MA2C03: TUTORIAL 5 PROBLEM SHEET

1) Let $A = \{3^p \mid p \in \mathbb{N}\}$ with the operation of multiplication.

(a) Is $(A, \cdot)$ a semigroup? Justify your answer.
(b) Is $(A, \cdot)$ a monoid? Justify your answer.
(c) Is $(A, \cdot)$ a group? Justify your answer.

Solution: (a) Yes, $(A, \cdot)$ is a semi-group. $A \subset \mathbb{Q}^*$, and $\mathbb{Q}^* = \mathbb{Q} \setminus \{0\}$ is a monoid under the operation of multiplication. We proved in lecture that if $a \in M$ for M a monoid with operation $*$ and $m, n \in \mathbb{N}$, then $a^m * a^n = a^{m+n}$. Here $a = 3$ and since addition is a binary operation on $\mathbb{N}$ as we showed in class, multiplication is a binary operation on A. The associativity of multiplication on A follows from the associativity of addition on $\mathbb{N}$ and the theorem that if $a \in M$ for M a monoid with operation $*$ and $m, n \in \mathbb{N}$, then $a^m * a^n = a^{m+n}$.

(b) Yes, $(A, \cdot)$ is a monoid. $3^0 = 1$ is the identity element on A because any $b \in A$ is of the form 3^p, so $b \cdot 1 = a^p \cdot a^0 = a^{p+0} = a^{0+p} = 1 \cdot b = a^p = b$.

(c) No, $(A, \cdot)$ is not a group. By the theorem on powers we proved in lecture, 3^{-p} would have to be the inverse of 3^p for $p \in \mathbb{N}$ because $3^{-p} \cdot 3^p = 3^p \cdot 3^{-p} = 3^{p-p} = 3^0 = 1$, but if $p \in \mathbb{N}$, then $p \geq 0$, so $-p \notin \mathbb{N}$ when p is negative. So if $p < 0$, $3^{-p} \notin A$. Therefore, the only invertible element in A is the identity element $3^0 = 1$.

2) (Slightly modified question from the annual exam 2017-2018) Let $A = \{(x, y) \in \mathbb{R}^2 \mid x + 2y = 0\}$ with the operation of addition given by

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2).$$

(a) Is $(A, +)$ a semigroup? Justify your answer.
(b) Is $(A, +)$ a monoid? Justify your answer.
(c) Is $(A, +)$ a group? Justify your answer.
(d) What geometric object is the set A in $\mathbb{R}^2$?

Solution: (a) Yes, $(A, +)$ is a semi-group. If $x_1 = -2y_1$ and $x_2 = -2y_2$, then $x_1 + x_2 = -2y_1 - 2y_2 = -2(y_1 + y_2)$, so $+$ is a binary operation on A. We proved in lecture that addition is an associative binary operation on $\mathbb{R}$, so $+$ is associative on A as associativity will function component by component in the vector (x, y).

(b) Yes, $(A, +)$ is a monoid. $(0, 0)$ is the identity element on A because for any $(x, y) \in A$,

$$(x, y) + (0, 0) = (x + 0, y + 0) = (0 + x, 0 + y) = (0, 0) + (x, y) = (x, y).$$

(c) Yes, $(A, +)$ is a group. For any $(x, y) \in A$, $(-x, -y)$ is its inverse because $(x, y) + (-x, -y) = (-x, -y) + (x, y) = (0, 0)$. Therefore, all elements of A are invertible.

(d) A is the line passing through the origin $(0, 0)$ and the point $(2, -1)$ as $2 + 2(-1) = 0$.