MA2C03: TUTORIAL 3 PROBLEM SHEET

1) (From the 2016-2017 Annual Exam) Let Q denote the relation on the set $\mathbb{Z}$ of integers, where integers x and y satisfy xQy if and only if

$$x - y = (x - y)(x + 2y).$$

Determine the following:

(i) Whether or not the relation R is *reflexive*;
(ii) Whether or not the relation R is *symmetric*;
(iii) Whether or not the relation R is *transitive*;
(iv) Whether or not the relation R is an *equivalence relation*;
(v) Whether or not the relation R is *anti-symmetric*;
(vi) Whether or not the relation R is a *partial order*.

Justify your answers.

Solution: $x, y \in \mathbb{Z}$ satisfy xRy iff $x - y = (x - y)(x + 2y)$, which is equivalent to $(x - y)(x + 2y - 1) = 0$, i.e., $x = y$ or $x + 2y - 1 = 0$.

(i) **Reflexivity:** The relation R is reflexive because xRx holds for all $x \in \mathbb{Z}$ as $x - x = (x - x)(x + 2x) = 0$.

(ii) **Symmetry:** The relation R is not symmetric because if $x \neq y$, then xRy holds if $x + 2y = 1$, thus for yRx we would need $y + 2x = 1$, which only holds at the same time with $x+2y = 1$ when $x = y = \frac{1}{3} \notin \mathbb{Z}$.

(iii) **Anti-symmetry:** The relation R is anti-symmetric. Having xRy and yRx when $x \neq y$ would imply $x + 2y = 1$ and $y + 2x = 1$ hold simultaneously, which gives $x = y = \frac{1}{3} \notin \mathbb{Z}$. Therefore, xRy and yRx can both be true only if $x = y$.

(iv) **Transitivity:** The relation R is not transitive. Assume xRy and yRz hold for $x, y, z \in \mathbb{Z}$. There are 4 cases to consider:
Case 1: $x = y$ and $y = z$, then $x = z$, so xRz as needed.
Case 2: $x = y$ and $y + 2z = 1$, then $x + 2z = 1$, so xRz as needed.
Case 3: $x + 2y = 1$ and $y = z$, then $x + 2z = 1$, so xRz as needed.
Case 4: $x + 2y = 1$ and $y + 2z = 1$, then $x + 2(1 - 2z) = 1$, so $x + 2 - 4z = 1$, i.e., $x - 4z = -1$. This last equation is satisfied for example for $x = 3$, $z = 1$. Take $y = -1$ in order to satisfy $x + 2y = 1$. We see that $x + 2z = 3 + 2 = 5 \neq 1$, so xRz fails. We have constructed a counterexample.

(v) **Equivalence relation:** The relation R is not an equivalence relation because while reflexive, it fails to be symmetric and transitive.

(vi) **Partial order:** The relation R is not a partial order because while reflexive and anti-symmetric, it fails to be transitive.

2) (From the 2016-2017 Annual Exam) Let $f : [-2,2] \to [-15,1]$ be the function defined by $f(x) = x^2 + 3x - 10$ for all $x \in [-2,2]$. Determine whether or not this function is injective and whether or not it is surjective. Justify your answers.

Injectivity: $f(x) = x^2 + 3x - 10 = (x-2)(x-5)$ This function is not injective on the interval $[-2,2]$. Acceptable justifications: drawing the graph, providing two values $x_1, x_2 \in [-2,2]$, $x_1 \neq x_2$ such that $f(x_1) = f(x_2)$, applying Rolle's theorem (noticing that $f'(x) = 2x + 3$ so $f'\left(-\dfrac{3}{2}\right) = 0$, and $\dfrac{3}{2} \in [-2,2]$), etc.

Surjectivity: $f(x) = x^2 + 3x - 10$ is not surjective on the interval $[-2,2]$. Acceptable justifications: drawing the graph, providing a value in $[-15,1]$ that $f(x)$ does not assume, showing the minimum value occurs at $\dfrac{3}{2}$, where $f\left(\dfrac{3}{2}\right) = -12.25 > -15$, etc.