MA2C03: ASSIGNMENT 1
DUE BY FRIDAY, NOVEMBER 18
SOLUTION SET

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1) We restated Russell's paradox in class as: "In a town, there was just one barber who was male. The barber shaved only the men who did not shave themselves. Who shaved the barber?" Consider the following: "In a town, there was just one barber. The barber shaved the men who did not shave themselves. Who shaved the barber?" Is the latter the same as Russell's paradox? Why or why not? Justify your answer.

Solution.
The latter is not the same as Russell's paradox. There are two differences that lead to the resolution of the 'paradox';

(1) The barber can be female.
(2) The former statement states "the barber shaved *only* the men who did not shave themselves". The *only* is missing from the latter statement, meaning there was never a problem to begin with - the barber can shave themselves regardless of gender.

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Grading rubric: 10 points total - 5 points if reason (1) or (2) is identified, 5 points for further justification as to why the 'paradox' is nullified.
Common mistakes: Being overly formal and attempting to prove the paradox and nullify the 'paradox' using functions or formulae not defined in one's answer.

2) Prove that $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$ for all sets A, B, and C using the proof methods employed in lecture. Venn diagrams, truth tables, or diagrams for simplifying statements in Boolean algebra such as Veitch diagrams are **NOT** acceptable and will not be awarded any credit.

Solution.
There were three accepted methods of proof for this question.

(1) **Set inclusion method.** Let $x \in A \setminus (B \cup C)$. This implies (using De Morgan's (DM) laws)

$$x \in A \cap (B \cup C)^c \Rightarrow x \in A \cap (B^c \cap C^c) \Rightarrow x \in A \text{ AND } (x \in B^c \text{ AND } x \in C^c)$$

By distributivity

$$\Rightarrow (x \in A \text{ AND } x \in B^c) \text{ AND } (x \in A \text{ AND } x \in C^c)$$

Then by definition

$$(x \in A \cap B^c) \text{ AND } (x \in A \cap C^c) \Rightarrow (x \in A \setminus B) \text{ AND } (x \in A \setminus C)$$

which means $x \in (A \setminus B) \cap (A \setminus C)$ meaning $A \setminus (B \cup C) \subseteq (A \setminus B) \cap (A \setminus C)$. The reverse statement is identical;

$$x \in (A \setminus B) \cap (A \setminus C) \Rightarrow (x \in A \setminus B) \text{ AND } (x \in A \setminus C)$$

$$\Rightarrow (x \in A \cap B^c) \text{ AND } (x \in A \cap C^c)$$

$$\Rightarrow (x \in A \text{ AND } x \in B^c) \text{ AND } (x \in A \text{ AND } x \in C^c) \qquad \text{(using DM)}$$

$$\Rightarrow x \in A \text{ AND } (x \in B^c \text{ AND } x \in C^c) \Rightarrow x \in A \cap (B^c \cap C^c)$$

$$\Rightarrow x \in A \cap (B \cup C)^c \Rightarrow x \in A \setminus (B \cup C)$$

so $(A \setminus B) \cap (A \setminus C) \subseteq A \setminus (B \cup C)$ and we conclude $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$ as required.
Note that logical AND (”$\wedge$”) can be used in place of the word AND.

(2) **Set operations method.**

$$A \setminus (B \cup C) = A \cap (B \cup C)^c = A \cap (B^c \cap C^c) \qquad \text{DM}$$

$$= (A \cap B^c) \cap (A \cap C^c) \qquad \text{Distributivity}$$

$$= (A \setminus B) \cap (A \setminus C)$$

as required.

(3) **Set membership conditions method.** We wish to determine the membership criteria for the left and right hand sides of the equation:

$$x \in A \setminus (B \cup C) \Rightarrow x \in A \cap (B \cup C)^c \Rightarrow x \in A \cap (B^c \cap C^c) \qquad \text{DM}$$

$$\Rightarrow x \in A \wedge x \in (B^c \cap C^c) \Rightarrow x \in A \wedge (x \in B^c \wedge x \in C^c)$$

$$\Rightarrow x \in A \wedge x \in B^c \wedge x \in C^c \qquad \text{Distributivity}$$

This last line is the membership conditions for $x \in A \setminus (B \cup C)$.
For the RHS of the equation;

$$x \in (A \setminus B) \cap (A \setminus C) \Rightarrow x \in (A \cap B^c) \wedge x \in (A \cap B^c)$$

$$\Rightarrow (x \in A \wedge x \in B^c) \wedge (x \in A \wedge x \in C^c) \Rightarrow x \in A \wedge x \in B^c \wedge x \in C^c$$

This last line is the membership criterion for $x \in (A \setminus B) \cap (A \setminus C)$. We see the two membership criterion are the same for $A \setminus (B \cup C)$ and $(A \setminus B) \cap (A \setminus C)$ thus the two sets must be equal, as required.

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Grading rubric: 10 points total - 5 points for each of the directions. Points are *deducted* for "creative" uses or manipulations of connectives. Venn/Veitch diagrams/anything that's a picture is not accepted as proof and gets 0 points (in either direction of proof).

Common mistakes: The most common mistake was stating one would prove the question via method (1), then attempting the question via method (3), but leaving out any mention of membership criteria, or the fact one was no longer trying to prove set inclusion. This is viewed as an incomplete proof. (I've marked "finish it!" on answers with this flaw). Another mistake was only completing the first half of proof (1), i.e. proving $A \setminus (B \cup C) \subseteq (A \setminus B) \cap (A \setminus C)$. A very puzzling mistake was students proving or concluding "$x \in A$". There are a number of scripts with something along the lines of "$\therefore x \in A$ in LHS and RHS as needed". The fact that x happens to be an element of A **is completely irrelevant to the question.** If one's proof relied on the fact $x \in A$, marks were deducted.

3) Let $\mathbb{N}$ be the set of all natural numbers, and let $\mathcal{P}(\mathbb{N})$ be its power set. For $\mathfrak{a}, \mathfrak{b} \in \mathcal{P}(\mathbb{N})$, $\mathfrak{a} R \mathfrak{b}$ if and only if $\mathfrak{a}$ and $\mathfrak{b}$ are not disjoint, i.e. $\mathfrak{a} \cap \mathfrak{b} \neq \emptyset$. Determine:

(1) Whether or not the relation R is *reflexive*;
(2) Whether or not the relation R is *symmetric*;
(3) Whether or not the relation R is *anti-symmetric*;
(4) Whether or not the relation R is *transitive*;
(5) Whether or not the relation R is an *equivalence relation*;
(6) Whether or not the relation R is a *partial order*.

Justify your answers.

Solution.

(1) R is not reflexive; choose $\mathfrak{a} = \emptyset$. $\emptyset \cap \emptyset = \emptyset$ so $\neg \mathfrak{a} R \mathfrak{a}$.
(2) R is symmetric;

$$\mathfrak{a} \cap \mathfrak{b} \neq \emptyset \Rightarrow \exists x (x \in \mathfrak{a} \wedge x \in \mathfrak{b}) \Rightarrow \exists x (x \in \mathfrak{b} \wedge x \in \mathfrak{a}) \Rightarrow \mathfrak{b} \cap \mathfrak{a} \neq \emptyset$$

(3) R is not antisymmetric; choose $\mathfrak{a} = \{1, 2, 3\}$ and $\mathfrak{b} = \{2, 3, 4\}$. Then $\mathfrak{a} R \mathfrak{b} \wedge \mathfrak{b} R \mathfrak{a}$ but $\mathfrak{a} \neq \mathfrak{b}$.

(4) R is not transitive; with $\mathfrak{a}$ and $\mathfrak{b}$ as above, choose $\mathfrak{c} = \{4, 5, 6\}$. Then $\mathfrak{a}R\mathfrak{b} \wedge \mathfrak{b}R\mathfrak{c}$ but $\mathfrak{a} \cap \mathfrak{c} = \emptyset$.
(5) R is not an equivalence relation due to (1) and (4).
(6) R is not a partial order due to (1), (3) and (4).

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Grading rubric: 10 points total - 2 points for (1) – (4) each (1 point for a correct statement, 1 point for a correct proof of that statement) and 1 point for (5) – (6) each.
Common mistakes: The most common mistake was not realising or ignoring the fact $\emptyset \in P(\mathbb{N})$, for (1). Many students also are not clear on the definitions of reflexivity and antisymmetry. Another common mistake was assuming Symmetric $\Rightarrow$ Not antisymmetric. **In general this is not true**. For example, the relation "equals" is both symmetric and antisymmetric. If one's proof for (3) relied on assuming Symmetric $\Rightarrow$ Not antisymmetric a mark was deducted.