}{2^n} x_n$$

Recall that $\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots = 1$. This means that

$$\frac{1}{2^k} \sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2^{k+1}} + \frac{1}{2^{k+2}} + \frac{1}{2^{k+3}} + \ldots = \frac{1}{2^k} \quad \forall\, k \geq 1.$$

Thus $0.\underbrace{1000\cdots}_{\text{all 0's}}$ and $0.0\underbrace{1111\cdots}_{\text{all 1's}}$ both represent $\frac{1}{2}$.

Similarly, any $x \in (0,1)$ that is a sum of the form $\frac{1}{2^{p_1}} + \frac{1}{2^{p_2}} + \ldots + \frac{1}{2^{p_k}}$ for $p_1, \ldots, p_k \in \mathbb{N}^*$, $p_1 < p_2 < \ldots < p_k$ has two binary representations.