MA2C03: ASSIGNMENT 1SOLUTIONS

1) For any three sets A, B, and C, prove using the proof methods employed in lecture that $A \times (B \cup C) = (A \times B) \cup (A \times C)$, where $\times$ is the Cartesian product. Venn diagrams, truth tables, or diagrams for simplifying statements in Boolean algebra such as Veitch diagrams are **NOT** acceptable and will not be awarded any credit.

Solution: We prove this equality of sets by proving inclusion in both directions:

Proving that $A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$: By the definition of the Cartesian product, $\forall z \in A \times (B \cup C)$, $z = (x, y)$, where $x \in A$ and $y \in (B \cup C)$. The latter means $y \in B$ or $y \in C$. Therefore, $z = (x, y) \in A \times B$ or $z = (x, y) \in A \times C$. Thus, $z \in (A \times B) \cup (A \times C)$ as needed.

Proving that $A \times (B \cup C) \supseteq (A \times B) \cup (A \times C)$: $\forall z \in (A \times B) \cup (A \times C)$, $z \in A \times B$ or $z \in A \times C$. By the definition of the Cartesian product, $z \in A \times B$ means $z = (x, y)$ with $x \in A$ and $y \in B$, whereas $z \in A \times B$ means $z = (x, y)$ with $x \in A$ and $y \in C$. Altogether, $y \in B$ or $y \in C$, hence $y \in B \cup C$ and $x \in A$. Therefore, $z = (x, y) \in A \times (B \cup C)$ as needed.

Grading rubric: 10 points total, 5 points for each of the directions.

2) Let $B = \{(x, y) \mid y = x\}$, where $x, y \in \mathbb{R}$. For each real number r, let $A_r = \{(x, y) \mid x^2 + y^2 = r^2\}$. Then $\{A_r \mid r \in \mathbb{R}\}$ is a collection of sets (circles) indexed by $\mathbb{R}$.

(a) Using $\mathbb{R}$ as the index set makes most of the circles in this collection be repeated. Name two smaller index sets that can be used to define this collection of circles without any repetition. Justify your answer.
(b) What is the union of all the $A_r's$? Justify your answer.
(c) Describe $B \cap A_r$. Justify your answer.
(d) Verify that $B \cap \left(\bigcup_{r \in \mathbb{R}} A_r\right) = \bigcup_{r \in \mathbb{R}} (B \cap A_r)$.

Solution: (a) If $r^2 > 0$, then both r and $-r$ give the same circle $x^2 + y^2 = r^2$ as $r^2 = (-r)^2$. As a result, we could use the index set $I' = \{r \in \mathbb{R} \mid r \geq 0\}$ or the index set $I'' = \{r \in \mathbb{R} \mid r \leq 0\}$. Either of these would define the collection of circles without repetition.

(b) $\bigcup_{r\in\mathbb{R}} A_r = \mathbb{R}^2$. The reason is that for any $(x,y)\in\mathbb{R}^2$, $x^2+y^2=p$ for some $p\geq 0$. We let $r=\sqrt{p}$, which is well-defined, and immediately see that $(x,y)\in A_r$.

(c) If $(x,y)\in B\cap A_r$, then $x^2+y^2=r^2$ and $x=y$. Therefore $x^2+y^2=2x^2=r^2$, so $x^2=\dfrac{r^2}{2}$, which means $x=\pm\dfrac{r\sqrt{2}}{2}$. Therefore,

$$B\cap A_r=\left\{\left(\frac{r\sqrt{2}}{2},\frac{r\sqrt{2}}{2}\right),\left(-\frac{r\sqrt{2}}{2},-\frac{r\sqrt{2}}{2}\right)\right\},$$

where we now assume that the indexing set is $I'=\{r\in\mathbb{R}\mid r\geq 0\}$, so $r\geq 0$.

(d) By part (b), $B\cap\left(\bigcup_{r\in\mathbb{R}} A_r\right)=B\cap\mathbb{R}^2=B$ as $\mathbb{R}^2$ is the universal set in this problem. To figure out what the right-hand side of the expression is, we pass to an indexing set that specifies the collection of circles uniquely such as I', which has the additional advantage of being exactly the one we used in part (c). Therefore,

$$\bigcup_{r\in\mathbb{R}}(B\cap A_r)=\bigcup_{I'}(B\cap A_r)=\bigcup_{r\geq 0}(B\cap A_r)$$
$$=\bigcup_{r\geq 0}\left\{\left(\frac{r\sqrt{2}}{2},\frac{r\sqrt{2}}{2}\right),\left(-\frac{r\sqrt{2}}{2},-\frac{r\sqrt{2}}{2}\right)\right\}=\bigcup_{r\in\mathbb{R}}\left\{\left(\frac{r\sqrt{2}}{2},\frac{r\sqrt{2}}{2}\right)\right\}.$$

Consider the function $f(x)=\dfrac{x\sqrt{2}}{2}$. $f:\mathbb{R}\to\mathbb{R}$ is a bijective function because it has inverse $f^{-1}:\mathbb{R}\to\mathbb{R}$ given by $f^{-1}(w)=w\sqrt{2}$. Thus, f must be a surjective function, so its image is all of $\mathbb{R}$. We conclude that

$$\bigcup_{r\in\mathbb{R}}(B\cap A_r)=\bigcup_{r\in\mathbb{R}}\left\{\left(\frac{r\sqrt{2}}{2},\frac{r\sqrt{2}}{2}\right)\right\}=\bigcup_{w\in\mathbb{R}}\{(w,w)\}=B$$

as needed.

Grading rubric: 20 points total, 5 points for each part.

3) Let $\mathbb{R}$ be the set of real numbers. For $x,y\in\mathbb{R}$, $x\sim y$ iff $x+y\in\mathbb{Z}$, i.e., if the sum $x+y$ is an integer. Determine:

(i) Whether or not the relation $\sim$ is *reflexive*;
(ii) Whether or not the relation $\sim$ is *symmetric*;
(iii) Whether or not the relation $\sim$ is *anti-symmetric*;
(iv) Whether or not the relation $\sim$ is *transitive*;
(v) Whether or not the relation $\sim$ is an *equivalence relation*;
(vi) Whether or not the relation $\sim$ is a *partial order*.

Justify your answers.

Solution: (i) $\sim$ is not reflexive. **Counterexample:** Let $x = \frac{1}{3}$. Then $x + x = \frac{1}{3} + \frac{1}{3} = \frac{2}{3} \notin \mathbb{Z}$. Therefore, $x \not\sim x$.

(ii) $\sim$ is symmetric because if $x \sim y$, then $x + y = q \in \mathbb{Z}$, and $y + x = x + y = q \in \mathbb{Z}$, which means $y \sim x$.

(iii) $\sim$ is not anti-symmetric. **Counterexample:** Let $x = 1$ and $y = 0$. Then $x + y = 1 + 0 = 1 \in \mathbb{Z}$, so $x \sim y$, while $y + x = 1 \in \mathbb{Z}$, so $y \sim x$ also holds. Clearly, $x \neq y$. Note that here it is NOT correct to say that $\sim$ fails to be anti-symmetric because it is symmetric as we know there exists one relation (equality), which is both symmetric AND anti-symmetric.

(iv) $\sim$ is not transitive. **Counterexample:** Let $x = \frac{1}{3}$, $y = \frac{2}{3}$, and $z = \frac{1}{3}$. $x + y = \frac{1}{3} + \frac{2}{3} = 1 \in \mathbb{Z}$, so $x \sim y$. $y + z = \frac{2}{3} + \frac{1}{3} = 1 \in \mathbb{Z}$, so $y \sim z$. $x + z = \frac{1}{3} + \frac{1}{3} = \frac{2}{3} \notin \mathbb{Z}$, so $x \not\sim z$.

(v) $\sim$ is a not equivalence relation because it fails to be reflexive and transitive.

(vi) $\sim$ is not a partial order because it fails to be reflexive, anti-symmetric, and transitive as we saw above.

Grading rubric: 10 points total, (i)-(iv) are 2 points each with 1 point for the answer and 1 point for the justification, while (v) and (vi) are 1 point each.

4) In the country of Tannu Tuva, a valid license plate consists of any digit except 0, followed by any two letters of the English alphabet, followed by any two digits.

(a) Let D be the set of all digits and L the set of all letters. With this notation, write the set of all possible license plates as a Cartesian product. Justify your answer

(b) How many possible license plates are there? Justify your answer.

Solution: (a) (5 points total) The set of all digits except 0 is given by $D \setminus \{0\}$ (2 points). Therefore, the Cartesian product $(D \setminus \{0\}) \times L \times L \times D \times D$ gives us the set of all license plates consisting of any digit except 0, followed by any two letters of the English alphabet, followed by any two digits (3 points).

(b) (5 points total) D has 10 elements, so $D \setminus \{0\}$ has 9 elements. (1 point). L has 26 elements (1 point). As we showed in class, if a set A has m elements, and a set B has p elements, then their Cartesian product has mp elements (1 point). Applying this result inductively, we obtain that $D \setminus \{0\} \times L$ has 9×26 elements, $D \setminus \{0\} \times L \times L$ has $9 \times 26 \times 26$ elements, $D \setminus \{0\} \times L \times L \times D$ has $9 \times 26 \times 26 \times 10$ elements, so the set of license plates $D \setminus \{0\} \times L \times L \times D \times D$ must have $9 \times 26 \times 26 \times 10 \times 10 = 608,400$ elements (2 points).

5) Use mathematical induction to prove that

$$\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \cdots + \frac{1}{n(n+1)} = 1 - \frac{1}{n+1}.$$

$\left(\text{Hint: Use the fact that } \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}.\right)$

Solution: We first prove the base case and then the inductive case.

Base case: For $n = 1$, $\frac{1}{2} = 1 - \frac{1}{2}$.

Inductive step: Assume that

$$\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \cdots + \frac{1}{n(n+1)} = 1 - \frac{1}{n+1}$$

and seek to prove that

$$\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \cdots + \frac{1}{n(n+1)} + \frac{1}{(n+1)(n+2)} = 1 - \frac{1}{n+2}.$$

By the hint, $\frac{1}{(n+1)(n+2)} = \frac{1}{n+1} - \frac{1}{n+2}$. Now using the inductive hypothesis, we see that

$$\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \cdots + \frac{1}{n(n+1)} + \frac{1}{(n+1)(n+2)} = 1 - \frac{1}{n+1} + \frac{1}{n+1} - \frac{1}{n+2} = 1 - \frac{1}{n+2}$$

as needed.

Grading rubric: 10 points total, 2 for the base case and 8 for the inductive step.

6) Let $f : (1,3) \to \mathbb{R}$ be the function defined by $f(x) = \frac{1}{x^2 - 4x + 3}$ for all x such that $1 < x < 3$. Determine whether or not this function is injective and whether or not it is surjective. Justify your answers.

Solution: The first observation to be made is that $f(x) = \frac{1}{x^2-4x+3} = \frac{1}{(x-1)(x-3)}$. We immediately notice that $x-1$ is everywhere positive on the interval $(1,3)$, which corresponds to $1 < x < 3$, whereas $x-3$ is everywhere negative on the same interval. We conclude that $f(x) = \frac{1}{(x-1)(x-3)}$ is everywhere negative on the interval $(1,3)$. As a result, our function f cannot achieve any positive value such as $y=1$, so f cannot be surjective as a function from $(1,3)$ to $\mathbb{R}$. We can either draw the graph or use some methods from calculus to show f is not injective. Let's use methods from calculus. We take the derivative $f'(x) = \left((x^2-4x+3)^{-1}\right)' = -\frac{2x-4}{(x^2-4x+3)^2}$. The denominator never vanishes on $(1,3)$, but the numerator is zero at $x=2$, so $f'(2)=0$. Since the function is everywhere negative on $(1,3)$ and has vertical asymptotes at $x=1$ and $x=3$, we conclude that $x=2$ must be a local maximum, so there must be values on both sides of $x=2$ at which the output of f is the same. Indeed, $f\left(\frac{3}{2}\right) = \frac{1}{-\frac{1}{2}\cdot\frac{3}{2}} = -\frac{4}{3} = f\left(\frac{5}{2}\right) = \frac{1}{\frac{3}{2}\cdot\left(-\frac{1}{2}\right)}$.

Grading rubric: 10 points total, 5 points for disproving injectivity with 2 for the answer and 3 for the justification and 5 points for disproving surjectivity with 2 for the answer and 3 for the justification.

7) Let $C = \{z \in \mathbb{C} \mid |z| = 1\}$. Let $f_\theta : C \to C$ be given by $f_\theta(z) = e^{i\theta}z$. Let $F = \{f_\theta \mid \theta \in \mathbb{R}\}$. Consider the set F under the operation of composition of functions $\circ$.

(a) Is $(F,\circ)$ a semigroup? Justify your answer.
(b) Is $(F,\circ)$ a monoid? Justify your answer.
(c) Is $(F,\circ)$ a group? Justify your answer.
(d) Is the map $\psi : (\mathbb{R},+) \to (F,\circ)$ given by $\psi(\theta) = f_\theta$ a homomorphism? Justify your answer.
(e) Is that map ψ from part (d) an isomorphism from $(\mathbb{R},+)$ to $(F,\circ)$? Justify your answer.

Solution: (a) Yes, $(F,\circ)$ is a semigroup (2 points). Let $\theta,\theta' \in \mathbb{R}$. $f_\theta \circ f_{\theta'}(z) = e^{i\theta} f_{\theta'}(z) = e^{i\theta}\cdot e^{i\theta'}\cdot z = e^{i(\theta+\theta')}z$. Since $(\mathbb{R},+)$ is a semigroup as proven in lecture, $\theta+\theta' \in \mathbb{R}$, so $f_\theta \circ f_{\theta'}(z) = f_{\theta+\theta'}(z) \in F$. We conclude that $\circ$ is a binary operation on F (1 point). To prove associativity of $\circ$, we consider $\theta,\theta',\theta'' \in \mathbb{R}$.

$$(f_\theta \circ f_{\theta'}) \circ f_{\theta''}(z) = f_{\theta+\theta'+\theta''}(z) = f_\theta \circ (f_{\theta'} \circ f_{\theta''})(z),$$

so $\circ$ is associative as needed (1 point).

(b) Yes, $(F, \circ)$ is a monoid (2 points). Let $\theta = 0 \in \mathbb{R}$. $f_0(z) = e^{i\cdot 0}z = 1 \cdot z = z$. For any $f_\theta \in F$, $f_\theta \circ f_0 = f_0 \circ f_\theta = f_{\theta+0} = f_\theta$ because $(\mathbb{R}, +)$ is a monoid with identity element 0 as proven in lecture. We conclude that F has identity element f_0 (1 point for stating which is the identity element and 1 point for proving that is indeed the case).

(c) Yes, $(F, \circ)$ is a group (2 points). Remember that we proved in lecture that $(\mathbb{R}, +, 0)$ is a group. For any $\theta \in \mathbb{R}$, $-\theta \in \mathbb{R}$ is its inverse. Now consider f_θ. We immediately see that $f_\theta \circ f_{-\theta}(z) = f_{\theta+(-\theta)}(z) = f_0(z) = f_{-\theta} \circ f_\theta$. Therefore, $f_{-\theta}$ is the inverse of f_θ in F, so every f_θ is invertible (1 point for writing down the inverse and 1 point for proving it is indeed the inverse).

(d) The map $\psi : (\mathbb{R}, +) \to (F, \circ)$ given by $\psi(\theta) = f_\theta$ is indeed a homomorphism (2 points). As we showed above, for any $\theta, \theta' \in \mathbb{R}$, $\psi(\theta + \theta') = f_{\theta+\theta'} = f_\theta \circ f_{\theta'} = \psi(\theta) \circ \psi(\theta')$ (2 points).

(e) The map $\psi : (\mathbb{R}, +) \to (F, \circ)$ given by $\psi(\theta) = f_\theta$ is not an isomomorphism (2 points) as it fails to be bijective. Indeed, $f_0(z) = f_{2\pi}(z) = z$, the identity on C, so ψ is not injective (2 points).

Grading rubric: 20 points total, 4 points for each part with two points for the correct answer and 2 points for the justification.