MA2C03: ASSIGNMENT 4 SOLUTIONS

1) (20 points)

(a) Is $\{(x \in \mathbb{R} \mid \cos x = 1\}$ finite, countably infinite, or uncountably infinite? Justify your answer.

(b) Is $\{x \in \mathbb{R}^+ \mid \log x \in \mathbb{R} \backslash \mathbb{Q}\}$ finite, countably infinite, or uncountably infinite? Justify your answer. The set $\mathbb{R}^+$ is the set of all positive real numbers.

(c) Is $\bigcup_{n=1}^{10} \left\{ \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = n^2\} \cap \{(x,y) \in \mathbb{R}^2 \mid y^2 - x^4 = 0\} \right\}$ finite, countably infinite, or uncountably infinite? Justify your answer.

(d) Is the language L_{DFA} defined in lecture finite, countably infinite, or uncountably infinite? Justify your answer. Recall that

$$L_{DFA} = \{\langle B, w\rangle \mid B \text{ is a DFA that accepts input string } w\}$$

2) (20 points)

(a) Consider the language over the binary alphabet $A = \{0, 1\}$ given by $L = \{0^m 1^{3m} \mid m \in \mathbb{N}\}$. Write down the algorithm of a Turing machine that recognizes L. Process the following strings according to your algorithm: ϵ, 01, 011, 0110, and 0111.

(b) Write down the transition diagram of the Turing machine from part (a) carefully labelling the initial state, the accept state, the reject state, and all the transitions specified in your algorithm.

3) (10 points) Write down the algorithm of an enumerator that prints out EXACTLY ONCE every string in the language $L = \{7m+2 \mid m \in \mathbb{N}\}$ over the alphabet $A = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

4) (20 points)

(a) Prove that the language

$$L_{RG} = \{\langle G, w\rangle \mid G \text{ is a regular grammar that generates string } w\}$$

is Turing-decidable.

(b) Prove that $L_{CFG} \setminus L_{RG}$ is Turing-decidable, where L_{CFG} was defined in lecture as

$$L_{CFG} = \{\langle G, w\rangle \mid G \text{ is a CFG that generates string } w\}.$$

Solutions: 1 (a) $\cos x = 1$ when $x = 2\pi m$ with $m \in \mathbb{Z}$. Therefore, there is a bijective correspondence between $\{2\pi m \mid m \in \mathbb{Z}\}$ and $\mathbb{Z}$. Since $\mathbb{Z}$ is countably infinite, the set $\{x \in \mathbb{R} \mid \cos x = 1\}$ is likewise countably infinite.

1 (b) The log function $log : \mathbb{R}^+ \to \mathbb{R}$ is bijective as you learned before coming to university, which means $\{x \in \mathbb{R}^+ \mid \log x \in \mathbb{R} \setminus \mathbb{Q}\}$ is in bijective correspondence with $\mathbb{R} \setminus \mathbb{Q}$. We showed in lecture that $\mathbb{R}$ is uncountably infinite, while $\mathbb{Q}$ is countably infinite. We also showed in lecture that taking out a countably infinite set from an uncountably infinite one leaves an uncountably infinite set. Therefore, $\{x \in \mathbb{R}^+ \mid \log x \in \mathbb{R} \setminus \mathbb{Q}\}$ is uncountably infinite.

1 (c) $y^2 - x^4 = (y - x^2)(y + x^2) = 0$ so for each n, we are intersecting the circle centered at the origin of radius n with the two parabolae $y = x^2$ and $y = -x^2$. This gives us four intersection points, and we have ten values for n. For different values of n, we get different intersection points, so we have a total of 40 points in the set $\bigcup_{n=1}^{10} \left\{ \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = n^2\} \cap \{(x, y) \in \mathbb{R}^2 \mid y^2 - x^4 = 0\} \right\}$, which is thus finite.

1 (d) $L_{DFA} = \{\langle B, w\rangle \mid B$ is a DFA that accepts input string $w\}$ so $L_{DFA} = \bigcup_B L(B)$. A language accepted by a DFA is a regular language. We showed the set of regular languages is countably infinite in lecture. In a previous tutorial, you showed $L(B)$ is either finite or countably infinite, so we have a countably infinite union of at most countably infinite sets, so L_{DFA} has to be countably infinite.

Grading Rubric: 5 points per part, 2 marks for a correct answer and 3 marks for an adequate justification.

2)

(a) Here is the algorithm for recognising $L = \{0^m 1^{3m} : m \in \mathbb{N}\}$.

(1) If there is a blank in the first cell, ACCEPT. If there is anything else, apart from 0, then REJECT.
(2) If 0 is in the first cell, delete it, then move right to the first 1.
(3) If there is no first 1, REJECT. Otherwise change 1 to x and move right.
(4) If there is anything apart from 1 in this cell, REJECT; otherwise, change 1 to y and more right.

(5) If there is anything apart from 1 in this cell, REJECT; otherwise, change 1 to z and go to the next step.
(6) Move to the leftmost non blank symbol. If 0, go to step 2. If 1, REJECT. If x, go to step 7.
(7) Move right to the rightmost non blank character. If anything except for z is found, REJECT. Otherwise, ACCEPT.

Here is how the following strings are treated:

- ϵ is accepted immediately at step 1.
- $01\sqcup \to \sqcup 1 \sqcup \to \sqcup x \sqcup \to$ REJECT at step 4.
- $011\sqcup \to \sqcup 11 \sqcup \to \sqcup x1 \sqcup \to \sqcup xy \sqcup \to$ REJECT at step 5.
- $0110\sqcup \to \sqcup 110 \sqcup \to \sqcup x10 \sqcup \to \sqcup xy0 \sqcup \to$ REJECT at step 5.
- $0111\sqcup \to \sqcup 111 \sqcup \to \sqcup x11 \sqcup \to \sqcup xy1 \sqcup \to \sqcup xyz \to$ ACCEPT at step 7.

(b) The transition diagram for

$$T = (\{i, s_1, s_2, s_3, s_4, s_5, s_{\text{acc}}, s_{\text{rej}}\}, \{0,1\}, \{0,1,x,y\}, t, i, s_{\text{acc}}, s_{\text{rej}})$$

is at the end of the solution set.

Grading Rubric & Remarks: 6 marks for the algorithm (-1 mark for not accepting ϵ), 4 marks for processing the strings and 10 marks for (b). If your algorithm from (a) is incorrect, but your transition diagram in (b) is faithful to what you wrote, no additional marks will be taken off.

For (b), large, clear, well labelled diagrams are a necessity. If it cannot be read, it cannot be awarded marks! Correct, consistent notation is also required.

3) Let us order the input tape as $\{w_1, w_2, w_3, \dots\} = \{0, 1, 2, 3, 4, \dots\}$.
$E =$ Given the input tape $\{w_1, w_2, w_3, \dots\}$

(1) If $i = 7m + 2$ for $m \in \mathbb{N}$, then print w_i.

Grading Rubric: 10 points; other correct solutions exist besides this one.

4 (a) In lecture, we proved that

$$L_{DFA} = \{\langle B, w\rangle \;\middle|\; B \text{ is a DFA that accepts input string } w\}$$

is Turing-decidable. As we showed in lecture during the unit on formal languages and grammars, if G is regular grammar, then there exists a corresponding DFA B such that $L(G) = L(B)$. Therefore, $\langle G, w\rangle \in$

$L_{RG} \iff \langle B, w\rangle \in L_{DFA}$. Since L_{DFA} is Turing-decidable, L_{RG} is likewise Turing-decidable.

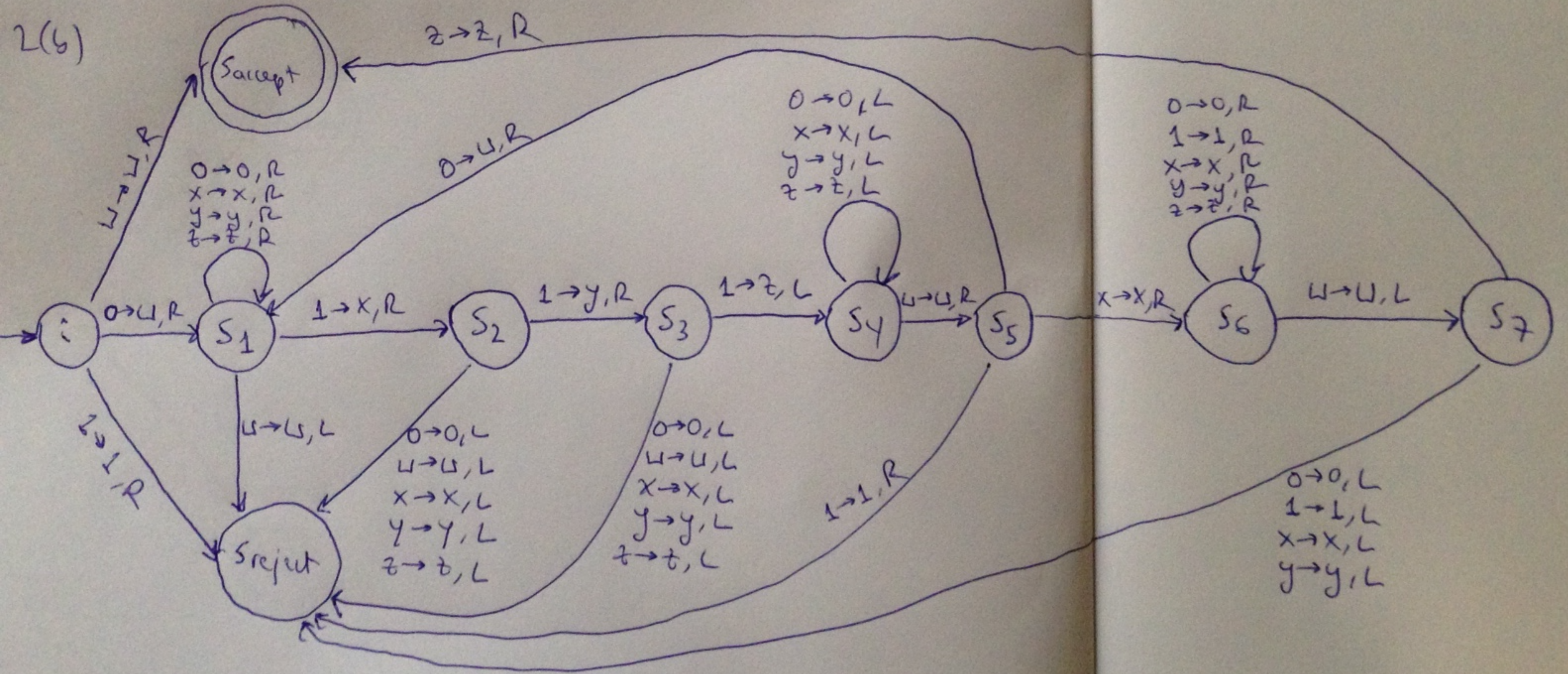
(b) L_{RG} is Turing-decidable implies that its complement $\overline{L_{RG}}$ must also be Turing-decidable as we take the decider of L_{RG} and swap its accept and reject states. Since $L_{CFG} \setminus L_{RG} = L_{CFG} \cap \overline{L_{RG}}$, we just need to show that the intersection of two Turing-decidable languages is Turing-decidable. Let M_1 be the decider for L_{CFG}, and let M_2 be the decider for $\overline{L_{RG}}$. We construct the decider M for $L_{CFG} \setminus L_{RG}$ as follows:

$M =$ on input $\langle G, w\rangle$

1. Run M_1 on $\langle G, w\rangle$, and run M_2 on $\langle G, w\rangle$. If both M_1 and M_2 accept, then ACCEPT; otherwise, REJECT.

Grading Rubric: 10 points per part. For part (a), both the given solution are directly constructing a Turing machine that decides L_{RG} are correct and will be given full marks. For part (b), proving that the intersection of two Turing-decidable languages is Turing-decidable is worth 5 points. The rest of the argument is worth the remaining 5 points.

2(b)



Saccept
Sreject
i
S1
S2
S3
S4
S5
S6
S7
z → z, R
0 → ⊔, R
⊔ → ⊔, R
0 → ⊔, R
1 → 1, R
0 → 0, R
x → x, R
y → y, R
z → z, R
⊔ → ⊔, L
1 → x, R
1 → y, R
1 → z, L
0 → 0, L
x → x, L
y → y, L
z → z, L
⊔ → ⊔, R
x → x, R
0 → 0, R
1 → 1, R
x → x, R
y → y, R
z → z, R
⊔ → ⊔, L
0 → 0, L
⊔ → ⊔, L
x → x, L
y → y, L
z → z, L
0 → 0, L
⊔ → ⊔, L
x → x, L
y → y, L
z → z, L
1 → 1, R
0 → 0, L
1 → 1, L
x → x, L
y → y, L