Primes

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Fundamental Property of Natural Numbers, $\mathbb{N}$

Any non empty set of Natural numbers has a least element.

A Natural number is a non-negative integer, i.e.
$\mathbb{N} = \{n | n \in \mathbb{Z} \land n \geq 0\}$

Prime Numbers

A natural number, p, is prime if it has exactly two divisors, 1 and p.

The first dozen primes are:
2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37.

A natural number, n, is composite if it is not prime, i.e. n has more than two factors.

Smallest Factor is Prime

The smallest factor ($\neq 1$) of a number, n, is prime.

The set of factors ($\neq 1$) of n is a non empty set of natural numbers. Let p be its least element, then $1 < p \leq n$.
If p is not prime, then p has a factor, k, such that $1 < k < p$. Since k a factor of p and p is a factor of n then k is a factor ($\neq 1$) of n that is less than p which was assumed to be the least factor of n.
$\therefore$ p is prime.

Infinity of Primes

The set of primes is infinite, i.e. there is no greatest prime.

Proof.

Assume the set of primes is finite i.e. the set of all primes is $\{p_1, \ldots, p_k\}$.
Let $n = 1 + p_1 * p_2 * \cdots * p_k$. Then $n > 1$ and has a least factor, p, which is prime.
The number, n, leaves a remainder, 1, when divided by $p_1$and p_2 $\ldots$ and p_k and so none of the elements in $\{p_1, \ldots, p_k\}$ is a factor of n and so p is a prime not in the set $\{p_1, \ldots, p_k\}$, a contradiction as $\{p_1, \ldots, p_k\}$ was assumed the set of all primes $\square$

Determine if n is prime

To determine if a natural number, n, is prime, check for (prime) divisors/factors (> 1) up to the $\sqrt{n}$.
If there are no factors (> 1) then the number, n, is prime.

Example: Check if 199 is prime. Since $14^2 = 196 < 199 < 15^2 = 225$, check all the primes up to 14 as divisors of 199.

Notation: $a|b$ "a divides b" or "a is a factor of b"
Also, $a \not| b$ "a does not divide b".

$2 \not| 199$, $3 \not| 199$, $5 \not| 199$,

$$\begin{aligned} 199 &= 7 * 28 + 3 \therefore 7 \not| 199, \\ 199 &= 11 * 18 + 1 \therefore 11 \not| 199, \\ 199 &= 13 * 15 + 4 \therefore 13 \not| 199, \end{aligned}$$

$\therefore$ 199 is prime.

Find prime factors of n

Find Prime Factors

To determine the prime factors of a natural number, n : start with the the lowest prime, 2, and divide the number by the lowest prime divisor giving a quotient and repeatedly check the prime divisors of the quotient.

Example: Find prime factors of 980.

$\frac{980}{2} = 490$, $\frac{490}{2} = 245$, $\frac{245}{5} = 49$, $\frac{49}{7} = 7$, $\frac{7}{7} = 1 \therefore$

$980 = 2 * 490 = 2^2 * 245 = 2^2 * 5 * 49 = 2^2 * 5 * 7^2$ i.e.

$$980 = 2^2 * 5 * 7^2$$

Rules of Thumb for divisibility

- n is divisible by 2 or 2 is a factor of n
  if the last (least significant) digit is divisible by 2
- n is divisible by 3
  if the digit sum is divisible by 3.
- n is divisible by 5
  if the last digit is divisible by 5 i.e. the last digit is 0 or 5.
- n is divisible by 7
  Check if n leaves a 0 remainder i.e. $n \, mod \, 7 = 0$.
- n is divisible by 11
  Take the alternating sum of the digits in the number, reading from left to right. If that is divisible by 11, so is the original number.
  Example: $n = 2728$ has alternating sum of digits $2 - 7 + 2 - 8 = -11$. Since -11 is divisible by 11, so is 2728.