MA2C03: ASSIGNMENT 1
DUE BY TUESDAY, OCTOBER 30 BY 5PM
AT LECTURE OR IN THE MATHS OFFICE ROOM 0.6

Please write down clearly both your name and your student ID number on everything you hand in. Please attach a cover sheet with a declaration confirming that you know and understand College rules on plagiarism. Details can be found on `http://tcd-ie.libguides.com/plagiarism/declaration`.

1) For any three sets A, B, and C, prove using the proof methods employed in lecture that $A \times (B \cup C) = (A \times B) \cup (A \times C)$, where $\times$ is the Cartesian product. Venn diagrams, truth tables, or diagrams for simplifying statements in Boolean algebra such as Veitch diagrams are **NOT** acceptable and will not be awarded any credit.

2) Let $B = \{(x, y) \,|\, y = x\}$, where $x, y \in \mathbb{R}$. For each real number r, let $A_r = \{(x, y) \mid x^2 + y^2 = r^2\}$. Then $\{A_r \mid r \in \mathbb{R}\}$ is a collection of sets (circles) indexed by $\mathbb{R}$.

(a) Using $\mathbb{R}$ as the index set makes most of the circles in this collection be repeated. Name two smaller index sets that can be used to define this collection of circles without any repetition. Justify your answer.
(b) What is the union of all the $A_r's$? Justify your answer.
(c) Describe $B \cap A_r$. Justify your answer.
(d) Verify that $B \cap \left(\bigcup_{r\in\mathbb{R}} A_r\right) = \bigcup_{r\in\mathbb{R}} (B \cap A_r)$.

3) Let $\mathbb{R}$ be the set of real numbers. For $x, y \in \mathbb{R}$, $x \sim y$ iff $x + y \in \mathbb{Z}$, i.e., if the sum $x + y$ is an integer. Determine:

(i) Whether or not the relation $\sim$ is *reflexive*;
(ii) Whether or not the relation $\sim$ is *symmetric*;
(iii) Whether or not the relation $\sim$ is *anti-symmetric*;
(iv) Whether or not the relation $\sim$ is *transitive*;
(v) Whether or not the relation $\sim$ is an *equivalence relation*;
(vi) Whether or not the relation $\sim$ is a *partial order*.

Justify your answers.

4) In the country of Tannu Tuva, a valid license plate consists of any digit except 0, followed by any two letters of the English alphabet, followed by any two digits.

(a) Let D be the set of all digits and L the set of all letters. With this notation, write the set of all possible license plates as a Cartesian product. Justify your answer

(b) How many possible license plates are there? Justify your answer.

5) Use mathematical induction to prove that

$$\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\cdots+\frac{1}{n(n+1)}=1-\frac{1}{n+1}.$$

$\left(\text{Hint: Use the fact that } \frac{1}{n(n+1)}=\frac{1}{n}-\frac{1}{n+1}.\right)$

6) Let $f:(1,3)\to\mathbb{R}$ be the function defined by $f(x)=\dfrac{1}{x^2-4x+3}$ for all x such that $1<x<3$. Determine whether or not this function is injective and whether or not it is surjective. Justify your answers.

7) Let $C=\{z\in\mathbb{C}\mid |z|=1\}$. Let $f_\theta : C\to C$ be given by $f_\theta(z)=e^{i\theta}z$. Let $F=\{f_\theta \mid \theta\in\mathbb{R}\}$. Consider the set F under the operation of composition of functions $\circ$.

(a) Is $(F,\circ)$ a semigroup? Justify your answer.

(b) Is $(F,\circ)$ a monoid? Justify your answer.

(c) Is $(F,\circ)$ a group? Justify your answer.

(d) Is the map $\psi:(\mathbb{R},+)\to(F,\circ)$ given by $\psi(\theta)=f_\theta$ a homomorphism? Justify your answer.

(e) Is that map ψ from part (d) an isomorphism from $(\mathbb{R},+)$ to $(F,\circ)$? Justify your answer.