MA2C03: TUTORIAL 14 PROBLEMS
GRAPH THEORY

1) Let (V, E) be the graph with vertices a, b, c, d, and e and edges ab, bd, be, ac, cd, and ae.

(a) Is this graph a tree? Justify your answer.

(b) If it is not a tree, how many distinct spanning trees does it have?

2) Consider the statement "A graph (V, E) is a tree $\iff$ $\#(E) = \#(V) - 1$." What hypothesis is needed for this equivalence to be true? Give an example to show why this hypothesis is necessary.

Recall that

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

read as " n choose k" gives the number of distinct combinations of k objects taken out of a possible n objects for $n \geq k \geq 0$ with the convention $0! = 1$.

3) Consider the complete graph K_n for $n = 2, 3, 4, 5$. In each of the four cases

(a) Is this graph a tree? Justify your answer.

(b) If it is not a tree, how many distinct spanning trees does it have?

(Hint: How many edges does K_n have?)

Solution: 1)(a) The graph is at the end of the solutions. It is not a tree as it contains circuits abea and abdca.

(b) There are 11 possibilities total as follows:

- Eliminate ab and one of ae, ac, cd, bd, or be (5 possibilities).
- Eliminate ae and one of ac, cd, or bd (eliminating ab gives a previously counted tree), so we have 3 possibilities in this case.
- Eliminate be and one of ac, cd, or bd (eliminating ab gives a previously counted tree), so we have 3 possibilities in this case as well.

2) The missing hypothesis is "connected." If the graph (V, E) is not connected we could have something like the graph with vertices a, b, c, d, and e and edges ab, bc, cd, and da, where the vertex e is isolated. This graph has 5 vertices and 4 edges, but it contains the circuit abcda, so it is not acyclical, and it has two connected components, so it is not connected. Therefore, it cannot be a tree.

3) In a complete graph K_n every vertex is connected to every other vertex, so the degree of every vertex is $n-1$. We have n vertices, so the number of edges in K_n is $\frac{n(n-1)}{2}$ as each edge is counted twice. Out of $\frac{n(n-1)}{2}$ edges, we are supposed to choose $n-1$ to construct a spanning tree as we have n vertices, so a tree connecting them has $n-1$ edges. Therefore, we first check whether our K_n has any circuits. If it does not, it is a tree. If it does, then the count

$$\binom{\frac{n(n-1)}{2}}{n-1}$$

gives the number of ways $n-1$ edges can be chosen, but in certain configurations depending on n, we can get graphs (V, E) satisfying $\#(E) = \#(V) - 1$ that are not connected (as we saw in the previous problem). We have to count those and subtract them from

$$\binom{\frac{n(n-1)}{2}}{n-1}$$

in order to get the number of distinct spanning trees.

$n = 2$ We have 2 vertices and 1 edge, so K_2 is a tree and hence its own spanning tree (1 choice of spanning tree).

$n = 3$ We have 3 vertices and 3 edges, K_3 contains a circuit, so it is not a tree. The number of distinct spanning trees is

$$\binom{3}{2} = \frac{3!}{1!\,2!} = 3$$

as it is not possible in this case to construct subgraphs of K_3 with 3 vertices and 2 edges that are disconnected.

$n = 4$ We have 4 vertices and $\frac{4 \cdot 3}{2} = 6$ edges, K_4 contains a number of circuits, so it is not a tree. The number of ways we can choose 3 edges out of 6 is

$$\binom{6}{3} = \frac{6!}{3!\,3!} = 20,$$

but there are

$$4 = \binom{4}{1}$$

different disconnected subgraphs of K_4 consisting of a triangle plus an isolated point. Those are not spanning trees of K_4, so the number of distinct spanning trees is

$$\binom{6}{3} - \binom{4}{1} = \frac{6!}{3!\,3!} - 4 = 20 - 4 = 16.$$

$n = 5$ We have 5 vertices and $\dfrac{5 \cdot 4}{2} = 10$ edges, K_5 contains a number of circuits, so it is not a tree. First of all,

$$\binom{10}{4} = \frac{10!}{4!\,6!} = \frac{7 \cdot 8 \cdot 8 \cdot 10}{1 \cdot 2 \cdot 3 \cdot 4} = 210.$$

Now, we have to figure out how many subgraphs of K_5 with 5 vertices and 4 edges are there that are disconnected. We could have a connected component with four vertices and four edges plus an isolated vertex. There are

$$5 = \binom{5}{1}$$

ways to choose a vertex, and the remaining four vertices form a copy of K_4, so that subgraph has four vertices and six edges. We must choose four of those six edges, so we have

$$\binom{6}{4} = \frac{6!}{4!\,2!} = \frac{6 \cdot 5}{2} = 15$$

possibilities. Altogether, we have

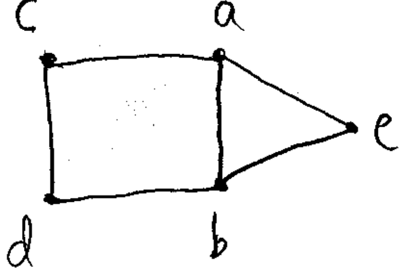
$$\binom{5}{1} \cdot \binom{6}{4} = 5 \cdot 15 = 75$$

subgraphs of K_5 consisting of a vertex and another component with four vertices and four edges. Also, we could have a triangle plus two vertices connected by an edge. There are

$$\binom{5}{2} = \frac{4 \cdot 5}{1 \cdot 2} = 10$$

such subgraphs of K_5. Therefore, the number of distinct spanning trees is

$$\binom{10}{4} - \binom{5}{1} - \binom{5}{2} = 210 - 75 - 10 = 125.$$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