Mathematics CS1003

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Linear Systems I

Definition:

A linear equation in n unknowns $x_1, x_2, \cdots, x_n$ is an equation of the form:

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

where $a_1, a_2, \cdots, a_n, b$ are given real numbers.

Definition:

A system of m linear equations in n unknowns $x_1, x_2, \cdots, x_n$ is a family of linear equations:

$$\begin{aligned}
&a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\
&a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\
&\vdots \\
&a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m
\end{aligned}$$

Linear Systems II

Definition:

A system of m linear equations in n unknowns is said to be:

- consistent if it has (at least) one solution,
- inconsistent otherwise.

Each equation of a system can be rewritten for $i = 1, 2, \cdots, m$:

$$\sum_{j=1}^{n} a_{ij} x_j = b_i$$

and the system can be written $A\mathbf{x} = \mathbf{b}$ considering the matrices:

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} \quad \text{and} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

Augmented Matrix

Definition:

The augmented matrix of the system is the concatenated matrix $[A\mathbf{b}]$ or:

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{pmatrix}$$

Write the augmented matrix of the following system of linear equations:

$$\begin{cases} 6x_3 + 2x_4 - 4x_5 - 8x_6 = 8 \\ 3x_3 + x_4 - 2x_5 - 4x_6 = 4 \\ 2x_1 - 3x_2 + x_3 + 4x_4 - 7x_5 + x_6 = 2 \\ 6x_1 - 9x_2 + 11x_4 - 19x_5 + 3x_6 = 1 \end{cases}$$

Row Echelon Form

Definition:

A matrix is in row-echelon form if

1. all zero rows (if any) are at the bottom of the matrix and
2. if two successive rows are non-zero, the second row starts with more zeros than the first (moving from left to right).

Example of row-echelon form

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

This is not a row echelon form:

$$\begin{pmatrix} 0 & 3 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Reduced Row Echelon Form

Definition:

A matrix is in reduced row-echelon form if

1. it is in row-echelon form,
2. the leading (leftmost non-zero) entry in each non-zero row is 1,
3. all other elements of the column in which the leading entry 1 occurs are zeros.

Reduced row-echelon matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 2 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Non-reduced row-echelon matrices

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Elementary Row Operations

Definition:

There are three types of elementary row operations that can be performed on matrices:

1. Interchanging two rows: $\mathbf{r_i} \leftrightarrow \mathbf{r_j}$ interchanges rows i and j.
2. Multiplying a row by a non-zero scalar: $\mathbf{r_i} \to \alpha \mathbf{r_i}$ multiplies row i by the non-zero scalar α.
3. Adding a multiple of one row to another row: $\mathbf{r_j} \to \mathbf{r_j} + \alpha \mathbf{r_i}$ adds α times row i to row j.

Row Equivalent Matrices

Definition:

A matrix A is row-equivalent to a matrix B if B is obtained from A by a sequence of elementary row operations.

If A and B are row equivalent augmented matrices of two systems of linear equations, then the two systems have the same solution sets (i.e. a solution of one system is a solution of the other).

Compare the systems:

$$\begin{cases} x_1 + 2x_2 = 0 \\ 2x_1 + x_2 = 1 \\ x_1 - x_2 = 2 \end{cases} \quad \text{and} \quad \begin{cases} 2x_1 + 4x_2 = 0 \\ x_1 - x_2 = 2 \\ 4x_1 - x_2 = 5 \end{cases}$$

Gauss-Jordan Elimination I

Gauss-Jordan Algorithm:

This is a process for computing a matrix in reduced row-echelon form B starting from a given matrix A. It is performed using the elementary row operations.

Here's how we implement the algorithm, step by step:

Label the rows in the matrix: **row 1** as **R1**, **row 2** as **R2** etc.

STEP 1:

Interchange the top row with another (if necessary) so as to have a **non-zero entry** as **far left as possible** in the first row.

STEP 2:

Multiply the top row by a suitable **constant** to get a **leading entry of 1**.

STEP 3:

Add multiples of the top row to each row below so that **all entries below the leading 1 are zero**.

Gauss-Jordan Elimination II

We have now dealt with the first non-zero element in the first row (it is a “1” and all the entries below it are zero), we turn our attention to the second row:

STEP 4:
Ignore the top row and move on to the second row.

STEP 5:
If necessary, **interchange** the second row with another row **below** it so as to get a **non-zero entry** as **far to the left as possible** in the second row.

STEP 6:
If necessary, **multiply** the second row by a **constant** to get a **leading entry of 1**.

STEP 7:
Add multiples of the second row to each row above and below so that **all entries above and below the leading 1 in the second row are zero**.

Gauss-Jordan Elimination III

We have now dealt with the first non-zero element in the first row and the first non-zero entry in the second row, we turn our attention to the third row:

STEP 8:
Move onto the third row and **repeat steps 5, 6 and 7** (but **this time looking at the third row, rather than the second row**). Keep going until you have dealt with all rows in the matrix.

STEP 9:
Stop when the matrix is in **reduced row echelon form**. Remember if there is a row that consists of all 0's then these rows should come at the bottom of the matrix. If your matrix represents a linear system of equations then you can read off your solution from this final matrix.

Gauss-Jordan Elimination in Operation I

Use Gauss-Jordan Elimination to transform the following matrix into reduced row echelon form:

$$A = \begin{pmatrix} 0 & 0 & 4 & 0 \\ 5 & 5 & -1 & 5 \\ 2 & 2 & -2 & 5 \end{pmatrix}$$

1. The first entry in the first row is zero, we see that the first entry in both of the rows below it is non-zero. So we interchange row 1 with one of these rows.

$$A \text{ with } \quad \mathbf{r_1} \leftrightarrow \mathbf{r_3} \text{ becomes } \quad \begin{pmatrix} 2 & 2 & -2 & 5 \\ 5 & 5 & -1 & 5 \\ 0 & 0 & 4 & 0 \end{pmatrix}$$

Gauss-Jordan Elimination in Operation II

2. We want the first entry in row 1 to be 1, so we divide by 2 i.e. $\mathbf{r_1} \rightarrow \frac{\mathbf{r_1}}{2}$

$$\begin{pmatrix} 1 & 1 & -1 & \frac{5}{2} \\ 5 & 5 & -1 & 5 \\ 0 & 0 & 4 & 0 \end{pmatrix}$$

3. We use the first row to “eliminate” the 5 at the start of the second row: $\mathbf{r_2} \rightarrow \mathbf{r_2} - 5\mathbf{r_1}$. Note the third row already starts with zero so we don’t need to do anything with it.

$$\begin{pmatrix} 1 & 1 & -1 & \frac{5}{2} \\ 0 & 0 & 4 & \frac{-15}{2} \\ 0 & 0 & 4 & 0 \end{pmatrix}$$

Gauss-Jordan Elimination in Operation III

4. We have completed steps 1, 2 and 3 of the Gauss-Jordan Algorithm. We move our focus to the second row in step 4. Step 5 requires us to get a "1" as far to the left as we can in the second row: $\mathbf{r_2} \to \frac{\mathbf{r_2}}{4}$

$$\begin{pmatrix} 1 & 1 & -1 & \frac{5}{2} \\ 0 & 0 & 1 & \frac{-15}{8} \\ 0 & 0 & 4 & 0 \end{pmatrix}$$

5. We now eliminate the terms above and below this 1: $\mathbf{r_1} \to \mathbf{r_1} + \mathbf{r_2}$ and $\mathbf{r_3} \to \mathbf{r_3} - 4\mathbf{r_2}$

$$\begin{pmatrix} 1 & 1 & 0 & \frac{5}{8} \\ 0 & 0 & 1 & -\frac{15}{8} \\ 0 & 0 & 0 & \frac{15}{2} \end{pmatrix}$$

6. We now shift our focus to the last row and we want the first non-zero entry in it to be a "1": $\mathbf{r_3} \to \mathbf{r_3} \times \frac{2}{15}$

$$\begin{pmatrix} 1 & 1 & 0 & \frac{5}{2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Gauss-Jordan Elimination in Operation IV

7. To finish $\mathbf{r_1} \to \mathbf{r_1} - \frac{5}{8}\mathbf{r_3}$

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

This final matrix is now in reduced row echelon form since

- the first non-zero entry in each row is a 1,
- the entries above and below these 1's are zero,
- the leading 1's move progressively to the right as we step down though the rows of the matrix.

Aside: If the matrix above represents a linear system, what might this last row tell us about the solution of that system?

Gauss-Jordan Elimination in Operation V

Using Gauss-Jordan Algorithm to find solutions to Linear Systems :

Put in matrix form, to solve the system $A\mathbf{x} = \mathbf{b}$:

1. Create the augmented matrix $(A\mathbf{b})$,
2. Compute its reduced row echelon form B,
3. We can break B up, separating off the last column $B = (B'\mathbf{b}')$. The original linear system is equivalent to the following set of equations:

$$B'\mathbf{x} = \mathbf{b}'$$

Gauss-Jordan Elimination in Operation VI

Let's go back to the matrix we've just worked with

$$A = \begin{pmatrix} 0 & 0 & 4 & 0 \\ 5 & 5 & -1 & 5 \\ 2 & 2 & -2 & 5 \end{pmatrix}$$

supposing it represents the system of equations

$$\begin{cases} 4x_3 = 0 \\ 5x_1 + 5x_2 - x_3 = 5 \\ 2x_1 + 2x_2 - 2x_3 = 5 \end{cases}$$

We found the reduced row echelon form of the matrix to be:

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Gauss-Jordan Elimination in Operation VII

this can be rewritten as the equations:

$$\begin{cases} x_1 + x_2 = 0 \\ x_3 = 0 \\ 0 = 1 \end{cases}$$

The solution (or lack thereof) of this system of equations is the same as that of the original equations.
The last equation shows explicitly that this system is inconsistent i.e. that the equations have no solution as clearly we can't have 0=1.

Gauss-Jordan Elimination in Operation VIII

Historical context:

- The method of Gaussian elimination is mentioned in the Chinese mathematical text “Jiuzhang suanshu“ or “The Nine Chapters on the Mathematical Art“. It dates to between 150BC and 179AD.
- In Europe it appears in the notes of Isaac Newton(1642 -1727) in 1670, where he commented that all the algebra books he knew of lacked a lesson for solving simultaneous equations – which he then supplied.
- Cambridge University published his notes as “Arithmetica Universalis” in 1707.
- By the end of the 18th century the method was a standard lesson in algebra textbooks.
- In 1810 Carl Friedrich Gauss devised a notation for symmetric elimination that was adopted in the 19th century by professional hand computers to solve the normal equations of least-squares problems (a methodology used for curve fitting to a set of points) .
- The algorithm we know was named after Gauss only in the 1950s as a result of confusion over the history of the subject

Linear Systems with an Infinite Number of Solutions I

The examples we've considered so far all have **unique** solutions or no solution. Can the procedure we have developed account for the possibility of **infinitely many** solutions?

Yes! Modifying the **Back Substitution** stage slightly shows how infinitely many solutions can arise **AND** determines structure of the solutions.

Let's see how it's done.....

Linear Systems with an Infinite Number of Solutions II

EXAMPLE 1: Find all solutions of the linear system

$$
\begin{aligned}
2x_1 + 2x_2 \quad & -6x_3 \quad & +6x_5 &= 2 \\
4x_1 + 4x_2 \quad & -11x_3 + 2x_4 \quad & +11x_5 &= 6 \\
x_1 + x_2 \quad & +x_3 + 8x_4 \quad & +x_5 &= 11 \\
-3x_1 - 3x_2 \quad & +11x_3 + 4x_4 \quad & &= 12.
\end{aligned}
$$

(I know this is written a bit oddly, but I was trying to get all the terms involving x_1 alligned, all the terms involving x_2 alligned etc – that way we can form the augmented matrix more easily)

SOLUTION: The augmented matrix of this linear system is

$$
\begin{pmatrix}
2 & 2 & -6 & 0 & 6 & 2 \\
4 & 4 & -11 & 2 & 11 & 6 \\
1 & 1 & 1 & 8 & 1 & 11 \\
-3 & -3 & 11 & 4 & 0 & 12
\end{pmatrix}.
$$

Linear Systems with an Infinite Number of Solutions III

Converting this to row echelon form:

$$\begin{array}{cl} \begin{pmatrix} 2 & 2 & -6 & 0 & 6 & 2 \\ 4 & 4 & -11 & 2 & 11 & 6 \\ 1 & 1 & 1 & 8 & 1 & 11 \\ -3 & -3 & 11 & 4 & 0 & 12 \end{pmatrix} & \begin{array}{l} \text{R1} \\ \text{R2} \\ \text{R3} \\ \text{R4} \end{array} \end{array}$$

$$\xrightarrow{R1 \to 1/2 \times R1} \begin{array}{cl} \begin{pmatrix} 1 & 1 & -3 & 0 & 3 & 1 \\ 4 & 4 & -11 & 2 & 11 & 6 \\ 1 & 1 & 1 & 8 & 1 & 11 \\ -3 & -3 & 11 & 4 & 0 & 12 \end{pmatrix} & \begin{array}{l} \text{R1} \\ \text{R2} \\ \text{R3} \\ \text{R4} \end{array} \end{array}$$

$$\xrightarrow{\substack{R2 \to R2 - 4 \times R1 \\ R3 \to R3 - R1 \\ R4 \to R4 + 3 \times R1}} \begin{array}{cl} \begin{pmatrix} 1 & 1 & -3 & 0 & 3 & 1 \\ 0 & 0 & 1 & 2 & -1 & 2 \\ 0 & 0 & 4 & 8 & -2 & 10 \\ 0 & 0 & 2 & 4 & 9 & 15 \end{pmatrix} & \begin{array}{l} \text{R1} \\ \text{R2} \\ \text{R3} \\ \text{R4} \end{array} \end{array}$$

Linear Systems with an Infinite Number of Solutions IV

$$\xrightarrow[\substack{R3 \to R3 - 4 \times R2 \\ R4 \to R4 - 2 \times R2}]{R1 \to R1 + 3R2} \begin{array}{c} \begin{pmatrix} 1 & 1 & 0 & 6 & 0 & 7 \\ 0 & 0 & 1 & 2 & -1 & 2 \\ 0 & 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 & 11 & 11 \end{pmatrix} \end{array} \begin{array}{l} \text{R1} \\ \text{R2} \\ \text{R3} \\ \text{R4} \end{array}$$

$$\xrightarrow[R3 \to 1/2 \times R3]{} \begin{pmatrix} 1 & 1 & 0 & 6 & 0 & 7 \\ 0 & 0 & 1 & 2 & -1 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 11 & 11 \end{pmatrix} \begin{array}{l} \text{R1} \\ \text{R2} \\ \text{R3} \\ \text{R4} \end{array}$$

$$\xrightarrow[R4 \to R4 - 11 \times R3]{R2 + R3} \begin{pmatrix} 1 & 1 & 0 & 6 & 0 & 7 \\ 0 & 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{array}{l} \text{R1} \\ \text{R2} \\ \text{R3} \\ \text{R4} \end{array}$$

The final matrix is in **reduced row echelon form**.

Linear Systems with an Infinite Number of Solutions V

- The linear system corresponding to this matrix is

$$\begin{aligned} x_1 + x_2 - 0x_3 + \qquad 6x_4 &= 7 \qquad \text{(E1)} \\ x_3 + 2x_4 &= 3 \qquad \text{(E2)} \\ x_5 &= 1 \qquad \text{(E3)} \end{aligned}$$

- **N.B.**: The **last row** of the row echelon matrix has equation

$$0x_1 + 0x_2 + 0x_3 + 0x_4 + 0x_5 = 0 \quad \text{i.e.} 0 = 0,$$

which imparts no additional information.

- Solve from bottom up: First, (E3) gives $x_5 = 1$.

Linear Systems with an Infinite Number of Solutions VI

A DIFFICULTY: Next variable is x_4. But (E2) is written so as to determine x_3 given x_4 and x_5. Moreover, there is **no equation** involving x_4 **in terms of** x_5, **NOR** one for x_2 **in terms of** x_3, x_4, x_5.

SOURCE OF OUR DIFFICULTY: We need to solve for **five unknowns**

$$x_1, x_2, \ldots x_5$$

in terms of **three** pieces of relevant information (**equations** (E1)–(E3))

However, we cannot determine $x_1, x_2, \ldots x_5$ **UNIQUELY**.

Linear Systems with an Infinite Number of Solutions VII

RESOLUTION: Assigning **values** to x_4, x_2 determines x_1, x_3, x_5. Values can be assigned in **infinitely many ways**. So there are infinitely many solutions to this linear system.

The values assigned to x_4, x_2 are called **PARAMETERS**. For x_4 we express this mathematically by letting

$$x_4 = t, \qquad t \in \mathbb{R}$$

where t is the **parameter**. (We are assuming that all our equations involve real numbers, so we note that $t \in \mathbb{R}$).

- Substitute expressions for x_4, x_5 into (E2) to obtain x_3:

$$x_3 = 3 - 2t.$$

- There is **no equation** for x_2 in terms of x_3, x_4, x_5, so we **introduce** another parameter for x_2. Mathematically, we let

$$x_2 = s, \qquad s \in \mathbb{R}$$

Linear Systems with an Infinite Number of Solutions VIII

- Substituting expressions for $x_2, \ldots x_5$ into (E1), we get x_1:

$$x_1 = 7 - 6x_4 - x_2 = 7 - 6t - s.$$

ANSWER: $x_1 = 7 - 6t - s$, $x_2 = s$, $x_3 = 3 - 2t$, $x_4 = t$, $x_5 = 1$ for any $s, t \in \mathbb{R}$.

Linear Systems with an Infinite Number of Solutions IX

Remarks:

- As there are **infinitely many** choices for s, t, so there are **infinitely many** solutions to system.
- Once s, t are **fixed**, the solution is determined. An **arbitrary** set of 5 values is **NOT** a solution: solutions have a particular form.

Introducing Parameters:

- Introduce parameters whenever the **reduced row echelon form** has **fewer equations than unknowns**.
- Introduce a **new parameter** for each **variable** that **does NOT appear** as the **first unknown** in an equation e.g., x_2, x_4 above.
- When we introduction of one or more parameters, it automatically means we have **infinitely many** solutions.

Linear Systems with No Solutions I

- Linear systems which have **NO SOLUTION** (called **INCONSISTENT** linear systems) can be detected by Gaussian Elimination by making some minor changes to the **Back Substitution** stage.

- Consider the linear system

$$\begin{aligned} 2x_1 + x_2 &= 1 \\ 4x_1 + 2x_2 &= 0. \end{aligned}$$

- This can be re-written as

$$\begin{aligned} x_1 + \frac{1}{2}x_2 &= \frac{1}{2} \\ x_1 + \frac{1}{2}x_2 &= 0, \end{aligned}$$

so clearly no pair of numbers x_1, x_2 can satisfy these equations.

Linear Systems with No Solutions II

- Subtracting the first equation from the second gives

$$\begin{aligned} x_1 + 1/2 \quad x_2 &= 1/2 \\ 0 &= -1/2 \end{aligned}$$

- Thus the **signature** of an **INCONSISTENT** linear system is a **contradictory** expression such as $0 = -1/2$.

Detecting Inconsistent Linear Systems with Gaussian Elimination

- First simplify the augmented matrix of the linear system using **elementary row operations**.
- If one of the **equations** of the linear system associated with the **simplified matrix** is **contradictory** then the system has **NO** solution.

Linear Systems with No Solutions III

- Using Gaussian Elimination, we show that the above example has no solution: performing row operations gives

$$\begin{array}{ll} \begin{pmatrix} 2 & 1 & 1 \\ 4 & 2 & 0 \end{pmatrix} & \begin{array}{l} \text{R1} \\ \text{R2} \end{array} \end{array}$$

$$\xrightarrow{R1 \to 1/2 \times R1} \begin{array}{ll} \begin{pmatrix} 1 & 1/2 & 1/2 \\ 4 & 2 & 0 \end{pmatrix} & \begin{array}{l} \text{R1} \\ \text{R2} \end{array} \end{array}$$

$$\xrightarrow{R2 \to R2 - 4 \times R1} \begin{array}{ll} \begin{pmatrix} 1 & 1/2 & 1/2 \\ 0 & 0 & -2 \end{pmatrix} & \begin{array}{l} \text{R1} \\ \text{R2} \end{array} \end{array}$$

- Last row of final matrix tells us:
  $0x_1 + 0x_2 = -2$, or $0 = -2$, a **contradiction**, so the system has **no solution**.

A More Challenging Example I

Before we move on from using Gaussian elimination, let's have a look at a harder example:

EXAMPLE: For which values of the constant a will the following linear system have:

(i) No solutions

(ii) Infinitely many solutions

(iii) A unique solution

$$
\begin{aligned}
x_1 + 2x_2 - 3x_3 &= 4 \\
5x_1 + 3x_2 - x_3 &= 10 \\
9x_1 + 4x_2 + (a^2 - 15)x_3 &= a + 12.
\end{aligned}
$$

A More Challenging Example II

SOLUTION:

- Simplify the augmented matrix using elementary row operations:

$$\begin{pmatrix} 1 & 2 & -3 & 4 \\ 5 & 3 & -1 & 10 \\ 9 & 4 & a^2-15 & a+12 \end{pmatrix} \begin{array}{l} \text{R1} \\ \text{R2} \\ \text{R3} \end{array}$$

$$\xrightarrow[R3 \to R3 - 9 \times R1]{R2 \to R2 - 5 \times R1} \begin{pmatrix} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & -14 & a^2+12 & a-24 \end{pmatrix} \begin{array}{l} \text{R1} \\ \text{R2} \\ \text{R3} \end{array}$$

$$\xrightarrow{R2 \to R2 - 1/7 \times R1} \begin{pmatrix} 1 & 2 & -3 & 4 \\ 0 & 1 & -2 & \frac{10}{7} \\ 0 & -14 & a^2+12 & a-24 \end{pmatrix} \begin{array}{l} \text{R1} \\ \text{R2} \\ \text{R3} \end{array}$$

$$\xrightarrow[R3 \to R3 + 14 \times R2]{R1 \to R1 - 2R2} \begin{pmatrix} 1 & 0 & 1 & \frac{8}{7} \\ 0 & 1 & -2 & \frac{10}{7} \\ 0 & 0 & a^2-16 & a-4 \end{pmatrix} \begin{array}{l} \text{R1} \\ \text{R2} \\ \text{R3} \end{array}$$

A More Challenging Example III

- Rewrite as a linear system.

$$\begin{aligned} x_1 - x_3 &= \frac{8}{7} && (1) \\ x_2 - 2x_3 &= \frac{10}{7} && (2) \\ (a^2 - 16)x_3 &= a - 4 && (3) \end{aligned}$$

(i) No solution:

- The only possible inconsistency is in (3) (a doesn't appear elsewhere).
- For the system to be **inconsistent**, (3) must have form

$$0x_3 = c \quad \text{where} \quad c \neq 0,$$

so that (3) reads $0 = c$ (**contradictory**).

A More Challenging Example IV

So there are no solutions if:

$$a^2 - 16 = 0 \quad \text{and} \quad a - 4 \neq 0,$$

Solving $a^2 - 16 = 0$ gives $a = +4$ or $a = -4$.
However $a - 4 \neq 0$ tells us $a \neq 4$.
Putting these two pieces of information together tells us that the only acceptable value is $a = -4$.

- **ANSWER**: If $a = -4$, there are **no solutions**.

A More Challenging Example V

(ii) Infinitely many solutions:

- There are infinitely many solutions when we find ourselves **introducing parameters**.
- Here this can only happen if a parameter is introduced for x_3, so that there is **no equation** in which x_3 is the **first unknown**
  So (3) must read $0x_3 = 0$.
  This means there are infinitely many solutions if $a^2 - 16 = 0$ and $a - 4 = 0$. Using similar calculations and reasoning to (i) above, we find that $a = 4$.
- **ANSWER**: If $a = 4$, there are **infinitely many solutions**.

A More Challenging Example VI

(iii) Unique solution:

- If $a^2 - 16 \neq 0$, we can divide (3) by $a^2 - 16$ so (3) now reads

$$x_3 = \frac{a-4}{a^2-16}.$$

- We can substitute this expression into (2) to solve for x_2, and then substitute the expressions for x_2, x_3 into (1) to obtain x_1
  So there is a unique solution if $a^2 - 16 \neq 0$.
- **ANSWER**: If $a \neq 4$ or $a \neq -4$, there is a **unique solution**.