Applications of minimal spanning trees

- design of networks such as computer networks, transportation networks, telecommunication networks, water supply networks, electrical grids, etc.
- computing minimal spanning trees appears as a subroutine in algorithms such as algorithms approximating NP-hard problems such as the traveling salesman problem.
- minimal spanning trees can be used to describe financial markets, in particular how stocks are correlated.
- various other problems in computer science and engineering.

Directed graphs

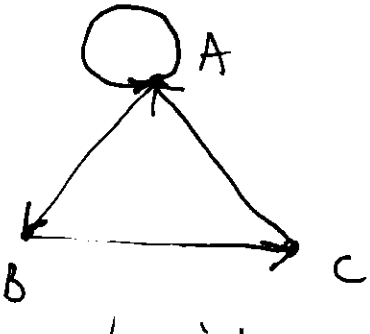
Task: Introduce a new category of graphs where the edges have directions and loops are allowed.

Def: A directed graph or digraph (V, E) consists of a finite set V together w/ a subset E of $V \times V$. The elements of V are the vertices of the digraph, whereas the elements of E are the edges of the digraph.

Remark: Recall that when we defined undirected graphs (V, E), the set of vertices E was a subset of V_2, where V_2 was the set consisting of all subsets of V with exactly two elements. Note that $\{v, w\} = \{w, v\} \in V_2$ if $v \neq w$, whereas $(v, w) \neq (w, v) \in V \times V$. The pairs in $V \times V$ are ordered. Also $(v, v) \in V \times V \Rightarrow$ loops are allowed as edges of a digraph, whereas they weren't allowed as edges of an undirected graph.

Def: Let $(v, w) \in E$ be the edge of a digraph (V, E). We say that v is the initial vertex and w is the terminal vertex of the edge. Furthermore, we say that the vertex w is adjacent from the vertex v and vertex v is adjacent to the vertex w, whereas the edge (v, w) is incident from the vertex v and incident to the vertex w.

Example



is a digraph with $V = \{A, B, C\}$ and $E = \{(A, A), (A, B), (B, C), (C, A)\}$

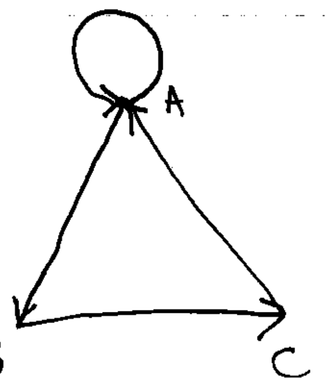
We use arrows to indicate the directions of the edges of a digraph (directed graph).

Just like an undirected graph, a directed graph has an adjacency matrix associated w/ it.

Let (V, E) be a directed graph, and let the vertices in V be ordered $v_1, v_2, \ldots, v_m$. The adjacency matrix of (V, E) is the $m \times m$ matrix (b_{ij})

$$\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ b_{21} & b_{22} & \cdots & b_{2m} \\ \vdots & & & \\ b_{m1} & b_{m2} & \cdots & b_{mm} \end{pmatrix} \quad \text{where } b_{ij} = \begin{cases} 1 & \text{if } (v_i, v_j) \in E \\ 0 & \text{otherwise} \end{cases}$$

<u>Example</u>



Let $v_1 = A \quad v_2 = B \quad v_3 = C$

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$(A, C) \notin E$ but $(C, A) \in E$

<u>Remark</u> The adjacency matrix of an undirected graph always had 0's on the diagonal, whereas the adjacency matrix of a directed graph could have some 1's on the diagonal due to the presence of loops.

Directed graphs and binary relations

<u>Task</u> Describe the one-to-one correspondence between directed graphs and binary relations on finite sets.

Let V be a finite set.

<u>To every relation R on V, there corresponds a directed graph</u>

Indeed, set $E = \{(v, w) \in V \times V \mid vRw\}$, Then (V, E) is a directed graph.

To every directed graph (V, E), there corresponds a relation R on V

Indeed, we define the relation R on V as follows: $\forall\, v, w \in V$,
$vRw \Leftrightarrow (v, w) \in E$.

Moral of the story

We can use directed graphs to visually represent binary relations on finite sets.

Countability of sets

__Task__ Understand what it means for a set to be countable, countably infinite uncountably infinite. Show that the set of all languages over a finite alphabet is uncountably infinite, whereas the set of all regular languages over a finite alphabet is countably infinite.

We want to understand sizes of sets. In the unit on functions last term, when we looked at functions defined on finite sets, we wrote down a set A of n elements as $A = \{a_1, \dots, a_n\}$. This notation mimics the process of counting: a_1 is the first element of A, a_2 is the second element of A, and so on up to a_n is the n^{th} element of A. In other words, another way of saying A is a set of n elements is that there exists a bijective function $f: A \to \{1, 2, \dots, n\}$. Let $J_n = \{1, 2, \dots, n\}$.

Def A set A has n elements $\iff \exists f: A \to J_n$ a bijection

NB This definition works $\forall n \geq 1$, $n \in \mathbb{N}^*$

Notation $\exists f: A \to J_n$ a bijection is denoted as $A \sim J_n$.

More generally, $A \sim B$ means $\exists f: A \to B$ a bijection, and it is a relation on sets. In fact, it is an equivalence relation (check!). $[J_n]$ is the equivalence class of all sets A of size n, i.e. $\#(A) = n$.

Def A set A is finite if $A \sim J_n$ for some $n \in \mathbb{N}^*$ or $A = \emptyset$.

Def A set A is infinite if A is not finite.

Examples $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{R}$, etc.

To understand sizes of infinite sets, generalize the construction above. Let $J = \mathbb{N}^* = \{1, 2, \dots\}$

Def A set A is countably infinite if $A \sim J$.

Def A set A is uncountably infinite if A is neither finite nor countably infinite.

In fact, we often treat together the cases A is finite or A is countably infinite since in both of these cases the counting mechanism that is so familiar to us works. Therefore, we have the following definition:

Def A set A is countable if A is finite ($A \sim J_n$ or $A = \emptyset$) or A is countably infinite ($A \sim J$).

There is a difference in terminology regarding countability between CS sources (textbooks, articles, etc.) and maths sources. Here is the dictionary:

CS	Maths
countable	at most countable
countably infinite	countable
uncountably infinite	uncountable

It's best to double check which terminology a source is using.

Goal Characterize $[\mathbb{N}]$, the equivalence class of countably infinite sets, and $[\mathbb{R}]$, the equivalence class of uncountably infinite sets the same size as $\mathbb{R}$.

Bad news Both $[\mathbb{N}]$ and $[\mathbb{R}]$ consist of infinite sets

Good news We only care about these two equivalence classes.

NB There are uncountably infinite sets of size bigger than $[\mathbb{R}]$ that can be obtained from the power set construction, but it is unlikely you will see them in your CS coursework.

To characterize $[\mathbb{N}]$, we need to recall the notion of a sequence:

Def A sequence is a set of elements $\{x_1, x_2, \ldots\}$ indexed by J, i.e. $\exists f: J \to \{x_1, x_2, \ldots\}$ s.t. $f(n) = x_n \ \forall n \in J$.

Recall that sequences and their limits were used to define various notions in calculus (differentiation, integration, etc.)

Also, calculators use sequences in order to compute with various rational and irrational numbers.