Mathematics CS1003

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Matrices I

In this section, we will learn about:

- Matrix Algebra
- How to compute the inverse of a matrix
  - by defining an augmented matrix,
  - and by computing its reduced row echelon form.
- Some applications

Matrices II

A matrix is a rectangular array of numbers.

Definition:

Matrices A and B are said to be equal if $A = (a_{ij})$ and $B = (b_{ij})$ are the same size and corresponding elements are equals: $a_{ij} = b_{ij}$ for all (i, j).

Remark: The notation $A = (a_{ij})$ can be rewritten as

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix}$$

when A is of order $n \times m$ (i.e. having n rows and m columns). The indexes i, j are respectively varying from 1 to n, and from 1 to m.

Matrices III

Operation on matrices:

Lets consider $A = [a_{ij}]$ and $B = [b_{ij}]$, two matrices of order $(n \times m)$

- **Addition:** we add the corresponding elements

$$A + B = [a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}]$$

- **Multiplication by a scalar α:** we multiply each element by the scalar

$$\alpha A = \alpha[a_{ij}] = [\alpha \times a_{ij}]$$

Matrices IV

matrix product:

Let $C = AB$ with A a $(n \times m)$ matrix and B a $(m \times p)$ matrix defined as

$$A = \begin{pmatrix} a_{1,1} & \cdots & a_{1,m} \\ \vdots & \ddots & \vdots \\ a_{i,1} & \ddots & a_{i,m} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,m} \end{pmatrix} \qquad B = \begin{pmatrix} b_{1,1} & \cdots & b_{1,j} & \cdots & b_{1,p} \\ \vdots & & \vdots & & \vdots \\ b_{m,1} & \cdots & b_{m,j} & \cdots & b_{m,p} \end{pmatrix}$$

Then

$$c_{ij} = a_{i1} \cdot b_{1j} + a_{i2} \cdot b_{2j} + \cdots + a_{im} \cdot b_{mj} = \sum_{k=1}^{m} a_{ik} \cdot b_{kj}$$

and $C = [c_{ij}]$ is a matrix of order $(n \times p)$.

Example:

$$\begin{pmatrix} 1 & 1 \\ \boxed{0 \quad 1} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 4 & 6 & \boxed{\begin{matrix} 8 \\ 9 \end{matrix}} & 2 \\ 5 & 7 & & 3 \end{pmatrix} = \begin{pmatrix} 9 & 13 & 17 & 5 \\ 5 & 7 & \boxed{9} & 3 \\ 4 & 6 & 8 & 2 \end{pmatrix}$$

Matrices V

Properties:

Provided that the matrices A, B, C have the apropriate order in each of the following operations, the properties of the operations on matrices are:

- Associativity
  - of the addition $(A + B) + C = A + (B + C) = A + B + C$
  - of the multiplication $(AB)C = A(BC) = ABC$
- Distributivity of multiplication over the addition:

$$(A + B)C = AC + BC$$

- Commutativity
  - of the addition $A + B = B + A$
  - but NOT the multiplication: $AB \neq BA$

Matrices VI

Theorem:

If A has an inverse, it is unique.

Proof: Imagine we have two inverses B and C of the matrix A then:

$$\begin{cases} BA = AB = I \\ AC = CA = I \end{cases}$$

Then multiplying by C the first equation on the right hand side, and multiplying by B the second equation by the left hand side:

$$\begin{cases} (BA)C = (AB)C = (I)C \\ B(AC) = B(CA) = B(I) \end{cases}$$

By the property of the product of matrices (associativity) and by the property of the identity matrix I, we rewrite the system as:

$$\begin{cases} BAC = ABC = C \\ BAC = BCA = B \end{cases} \quad \text{together these tell us} \quad B = C$$

Matrices VII

Finding Inverses:

To find the inverse of the matrix A, we reduce A to the identity matrix I by elementary row operations, while simultaneously applying the same operations to I.

1. Create the augmented matrix
$$\left[\begin{array}{c|c} A & I \end{array}\right]$$
and reduce it to its reduced row echelon form.

2. Now the matrix is in the form
$$\left[\begin{array}{c|c} I & B \end{array}\right]$$
and we have $B = A^{-1}$.

3. A good verification is to compute BA or AB to check it is equal to the identity matrix.

Matrices VIII

Find the inverse of

$$A = \begin{pmatrix} 2 & 6 & 6 \\ 2 & 7 & 6 \\ 2 & 7 & 7 \end{pmatrix}.$$

Matrices IX

First, we construct the desired augmented matrix and convert to row echelon form:

$$\begin{array}{cc}\left(\begin{array}{ccc|ccc} 2 & 6 & 6 & 1 & 0 & 0 \\ 2 & 7 & 6 & 0 & 1 & 0 \\ 2 & 7 & 7 & 0 & 0 & 1 \end{array}\right) & \begin{array}{c}\text{R1}\\ \text{R2}\\ \text{R3}\end{array}\end{array}$$

$$\underset{\longrightarrow}{R1 \to 1/2 \times R1} \quad \begin{array}{cc}\left(\begin{array}{ccc|ccc} 1 & 3 & 3 & 1/2 & 0 & 0 \\ 2 & 7 & 6 & 0 & 1 & 0 \\ 2 & 7 & 7 & 0 & 0 & 1 \end{array}\right) & \begin{array}{c}\text{R1}\\ \text{R2}\\ \text{R3}\end{array}\end{array}$$

$$\underset{\longrightarrow}{\begin{array}{c}R2 \to R2 - 2 \times R1\\ R3 \to R3 - 2 \times R1\end{array}} \quad \begin{array}{cc}\left(\begin{array}{ccc|ccc} 1 & 3 & 3 & 1/2 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \end{array}\right) & \begin{array}{c}\text{R1}\\ \text{R2}\\ \text{R3}\end{array}\end{array}$$

$$\begin{array}{c}R1 \to R1 - 3 \times R2\\ \longrightarrow \\ R3 \to R3 - R2\end{array} \quad \begin{array}{cc}\left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 7/2 & -3 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right) & \begin{array}{c}\text{R1}\\ \text{R2}\\ \text{R3}\end{array}\end{array}$$

Matrices X

$$\xrightarrow{R1 \to R1 - 3 \times R3} \begin{array}{cl} \left(\begin{array}{ccccccc} 1 & 0 & 0 & \vdots & 7/2 & 0 & -3 \\ 0 & 1 & 0 & \vdots & -1 & 1 & 0 \\ 0 & 0 & 1 & \vdots & 0 & -1 & 1 \end{array}\right) & \begin{array}{l} \text{R1} \\ \text{R2} \\ \text{R3} \end{array} \end{array}$$

A^{-1} is the matrix on the r.h.s of dots:

$$A^{-1} = \left(\begin{array}{ccc} 7/2 & 0 & -3 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{array}\right).$$

Check that $AA^{-1} = I$.

Matrices XI

Find the inverse of

1.
$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{pmatrix}.$$

2.
$$A = \begin{pmatrix} 1 & a & a^2 & a^3 \\ 0 & 1 & a & a^2 \\ 0 & 0 & 1 & a \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

   where $a \neq 0$.

3.
$$A = \begin{pmatrix} 1 & 3 & 3 \\ 1 & 3 & 4 \\ 1 & 4 & 3 \end{pmatrix}.$$

Matrices XII

For non-invertible matrices:

This method for finding the inverse of an invertible matrix also detects when a matrix is non-invertible (i.e., when its inverse does not exist).

Show that the matrix

$$A = \begin{pmatrix} 1 & 6 & 4 \\ 2 & 4 & -1 \\ -1 & 2 & 5 \end{pmatrix}$$

has no inverse.

Matrices XIII

SOLUTION: Construct the augmented matrix as above, and simplify it using elementary row operations:

$$\begin{array}{cc} \left(\begin{array}{ccc:ccc} 1 & 6 & 4 & 1 & 0 & 0 \\ 2 & 4 & -1 & 0 & 1 & 0 \\ -1 & 2 & 5 & 0 & 0 & 1 \end{array}\right) & \begin{array}{l} \text{R1} \\ \text{R2} \\ \text{R3} \end{array} \end{array}$$

Matrices XIV

$$\xrightarrow[R3 \to R3 + R1]{R2 \to R2 - 2 \times R1} \left(\begin{array}{ccccccc} 1 & 6 & 4 & \vdots & 1 & 0 & 0 \\ 0 & -8 & -9 & \vdots & -2 & 1 & 0 \\ 0 & 8 & 9 & \vdots & 1 & 0 & 1 \end{array}\right) \begin{array}{l} \text{R1} \\ \text{R2} \\ \text{R3} \end{array}$$

$$\xrightarrow{R2 \to -1/8R2} \left(\begin{array}{ccccccc} 1 & 6 & 4 & \vdots & 1 & 0 & 0 \\ 0 & 1 & 9/8 & \vdots & 1/4 & -1/8 & 0 \\ 0 & 8 & 9 & \vdots & 1 & 0 & 1 \end{array}\right) \begin{array}{l} \text{R1} \\ \text{R2} \\ \text{R3} \end{array}$$

$$\xrightarrow{R3 \to R3 - 8R2} \left(\begin{array}{ccccccc} 1 & 6 & 4 & \vdots & 1 & 0 & 0 \\ 0 & 1 & 9/8 & \vdots & 1/4 & -1/8 & 0 \\ 0 & 0 & 0 & \vdots & -1 & 1 & 1 \end{array}\right) \begin{array}{l} \text{R1} \\ \text{R2} \\ \text{R3} \end{array}$$

- Since R3 comprises **only** 0s on the l.h.s. of the dots, we **cannot convert** the **non-zero** entries above the zero in row 3, column 3 **to a zero** by elementary row operations, and therefore **cannot convert the l.h.s.** of the matrix to the **identity**

We **cannot invert** the matrix A, so A^{-1} does not exist.

Matrices XV

Linear Systems and invertible matrices:

We know that a linear system can be written as a matrix equation:

$$A\mathbf{x} = \mathbf{b}$$

Now if A is invertible (you need at least A to be square!), you can solve the system by:

1. Computing A^{-1}
2. Compute the solution $\mathbf{x} = A^{-1}\mathbf{b}$. (This is the solution because if we take $A\mathbf{x} = \mathbf{b}$ and multiply on the right on both sides by A^{-1} then we get $A^{-1}A\mathbf{x} = A^{-1}\mathbf{b}$. We then use the fact that $A^{-1}A = I$ to give the solution $\mathbf{x} = A^{-1}\mathbf{b}$)

Solve the linear system given by

$$\begin{aligned} x_1 + 3x_2 + 3x_3 &= 1 \\ x_1 + 3x_2 + 4x_3 &= -1 \\ x_1 + 4x_2 + 3x_3 &= 2. \end{aligned}$$

Matrices XVI

SOLUTION: We can rewrite this as the matrix system:

$$\underbrace{\begin{pmatrix} 1 & 3 & 3 \\ 1 & 3 & 4 \\ 1 & 4 & 3 \end{pmatrix}}_{A}\underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}_{\mathbf{x}} = \underbrace{\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}}_{\mathbf{b}},$$

We compute

$$A^{-1} = \begin{pmatrix} 7 & -3 & -3 \\ -1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix},$$

and so

$$\begin{aligned} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} &= A^{-1}\mathbf{b} = \begin{pmatrix} 7 & -3 & -3 \\ -1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} 7+3-6 \\ -1+0+2 \\ -1-1+0 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix}. \end{aligned}$$

Therefore $x_1 = 4$, $x_2 = 1$ and $x_3 = -2$.

ANSWER: The solution to the system is given by $x_1 = 4$, $x_2 = 1$ and $x_3 = -2$.

EXERCISE: Check that this is the solution to the linear system.

Matrices XVII

What is the solution of

$$
\begin{aligned}
x_1 + 3x_2 + 3x_3 &= -2 \\
x_1 + 3x_2 + 4x_3 &= 1 \\
x_1 + 4x_2 + 3x_3 &= -3 \text{ ?}
\end{aligned}
$$

Matrices XVIII

Example of application of Matrices:

- Matrix as an object in programming language.
- Useful for signal/image processing
- Next exercise: application to fitting a specific curve to a set of points.

Matrices XIX

The curve $y = a \cdot x^2 + b \cdot x + c$ passes through the points $(0, 0)$, $(1, 1)$ and (-2, 4) i.e. through the points $(x_1 = 0, y_1 = 0)$, $(x_2 = 1, y_2 = 1)$ and $(x_3 = -2, y_3 = 4)$.

Find the coefficients a, b and c.