KE Deduction

Valid Argument

An argument consists of premises, $P_1, \ldots, P_n$ and a conclusion, Q.
An argument is **valid** if the conclusion Q is a **Logical Consequence** of the premises, $P_1, \ldots, P_n$ i.e.

$$P_1, \ldots, P_n \models Q$$

i.e. Q can be deduced or derived from $P_1, \ldots, P_n$.
This can also be written as:

$$\frac{P_1, \ldots, P_n}{Q}$$

Cont'd

A conclusion is a Logical Consequence of the premises whenever all the premises are True then so is the conclusion i.e. it is not possible for all the premises to be True and the conclusion False. To show that $P_1, \ldots, P_n \models Q$ is valid (i.e. to show Q is a logical consequence of $P_1, \ldots, P_n$) via Truth Tables one shows

$$\models P_1 \to \cdots \to P_n \to Q$$

i.e. Show that $P_1 \to \cdots \to P_n \to Q$ is a Tautology.

$P \models Q \rightarrow P$

A simple valid argument is $P \models P$
i.e. P is a logical consequence of P .
Also, $P, Q \models P$ as it is not possible for both P and Q to be True and P False.
Since

$$P, Q \models P \textit{ iff } P \models Q \rightarrow P$$

then $P \models Q \rightarrow P$ i.e. if we assume P is True then so is $Q \rightarrow P$.
Also, $P \models Q \rightarrow P$ *iff* $\models P \rightarrow Q \rightarrow P$ and so $\models P \rightarrow Q \rightarrow P$ i.e. $P \rightarrow Q \rightarrow P$ is a Tautology which can be checked by Truth Table.

Invalid Argument

Invalid Argument

An argument is not valid (invalid), if in some situation or state all the premises are True and the conclusion False, i.e. the situation or state provides a counter example to the argument being valid.
Example: Q is not a logical consequence of $P \vee Q$ as it possible to $P \vee Q$ to be True and Q False. Let the state, s

var	P	Q
$s(var)$	T	F

In this state, s, the sentence $P \vee Q$ is True but the conclusion, Q, is False.

A propositional expression (i.e. sentence) is a Tautology if its negation is a Contradiction. A sentence is said to be Contingent if it is neither a Tautology nor a Contradiction, i.e. for some states the sentence is True and for some other states the sentence is False. For example, $P \vee \neg P$ is a Tautology, $P \wedge \neg P$ is a Contradiction and $P \vee Q$ is Contingent.

Satisfiable Sentence

A sentence, P, is **satisfiable** if there is a state in which P is True. A Tautology is satisfiable as it is True in all states. A Contingent sentence is satisfiable as it is True is some states. A contradiction is not satisfiable as it not True in any state. A Contradiction is an **unsatisfiable** sentence.

While using Truth Tables is an effective way of determining a valid argument, Logical Deduction can also be used to determine when an argument is valid. Logical Deductive proofs are in general more efficient in determining when an argument is valid.

Proofs by (Logical) Deduction use various proof rules, e.g. the **Detachment Rule** (**Modus Ponens**):

$$\frac{P \to Q,\ P}{Q}$$

which are used to deduce or derive the conclusion from the premises.
It is the clear that this rule is valid, i.e. $P \to Q,\ P \models Q$ as $P \to Q,\ P \models Q \ = \ P \to Q \models P \to Q$.

Natural Deduction

Gentzen and Jaskowski independently published their results concerning Natural Deduction (ND for short) in 1934. Later in the 1960's Raymond Smullyan created a simpler variation of Natural Deduction based on Proof by Refutation. This system was based on Truth Trees or Truth Tableaux. In the 1990's Marco Mondadori and Marcello D'Agostino created a simpler version of Smullyan's Truth Tableaux which they referred to as KE Deduction (KE). (The 'K' refers to Gentzen classical logic system, K, and the 'E' refers to the Elimination rules).

The ND system of Gentzen has deduction rules for the various propositional calculus operators: $\neg, \rightarrow, \wedge, \vee, \equiv$. In the ND system there are both Elimination Rules and Introduction Rules but the KE Deduction system has just Elimination Rules.

Natural Deduction (Cont'd)

In principle, the deduction rules for $\rightarrow$ and $\neg$ would suffice as each of the other operators can be defined in terms of $\rightarrow$ and $\neg$, i.e.

$$
\begin{aligned}
P \wedge Q &= \neg(P \rightarrow \neg Q) \\
P \vee Q &= \neg P \rightarrow Q \\
P \equiv Q &= (P \rightarrow Q) \wedge (Q \rightarrow P)
\end{aligned}
$$

Notations $\models$ and $\vdash$

In the context of checking the validity of arguments, both the symbols $\models$ and $\vdash$ have the meaning of 'therefore' i.e. assume premises $P_1, \ldots P_n$ therefore the conclusion Q i.e. $P_1, \ldots P_n \vdash Q$.
Some Logic Textbooks use the symbol $\therefore$ (therefore) instead of the symbols $\models$ or $\vdash$.
The symbol $\models$ is used when referring to Tautology related proofs which use Truth Tables and
the symbol $\vdash$ is used when referring to Deductive proofs which use proof rules. For example,

read $\models P$ as "P is a Tautology"

and

read $\vdash P$ as "P is a Theorem".

$\models$ and $\vdash$ (Cont'd)

In Propositional Logic, All Tautologies are Theorems and all Theorems are Tautologies and so

$$\models P \;=\; \vdash P$$

This is the **Completeness Theorem** for Propositional Logic. Predicate Logic is also Complete in that all Logically True sentences are Theorems and all Theorems are Logically True.

Proof by Refutation (Reductio Ad Absurdum, RAA)

Since $\neg P \; = \; P \rightarrow \bot$, to show $\vdash \neg P$, instead show $\vdash P \rightarrow \bot$ i.e. $P \vdash \bot$.
To derive $\bot$ from P, derive a contradiction from P i.e. derive both Q and $\neg Q$, for some Q.
In showing $\vdash \neg P$, by showing $P \vdash \bot$, the proof has the form, for some Q :

1	P	Negation of $\neg P$
$\vdots$	$\vdots$	
j	Q	
$\vdots$	$\vdots$	
k	$\neg Q$	
	$\bot$	Contradiction j,k

(from P conclude $\bot$ (a contradiction) e.g. both Q and $\neg Q$, for some Q) $\therefore \vdash \neg P$).

Showing $\vdash P$ by Proof by Refutation

The strategy of Proof by Refutation can be used to show $\vdash P$.
Since $P = \neg\neg P$, show $\vdash \neg\neg P$.
To show $\vdash \neg\neg P$, show $\vdash \neg P \to \bot$ i.e. $\neg P \vdash \bot$
$\therefore$ to show $\vdash P$,
assume $\neg P$ and
derive contradictory sentences, Q and $\neg Q$, some Q.
In showing $\vdash P$, by showing $\neg P \vdash \bot$, the proof has the form, for some Q :

1	$\neg P$	Negated Conclusion
$\vdots$	$\vdots$	
j	Q	
$\vdots$	$\vdots$	
k	$\neg Q$	
	$\bot$	Contradiction j,k

(from $\neg P$ conclude $\bot$ (a contradiction) e.g. both Q and $\neg Q$, for some Q) $\therefore \vdash P$).

Proof by Refutation

More generally, to show $P_1, \ldots P_n \vdash Q$, instead show $P_1, \ldots P_n, \neg Q \vdash \perp$, i.e. derive, for some R, both R and $\neg R$ from $P_1, \ldots P_n, \neg Q$.

1	P_1	Premise
$\vdots$	$\vdots$	
n	P_n	Premise
n+1	$\neg Q$	Negate Conclusion, Q
$\vdots$	$\vdots$	
j	R	
$\vdots$	$\vdots$	
k	$\neg R$	
	$\perp$	Contradiction j,k

It is not possible for the premises, $P_1, \ldots P_n$ to be True and the conclusion, Q to be False. i.e. the attempt to make the premises True and conclusion False fails as a contradiction is derived.

An argument is valid if it is not possible for the premises to be True and the conclusion False. A Proof by Refutation assumes the negation of the conclusion along with the premises and if this leads to a contradiction then the original conclusion is derivable from the premises.

Example: $(P \to Q) \to R \vdash \neg R \to P$

Consider whether $(P \to Q) \to R \vdash \neg R \to P$. This can be shown by an 8 row Truth Table but alternatively it can be shown by a Proof by Refutation using KE Deduction. Use is made of the following KE Deduction Rules:

$$\frac{\neg(P \to Q)}{P} \quad \text{and} \quad \frac{\neg(P \to Q)}{\neg Q} \quad \text{and} \quad \frac{\begin{matrix} P \to Q \\ \neg Q \end{matrix}}{\neg P}$$

Example (Cont'd)

To show a rule such as $\dfrac{\neg(P \to Q)}{P}$ i.e. $\neg(P \to Q) \models P$ is logically valid, show $\models \neg(P \to Q) \to P$. This can be done by Truth Table.

P	Q	$\neg$	$(P \to Q)$	$\to$	P
F	F	F		$\boldsymbol{T}$	
F	T	F		$\boldsymbol{T}$	
T	F			$\boldsymbol{T}$	T
T	T			$\boldsymbol{T}$	T

Similarly, $\models \neg(P \to Q) \to \neg Q$ can be shown by Truth Table.

Example (Cont'd)

To show $\dfrac{\begin{array}{c} P \to Q \\ \neg Q \end{array}}{\neg P}$ i.e. $P \to Q, \neg Q \models \neg P$ is logically valid, show either

$$\models (P \to Q) \to \neg Q \to \neg P$$

(recall: $\to$ associates to the right)
or alternatively,

$$\models (P \to Q) \land \neg Q \to \neg P$$

Example (Cont'd)

P	Q	$(P \to Q)$	$\wedge$	$\neg Q$	$\to$	$\neg P$
F	F				$\boldsymbol{T}$	T
F	T				$\boldsymbol{T}$	T
T	F	F	F	T	$\boldsymbol{T}$	
T	T		F	F	$\boldsymbol{T}$	

Example KE Proof

Show $(P \to Q) \to R \vdash \neg R \to P$ via a Proof by Refutation.

1	$(P \to Q) \to R$	Premise
2	$\neg(\neg R \to P)$	Negate Conclusion
3	$\neg R$	from 2 via Rule
4	$\neg P$	from 2 via Rule
5	$\neg(P \to Q)$	from(1,3) via Rule
6	P	from 5 via Rule
7	$\perp$	Contradiction(4,6)

Deduction Rules

$$\frac{\neg(P \to Q)}{P}$$

$$\frac{\neg(P \to Q)}{\neg Q}$$

$$\frac{\begin{array}{c} P \to Q \\ \neg Q \end{array}}{\neg P}$$

Given the premise, $(P \to Q) \to R$, it is not possible for the conclusion, $\neg R \to P$ to be False

$\therefore (P \to Q) \to R \vdash \neg R \to P$

Comment on Proof;

From line 2 $\neg(\neg R \to P)$, we can conclude both line 3 $\neg R$ and also line 4 $\neg P$.
From line 1 $(P \to Q) \to R$ and line 3 $\neg R$ conclude 5 $\neg(P \to Q)$.
From line 5 conclude 6 P.
A contradiction arises from lines 4 and 6.
Given the premise $(P \to Q) \to R$ and assuming $\neg(\neg R \to P)$ a contradiction is derived and therefore

$$(P \to Q) \to R \models \neg R \to P$$

From a contradiction, any sentence can be derived

From a contradiction, any sentence can be derived

i.e. For any Q,

$$\perp \vdash Q, \text{ i.e. } P, \neg P \vdash Q.$$

It is not possible for the premises to True and the conclusion False as in the case of the premises being contradictory, they are always False.

Note that $\perp \rightarrow Q$ is a Tautology i.e. $\models \perp \rightarrow Q \therefore \perp \models Q$.

Contraposition

Proof by Contraposition may be confused with Proof by Refutation. If the Conclusion is of the form $P \rightarrow Q$, then since $P \rightarrow Q = \neg Q \rightarrow \neg P$, to show $\vdash P \rightarrow Q$, show instead $\vdash \neg Q \rightarrow \neg P$. To show $\vdash \neg Q \rightarrow \neg P$ i.e. $\neg Q \vdash \neg P$ i.e. assume $\neg Q$ and deduce $\neg P$. To show $P \vdash Q$ show $\neg Q \vdash \neg P$.
Example: To show:

If n^2 is even then n is even,

show instead its contraposition

If n is odd then n^2 is odd.

Let P : n^2 is even $\therefore \neg P = n^2$ is odd and
let Q : n is even $\therefore \neg Q = n$ is odd.
$P \rightarrow Q = \neg Q \rightarrow \neg P$
If n^2 is even then n is even = If n is odd then n^2 is odd.

KE Deduction Rules

For convenience and reference, the KE Deduction rules are divided into categories.

α(alpha) Rules

α (alpha) Rules

$$\frac{P \wedge Q}{P} \qquad \frac{P \wedge Q}{Q} \qquad \frac{\neg(P \vee Q)}{\neg P} \qquad \frac{\neg(P \vee Q)}{\neg Q}$$

$$\frac{\neg(P \rightarrow Q)}{P} \qquad \frac{\neg(P \rightarrow Q)}{\neg Q}$$

Double Negation Rule (DN)

Double Negation (DN)

$$\frac{\neg\neg P}{P}$$

In proofs, the 'double negation' rule may be used implicitly, i.e. write $\neg\neg P$ as P.
For convenience, the following rule can also be used

$$\frac{P}{\neg\neg P}$$

β (beta) Rules

β-Rules

$$\frac{\begin{array}{c} P \vee Q \\ \neg P \end{array}}{Q} \qquad \frac{\begin{array}{c} \neg(P \wedge Q) \\ P \end{array}}{\neg Q} \qquad \frac{\begin{array}{c} P \rightarrow Q \\ P \end{array}}{Q} \qquad \frac{\begin{array}{c} P \rightarrow Q \\ \neg Q \end{array}}{\neg P}$$

Consider β-rule, $\dfrac{\begin{array}{c} \neg(P \wedge Q) \\ P \end{array}}{\neg Q}$, this may be read as:

If $P \wedge Q$ is False and P is True, then Q must be False.

Branching Rule (BR)

Branching Rule (BR)

$$\begin{array}{cc} & \\ P & \neg P \end{array}$$

or

$$\begin{array}{c|c} \hline P & \neg P \end{array}$$

The Branching Rules is used in proofs that need Case Analysis (Separation of Cases) i.e. the case when P is True and the case when P is False i.e. the case when P and the case when $\neg P$.

Example: The Butler Shot the Earl

Previously, the following was shown by Truth Table and by truth table refutation.

$$\frac{s \vee b,\ \ b \rightarrow \neg d,\ \ c \vee d}{c \vee s}$$

i.e.

$$s \vee b,\ \ b \rightarrow \neg d,\ \ c \vee d \models c \vee s$$

The argument

$$s \vee b,\ \ b \rightarrow \neg d,\ \ c \vee d \vdash c \vee s$$

can be proved via KE Deduction.

KE Proof Example (Cont'd)

1	$s \vee b$	Premise
2	$b \rightarrow \neg d,$	Premise
3	$c \vee d$	Premise
4	$\neg(c \vee s)$	Negated Conclusion
5	$\neg c$	$\alpha(4)$
6	$\neg s$	$\alpha(4)$
7	d	$\beta(3,5)$
8	b	$\beta(1,6)$
9	$\neg d$	$\beta(2,8)$
10	$\perp$	$\times(7,9)$

KE Proof Example (Cont'd)

Hence the argument

$$\frac{s \vee b, \;\; b \rightarrow \neg d, \;\; c \vee d}{c \vee s}$$

is valid.

Superman Argument

Consider the following argument that determines whether Superman exists.

If Superman was able and willing to prevent evil, he would do so.
If Superman was unable to prevent evil, he would be powerless.
Superman does not prevent evil.
if Superman was unwilling to prevent evil, he would be malevolent.
If Superman exists then he is neither powerless nor malevolent.

Therefore

Superman does not exist.

Abbreviating the Propositions

Abbreviate the various propositions used in the argument.

A: Superman is able to prevent evil
W: Superman is willing to prevent evil
P: Superman is powerless
M: Superman is malevolent
E: Superman prevents evil
S: Superman exists

Abbreviating the Argument

Using the proposition abbreviations, abbreviate the premises:

- If Superman was able and willing to prevent evil, he would do so.

$$P_1 : \quad A \wedge W \rightarrow E$$

- If Superman was unable to prevent evil, he would be powerless

$$P_2 : \; \neg A \rightarrow P$$

Superman Proof (Cont'd)

- Superman does not prevent evil

$$P_3 : \neg E$$

- if Superman was unwilling to prevent evil, he would be malevolent.

$$P_4 : (\neg W \rightarrow M)$$

- If Superman exists then he is neither powerless nor malevolent.

$$P_5 : S \rightarrow \neg P \wedge \neg M$$

Abbreviate the conclusion :

- Superman does not exist.

$$\neg S$$

Show

$$\vdash P_1 \wedge P_2 \wedge P_3 \wedge P_4 \wedge P_5 \rightarrow \neg S$$

The 5 premises P_1, P_2, P_3, P_4, P_5 and the conclusion $\neg S$ involve 6 basic propositions, A, W, P, M, E, S and the validity of the argument may be shown by Truth Table using the 6 basic propositions but this would need $2^6 (= 64)$ rows in the Truth Table. It is more efficient and instructive to prove it using KE Deduction.

KE Proof: Superman

Given are the premises:

$$A \wedge W \rightarrow E,$$

$$\neg A \rightarrow P,$$

$$\neg E,$$

$$\neg W \rightarrow M,$$

$$S \rightarrow \neg P \wedge \neg M,$$

and Conclusion

$$\neg S$$

KE Proof: Superman

1	$A \wedge W \to E$	Premise
2	$\neg A \to P$	Premise
3	$\neg E$	Premise
4	$\neg W \to M$	Premise
5	$S \to \neg P \wedge \neg M$	Premise
6	S	Negated Conclusion
7	$\neg P \wedge \neg M$	$\beta(5,6)$
8	$\neg P$	$\alpha(7)$
9	$\neg M$	$\alpha(7)$
10	$\neg(A \wedge W)$	$\beta(1,3)$
11	A	$\beta(2,8)$ and DN
12	$\neg W$	$\beta(10,11)$
13	M	$\beta(4,12)$
	$\perp$	$\times(9,13)$

Branching Rule Example

Determine: $(P \to Q) \to Q, P \to R, Q \to \neg Q \vdash \neg(R \to Q)$

1	$(P \to Q) \to Q$	Premise
2	$P \to R$	Premise
3	$Q \to \neg Q$	Premise
4	$R \to Q$	(neg.Concl)

<<Left>>

L1	$P \to Q$	$\beta(1, L1)$
L2	Q	$\beta(3, L2)$
L3	$\neg Q$	$\times(L2, L3)$

<<Right>>

R1	$\neg(P \to Q)$	$\alpha(R1)$
R2	P	$\alpha(R1)$
R3	$\neg Q$	$\beta(2, R2)$
R4	R	$\beta(4, R4)$
R5	Q	$\times(R3, R5)$

KE Deduction Rules for $\equiv$

Recall: $P \equiv Q = (P \rightarrow Q) \wedge (Q \rightarrow P) \therefore$
To prove $P \equiv Q$
Either

$$\text{prove both } P \rightarrow Q \text{ and } Q \rightarrow P \text{ i.e. } P \vdash Q \text{ and } Q \vdash P$$
$$\text{or prove both } P \rightarrow Q \text{ and } \neg P \rightarrow \neg Q \text{ i.e. } P \vdash Q \text{ and } \neg P \vdash \neg Q$$

or

use KE Deduction Rules for $\equiv$

Recall the Truth Table for $\equiv$

P	Q	$P \equiv Q$
F	F	T
F	T	F
T	F	F
T	T	T

KE Deduction Rules for $\equiv$(Cont'd)

η (eta) Rules

$$\frac{\begin{array}{c}P \equiv Q\\ P\end{array}}{Q} \qquad \frac{\begin{array}{c}P \equiv Q\\ Q\end{array}}{P} \qquad \frac{\begin{array}{c}\neg(P \equiv Q)\\ P\end{array}}{\neg Q} \qquad \frac{\begin{array}{c}\neg(P \equiv Q)\\ Q\end{array}}{\neg P}$$

$$\frac{\begin{array}{c}P \equiv Q\\ \neg P\end{array}}{\neg Q} \qquad \frac{\begin{array}{c}P \equiv Q\\ \neg Q\end{array}}{\neg P} \qquad \frac{\begin{array}{c}\neg(P \equiv Q)\\ \neg P\end{array}}{Q} \qquad \frac{\begin{array}{c}\neg(P \equiv Q)\\ \neg Q\end{array}}{P}$$

Properties of $\equiv$:

- Symmetric: $P \equiv Q = Q \equiv P$
- Negation: $\neg(P \equiv Q) = \neg P \equiv Q = P \equiv \neg Q$.
- Opposite: $P \equiv Q = \neg P \equiv \neg Q$

Since $\neg(P \equiv Q) = \neg P \equiv Q$ then the rule $\dfrac{\begin{array}{c}\neg(P \equiv Q)\\ \neg P\end{array}}{Q}$ may be rewritten as $\dfrac{\begin{array}{c}\neg P \equiv Q\\ \neg P\end{array}}{Q}$ which has the form of the rule $\dfrac{\begin{array}{c}R \equiv S\\ R\end{array}}{S}$.

Similarly for other rules.
In proofs, the 'double negation' rule may be implicitly used, i.e.$\neg\neg P$ may be rewritten as P.

KE Proof of $\vdash P \to Q \equiv \neg Q \to \neg P$

(The Branching Rule is required)

1	$\neg(P \to Q \equiv \neg Q \to \neg P)$	Negated Conclusion
i.e.	$\neg(P \to Q) \equiv \neg Q \to \neg P$	Negated Conclusion

	<<Left>>		<<Right>>
L1	$P \to Q$	R1	$\neg(P \to Q)$
	$\eta(1,\ L1)$		$\eta(1,\ R1)$
L2	$\neg(\neg Q \to \neg P)$	R2	$\neg Q \to \neg P$
	$\alpha(L2)$		$\alpha(R1)$
L3	$\neg Q$	R3	P
	$\alpha(L2)$		$\alpha(R1)$
L4	P	R4	$\neg Q$
	$\beta(L1,\ L4)$		$\beta(R2, R4)$
L5	Q	R5	$\neg P$
	$\times(L3,\ L5)$		$\times(R3, R5)$
L6	$\perp$	R6	$\perp$

De Morgan's Law

It is straightforward to show by Truth Tables that
$\vdash \neg(P \wedge Q) \equiv \neg P \vee \neg Q$.
It can also be proved by KE Deduction using the KE $\eta - \mathit{rules}$ or
From above, to prove $\vdash P \equiv Q$
prove both $P \rightarrow Q$ and $\neg P \rightarrow \neg Q$ i.e. $P \vdash Q$ and $\neg P \vdash \neg Q$.

To prove $\vdash \neg(P \wedge Q) \equiv \neg P \vee \neg Q$,
prove

- $\neg(P \wedge Q) \vdash \neg P \vee \neg Q$ and
- $\neg\neg(P \wedge Q) \vdash \neg(\neg P \vee \neg Q)$
  i.e. $P \wedge Q \vdash \neg(\neg P \vee \neg Q)$

De Morgan's Law: $\neg(P \wedge Q) \vdash \neg P \vee \neg Q$

Case: $\neg(P \wedge Q) \vdash \neg P \vee \neg Q$

1	$\neg(P \wedge Q)$	Premise
2	$\neg(\neg P \vee \neg Q)$	Neg. Concl.
	$\alpha(2)$	
3	P	
	$\alpha(2)$	
4	Q	
	$\beta(1,3)$	
5	$\neg Q$	
	$\times(4,5)$	
R6	$\perp$	

De Morgan's Law: $P \wedge Q \vdash \neg(\neg P \vee \neg Q)$

Case: $P \wedge Q \vdash \neg(\neg P \vee \neg Q)$

1	$P \wedge Q$	Premise
2	$\neg\neg(\neg P \vee \neg Q)$	Neg. Concl.
	$DN(2)$	
3	$\neg P \vee \neg Q$	
	$\alpha(1)$	
4	P	
	$\alpha(1)$	
5	Q	
	$\beta(3, 4)$	
6	$\neg Q$	
	$\times(5, 6)$	
7	$\bot$	

De Morgan's Law can be proved directly using the $\eta - rules$ in KE Deduction, i.e. assume $\neg(\neg(P \wedge Q) \equiv \neg P \vee \neg Q)$ and show that this leads to a contradiction.

De Morgan's Law Proof

(The Branching Rule is required)

1 $\neg(\neg(P \wedge Q) \equiv \neg P \vee \neg Q)$ Negated Conclusion

i.e. $P \wedge Q \equiv \neg P \vee \neg Q$ Negated Conclusion

	<<Left>>		<<Right>>
L1	$P \wedge Q$	R1	$\neg(P \wedge Q)$
	$\eta(1, L1)$		$\eta(1, R1)$
L2	$\neg P \vee \neg Q$	R2	$\neg(\neg P \vee \neg Q)$
	$\alpha(L1)$		$\alpha(R2)$
L3	P	R3	P
	$\alpha(L1)$		$\alpha(R2)$
L4	Q	R4	Q
	$\beta(L2, L3)$		$\beta(R1, R3)$
L5	$\neg Q$	R5	$\neg Q$
	$\times(L4, L5)$		$\times(R4, R5)$
L6	$\perp$	R6	$\perp$

Comment on KE Proof

This proof makes use of a deduction rule that is similar to a De Morgan Law.
The deduction rules

$$\frac{\neg(P \vee Q)}{\neg P} \qquad \frac{\neg(P \vee Q)}{\neg Q}$$

may be viewed as consequences of the De Morgan Law
$\neg(P \vee Q) = \neg P \wedge \neg Q$

Invalid argument

John or Joyce will go to the party.
If Joyce goes to the party then, Clare will go unless Stephen goes.
Stephen will go if John does not go.
Therefore,
Clare will go to the party.

[from J. Kelly 'Essence of Logic']

We use the abbreviations:

- J : John will go to the party
- Y : Joyce will go to the party
- C : Clare will go to the party
- S : Stephen will go to the party

Formalise argument

The argument can be abbreviated as:

Note: $Q \textit{ unless } P = Q \textit{ if not } P = \neg P \rightarrow Q = P \vee Q$

$$\begin{array}{ll} & J \vee Y \\ & Y \rightarrow S \vee C \\ & \neg J \rightarrow S \\ \therefore & C \end{array}$$

Argument Truth Tree

$J \vee Y \ldots 1$

$Y \rightarrow (S \vee C) \ldots 2$

$\neg J \rightarrow S \ldots 3$

$\neg C \ldots 4$

«L»

$J \ldots 5$

«LL»

$Y \ldots 6$

$\beta(2,6)$

$S \vee C \ldots 6$

$\beta(4,6)$

S

«Open»

«LR»

$\neg Y$

«R»

$\neg J$

Not a valid argument

A path in the KE Deduction (marked <<Open>>) cannot be closed hence the set of propositions $\{J \vee Y, Y \rightarrow (S \vee C), \neg J \rightarrow S, \neg C\}$ is consistent (i.e. all true).
Thus it is possible for the all the premises $\{J \vee Y, Y \rightarrow (S \vee C), \neg J \rightarrow S\}$ to be True and the conclusion, C, to be False. The argument is not valid.
The set $\{J \vee Y, Y \rightarrow (S \vee C), \neg J \rightarrow S, \neg C\}$ is True in the state:

C	J	S	Y
F	T	T	T

which can be obtained by traversing the left subtrees.
$\therefore$ in this state, the premises $\{J \vee Y, Y \rightarrow (S \vee C), \neg J \rightarrow S\}$ are all true and the conlclusion, C is false.

Check by Truth Table

Check whether

$$(J \vee Y) \wedge (Y \rightarrow S \vee C) \wedge (\neg J \rightarrow S) \rightarrow C$$

is a Tautology.
Consider the following row of the Truth Table:
(variables in alphabetical order)

C	J	S	Y	$(J \vee Y)$	$\wedge$	$(Y \rightarrow S \vee C)$	$\wedge$	$(\neg J \rightarrow S)$	$\rightarrow$	C
$\vdots$	$\vdots$	$\vdots$	$\vdots$							
F	T	T	T	T		T		T	**F**	F
$\vdots$	$\vdots$	$\vdots$	$\vdots$							

$(J \vee Y) \wedge (Y \rightarrow S \vee C) \wedge (\neg J \rightarrow S) \rightarrow C$ is not a Tautology.
The argument
$J \vee Y, Y \rightarrow (S \vee C), \neg J \rightarrow S \models C$ is not valid.