Set Properties

Set Properties

Sets have properties similar but not the same as Arithmetic.
Let U be the Universal set of elements of interest.
Let $X, Y, Z \subseteq U$
The basic operators on sets are:

- Complement: $\overline{X}$
- Intersection: $X \cap Y$
- Union $X \cup Y$

Fundamental Set Properties

Fundamental Properties of Set Theory Operators

Identity

$$X \cap U = X \qquad X \cup \{\} = X$$

Anihilation

$$X \cap \{\} = \{\} \qquad X \cup U = U$$

Complement

$$X \cap \overline{X} = \{\} \qquad X \cup \overline{X} = U$$

Idempotent

$$X \cap X = X \qquad X \cup X = X$$

Commutativity

$$X \cap Y = Y \cap X \qquad X \cup Y = Y \cup X$$

Fundamental Set Props Cont'd

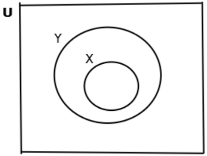
Associativity
$(X \cap Y) \cap Z = X \cap (Y \cap Z)$ $(X \cup Y) \cup Z = X \cup (Y \cup Z)$

Distributivity: $\cap$ **over** $\cup$
$X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z)$

Distributivity: $\cup$ **over** $\cap$
$X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z)$

Elementary Properties of Sets

- $\overline{\overline{X}} = X$
- $X \cap Y \subseteq X$ and $X \cap Y \subseteq Y$
- $X \subseteq X \cup Y$ and $Y \subseteq X \cup Y$
- $X \subseteq Y \equiv \overline{Y} \subseteq \overline{X}$



- Also $Y \subseteq X \equiv \overline{X} \subseteq \overline{Y}$

Elementary Properties (Cont'd)

- $X = Y \equiv \overline{X} = \overline{Y}$

Proof:

$X = Y$
$\equiv X \subseteq Y$ *and* $Y \subseteq X$
$\equiv \overline{Y} \subseteq \overline{X}$ *and* $\overline{X} \subseteq \overline{Y}$
$\equiv \overline{X} = \overline{Y}$

De Morgan Law $\overline{X \cap Y} = \overline{X} \cup \overline{Y}$ via Karnaugh Map

De Morgan Law: $\overline{X \cap Y} = \overline{X} \cup \overline{Y}$

$$X \cap Y = \begin{array}{c|c|c|} & & Y \\ \cline{2-3} & 0 & 0 \\ \cline{2-3} X & 0 & 1 \\ \cline{2-3} \end{array} \quad \therefore \quad \overline{X \cap Y} = \begin{array}{c|c|c|} & & Y \\ \cline{2-3} & 1 & 1 \\ \cline{2-3} X & 1 & 0 \\ \cline{2-3} \end{array}$$

$$\overline{X} = \begin{array}{c|c|c|} & & Y \\ \cline{2-3} & 1 & 1 \\ \cline{2-3} X & 0 & 0 \\ \cline{2-3} \end{array} \quad \overline{Y} = \begin{array}{c|c|c|} & & Y \\ \cline{2-3} & 1 & 0 \\ \cline{2-3} X & 1 & 0 \\ \cline{2-3} \end{array}$$

$\therefore$

$$\overline{X} \cup \overline{Y} = \begin{array}{c|c|c|} & & Y \\ \cline{2-3} & 1 & 1 \\ \cline{2-3} X & 1 & 0 \\ \cline{2-3} \end{array} = \overline{X \cap Y}$$

Cardinality of Sets, $|X|$

$|X|$ is defined as the size of set X,
i.e. $|X|$ is the number of elements in X .
Alternative Notation: $\#X$
With $A = \{2, 3, 5, 7, 11, 13, 17, 19\}$, $|A| = 8$.

Disjoint Sets

Sets X and Y are disjoint iff $X \cap Y = \{\}$.

Lemma 1

$$B \subseteq A \to |A - B| = |A| - |B|$$

Lemma 2

$$A \cap B = \{\} \to |A \cup B| = |A| + |B|$$

Property $A \cup B$

Recall: $A \cup B$ is the union of disjoint subsets:

$$A \cup B = (A \cap \overline{B}) \cup (A \cap B) \cup (\overline{A} \cap B)$$

Thm: $A - (A \cap B) = A \cap \overline{B}$
Pf:
$A - (A \cap B)$
$= A \cap \overline{A \cap B}$ by prop. of set difference
$= A \cap (\overline{A} \cup \overline{B})$ by De Morgan
$= (A \cap \overline{A}) \cup (A \cap \overline{B})$ by Distributivity
$= A \cap \overline{B}$ as $A \cap \overline{A} = \{\}$ and $\{\} \cup Y = Y$.

From this result:

$$A \cup B = (A - (A \cap B)) \cup (A \cap B) \cup (B - (A \cap B))$$

Cardinalty Cont'd

Theorem $|A \cup B| = |A| + |B| - |A \cap B|$

$A \cup B$ can be split into disjoint sets:
i.e. $A \cup B = (A - (A \cap B)) \cup (B - (A \cap B)) \cup (A \cap B)$

Proof.

$$
\begin{aligned}
|A \cup B| &= |(A - (A \cap B)) \cup (B - (A \cap B)) \cup (A \cap B)| \\
&\quad \{\textit{all these disjoint}\} \\
&= |A - (A \cap B)| + |B - (A \cap B)| + |A \cap B| \\
&\quad \{A \cap B \subseteq A \textit{ and } A \cap B \subseteq B\} \\
&= |A| - |A \cap B| + |B| - |A \cap B| + |A \cap B| \\
&= |A| + |B| - |A \cap B|
\end{aligned}
$$

$\square$

Cardinality of Sets

Cardinality $|A \cup B \cup C|$

$$
\begin{aligned}
|A \cup B \cup C| &= |(A \cup B) \cup C| \\
&= |A \cup B| + |C| - |(A \cup B) \cap C| \\
&= \{\mathit{Distributive\ law}\} \\
&\quad\ |A \cup B| + |C| - |(A \cap C) \cup (B \cap C)| \\
&= |A| + |B| - |A \cap B| + |C| \\
&\quad\ -(|A \cap C| + |B \cap C| - |(A \cap C) \cap (B \cap C)| \\
&= |A| + |B| + |C| \\
&\quad\ -(|A \cap B| + |A \cap C| + |B \cap C|) \\
&\quad\ +|A \cap B \cap C|
\end{aligned}
$$

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

Example

Students pass the year if they pass all 3 exams A, B, C.
For a particular year it was found that

- 3% failed all 3 papers
- 9% failed papers B and C
- 10% failed papers A and C
- 12% failed papers A and B
- 32% failed paper A
- 30% failed paper B
- 46% failed paper C

1. What percentage of students passed the year
2. What percentage failed exactly one paper.

Solution

Solution:

$$\begin{array}{c|c|c|c|c|}
 & & & \multicolumn{2}{c}{B} \\ \cline{2-5}
 & 20 & 30 & 6 & 12 \\ \cline{2-5}
A & 13 & 7 & 3 & 9 \\ \cline{2-5}
\multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{2}{c}{C} & \multicolumn{1}{c}{}
\end{array}$$

Example

A language college consists of students that study French, German or Spanish. In the college, 280 students study French, 254 students study German and 280 students study Spanish.
97 students study French as well as German,
152 students study French as well as Spanish and
138 students study German as well as Spanish.
73 students study all the three languages.

How many students are there in the language college?

Solution

Abbreviations:

F is the set of French students
G is the set of German students
S is the set of Spanish students.

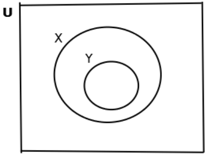
The number of language students $= |F \cup G \cup S|$
From the above property of Cardinality:

$$
\begin{aligned}
&|F \cup G \cup S| \\
&= |F| + |G| + |S| - (|F \cap G| + |F \cap S| + |G \cap S|) + |F \cap G \cap S| \\
&= 280 + 254 + 280 - (97 + 152 + 138) + 73 \\
&= 500
\end{aligned}
$$

Set Theory Theorems

Set Theory Theorems

- $Y \subseteq X \quad \equiv \quad X \cup Y = X$
- $Y \subseteq X \quad \equiv \quad X \cap Y = Y$



$Y \subseteq X \equiv X \cup Y = X$

Show $Y \subseteq X \equiv X \cup Y = X$

1. $Y \subseteq X \to X \cup Y = X$
2. $X \cup Y = X \to Y \subseteq X$

Proof.

(1.)
Assume $Y \subseteq X$,
show $X \cup Y = X$ i.e. $X \cup Y \subseteq X$ and $X \subseteq X \cup Y$
Show $X \cup Y \subseteq X$
let $z \in X \cup Y$
$\therefore z \in X$ or $z \in Y$ $\square$

Cont'd

Proof.

Case $z \in X$
$\therefore z \in X$
Case $z \in Y$
$\{$assuming $Y \subseteq X$ $\}$
$\therefore z \in X$.

Show $X \subseteq X \cup Y$
True, from properties of $\cup$. $\square$

$Y \subseteq X \equiv X \cup Y = X$ (Cont'd)

Show(2.) $X \cup Y = X \rightarrow$ Y$\subseteq$X

Proof.

(2.)
Assume $X \cup Y = X$, show $Y \subseteq X$
let $z \in Y$,
$\therefore z \in X \cup Y$
{assuming $X \cup Y = X$ }
$\therefore z \in X$ $\square$

$Y \subseteq X \equiv X \cap Y = Y$

Show $Y \subseteq X \equiv X \cap Y = Y$

i.e. Show

1. $Y \subseteq X \rightarrow X \cap Y = Y$
2. $X \cap Y = Y \rightarrow Y \subseteq X$

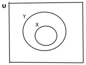
Proof.

Exercise □

$X \subseteq Y \equiv X \cap \overline{Y} = \{\}$

Theorem

$X \subseteq Y \equiv X \cap \overline{Y} = \{\}$



Show $X \subseteq Y \rightarrow X \cap \overline{Y} = \{\}$

Assume $X \subseteq Y$ i.e. from above (swapping X and Y): $X \cap Y = X$

$\therefore$

$X \cap \overline{Y}$

$= (X \cap Y) \cap \overline{Y}$ given $X \cap Y = X$

$= X \cap (Y \cap \overline{Y})$

$= X \cap \{\}$

$= \{\}$

Show $X \cap \overline{Y} = \{\} \to X \subseteq Y$

Show $X \cap \overline{Y} = \{\} \to X \subseteq Y$

Assume $X \cap \overline{Y} = \{\}$
As $X \subseteq Y \equiv X \cap Y = X$, show $X = X \cap Y$

$$\begin{aligned} X &= X \cap U \\ &= X \cap (Y \cup \overline{Y}) \\ &= (X \cap Y) \cup (X \cap \overline{Y}) \\ &= (X \cap Y) \cup \{\} \\ &= X \cap Y \end{aligned}$$

De Morgan's Laws

De Morgan's Laws

1. $\overline{(X \cap Y)} = \overline{X} \cup \overline{Y}$ – De Morgan 1
2. $\overline{(X \cup Y)} = \overline{X} \cap \overline{Y}$ – De Morgan 2

Proof of De Morgan's Law 1

Proof of De Morgan 1 $\overline{X \cap Y} = \overline{X} \cup \overline{Y}$

1. Show $\overline{X} \cup \overline{Y} \subseteq \overline{X \cap Y}$
2. Show $\overline{X \cap Y} \subseteq \overline{X} \cup \overline{Y}$

Show 1. $\overline{X} \cup \overline{Y} \subseteq \overline{X \cap Y}$

Theorem

$\overline{X} \cap \overline{X \cap Y} = \overline{X}$

Proof.

$X \cap Y \subseteq X$

$\{\ A \subseteq B \equiv \overline{B} \subseteq \overline{A}\ \}$

$\equiv \overline{X} \subseteq \overline{X \cap Y}$

$\{\ A \subseteq B \equiv A \cap B = A\ \}$

$\equiv \overline{X} \cap \overline{X \cap Y} = \overline{X}$ $\square$

Corollary

$\overline{Y} \cap \overline{X \cap Y} = \overline{Y}$

Show 1. $\overline{X} \cup \overline{Y} \subseteq \overline{X \cap Y}$ (Cont'd)

Theorem

$\overline{X} \cup \overline{Y} \subseteq \overline{X \cap Y}$

Recall: $A \subseteq B \equiv A \cap B = A$

Proof.

$\overline{X} \cup \overline{Y} \subseteq \overline{X \cap Y}$
$\equiv (\overline{X} \cup \overline{Y}) \cap \overline{X \cap Y} = \overline{X} \cup \overline{Y}$
{ $\cap$ Distributes over $\cup$ }
$\equiv (\overline{X} \cap \overline{X \cap Y}) \cup (\overline{Y} \cap \overline{X \cap Y}) = \overline{X} \cup \overline{Y}$
{ by Thms. $\overline{X} \cap \overline{X \cap Y} = \overline{X}$ and $\overline{Y} \cap \overline{X \cap Y} = \overline{Y}$ }
$\equiv \overline{X} \cup \overline{Y} = \overline{X} \cup \overline{Y}$
$\equiv$ *True* $\square$

Show 2. $\overline{X \cap Y} \subseteq \overline{X} \cup \overline{Y}$

Theorem

$\overline{\overline{X} \cup \overline{Y}} \cup X = X$

Proof.

$\overline{X} \subseteq \overline{X} \cup \overline{Y}$

$\equiv \overline{\overline{X} \cup \overline{Y}} \subseteq \overline{\overline{X}}$

$\equiv \overline{\overline{X} \cup \overline{Y}} \subseteq X$

$\equiv \overline{\overline{X} \cup \overline{Y}} \cup X = X$

Corollary: $\overline{\overline{X} \cup \overline{Y}} \cup Y = Y$ $\square$

Show 2. $\overline{X \cap Y} \subseteq \overline{X} \cup \overline{Y}$ (Cont'd)

Theorem

$\overline{X \cap Y} \subseteq \overline{X} \cup \overline{Y}$

Proof.

$\overline{X \cap Y} \subseteq \overline{X} \cup \overline{Y}$

$\{\, A \subseteq B \equiv \overline{B} \subseteq \overline{A} \,\}$

$\equiv \overline{\overline{X} \cup \overline{Y}} \subseteq \overline{\overline{X \cap Y}}$

$\equiv \overline{\overline{X} \cup \overline{Y}} \subseteq X \cap Y$

$\equiv \overline{\overline{X} \cup \overline{Y}} \cup (X \cap Y) = X \cap Y$

$\equiv \left(\overline{\overline{X} \cup \overline{Y}} \cup X\right) \cap \left(\overline{\overline{X} \cup \overline{Y}} \cup Y\right) = X \cap Y$

$\equiv X \cap Y = X \cap Y$ from above Thm.

$\equiv \mathit{True}$ $\square$

Prove De Morgan 2 $\overline{X \cup Y} = \overline{X} \cap \overline{Y}$

Theorem

$\overline{X \cup Y} = \overline{X} \cap \overline{Y}$

Proof.

$\overline{X \cup Y} = \overline{X} \cap \overline{Y}$

$\{\ A = B \equiv \overline{A} = \overline{B}\ \}$

$\equiv \overline{\overline{X \cup Y}} = \overline{\overline{X} \cap \overline{Y}}$

$\equiv X \cup Y = \overline{\overline{X} \cap \overline{Y}}$

$\{$ De Morgan 1 $\}$

$\equiv X \cup Y = \overline{\overline{X}} \cup \overline{\overline{Y}}$

$\equiv X \cup Y = X \cup Y$

$\equiv \mathit{True}$ $\square$

Power Set

The Power Set, $P(S)$, of a set S, is the set of subsets of S,
i.e. $x \in P(S) \equiv x \subseteq S$.
If $|S| = n$ then $|P(S)| = 2^n$.

Example

$S = \{a, b, c\}$
$P(S) = \{\,\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\,\}$
where $\emptyset$ is the empty set, i.e. $\emptyset = \{\}$.

In forming the subsets of S e.g.$S = \{0, 1, 2, 3, \ldots, n-1\}$, we have 2 choices for each element; to include it or exclude it.
2 choices for 0, 2 choices for 1, 2 choices for 2 etc.
Total #choices $= 2 * 2 * \cdots * 2$ (n times) $= 2^n$.
There is a natural correspondence between the subsets of $\{0, 1, 2, 3, \ldots, n-1\}$ and binary numbers.

Subsets and Binary

subset	$n-1$	$\ldots$	k	$\ldots$	3	2	1	0
$\{\}$	0	$\ldots$	0	$\ldots$	0	0	0	0
$\{0\}$	0	$\ldots$	0	$\ldots$	0	0	0	1
$\{1\}$	0	$\ldots$	0	$\ldots$	0	0	1	0
$\{0,1\}$	0	$\ldots$	0	$\ldots$	0	0	1	1
$\vdots$								
$\{\ldots,k,\ldots\}$		$\ldots$	1	$\ldots$				
$\vdots$								
$\{0,1,2,\ldots k,\ldots n-1\}$	1	$\ldots$	1	$\ldots$	1	1	1	1

- 0 in column, k, indicates that k is not in the subset
- 1 in column, k, indicates that k is in the subset.

Binary and Decimal

Binary			decimal
$0\dots0$	0	$=$	$0*2^{n-1}+\cdots+0*2^2+0*2^1+0*2^0$
$0\dots1$	1	$=$	$0*2^{n-1}+\cdots+0*2^2+0*2^1+1*2^0$
$0\dots10$	2	$=$	$0*2^{n-1}+\cdots+0*2^2+1*2^1+0*2^0$
$0\dots11$	3	$=$	$0*2^{n-1}+\cdots+0*2^2+1*2^1+1*2^0$
	$\vdots$		$\vdots$
$1\dots1$	2^n-1	$=$	$1*2^{n-1}+\cdots+1*2^2+1*2^1+1*2^0$

$|P(S)| = 2^{|S|}$ Proof by Induction

$|P(S)| = 2^{|S|}$

Let $|S| = n$. Proof by induction on n.
Base Case:
$n = 0$
If $|S| = 0$ then $S = \emptyset \therefore P(S) = \{\emptyset\}$.
$|\{\emptyset\}| = 1$ tf $|P(S)| = 1 = 2^0 = 2^{|S|}$.
Induction Step:
Assume true for n, show true for $n + 1$.
i.e. Assume (if $|A| = n$ then $|P(A)| = 2^n$),
show (if $|S| = n + 1$ then $|P(S)| = 2^{n+1}$).

Induction Step

Assume $|S| = n + 1$.
Consider an element, x, of S, i.e. $x \in S$.
Discard x , then we have $S - \{x\}$ and $\therefore |S - \{x\}| = n$.
By induction, $|P(S - \{x\})| = 2^n$.
The original subsets of S consist of

- those that do not have the element, x,
  i.e. the subsets of $S - \{x\}$. and $|P(S - \{x\})| = 2^n$.
- those that do have the element, x, which are the subsets of of $S - \{x\}$ with the element, x, added in, giving 2^n subsets.

$\therefore |P(S)| = 2^n + 2^n = 2^{n+1}$.

Cantor's Theorem, $|\mathbb{N}| \neq |P(\mathbb{N})|$

Cardinality of Sets

Let $S = \{0, 1, 2\}$ then
$P(S) = \{\emptyset, \{0\}, \{1\}, \{2\}, \{0, 1\}, \{0, 2\}, \{1, 2\}, \{0, 1, 2\}\}, \therefore$
$|S| = 3$ and $|P(S)| = 8$ and in this case $|S| \neq |P(S)|$.
For any finite set, S, $|S| \neq |P(S)|$.

Sets with same Cardinality

Two sets have the same cardinality iff there is a one to one, 1-1, correspondence between both sets.
Let $A = \{a, b, c, d, e, \ldots, x, y, z\}$and $B = \{1, 2, 3, \ldots 26\}$ then $|A| = |B|$ as we have the 1-1 correspondence

A	a	b	c	$\ldots$	y	z
B	1	2	3	$\ldots$	25	26

$|\mathbb{N}| = |Even|$

$|\mathbb{N}| = |Even|$

Consider infinite sets:
Infinite sets S_1 and S_2 have the same cardinality if there is a one to one, 1-1, correspondence between both sets.
Let *Even* be the set of even natural numbers then $|\mathbb{N}| = |Even|$ as:

$Even$	0	2	4	6	...	$2 * n$	...
$\mathbb{N}$	0	1	2	3	...	n	...

There is a 1-1 correspondence between the two sets $\mathbb{N}$ and *Even*.
The sets $\mathbb{N}$ and *Even* have the same cardinality i.e. $|\mathbb{N}| = |Even|$, even though $Even \subseteq \mathbb{N}$ and $Even \neq \mathbb{N}$.

$|\mathbb{N}| = |\mathbb{Z}|$

$|\mathbb{N}| = |\mathbb{Z}|$

Consider a 1-1 correspondence between $\mathbb{N}$ and $\mathbb{Z}$,

$\mathbb{N}$	n	$\dots$	$2*k-1$	$\dots$	3	1	0	2	4	$\dots$	$2*k$	$\dots$
$\mathbb{Z}$	$f(n)$	$\dots$	$-k$	$\dots$	-2	-1	0	1	2	$\dots$	k	$\dots$

The odd natural numbers are in 1-1 correspondence with the negative integers
and the even natural numbers are in 1-1 correspondence with the positive integers.
The function, $f(n)$, can be defined as:

$$f(n) = \mathit{if}\ \mathit{even}(n)\ \mathit{then}\ \frac{n}{2}\ \mathit{else}\ \frac{-(n+1)}{2}$$

e.g. $f(2*k-1) = \frac{-((2*k-1)+1)}{2} = \frac{-2*k}{2} = -k$

|Naturals|=|Positive Rationals|

Let $\mathbb{Q}^+$ be the set of positive Rational numbers (positive fractions).

$|\mathbb{N}| = |\mathbb{Q}^+|$

Let $f : \mathbb{N} \to \mathbb{Q}^+$ such that

$\mathbb{N}$	n	0	1	2	3	4	5	...	...
$\mathbb{Q}^+$	$\frac{a}{b}$	$\frac{1}{1}$	$\frac{1}{2}$	$\frac{2}{1}$	$\frac{1}{3}$	$\frac{2}{2}$	$\frac{3}{1}$	...	...

We can list all fractions using the following:

List all fractions $\frac{a}{b}$ such that $a + b = 2$

List all fractions $\frac{a}{b}$ such that $a + b = 3$

List all fractions $\frac{a}{b}$ such that $a + b = 4$

etc.

|Naturals|=|Positive Rationals| Cont'd

Consider listing the positive Rationals in matrix form:
Each row is infinite and there are infinite rows.

$$
\begin{array}{ccccc}
\frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots \\
\frac{2}{1} & \frac{2}{2} & \frac{2}{3} & \frac{2}{4} & \cdots \\
\frac{3}{1} & \frac{3}{2} & \frac{3}{3} & \frac{3}{4} & \cdots \\
\frac{4}{1} & \frac{4}{2} & \frac{4}{3} & \frac{4}{4} & \cdots \\
\vdots & \vdots & \vdots & \vdots &
\end{array}
$$

List the Rationals along the diagonals.

|Naturals|=|Naturals x Naturals|

$|\mathbb{N}| = |\mathbb{N} \times \mathbb{N}|$

Consider $f : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ such that

$\mathbb{N}$	n	0	1	2	3	4	...
$\mathbb{N}\times\mathbb{N}$	$f(n)$	(0,0)	(0,1)	(1,0)	(0,2)	(1,1)	...

We can list all pairs from $\mathbb{N}\times\mathbb{N}$ by:
listing all pairs (a,b) such that a+b = 0, i.e. (0,0)
listing all pairs (a,b) such that a+b = 1, i.e. (0,1), (1,0)
listing all pairs (a,b) such that a+b = 2, i.e. (0,2), (1,1), (2,0)
etc.

$|\mathbb{N}| = |\mathbb{N}|^k$

Note: *Consider a different 1-1 function*
The function, $g : \mathbb{N}^+ \times \mathbb{N}^+ \to \mathbb{N}^+$ such that

$$g(m, n) = 2^{m-1} * (2n - 1)$$

is a 1-1 function.
Exercise: Find m and n such that $g(m, n) = 80$.

$|\mathbb{N}| = |\mathbb{N}|^k$

Since $|\mathbb{N} \times \mathbb{N}| = |\mathbb{N}| * |\mathbb{N}|$ we get that $|\mathbb{N}| = |\mathbb{N}|^2$.
Similarly, $|\mathbb{N}|^3 = |\mathbb{N}| * |\mathbb{N}|^2$ therefore $|\mathbb{N}| = |\mathbb{N}|^3$,
therefore, for any finite natural $k > 0$, $|\mathbb{N}| = |\mathbb{N}|^k$.

Proof of Cantor's Theorem $|\mathbb{N}| \neq |P(\mathbb{N})|$

Cantor's Theorem

$|\mathbb{N}| \neq |P(\mathbb{N})|$

Proof is by contradiction.
Assume $|\mathbb{N}| = |P(\mathbb{N})|$ $\therefore$ there is a 1-1 correspondence between $\mathbb{N}$ and $P(\mathbb{N})$.

$\mathbb{N}$	0	1	$\dots$	n	$\dots$
$P(\mathbb{N})$	$sub(0)$	$sub(1)$	$\dots$	$sub(n)$	$\dots$

where $sub(n)$ is the subset corresponding to n.
Also, for each subset, S, of $\mathbb{N}$ there is a matching element in $\mathbb{N}$, i.e. for each element $S \in P(\mathbb{N})$, there is an element, $k \in \mathbb{N}$, such that $sub(k) = S$.
Recall: $S \in P(\mathbb{N})$ iff $S \subseteq \mathbb{N}$.

Cantor's Thm. (Cont'd)

For each subset, $sub(n)$, of $\mathbb{N}$, either $n \in sub(n)$ or $n \notin sub(n)$.
Define a subset D of $\mathbb{N}$, such that

$$D = \{k \in \mathbb{N} : \ k \notin sub(k)\}$$

i.e. for $k \in \mathbb{N}$,

$$k \in D \equiv k \notin sub(k)$$

Note similarity with Russell Set, R, where
$R = \{x \,|\, x \notin x\}$
i.e. $x \in R \equiv x \notin x$.

Cantor's Thm. (Cont'd)

Since $D \subseteq \mathbb{N}$, i.e. $D \in P(\mathbb{N})$,
there is an element, $d \in \mathbb{N}$, such that $sub(d) = D$, $\therefore$.

$$d \in sub(d) \equiv d \in D$$

but from the definition of D,

$$d \in D \equiv d \notin sub(d)$$

and so $d \in sub(d) \equiv d \notin sub(d)$, a contradiction.
This contradiction arose due to assuming that $|\mathbb{N}| = |P(\mathbb{N})|$ $\therefore$
$|\mathbb{N}| \neq |P(\mathbb{N})$.

$|(0,1)| = |P(\mathbb{N})|$

In Real Number Theory, the notation $(0,1)$ is used to denote the set of Real numbers between 0 and 1 i.e. $(0,1) = \{x \in \mathbb{R} \,|\, 0 < x < 1\}$. The notation, $(0,1)$, denotes an **open interval**, i.e. the end points are not included while the notation $[0,1]$ denotes the **closed interval** that does include both end points.

Consider $x \in (0,1)$ in binary notation.
0.5 *in decimal* $=$ 0.1 *in binary*
as 0.5 *in decimal* $= 5 * \frac{1}{10} = \frac{1}{2}$ and 0.1 *in binary* $= 1 * \frac{1}{2} = \frac{1}{2}$.
Every $x \in (0,1)$ can be written in binary as: $x = 0.b_0 b_1 b_2 \ldots$
where $b_i = 0$ *or* 1 .

$|(0,1)| = |P(\mathbb{N})|$ Cont'd

$|(0,1)| = |P(\mathbb{N})|$

The 1-1 function $s : (0,1) \to P(\mathbb{N})$ is defined as follows. For every (binary) $x \in (0,1)$ where $x = 0.b_0 b_1 b_2 \ldots$ there corresponds exactly one subset, $s(x) \subseteq \mathbb{N}$, where, for $k \in \mathbb{N}$,

$$k \in s(x) \quad \textit{iff} \quad b_k = 1$$

Corollary: It can be shown that $|\mathbb{R}| = |(0,1)|$ and therefore $|\mathbb{R}| = |P(\mathbb{N})|$

$|(0,1)| = |P(\mathbb{N})|$ Cont'd

Eample $x = \frac{5}{8}$

In binary,

$$x = 0.10100\ldots$$

therefore

$$s(x) = \{0,2\}$$

Example.

$$x = 0.00100100\ldots$$

therefore

$$s(x) = \{2,5\}$$