Mathematics CS1003

Taylor Polynomials

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Taylor Polynomials Introduction I

- Evaluating polynomial functions is easy. For example, if

$$f(x) = 6 - 3x + 2x^2 + x^3,$$

then

$$f(3) = 6 - 3 \times 3 + 2 \times 3^2 + 3^3 = 42$$

- However, many common functions, e.g. $\sin x$, e^x and $\ln x$, cannot be easily evaluated at given points in their domains except in a few special cases.

How does a calculator/computer approximate the value of $\ln 3$?
One method approximates the natural logarithm function by a polynomial function. This allows $\ln 3$ to be evaluated using just the simple arithmetical operations of addition, subtraction and multiplication. In reality computers use more efficient (but more complicated) polynomial methods, however by studying Taylor polynomials we learn about the basic ideas of polynomial approximation.

Taylor Polynomials Introduction II

- Polynomial approximations are useful as it is easy to multiply polynomials together, and to differentiate and integrate them.
- Polynomial approximations allow complex problems to be described by simple mathematical models, making these problems easier both to understand and to solve.
- Taylor polynomials are theoretically important, as they lead to a way of representing functions by infinite series called Taylor series (at least for some values in their domains).

Taylor Polynomials Introduction III

In this section of the module we will study

- The approximation of functions by linear and quadratic functions.
- An extension of this approximation to Taylor polynomials of higher degree
- Taylor series
- Using known Taylor series to derive Taylor series for further functions.

Linear and Quadratic Taylor Polynomials I

We start by approximating a given function f by a linear function close to some point a in the domain of f .

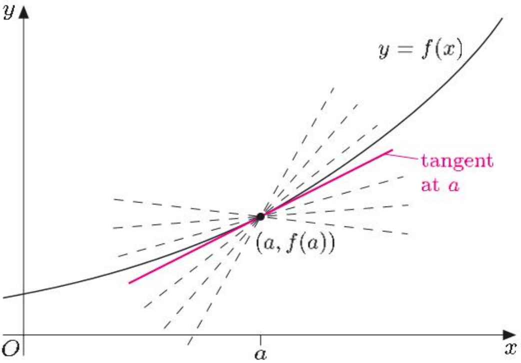

Figure 1.1 A smooth function f and lines through the point $(a, f(a))$

Linear and Quadratic Taylor Polynomials II

We seek a straight line, with equation $y = a_0 + a_1 x$, that approximates the curve $y = f(x)$ close to the point $(a, f(a))$.

The most suitable line appears to be the tangent to the graph of f at a; this is the line whose gradient is the same as that of the graph of f at $(a, f(a))$.

Why the tangent?

Any other line through the point "moves away more quickly" from the graph of f. The linear function whose slope is this tangent is called the linear Taylor polynomial about a for f. For any point x close to a, the value of this polynomial at x is an approximation for $f(x)$.

Linear and Quadratic Taylor Polynomials III

Example 1.1 Approximating the exponential function

(a) Find the linear Taylor polynomial about 0 for the function $f(x) = e^x$.

(b) Use this polynomial to find an approximation for $e^{0.1}$.

(a) We need to find the line that passes through the point $(0, f(0))$ and has slope equal to that of the curve $y = f(x)$ at $(0, f(0))$.

Now $(0, f(0)) = (0, e^0) = (0, 1)$ and $f'(x) = e^x$, so the slope of the line at $(0, 1)$ is $f'(0) = e^0 = 1$.
Thus the required line passes through the point $(0, 1)$ and has gradient (slope) 1.
The slope is 1, so the equation has the form $y = a_0 + x$.
It passes through $(0, 1)$, so we have $1 = a_0 + 0$; that is $a_0 = 1$.
Thus, the line is $y = 1 + x$, and so the linear Taylor polynomial is $p(x) = 1 + x$

(b) The approximation for $e^{0.1}$ given by the polynomial p is $p(0.1) = 1 + 0.1 = 1.1$.

Linear and Quadratic Taylor Polynomials IV

Function $f(x) = e^x$ and its linear approximation, $p(x) = 1 + x$, are illustrated below. The linear approximation appears to be good for values of x close to zero, but its accuracy decreases as the distance of x from 0 increases.

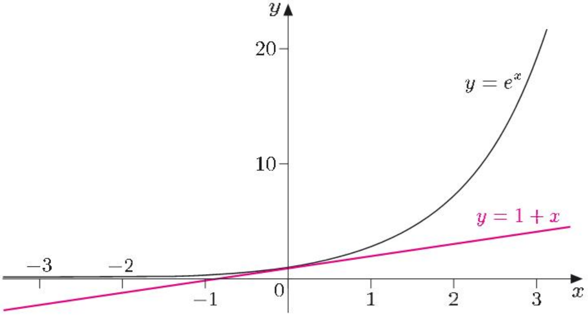

Figure 1.2 Linear Taylor polynomial about 0 for $f(x) = e^x$

Linear and Quadratic Taylor Polynomials V

We use a remainder function to judge the acccuracy of the polynomial approximation

If f is a function, and p is its polynomial approximation, then we define the remainder function $r(x)$ as:

$$r(x) = f(x) - p(x),$$

for all x in the domain of f.

For a given value of x, the closer the value of $r(x)$ is to zero, the better $p(x)$ is as an approximation for $f(x)$. A value of a remainder function is often referred to simply as a remainder.

Linear and Quadratic Taylor Polynomials VI

x	$f(x) = e^x$	$p(x) = 1 + x$	$r(x) = f(x) - p(x)$
-1	0.3679	0	0.3679
-0.75	0.4724	0.25	0.2224
-0.5	0.6065	0.5	0.1065
-0.25	0.7788	0.75	0.0288
0	1	1	0
0.25	1.2840	1.25	0.0340
0.5	1.6487	1.5	0.1487
0.75	2.1170	1.75	0.3670
1	2.7183	2	0.7183

From the table we see $r(x)$ is small for values of x close to 0 (indicating that $1 + x$ is a fairly good approximation for $f(x) = e^x$ near $x = 0$)

Linear and Quadratic Taylor Polynomials VII

A linear Taylor polynomial for $\sin$:

(a) Find the linear Taylor polynomial about 0 for the function $f(x) = \sin x$.

We need to find the line that passes through the point $(0, f(0))$ (i.e. $(0, \sin(x)) = (0, 0)$) and has slope equal to that of the curve $y = \sin x$ at this point. The derivative tells us the slope of the curve is $f'(x) = \cos(x)$, so at the point $x = 0$ the line has slope 1 (since $\cos(0) = 1$).

The linear Taylor polynomial about 0 for the sine function contains no constant term; it is $p(x) = x$. This happens because the graph of the sine function passes through the origin.

For you to do at home:

Investigate the accuracy of this approximation by using your calculator to compile a remainder table. Comment on what you notice about the values of the remainder function.

Linear and Quadratic Taylor Polynomials VIII

The graphs of $f(x) = \sin x$ and $p(x) = x$ are shown in Figure 1.3. The graph of p lies below that of f for $x < 0$ and above it for $x > 0$. You can confirm this by looking at the signs of the remainders in your table.

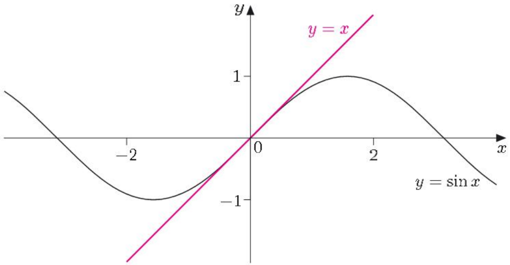

Figure 1.3 Linear Taylor polynomial about 0 for $f(x) = \sin x$

Linear and Quadratic Taylor Polynomials IX

We saw that the linear function $p(x) = x$ is an approximation for the function $f(x) = \sin x$ for values of x close to 0. This approximation is consistent with the formula

$$\lim_{x \to 0} \left(\frac{\sin x}{x} \right) = 1.$$

that you may have met before.

Linear and Quadratic Taylor Polynomials X

A linear Taylor polynomial for $\cos$:

(a) Find the linear Taylor polynomial about 0 for the function $f(x) = \cos x$.

(b) Use this polynomial to find an approximation for $\cos(0.2)$, and use your calculator to find the value of the associated remainder to six decimal places.

(a) We need to find the line that passes through the point $(0, \cos(0))$ (i.e. $(0, 1)$) and has slope equal to that of the curve $y = \cos x$ at this point. The derivative tells us the slope of the curve is $f'(x) = -\sin(x)$, so at the point $x = 0$ the line has slope 0 (since $\sin(0) = 0$).

(b) The line through $(0, 1)$ with slope 0 is $p(x) = 1$.
$p(0.2) = 1$, so our polynomial give an approximate value of $\cos(0.2)$ is 1.
$r(x) = \cos(0.2) - 1 = -0.019933$.

Linear and Quadratic Taylor Polynomials XI

The graphs of $f(x) = \cos x$ and $p(x) = 1$ are shown in Figure 1.4.

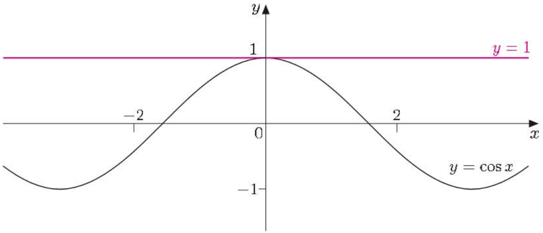

Figure 1.4 Linear Taylor polynomial about 0 for $f(x) = \cos x$

Linear and Quadratic Taylor Polynomials XII

All the linear Taylor polynomials that you have seen so far in this subsection have been about 0. In the next activity you are asked to find linear Taylor polynomials about another point.

More linear Taylor polynomials

Find the linear Taylor polynomial about 1 for each of the following functions.

(a) $f(x) = \ln x - \frac{1}{x}$

(b) $f(x) = e^x$

Linear and Quadratic Taylor Polynomials XIII

(a) $f(x) = \ln x - \frac{1}{x}$

Here $f(x) = \ln x - \frac{1}{x}$,so $(1, f(1)) = (1, \ln 1 - \frac{1}{1}) = (1, -1)$.

Also $f'(x) = \frac{1}{x} + \frac{1}{x^2}$, so the gradient of the curve at $(1, -1)$ is $f'(1) = \frac{1}{1} + \frac{1}{1^2} = 2$. Thus the required line passes through the point $(1, -1)$ and has slope 2.

Since its slope is 2, the line has an equation of the form $y_0 = a_0 + 2x$. Since it passes through $(1, -1)$, we have $-1 = a_0 + 2 \times 1$; that is. $a_0 = -3$. Thus the line has equation $y = -3 + 2x$, and so the linear Taylor polynomial is $p(x) = -3 + 2x$.

Linear and Quadratic Taylor Polynomials XIV

(b) $f(x) = e^x$

Here $f(x) = e^x$, so $(1, f(1)) = (1, e^1) = (1, e)$.

Also $f'(x) = e^x$, so the slope of the curve at $(1, e)$ is $f'(1) = e^1 = e$. Thus the required line passes through the point $(1, e)$ and has slope$e$.

Since its slope is e, the line has an equation $y = a_0 + ex$. Since it passes through $(1, e)$, we have $e = a_0 + e \times 1$; that is $a_0 = 0$. Thus the line has equation $y = ex$, and so the linear Taylor polynomial is $p(x) = ex$.

Linear and Quadratic Taylor Polynomials XV

We have found the linear Taylor polynomial about 1 for the function $f(x) = e^x$ and the linear Taylor polynomial about 0 for the same function.

The graphs of these linear Taylor polynomials are shown in Figure 1.5. As expected, it appears that one of these polynomials approximates e^x for values of x close to 1, while the other approximates e^x for values of x close to 0. This illustrates that, in general, polynomial approximations about different points are different polynomials.

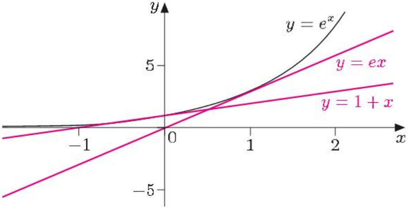

Figure 1.5 Linear Taylor polynomials about 0 and 1 for $f(x) = e^x$

Linear and Quadratic Taylor Polynomials XVI

Linear Taylor polynomials can be used to approximate a square root.

(a) Show that the linear Taylor polynomial about 0 for the function

$$f(x) = (1+x)^k, \qquad \text{where} \quad k > 0,$$

is

$$p(x) = 1 + kx.$$

Here $f(x) = (1+x)^k$, so $(0, f(0)) = (0, (1+0)^k) = (0, 1)$.

Also $f'(x) = k(1+x)^{k-1}$, so the slope of the curve at $(0, 1)$ is $f'(0) = k(1+0)^{k-1} = k$. Thus the required line passes through the point $(0, 1)$ and has slope k.

Since its slope is k, the line has an equation of the form $y = a_0 + kx$.

Since it passes through $(0, 1)$, we have $1 = a_0 + k \times 0$; that is, $a_0 = 1$. Thus the line has equation $y = 1 + kx$, and so the linear Taylor polynomial is $p(x) = 1 + kx$.

Linear and Quadratic Taylor Polynomials XVII

(b) Use the polynomial p from part (a) with $k = \frac{1}{2}$ and $x = 0.01$ to find an approximate value for $\sqrt{1.01}$. Use your calculator to find, to six decimal places, the value of the associated remainder.

Taking $k = \frac{1}{2}$, we have

$$\sqrt{1.01} = (1 + 0.01)^{\frac{1}{2}} = f(0.01)$$

and

$$p(0.01) = 1 + \frac{1}{2} \times 0.01 = 1.005.$$

Hence $\sqrt{1.01} \approx 1.005$. To six decimal places, the remainder is

$$\begin{aligned} r(0.01) &= f(0.01) - p(0.01) \\ &= 1.004988 - 1.005 \\ &= -0.000012. \end{aligned}$$

Quadratic Taylor polynomials about Zero I

If f is a differentiable function whose domain contains 0 then we approximated f close to 0 by $p(x) = a_0 + a_1 x$, which was chosen to be the function whose graph is the tangent line to the graph of f at 0.

In other words, p was chosen to be the linear function that satisfies the following two conditions:

(i) the values of the function and its approximation are equal at 0; that is, $p(0) = f(0)$;

(ii) the values of the first derivatives of the function and its approximation are equal at 0; that is, $p'(0) = f'(0)$.

Quadratic Taylor polynomials about Zero II

Suppose we approximate f close to 0 by a function p of the form

$$p(x) = a_0 + a_1 x + a_2 x^2.$$

It seems sensible to require this new function p to satisfy conditions (i) and (ii). As p now has three coefficients, we can also impose a third condition, and a natural one to choose is:

(iii) the values of the second derivatives of the function and the approximation are equal at 0; that is, $p''(0) = f''(0)$.

We can impose this condition provided that $f(0)$ exists; that is, provided that f is twice-differentaible at 0.

Quadratic Taylor polynomials about Zero III

(i) and (ii) ensure that the function and its approximation both pass through $(0, f(0))$ and have the same gradient at that point.

Condition (iii) ensures that the function and the approximation also have the same rate of change of gradient at 0. (Roughly speaking, this means that their graphs have the same "curvature" at the point $(0, f(0))$.)

The polynomial p that satisfies conditions (i), (ii) and (iii) is called the quadratic Taylor polynomial about 0 for f. For any point x close to 0, the value of $p(x)$ is an approximation for $f(x)$.

Sometimes the polynomial $p(x) = a_0 + a_1x + a_2x^2$ that satisfies conditions (i), (ii) and (iii) has $a_2 = 0$ and so is not a quadratic polynomial, but has degree 1 or less. Nevertheless we still refer to the approximating polynomial as the quadratic Taylor polynomial about 0 for f. So a quadratic Taylor polynomial is not necessarily a quadratic polynomial!

Quadratic Taylor polynomials about Zero IV

Example

Find the quadratic Taylor polynomial about 0 for the function $f(x) = e^x$.

Solution

We want $p(x) = a_0 + a_1x + a_2x^2$.

We need to find a_0, a_1 and a_2 so that the value of p at 0, and the first and second derivative values of p at 0, are the same as those of f.First we ensure that $p(0) = f(0)$. We have

$$f(x) = e^x \qquad \text{and} \qquad p(x) = a_0 + a_1x + a_2x^2,$$

so

$$f(0) = e^0 = 1 \qquad \text{and} \qquad p(0) = a_0.$$

So $a_0 = 1$.

Quadratic Taylor polynomials about Zero V

Next we ensure that $p'(0) = f'(0)$.

$$f'(x) = e^x \qquad \text{and} \qquad p'(x) = a_1 + 2a_2x,$$

so

$$f'(0) = e^0 = 1 \qquad \text{and} \qquad p'(0) = a_1.$$

So $a_1 = 1$.

Finally we must have $p''(0) = f''(0)$. We have

$$f''(x) = e^x \qquad \text{and} \qquad p''(x) = 2a_2,$$

so

$$f''(0) = e^0 = 1 \qquad \text{and} \qquad p''(0) = 2a_2.$$

Thus we must have $2a_2 = 1$; that is, $a_2 = \dfrac{1}{2}$. Hence the quadratic Taylor polynomial about 0 for $f(x) = e^x$ is

$$p(x) = 1 + x + \frac{1}{2}x^2.$$

Quadratic Taylor polynomials about Zero VI

Graph of $f(x) = e^x$ and $p(x) = 1 + x + \frac{1}{2}x^2$. The quadratic function $p(x)$ appears to be a more accurate approximation for $f(x) = e^x$ for values of x close to 0 than the linear function ($p(x) = 1 + x$) we found previously.

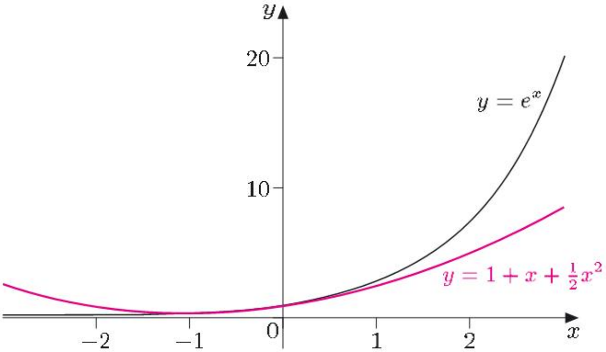

Figure 1.6 Quadratic Taylor polynomial about 0 for $f(x) = e^x$

Quadratic Taylor polynomials about Zero VII

How good an approximation is it? Let's look at some remainders:

x	$f(x) = e^x$	$p(x) = 1 + x + \frac{1}{2}x^2$	$r(x) = f(x) - p(x)$
-1	0.3679	0.5	-0.1321
-0.75	0.4724	0.5313	-0.0589
-0.5	0.6065	0.625	-0.0185
-0.25	0.7788	0.7813	-0.0024
0	1	1	0
0.25	1.2840	1.2813	0.0028
0.5	1.6487	1.625	0.0237
0.75	2.1170	2.0313	0.0858
1	2.7183	2.5	0.2183

Quadratic Taylor polynomials about Zero VIII

A quadratic Taylor polynomial for cos

(a) Find the quadratic Taylor polynomial about 0 for the function $f(x) = \cos x$.

(b) Use this polynomial to find an approximation for cos(0.2), and use your calculator to find the value of the associated remainder to six decimal places. Compare this approximation for cos(0.2) with the linear one found before. Which is better?

Let the polynomial that we seek be

$$p(x) = a_0 + a_1 x + a_2 x^2.$$

First we ensure that $p(0) = f(0)$. We have

$$\begin{aligned} f(x) &= \cos x, & p(x) &= a_0 + a_1 x + a_2 x^2 \\ f(0) &= \cos 0 = 1, & p(0) &= a_0. \end{aligned}$$

Thus we must have $a_0 = 1$.

Quadratic Taylor polynomials about Zero IX

Next we ensure that $p'(0) = f'(0)$. We have

$$\begin{aligned} f'(x) &= -\sin x, & p'(x) &= a_1 + 2a_2 x \\ f'(0) &= -\sin 0 = 0, & p'(0) &= a_1 \end{aligned}$$

Thus we must have $a_1 = 0$.

Finally we ensure that $p''(0) = f''(0)$. We have

$$\begin{aligned} f''(x) &= -\cos x, & p''(x) &= 2a_2 \\ f''(0) &= -\cos 0 = -1, & p'(0) &= 2a_2 \end{aligned}$$

Thus we must have $2a_2 = -1$; that is, $a_2 = -\frac{1}{2}$.

The quadratic Taylor polynomial about 0 for the cosine function is

$$p(x) = 1 - \frac{1}{2}x^2.$$

Quadratic Taylor polynomials about Zero X

We have

$$\cos(0.2) \approx p(0.2) = 1 - \frac{1}{2}(0.2)^2 = 0.98.$$

To six decimal places, the remainder is

$$\begin{aligned} r(0.2) &= \cos(0.2) - p(0.2) \\ &= 0.980067 - 0.98 \\ &= 0.000067. \end{aligned}$$

This remainder has smaller magnitude than the remainder found when we looked at the linear approximation for $\cos(x)$, so the approximation $\cos(0.2) \approx 0.98$ is better than $\cos(0.2) \approx 1$, as expected.

Quadratic Taylor polynomials about Zero XI

A problem for you to do at home

Find the quadratic Taylor polynomial about 0 for the function $f(x) = \sin x$.

You will find that

$$p(x) = x$$

is the quadractic Taylor polynomial about 0 for the sine function.

Quadratic Taylor polynomials about Zero XII

The graphs of $f(x) = \cos x$ and the approximation $p(x) = 1 - \frac{1}{2}x^2$ found previously are shown below

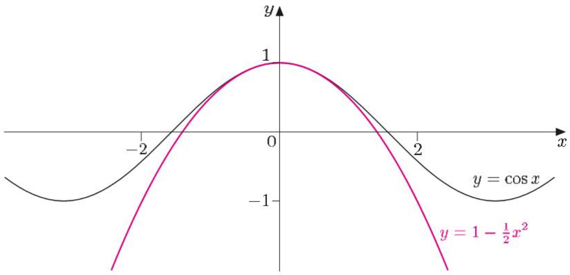

Figure 1.7 Quadratic Taylor polynomial about 0 for $f(x) = \cos x$

Quadratic Taylor polynomials about Zero XIII

We now have linear and quadratic Taylor polynomials about 0 for each of the functions $\exp$, $\sin$ and $\cos$.

Function	Linear taylor polynomial about 0	Quadratic Taylor polynomial about 0
e^x	$1+x$	$1+x+\frac{1}{2}x^2$
$\sin x$	x	x
$\cos x$	1	$1-\frac{1}{2}x^2$

Note: if we take the linear Taylor polynomial and add the appropriate term in x^2 (the term is $0x^2$ in the case of $\sin$) we get the quadratic Taylor polynomial.

This holds for every function f for which these approximations can be found, and it extends to higher degree Taylor polynomials.

Taylor polynomials of degree n about 0 I

From what we've seen, you might guess that for any suitable function f whose domain contains 0, we can obtain more and more accurate approximations for f close to 0 by taking polynomials of higher and higher degree, and choosing the coefficients to ensure that the values of higher and higher derivatives of the polynomial at 0 are the same as those of the corresponding derivatives of f.

Taylor polynomials of degree n about 0 II

For any value of n, it is possible to find a polynomial

$$p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_nx^n$$

whose value is the same as that of f at 0, and whose first, second, third, .. . , nth derivatives have the same values at 0 as the corresponding derivatives of f.

We look at a general function f and general value of n. Thus $f(0), f'(0), f''(0),$ $f^{(3)}(0), \ldots, f^{(n)}(0)$ will not be evaluated. The result will be a general formula for $p(x)$ that we can apply in particular cases.

The values of the constants $a_0, a_1, a_2, \ldots, a_n$ in the polynomial $p(x)$ must be such that the value of p at 0, and the first, second, third, . . . , nth derivative values of p at 0, are the same as those of f.

Taylor polynomials of degree n about 0 III

First we ensure that $p(0) = f(0)$. We have

$$p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n \quad \text{so} \quad p(0) = a_0.$$

Thus we must have $a_0 = f(0)$.

Next we ensure that $p'(0) = f'(0)$. We have

$$p'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \cdots + n a_n x^{n-1}. \quad \text{so} \quad p'(0) = a_1.$$

Thus we must have $a_1 = f'(0)$.

Taylor polynomials of degree n about 0 IV

Then we ensure that $p''(0) = f''(0)$. We have

$$p''(x) = 2a_2 + 3 \times 2a_3x + 4 \times 3a_4x^2 + \cdots + n(n-1)a_nx^{n-2}, \quad \text{so} \quad p''(0) = 2a_2.$$

Thus we must have $2a_2 = f''(0)$; that is, $a_2 = \frac{1}{2!}f''(0)$.

Then we ensure that $p^{(3)}(0) = f^{(3)}(0)$. We have

$$p^{(3)}(x) = 3 \times 2a_3 + 4 \times 3 \times 2a_4x + \cdots + n(n-1)(n-2)a_nx^{n-3}, \quad \text{so} \quad p^{(3)}(0) = 3! \times a_3.$$

(We've written 2 as 2! and $3 \times 2 \times 1$ as $3!$ rather than 6, to highlight the emerging pattern. Recall that in general: $k! = k(k-1) \times \cdots \times 2 \times 1$.)

Thus we must have $3! \times a_3 = f^{(3)}(0)$; that is, $a_3 = \frac{1}{3!}f^{(3)}(0)$.

Taylor polynomials of degree n about 0 V

Continuing in this way, we find that we must have $a_4 = \frac{1}{4!}f^{(4)}(0)$, $a_5 = \frac{1}{5!}f^{(5)}(0)$, and so on, until finally, to ensure that $p^{(n)}(0) = \frac{1}{n!}f^{(n)}(0)$, we must have $a_n = \frac{1}{n!}f^{(n)}(0)$.

The resulting formula for the polynomial $p(x)$ is given below.

Taylor polynomials about 0

Let f be a function that is n-times differentiable at 0. The **Taylor polynomial** of degree n about 0 for f is

$$p(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n$$

From this we can find the Taylor polynomial $p(x)$ of degree n about 0 for any suitable function f, by calculating the value of f at 0, and the values of the first, second, third, ..., nth derivatives of f at 0, and substituting these into the formula.

Taylor polynomials of degree n about 0 VI

All the Taylor polynomials we have seen until now have had degree 1 or above, you can take $n = 0$ in the formula to obtain a Taylor polynomial of degree 0 about 0 for a function f. This is the constant function whose graph is the horizontal line through the point $(0, f(0))$. **Taylor polynomials of degree 0 are called constant Taylor polynomials.**

You have seen the terms “linear” and “quadratic” used to describe Taylor polynomials of degree 1 and degree 2, respectively. The terms cubic, quartic and quintic will be used to refer to Taylor polynomials of degree 3, degree 4 and degree 5, respectively.

Taylor polynomials of degree n about 0 VII

Find the quartic Taylor polynomial about 0 for the function $f(x) = e^x$.

Solution

The $n-th$ derivative of the function $f(x) = e^x$ is $f^{(n)}(x) = e^x$, so $f^{(n)} = e^0 = 1$ for all positive integers n. So the required polynomial is

$$p(x) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4;$$

that is,

$$p(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4;$$

Comment

Since $f^{(n)} = 1$ for all positive integers n, it follows that for any n the Taylor polynomial of degree n about 0 for the exponential function is

$$p(x) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots + \frac{1}{n!}x^n.$$

Taylor polynomials of degree n about 0 VIII

The graphs of the function $f(x) = e^x$ and the quartic Taylor polynomial about 0 found before are shown below. Notice how much better this approximation is than the linear and quadratic ones found before:

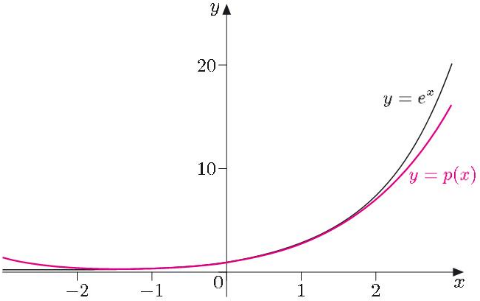

Figure 2.1 Quartic Taylor polynomial about 0 for $f(x) = e^x$

Taylor polynomials of degree n about 0 IX

Problem to do at home:

Find the quartic Taylor polynomials about 0 for each of the following functions.

(a) $f(x) = \cos x$

(b) $f(x) = \sin x$

Compare these to the linear and quadratic approximations found before.

You will find the quartic Taylor polynomial about 0 for the cosine function is

$$p(x) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 \dots$$

You will find the quartic Taylor polynomial about 0 for the sine function is

$$p(x) = x - \frac{1}{6}x^3$$

This quartic Taylor polynomial about 0 for the sine function is an example of a Taylor polynomial of degree n whose polynomial degree is less than n.

Taylor polynomials of degree n about 0 X

The graphs of the cosine and sine functions, and the quartic Taylor polynomials for these functions are shown in Figure 2.2(a) and (b), respectively.

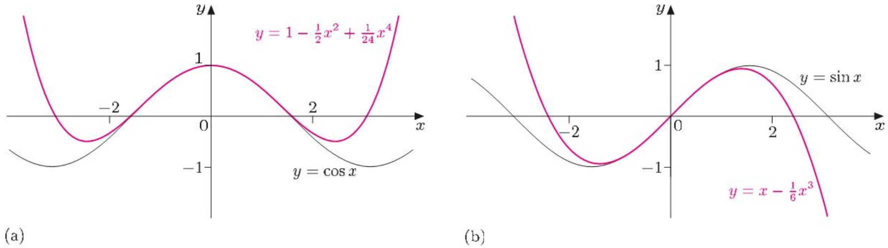

Figure 2.2 Quartic Taylor polynomials about 0 for cos and sin

Taylor polynomials of degree n about 0 XI

You may have noticed that the quartic Taylor polynomial about 0 for the cosine function contains terms in even powers of x only, whereas that for the sine function contains terms in odd powers of x only.

These observations are explained by the facts that $\cos$ is an even function (i.e. $\cos(x) = \cos(-x)$) and $\sin$ is an odd function (i.e. $\sin(-x) = -\sin(x)$).

In fact, any Taylor polynomial about 0 for an even function (where $f(x) = f(-x)$) contains terms in even powers of x only, whereas any Taylor polynomial about 0 for an odd function (where $f(-x) = -f(x)$) contains terms in odd powers of x only.

Taylor polynomials of degree n about 0 XII

To prove the even function result, suppose that f is an even function. Thus the graph of f is unchanged under reflection in the y-axis, or, equivalently,

$$f(-x) = f(x),$$

for all x in the domain of f .

If we differentiate both sides of this equation once, twice, three times, and so on, then we obtain

$$\begin{aligned} -f'(-x) &= f'(x), \\ f''(-x) &= f''(x), \\ -f^{(3)}(-x) &= f^{(3)}(x), \\ f^{(4)}(-x) &= f^{(4)}(x), \end{aligned}$$

and so on.

Taylor polynomials of degree n about 0 XIII

Whenever k is odd we have

$$-f^{(k)}(-x) = f^{(k)}(x),$$

and putting $x = 0$ gives $-f^{(k)}(0) = f^{(k)}(0)$; that is, $f^{(k)}(0) = 0$ (since the LHS is minus the RHS, the only way they can be equal is if the function is zero.

Now the coefficient of x_k in the Taylor series about 0 for f is $\dfrac{f^{(k)}(0)}{k!}$, so this coefficient is zero when k is odd, and hence the Taylor series contains only terms in even powers of x, as claimed.

The result for an odd function can be proved in a similar manner, but the details are omitted here.

Taylor polynomials of degree n about 0 XIV

Find the Taylor polynomial of degree n about 0 for the function

$$f(x) = \frac{1}{1-x}.$$

Hint: You need to differentiate $f(x)$ once, twice, three times, ... ,until a pattern is clear and you can write down a formula for the nth derivative.

To find the Taylor polynomial of degree n about 0 for $f(x) = \frac{1}{(1-x)}$ we evaluate $f(0), f'(0), f''(0), \ldots, f^{(n)}(0)$. The pattern is as follows:

Taylor polynomials of degree n about 0 XV

$$
\begin{aligned}
f(x) &= \frac{1}{1-x}, & f(0) &= 1;\\
f'(x) &= \frac{1}{(1-x)^2}, & f'(0) &= 1;\\
f''(x) &= \frac{2}{(1-x)^3}, & f''(0) &= 2;\\
f^{(3)}(x) = \frac{3\times 2}{(1-x)^4} &= \frac{3!}{(1-x)^4}, & f^{(3)}(0) &= 3!;\\
f^{(4)}(x) = \frac{4\times 3\times 2}{(1-x)^5} &= \frac{4!}{(1-x)^5}, & f^{(4)}(0) &= 4!;
\end{aligned}
$$

and so on, terminating with

$$
f^{(n)}(x) = \frac{n!}{(1-x)^{n+1}}, \qquad f^{(n)}(0) = n!.
$$

Taylor polynomials of degree n about 0 XVI

Hence the Taylor polynomial of degree n about 0 for $f(x) = \frac{1}{(1-x)}$ is

$$
\begin{aligned}
p(x) &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n \\
&= 1 + x + \frac{2}{2!}x^2 + \frac{3!}{3!}x^3 + \cdots + \frac{n!}{n!}x^n \\
&= 1 + x + x^2 + x^3 + \cdots + x^n.
\end{aligned}
$$

Taylor polynomials of degree n about 0 XVII

The graphs of the function $f(x) = \dfrac{1}{(1-x)}$ and the Taylor polynomial found above, in the case $n = 3$, are shown below

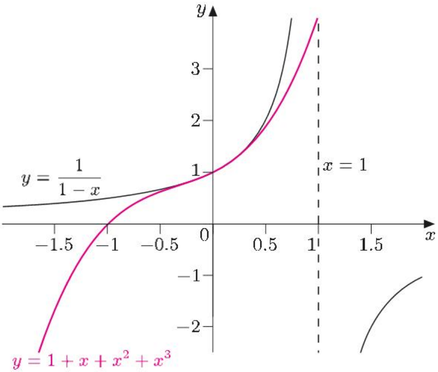

Figure 2.3 Cubic Taylor polynomial about 0 for $f(x) = 1/(1-x)$

Taylor polynomials of degree n about a I

It's possible to generalise the derivation of the formula for a Taylor polynomial about 0 to obtain a formula for a polynomial that approximates a suitable function f close to any chosen point a in its domain. The required formula is given below.

Taylor polynomials about a

Let f be a function that is n-times differentiable at a.The **Taylor polynomial** of degree n about a for f is

$$p(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Taylor polynomials of degree n about a II

Some points to note:

- As in the case $a = 0$, the formula shows that any Taylor polynomial about a point a for a function f can be obtained from a Taylor polynomial of lower degree about a for f by adding the appropriate further terms.
- Also, again as in the case $a = 0$, it is possible for the Taylor polynomial of degree n about a for f to be a polynomial of degree less than n; this happens when $f^{(n)}(a) = 0$.
- If p is a Taylor polynomial for a function f about a point a in its domain, and x is a point close to a, then $p(x)$ is an approximation for $f(x)$. Usually, the greater the degree of the Taylor polynomial, and the closer x is to a, the more accurate the approximation.

Taylor polynomials of degree n about a III

Find the quartic Taylor polynomial about 1 for the function $f(x) = \ln x$.

The first four derivatives of the function $f(x) = \ln x$ and their values at $x = 1$ are as follows:

$$\begin{aligned} f'(x) &= \frac{1}{x} & f'(1) &= 1, \\ f''(x) &= -\frac{1}{x^2} & f''(1) &= -1, \\ f^{(3)}(x) &= \frac{2}{x^3} & f^{(3)}(1) &= 2, \\ f^{(4)}(x) = -\frac{3 \times 2}{x^4} &= -\frac{3!}{x^4} & f^{(4)}(1) &= -3!. \end{aligned}$$

So, from the formula , the quartic Taylor polynomial about 1 for the function $f(x) = \ln x$ is

$$0 + 1 \times (x-1) + \frac{(-1)}{2!}(x-1)^2 + \frac{2}{3!}(x-1)^3 + \frac{(-3!)}{4!}(x-1)^4;$$

that is,

$$(x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4.$$

Taylor polynomials of degree n about a IV

$$(x-1)-\frac{1}{2}(x-1)^2+\frac{1}{3}(x-1)^3-\frac{1}{4}(x-1)^4.$$

Comment

If you look at the patterns in $f(x), f'(x), f''(x), f^{(3)}(x), \ldots$ in the solution, then you will see that the pattern of terms continues as the degree of the Taylor polynomial increases. Thus the Taylor polynomial of degree n about 1 for $f(x) = \ln x$ is

$$(x-1)-\frac{1}{2}(x-1)^2+\frac{1}{3}(x-1)^3-\cdots+(-1)^{(n-1)}\frac{1}{n}(x-1)^n.$$

(Note: The expression $(-1)^{(n-1}$ in the final term is just a neat way to give the term a negative sign when n is even and a positive sign when n is odd.)

Taylor polynomials of degree n about a V

The graph shows the function $f(x) = \ln x$ and the quartic Taylor polynomial about 1 found above.

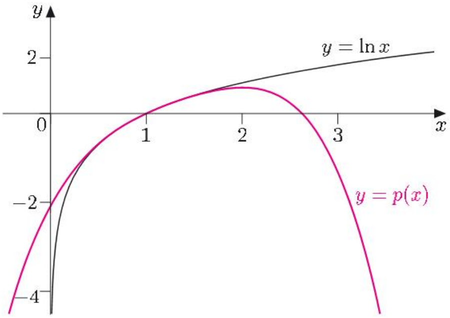

Figure 2.4 Quartic Taylor polynomial about 1 for $f(x) = \ln x$

Taylor polynomials of degree n about a VI

Find the cubic Taylor polynomial about $\frac{1}{6}\pi$ for the function $f(x) = \sin x$.

We must evaluate $f(\frac{1}{6}\pi), f'(\frac{1}{6}\pi), f''(\frac{1}{6}\pi)$ and $f^{(3)}(\frac{1}{6}\pi)$ as follows:

$$
\begin{aligned}
f(x) &= \sin x, & f(\frac{1}{6}\pi) &= \frac{1}{2}, \\
f'(x) &= \cos x, & f'(\frac{1}{6}\pi) &= \frac{\sqrt{3}}{2}, \\
f''(x) &= -\sin x, & f''(\frac{1}{6}\pi) &= -\frac{1}{2}, \\
f^{(3)}(x) &= -\cos x, & f^{(3)}(\frac{1}{6}\pi) &= -\frac{\sqrt{3}}{2},
\end{aligned}
$$

Hence the cubic Taylor polynomial about $\frac{1}{6}\pi$ for the function $f(x) = \sin x$ is

$$
\begin{aligned}
p(x) &= f(\frac{1}{6}\pi) + f'(\frac{1}{6}\pi)(x - \frac{1}{6}\pi) + \frac{f''(\frac{1}{6}\pi)}{2!}(x - \frac{1}{6}\pi)^2 + \frac{f^{(3)}(\frac{1}{6}\pi)}{3!}(x - \frac{1}{6}\pi)^3 \\
&= \frac{1}{2} + \frac{\sqrt{3}}{2}(x - \frac{1}{6}\pi) - \frac{1}{4}(x - \frac{1}{6}\pi)^2 - \frac{\sqrt{3}}{12}(x - \frac{1}{6}\pi)^3.
\end{aligned}
$$

Taylor polynomials of degree n about a VII

Comment

The Taylor polynomial about $\frac{1}{6}\pi$ for the function $f(x) = \sin x$ contains terms in $(x - \frac{1}{6}\pi)^k$ with k even as well as with k odd.

This is not surprising; the previous discussion about even and odd functions applies to Taylor polynomials about 0 only.

Taylor polynomials of degree n about a VIII

The graph shows the function $f(x) = \sin x$ and the cubic Taylor polynomial about n found.

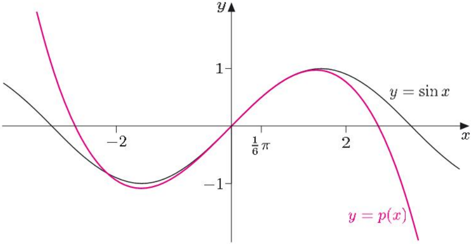

Figure 2.5 Cubic Taylor polynomial about $\frac{1}{6}\pi$ for $f(x) = \sin x$

Taylor polynomials of degree n about a IX

Sigma notation

Sigma notation provides a concise way to write down Taylor polynomials. Our formula for the Taylor polynomial of degree n about 0 for a function f is

$$p(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n.$$

In sigma notation this polynomial is

$$p(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}x^k.$$

For example, the quartic Taylor polynomial about 0 for the function $f(x) = e^x$, we found before is

$$p(x) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 = \sum_{k=0}^{4} \frac{1}{k!}x^k.$$

(Recall that $0! = 1$. Also note that $f^{(0)}$ is interpreted to mean f itself, and that by convention $0^0 = 1$ in series of this type.)

Using Taylor polynomials for approximation I

First some notation:

Write $p_n(x)$, for the Taylor polynomial of degree n e.g. the Taylor polynomials of degrees 1 and 2 about 0 for the function $f(x) = e^x$ are $p_1(x) = 1 + x$ and $p_2(x) = 1 + x + \frac{1}{2}x^2$, respectively.

The assciated remainder functions are written as e.g. $r_1(x) = f(x) - p_1(x)$ and $r_2(x) = f(x) - p_2(x)$.

Using Taylor polynomials for approximation II

Graphs of the Taylor polynomials of degrees 1, 2, 3 and 4 about 0 for the function $f(x) = e^x$, together with the graph of $f(x) = e^x$ itself. As expected, it appears that as the degree of the Taylor polynomial increases, its graph approximates the graph of f near 0 more and more closely.

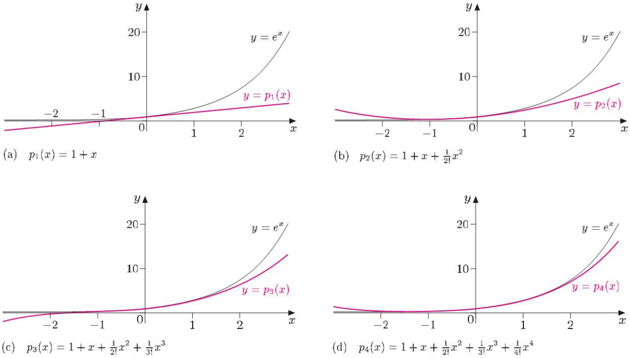

Figure 2.6 Taylor polynomials about 0 for $f(x) = e^x$

Using Taylor polynomials for approximation III

Consider, $f(x) = e^x$, for a particular value of x near 0, namely $x = 0.25$.
$e^{0.25} = 1.2840254167$ to ten decimal places.
As the degree n of the Taylor polynomial increases, the accuracy of $p_n(0.25)$ as an approximation for $e^{0.25}$ improves.

n	$p_n(x)$	$p_n(0.25)$	$r_n(0.25) = e^{0.25} - p_n(0.25)$
1	$1 + x$	1.25	0.034 025 4167
2	$1 + x + x^2/2!$	1.281 25	0.002 775 4167
3	$1 + x + x^2/2! + x^3/3!$	1.283 854 1667	0.000 171 2500
4	$1 + x + x^2/2! + \ldots + x^4/4!$	1.284 016 9271	0.000 008 4896
5	$1 + x + x^2/2! + \ldots + x^5/5!$	1.284 025 0651	0.000 000 3516
6	$1 + x + x^2/2! + \ldots + x^6/6!$	1.284 025 4042	0.000 000 0125
7	$1 + x + x^2/2! + \ldots + x^7/7!$	1.284 025 4163	0.000 000 0004
8	$1 + x + x^2/2! + \ldots + x^8/8!$	1.284 025 4167	0.000 000 0000

Using Taylor polynomials for approximation IV

Taylor polynomials can be used to calculate approximations for values of functions to any desired accuracy.

Problem: There is no easy method for determining a suitable degree for the Taylor polynomial in any given case.

“Rule of thumb” : To approximate to m decimal places, then calculate approximations using Taylor polynomials of degree 1, 2, 3, and so on, until two successive approximations agree to $m + 2$ decimal places.

Note: Two numbers agree to a given number of decimal places, if the values resulting from rounding them to that number of decimal places are equal e.g. 0.237 and 0.241 agree to two decimal places, since in each case rounding to two decimal places gives 0.24. Similarly, 0.241 and 0.247 do not agree to two decimal places, since rounding to two decimal places gives 0.24 in the first case and 0.25 in the second.

Using Taylor polynomials for approximation V

Example

Use Taylor polynomials about 1 to evaluate $\ln(1.1)$ to four decimal places.

Solution

We noted that the Taylor polynomial of degree n about 1 for $f(x) = \ln x$ is

$$p_n(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \cdots (-1)^{n-1}\frac{1}{n}(x-1)^n.$$

Using Taylor polynomials for approximation VI

Using this fact, and calculating values to six decimal places, we obtain

$$\begin{aligned}
p_1(1.1) &= 1.1 - 1 = 0.1, \\
p_2(1.1) &= p_1(1.1) - \frac{1}{2}(1.1-1)^2 = 0.095, \\
p_3(1.1) &= p_2(1.1) + \frac{1}{3}(1.1-1)^3 = 0.095333, \\
p_4(1.1) &= p_3(1.1) - \frac{1}{4}(1.1-1)^4 = 0.095308, \\
p_5(1.1) &= p_4(1.1) + \frac{1}{5}(1.1-1)^5 = 0.095310, \\
p_6(1.1) &= p_5(1.1) - \frac{1}{6}(1.1-1)^6 = 0.095310.
\end{aligned}$$

Using Taylor polynomials for approximation VII

The values of $p_5(1.1)$ and $p_6(1.1)$ agree to six decimal places, so it is likely that

$$\ln(1.1) = 0.0953,$$

to four decimal places. (This is the case.)

We calculate values to six decimal places because we wish to find a pair of successive values that agree to 4 + 2 = 6 decimal places.

Using Taylor polynomials for approximation VIII

The next example involves Taylor polynomials about 0 for the sine function.

Since in this case each Taylor polynomial of even degree is identical to the Taylor polynomial of degree one less (that is, $p_2(x) = p_1(x), p_4(x) = p_3(x)$, and so on), we would quickly find two successive approximations that agree to any specified number of decimal places, but this would tell us nothing about the accuracy of the approximation! For this reason we consider only the Taylor polynomials of odd degree.

(This is because $\sin$ is an odd function as we discussed before)

Using Taylor polynomials for approximation IX

Example: Use Taylor polynomials about 0 to find the value of $\sin(0.2)$ to six decimal places.

Solution

We have found the quartic Taylor polynomial about 0 for the sine function is

$$p_4(x) = x - \frac{1}{3!}x^3.$$

We can use the formula to calculate further terms to obtain Taylor polynomials of higher degree. For example, the Taylor polynomial of degree 7 about 0 for the sine function is:

$$p_7(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7.$$

Using Taylor polynomials for approximation X

Using this fact, and calculating values to eight decimal places, we obtain

$$\begin{aligned}
p_1(0.2) &= 0.2, \\
p_3(0.2) &= p_1(0.2) - \frac{1}{3!}(0.2)^3 = 0.19866667, \\
p_5(0.2) &= p_3(0.2) + \frac{1}{5!}(0.2)^5 = 0.19866933, \\
p_7(0.2) &= p_5(0.2) - \frac{1}{7!}(0.2)^7 = 0.19866933.
\end{aligned}$$

The values of $p_5(0.2)$ and $p_7(0.2)$ agree to eight decimal places, so it is likely that

$$\sin(0.2) = 0.198669,$$

to six decimal places. (This is the case.)

(Note: We calculate values to 6 + 2 = 8 decimal places.)

Using Taylor polynomials for approximation XI

Activity: Finding approximate values of functions

Use Taylor polynomials about 0 to find

(a) the value of $e^{0.05}$ to four decimal places;

(b) the value of $\cos(0.2)$ to six decimal places.

Accuracy of approximations from Taylor polynomials I

Recall: Let f be a function that is n-times differentiable at a.

The **Taylor polynomial** of degree n about a for f is

$$p(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Taylor's Theorem tells us about the accuracy of approximations found by evaluating Taylor polynomials:

Let f be a function that is $(n+1)$-times differentiable on an open interval I containing the point a. Let p_n be the Taylor polynomial of degree n about a for f , and let r_n be the associated remainder function; that is, $r_n(x) = f(x) - p_n(x)$. Then, for each number x in the interval I , we have

$$r_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1},$$

for some number c between a and x.

Accuracy of approximations from Taylor polynomials II

Note: the formula for $r_n(x)$ is just the formula for the term in $(x-a)^{n+1}$ in the Taylor polynomial of degree $n+1$ about a for f (that is, the "next term"), but with a replaced by c in the expression for the coefficient of $(x-a)^{n+1}$.

What about the number c?

c depends on x (as well as on f and n). In general different values of c are needed for different values of x.

Since we do not know which value of c is required for any given value of x, we cannot use the formula directly. However we can often use it to show that the magnitude of $r_n(x)$ is less than a certain value, and this tells us about the accuracy of the approximation provided by the Taylor polynomial $p_n(x)$.

Accuracy of approximations from Taylor polynomials III

Consider the Taylor polynomial of degree 4 about 0 for the function $f(x) = \sin x$, which is $x - \frac{1}{6}x^3$. The formula for the remainder function in the case of this function and its quartic Taylor polynomial is

$$r_4(x) = \frac{f^{(5)}(c)}{5!} \times (x-0)^5 = \frac{x^5 \cos c}{120},$$

for some value of c between 0 and x.

To obtain an approximation for $\sin(0.3)$, we take $x = 0.3$ in the Taylor polynomial:

$$\sin(0.3) \approx 0.3 - \frac{1}{6}(0.3)^3 = 0.2955.$$

Accuracy of approximations from Taylor polynomials IV

Taking $x = 0.3$ in the formula for the remainder shows that the difference between the value of $\sin(0.3)$ and this approximation is

$$r_4(0.3) = \frac{(0.3)^5 \cos c}{120} = 0.00002025 \cos c,$$

for some value of c between 0 and 0.3. We do not know the exact value of c, but we do know that $|\cos c| \leqslant 1$, so certainly

$$|r_4(0.3)| = 0.00002025 |\cos c| \leqslant 0.00002025,$$

which tells us that the approximation 0.2955 for $\sin(0.3)$ is accurate to at least four decimal places.

Taylor series I

We have seen that it is usually the case that the greater the degree of a Taylor polynomial about a point a for a function f , the more accurate the Taylor polynomial is as an approximation for f close to a.

But what happens if we take a Taylor polynomial of “infinite degree”; that is, if we add on all possible terms? This question is considered next.

What is a Taylor series? I

Taylor polynomial of degree n about 0 for the function $f(x) = \frac{1}{(1-x)}$ is

$$p_n(x) = 1 + x + x^2 + x^3 + \cdots + x^n.$$

Let us now consider the infinite series obtained by letting n tend to infinity, which is

$$1 + x + x^2 + x^3 + \cdots .$$

If an infinite series has the property that, as we add on more and more terms, the sequence of sums obtained settles down in the long term to a particular value s, then we say that s is the sum of the series.

We know that as n becomes large, $p_n(x)$ settles down to $\frac{1}{(1-x)}$, at least for values of x near 0. That is, we expect the sum of the infinite series to be $\frac{1}{(1-x)}$ for such x. But for precisely which values of x does this happen?

What is a Taylor series? II

The answer is that infinite series sums to $\frac{1}{(1-x)}$ for all values of x in the range $-1 < x < 1$. For example, with $x = 0.5$ the series

$$1 + 0.5 + (0.5)^2 + (0.5)^3 + \cdots = 1 + 0.5 + 0.25 + 0.125 + \cdots$$

has sum

$$\frac{1}{1 - 0.5} = 2.$$

For values of x outside the range $-1 < x < 1$, the series does not sum to $\frac{1}{(1-x)}$; in fact, it does not have a finite sum for such x.

For example, if $x = 2$ then the function $f(x) = \frac{1}{(1-x)}$ has value -1, but the series is

$$1 + 2 + 2^2 + 2^3 + 2^4 + \cdots = 1 + 2 + 4 + 8 + 16 + \cdots ,$$

which does not sum to any real number, since as more and more terms are added on, the sum increases without limit.

What is a Taylor series? III

We can confirm that our series sums to $\frac{1}{(1-x)}$ for all x in the range $-1 < x < 1$ by noting that it is an infinite geometric series; that is, it is of the form $a + ar + ar^2 + ar^3 + \cdots$.

By a standard result, such a series has sum $\frac{a}{(1-r)}$, provided that $|r| < 1$.
$1 + x + x^2 + x^3 + \cdots$ is a series of this form with $a = 1$ and $r = x$, so it sums to $\frac{1}{(1-x)}$, provided that $|x| < 1$; that is, provided that $-1 < x < 1$, as claimed above.

What is a Taylor series? IV

If f is a function that is differentiable infinitely many times at a point a in its domain, then the infinite series obtained by letting the degree n of a Taylor polynomial about a for f tend to infinity is called the **Taylor series** about a for f.

Taylor series about a

Let f be a function that is differentiable infinitely many times at a. The **Taylor series** about a for f is

$$f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\frac{f^{(3)}(a)}{3!}(x-a)^3+\cdots+\frac{f^{(k)}(a)}{k!}(x-a)^k+\cdots,$$

The point a is called the centre of the Taylor series.

If $a=0$, then the Taylor series reduces to

$$f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f^{(3)}(0)}{3!}x^3+\cdots+\frac{f^{(k)}(0)}{k!}x^k+\cdots,$$

What is a Taylor series? V

If x is a point for which the Taylor series in the formula sums to $f(x)$, then we say that the Taylor series is valid for the point x. For example, we saw that the Taylor series about 0 for the function $f(x) = \frac{1}{(1-x)}$ is valid for all x in the range $-1 < x < 1$, but not for values of x outside this range.

Any range of values of x for which a Taylor series is valid is called a range of validity for the series, and the series is said to represent the function on any range of validity.

Thus $-1 < x < 1$ is a range of validity for the Taylor series about 0 for the function $f(x) = \frac{1}{(1-x)}$. Hence $0 < x < 1$, for example, is also a range of validity for this series since each such value of x lies in the range $-1 < x < 1$; but $0 < x < 3$ is not, since it includes values outside the range $-1 < x < 1$.

What is a Taylor series? VI

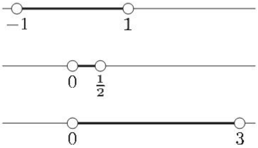

Figure 3.1 Intervals

A range of validity is a subset of the set $\mathbf{R}$ of real numbers, so it should really be denoted using appropriate set notation. For example, the range of validity $-1 < x < 1$ could be denoted by $(-1, 1)$ using the usual notation for an open interval. However, in practice the double inequality notation is convenient, and we will use that instead.

What is a Taylor series? VII

A Taylor series about a is always valid for $x = a$. This is because if we set $x = a$ in the Taylor Series formula, then all the terms except the first are equal to zero, so the sum of the series is just the first term $f(a)$, which is precisely the value of f at a.

This is by design, since the first term of a Taylor polynomial is chosen to be $f(a)$ to ensure that the value of the polynomial at a is the same as that of f.

What is a Taylor series? VIII

Example: A Taylor series for exp

Find the Taylor series about 0 for the function $f(x) = e^x$.

Solution

The nth derivative of the function $f(x) = e^x$ is $f^{(n)}(x) = e^x$, so $f^{(n)}(0) = 1$ for all positive integers n. Hence, by using our formula, the required Taylor series is

$$1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \ldots,$$

What is a Taylor series? IX

Activity: Finding Taylor series

Find the Taylor series about 0 for each of the following functions, writing down enough terms to make the general pattern clear.

(a) $f(x) = \cos x$,

(b) $f(x) = \sin x$

In each case you should be able to see the pattern in the values $f(0), f'(0), f''(0), f^{(3)}(0), \ldots$ from when we looked at $\cos$ and $\sin$ before. You are not expected to find ranges of validity for these series.

What is a Taylor series? X

As before, you will find that the Taylor series about 0 for the even function $\cos$ contains terms in even powers of x only, while the Taylor series for the odd function $\sin$ contains terms in odd powers of x only.

You may be surprised to learn that the Taylor series about 0 for the exponential, sine and cosine functions are all valid for every real number x. In other words, the equations

$$\begin{aligned} e^x &= 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \dots \\ \sin x &= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 + \dots \\ \cos x &= 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \dots, \end{aligned}$$

are true for all $x \in \mathbf{R}$.

When you remember that the coefficients of a Taylor series for a function are chosen by taking into account the value of the function and its derivatives at a single point a, it may seem amazing that the resulting series can turn out to be equal to the function for every real number x.

What is a Taylor series? XI

All the Taylor series so far have had centre 0. Let's find a Taylor series with a different centre:

Use the Taylor Series formula to find the Taylor series about 1 for the function $f(x) = \sqrt{x}$, writing down enough terms to make the general pattern clear. You are not expected to find a range of validity for this series.

What is a Taylor series? XII

To find the Taylor series about 1 for the function $f(x) = \sqrt{x} = x^{\frac{1}{2}}$, we evaluate $f(1), f'(1), f''(1), f^{(3)}(1)$, and so on. We have

$$
\begin{aligned}
f(x) &= x^{\frac{1}{2}}, & f(1) &= 1;\\
f'(x) &= \frac{1}{2}x^{-\frac{1}{2}}, & f'(1) &= \frac{1}{2};\\
f''(x) &= -\frac{1}{2} \times \frac{1}{2}x^{-\frac{3}{2}}, & f''(1) &= -\frac{1}{2} \times \frac{1}{2};\\
f^{(3)}(x) &= \frac{3}{2} \times \frac{1}{2} \times \frac{1}{2}x^{-\frac{5}{2}}, & f^{(3)}(1) &= \frac{3}{2} \times \frac{1}{2} \times \frac{1}{2};\\
f^{(4)}(x) &= -\frac{5}{2} \times \frac{3}{2} \times \frac{1}{2} \times \frac{1}{2}x^{-\frac{7}{2}}, & f^{(4)}(1) &= -\frac{5}{2} \times \frac{3}{2} \times \frac{1}{2} \times \frac{1}{2};\\
f^{(5)}(x) &= \frac{7}{2} \times \frac{5}{2} \times \frac{3}{2} \times \frac{1}{2} \times \frac{1}{2}x^{-\frac{9}{2}}, & f^{(4)}(1) &= \frac{7}{2} \times \frac{5}{2} \times \frac{3}{2} \times \frac{1}{2} \times \frac{1}{2};
\end{aligned}
$$

What is a Taylor series? XIII

Hence the Taylor series about 1 for the function $f(x) = \sqrt{x}$ is

$$\begin{aligned} & f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f^{(3)}(1)}{3!}(x-1)^3 + \ldots \\ = \quad & 1 + \frac{1}{2}(x-1) - \frac{1}{2^2 2!}(x-1)^2 + \frac{3 \times 1}{2^3 3!}(x-1)^3 - \frac{5 \times 3 \times 1}{2^4 4!}(x-1)^4 + \frac{7 \times 5 \times 3 \times 1}{2^5 5!}(x - \end{aligned}$$

The general pattern of terms is clear.

Comment

This Taylor series is valid for $0 < x < 2$. The function $f(x) = \sqrt{x}$ has no Taylor series about 0, since it is not differentiable at 0. (Its derived function $f'(x) = \frac{1}{2}x^{-1/2}$ is not defined at 0.)

What is a Taylor series? XIV

Sigma notation

The formula for the Taylor series about a for a function f can be written concisely in sigma notation as

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

For example, the Taylor series about 0 for the function $f(x) = e^x$ is

$$\sum_{k=0}^{\infty} \frac{1}{k!}x^k.$$

In the cases of odd and even functions it is more awkward to write down the Taylor series about 0 in sigma notation, because for even functions such series contain only even powers of x and for odd functions only odd powers are involved. For example, the Taylor series for the function $f(x) = \sin x$, given above, is usually written as

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}x^{2k+1}.$$

What is a Taylor series? XV

For $k = 0, 1, 2, \ldots$, the expression x^{2k+1} equals $x, x^3, x^5, \ldots$, and the expression $(-1)^k$ looks after the alternating signs. The Taylor series for the function $f(x) = \cos x$ is usually written as

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}.$$

For $k = 0, 1, 2, \ldots$, the expression x^{2k} equals $1, x^2, x^4, \ldots$

Taylor series about 0 for other odd and even functions can be written in a similar way.

Some standard Taylor series I

The Taylor series from some standard functions are given below. The ranges of validity for the Taylor series are given (but the proof of these ranges is beyond the scope of this course).

Standard Taylor series about 0

$$
\begin{aligned}
\sin x &= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 + \ldots, \quad \text{for } x \in \mathbf{R} \\
\cos x &= 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \ldots, \quad \text{for } x \in \mathbf{R} \\
e^x &= 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \ldots, \quad \text{for } x \in \mathbf{R} \\
\ln(1+x) &= x - \frac{1}{2}x^2 + \frac{1}{3!}x^3 - \frac{1}{4!}x^4 + \ldots, \quad \text{for } -1 < x < 1 \\
\frac{1}{1-x} &= 1 + x + x^2 + x^3 + x^4 + \ldots, \quad \text{for } -1 < x < 1 \\
(1+x)^\alpha &= 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \ldots, \\
&\qquad \text{for } -1 < x < 1 \text{and for any} \alpha \in \mathbf{R}.
\end{aligned}
$$

Some standard Taylor series II

- The last series given is the binomial series.
- Each of the ranges of validity given in the box is an open interval that is symmetric about the centre 0 of the series.
- These ranges of validity are the 'maximum' ranges for which the series are valid, with two exceptions. The series for $\ln(1+x)$ is also valid when $x = 1$, but the box gives the range $-1 < x < 1$ because it is often convenient to work with ranges that are open intervals. The other exception involves the binomial series. For $-1 < x < 1$, this series sums to $(1+x)^\alpha$ for any real number α, including negative and fractional numbers. For most values of α, the maximum range of validity is $-1 < x < 1$, but when α is a non-negative integer, the series is valid for every real number x. We will see why this is so later on.
- We give the Taylor series about 0 for the function $\ln(1+x)$, rather than for the standard function $\ln x$. This is because the function $\ln x$ has no Taylor series about 0, since its domain does not include 0.

Some standard Taylor series III

Example: Finding binomial series

Use the binomial series to find the Taylor series about 0 for each of the following functions. In each case state a range of validity for the series.

(a) $f(x) = \dfrac{1}{(1+x)}$

(b) $f(x) = (1+x)^4$

Solution

(a) Since $\frac{1}{(1+x)} = (1+x)^{-1}$, we can take $\alpha = -1$ in the binomial series to give

$$\begin{aligned} \frac{1}{(1+x)} &= 1 + (-1)x + \frac{(-1)(-2)}{2!}x^2 + \frac{(-1)(-2)(-3)}{3!}x^3 + \ldots \\ &= 1 - x + x^2 - x^3 + x^4 - \ldots. \end{aligned}$$

This Taylor series is valid for $-1 < x < 1$.

Some standard Taylor series IV

(b) Taking $\alpha = 4$ in the binomal series, we obtain

$$\begin{aligned}
(1+x)^4 &= 1 + 4x + \frac{4\times 3}{2!}x^2 + \frac{4\times 3\times 2}{3!}x^3 + \frac{4\times 3\times 2\times 1}{4!}x^4 \\
&\quad + \frac{4\times 3\times 2\times 1\times 0}{5!}x^5 + \frac{4\times 3\times 2\times 1\times 0\times(-1)}{6!}x^6 + \ldots \\
&= 1 + 4x + 6x^2 + 4x^3 + x^4.
\end{aligned}$$

This Taylor series is valid for $-1 < x < 1$. (In fact it is valid for $x \in \mathbf{R}$ which would be an equally appropriate answer.)
This shows that it is possible for a Taylor series to have a finite number of terms. This occurs when the coefficients of all terms from some term onwards are zero.

The series in this example probably also looks rather familiar to you: it is the binomial expansion of $(1+x)^4$.

Some standard Taylor series V

If α is any positive integer, then all terms after the term in x^α in the binomial series for $(1+x)^\alpha$ contain the factor $\alpha - \alpha = 0$ and are therefore equal to zero. The series is then the same as the binomial expansion of $(1+x)^\alpha$, which is valid for all $x \in \mathbf{R}$. The binomial series therefore generalises the binomial expansion of $(1+x)^\alpha$ from cases where α is a positive integer to cases where α can be any real number.

Some standard Taylor series VI

Activity: Finding a binomial series

Use the binomial series to find the Taylor series about 0 for the function $f(x) = (1+x)^{\frac{1}{2}}$. (Write down enough terms to make the general pattern clear.) State a range of validity for the series.

Taking $\alpha = \frac{1}{2}$ in the binomial series gives

$$\begin{aligned} & 1 + \frac{1}{2}x + \frac{\frac{1}{2}\left(\frac{1}{2}\right)}{2!}x^2 + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}x^3 + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{4!}x^4 + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(-\frac{7}{2}\right)}{5!}x^5 + \dots \\ = \; & 1 + \frac{1}{2}x - \frac{1}{2^2 2!}x^2 + \frac{1 \times 3}{2^3 3!}x^3 + \frac{1 \times 3 \times 5}{2^4 4!}x^4 + \frac{1 \times 3 \times 5 \times 7}{2^5 5!}x^5 + \dots \end{aligned}$$

The series is valid for $-1 < x < 1$.

(Have a look back at the series we found about 1 for $\sqrt{x}$. You will see that it can be obtained from the above series about 0 for $\sqrt{1+x}$ by replacing each occurence of x by $x - 1$.)

Some standard Taylor series VII

Once a Taylor series for a function f is known, we can find the corresponding Taylor polynomial of any degree n by truncating the series at an appropriate term. (To truncate a series at a term is to delete all subsequent terms.)

Some standard Taylor series VIII

Activity Taylor polynomials from Taylor series

Using the Taylor series about 0 for the function $f(x) = \ln(1+x)$, write down the cubic Taylor polynomial about 0 for f.

The cubic Taylor polynomial about 0 for the function $f(x) = \ln(1+x)$ is obtained from the Taylor series for $\ln(1+x)$ by deleting all the terms after $\frac{1}{3}x^3$ to give

$$x - \frac{1}{2}x^2 + \frac{1}{3}x^3.$$

Some standard Taylor series IX

Taylor polynomials obtained by truncating a Taylor series for a function f can, in principle, be used to find approximations for $f(x)$ for all values of x for which the series is valid.

In general, the further x is from the centre a of the series, the greater the degree of the Taylor polynomial needed to provide a particular level of accuracy, so the method is usually not practicable for distant values of x.

For example, you have seen that the Taylor series about 0 for the function $f(x) = \ln(1 + x)$ is valid for all values of x in the range $-1 < x < 1$. This means that, in principle, Taylor polynomials about 0 can be used to find an approximation for $\ln(1 + x)$ for any value of x in the interval $(-1, 1)$.
(Thus such polynomials can be used to find an approximation for $\ln x$ for any value of x in the interval $(0, 2)$.)

Some standard Taylor series X

Activity: Finding an approximate value for a function

By writing 1.1 as $1+0.1$, use the series found for $(1+x)^{\frac{1}{2}}$ to find a value for $\sqrt{1.1}$ to three decimal places. (Notice that $x = 0.1$ lies within the range of validity $-1 < x < 1$ for

the Taylor series about 0 for the function $f(x) = (1+x)^{\frac{1}{2}}$.
We use the series we found before for $f(x) = (1+x)^{\frac{1}{2}}$ i.e.

$$p(x) = 1 + \frac{1}{2}x - \frac{1}{2^2 2!}x^2 + \frac{1\times 3}{2^3 3!}x^3 + \frac{1\times 3\times 5}{2^4 4!}x^4 + \frac{1\times 3\times 5\times 7}{2^5 5!}x^5 + \dots$$

We obtain to five decimal places

$$\begin{aligned}
p_1(0.1) &= 1 + \frac{1}{2}(0.1) = 1.05, \\
p_2(0.1) &= p_1(0.1) - \frac{1}{2^2 2!}(0.1)^2 = 1.04875, \\
p_3(0.1) &= p_2(0.1) + \frac{1\times 3}{2^3 3!}(0.1)^3 = 1.04881,
\end{aligned}$$

Some standard Taylor series XI

The values of $p_3(0.1)$ and $p_4(0.1)$ agree to five decimal places, so it is likely that

$$\sqrt{1.1} = 1.049,$$

to three decimal places. (This is the case.)

Some standard Taylor series XII

All the standard Taylor series given have centre 0.

Taylor series (and Taylor polynomials) about 0 are often called Maclaurin series (and Maclaurin polynomials).

For example, rather than saying that $x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \dots$ is the Taylor series about 0 for the function $f(x) = \ln(1 + x)$, we can say that it is the Maclaurin series for the function $f(x) = \ln(1 + x)$, from which it is understood that the centre is 0.

Manipulating Taylor series I

Let’s look at some methods that allow us to obtain Taylor series for many functions from a few known Taylor series such as the standard ones mentioned before.

This usually involves much less work than obtaining the required Taylor series directly, using the given formulae.

When finding a Taylor series for a function, you may make use of any of the standard Taylor series. You are not expected to derive any of the standard series unless explicitly asked to do so.

Substituting for the variable in a Taylor series I

You have seen that the Taylor series about 0 for the function$g(x) = \frac{1}{(1-x)}$ is given by

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots, \qquad \text{for } -1 < x < 1$$

Let us consider the effect of substituting $x = 2t$ in this equation. We obtain

$$\frac{1}{1-2t} = 1 + 2t + (2t)^2 + (2t)^3 + \cdots = 1 + 2t + 4t^2 + 8t^3 + \dots$$

We have obtained a series equal to $\frac{1}{(1-2t)}$. Since the Taylor series for $g(x) = \frac{1}{(1-x)}$ is valid for $-1 < x < 1$, the above series in t is equal to $\frac{1}{(1-2t)}$ for $-1 < 2t < 1$, that is, for $-\frac{1}{2} < t < \frac{1}{2}$. We now replace t by x (since it is more usual to use x rather than t for the variable) to give

$$\frac{1}{1-2x} = 1 + 2x + 4x^2 + 8x^3 + \dots \text{for } -\frac{1}{2} < x < \frac{1}{2}.$$

The series in the above equation is equal to the function $f(x) = \frac{1}{(1-2x)}$ for $-\frac{1}{2} < t < \frac{1}{2}$, but is it a Taylor series? It is of the right form to be a Taylor series about 0, since each of its terms is a constant mutiplied by a power of x.

Substituting for the variable in a Taylor series II

If we were to use the usual formula to find the Taylor series about 0 for f, would we obtain the same series? Yes:

Let f be a real function f. If we can find by any means a series

$$a_0 + a_1(x-a) + a_2(x-a)^2 + a_3(x-a)^3 + \dots$$

that is equal to $f(x)$ for all x in some open interval containing a, then this series is the Taylor series about a for f, and hence it is the only series of this form that is equal to $f(x)$ for all x in that interval.

The proof of this result is beyond the scope of this course and we assume this result throughout the rest of this section.

Substituting for the variable in a Taylor series III

Taylor series can be found for many functions f by substituting for the variable in a known Taylor series. A range of validity for the new series can often be deduced from a range of validity for the original Taylor series.

Example 4.1 Substituting into a Taylor series

Find the Taylor series about 0 for the function

$$f(x) = \frac{1}{1+x^2}$$

and determine a range of validity for this series.

Solution

The Taylor series about 0 for $\frac{1}{(1-x)^2}$ is,

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots, \quad \text{for } -1 < x < 1.$$

Therefore the Taylor series about 0 for $\frac{1}{1+x^2} = \frac{1}{(1-(-x^2))}$ is

$$\frac{1}{1+x^2} = 1 + (-x^2) + (-x^2)^2 + (-x^2)^3 + \dots = 1 - x^2 + x^4 - x^6 + \dots.$$

Substituting for the variable in a Taylor series IV

The Taylor series for $\frac{1}{1-x}$ is valid for $-1 < x < 1$, so the series for $\frac{1}{(1-(-x^2))}$ is valid for $-1 < -x < 1$.

The left-hand inequality is $-1 < -x^2$, which is equivalent to $1 > x^2$; that is, $-1 < x < 1$. The right-hand inequality is $-x^2 < 1$, which is equivalent to $x^2 > -1$ and therefore does not place any restriction on x, since the square of any real number is non-negative. Thus the Taylor series for $\frac{1}{(1+x^2)}$ is valid for $-1 < x < 1$.

(Here we replace x by $-x$, which is equivalent to making the substitution $x = -t^2$ and then replacing t by x. Also note that in this solution and elsewhere it is often convenient to refer to a function by just giving its rule rather than by assigning a name.

Substituting for the variable in a Taylor series V

Activity 4.1 Substituting into a Taylor series

By substituting for the variable in a standard Taylor series, find the Taylor series about 0 for each of the following functions. In each case so far determine a range of validity for the series.

Although the series in part (a) can be found by taking $\alpha = -1$ in the binomial series, as was done in before it is more convenient to make a suitable substitution in the Taylor series for $\frac{1}{(1-x)}$, and this is what you are asked to do here.

(a)$f(x) = \frac{1}{(1+x)}$

(b) $f(x) = \ln(1-x)$

(c) $f(x) = \ln(1+3x)$

(d) $f(x) = e^{x^3}$

Substituting for the variable in a Taylor series VI

Some substitutions can lead to a new Taylor series with a different centre.

Activity 4.2 Changing the centre of a Taylor series

By replacing x by $x-1$ in the Taylor series about 0 for $\frac{1}{(1+x)}$, find the Taylor series about 1 for the function $f(x)=\frac{1}{x}$. Determine a range of validity for this series.

Comment

We know that the series found in the solution is the Taylor series about 1 for $\frac{1}{x}$ because of the result we found previously.

Adding, subtracting and multiplying Taylor series I

We can find Taylor series for some functions f is by applying standard arithmetic operations to known Taylor series. For example, we know that

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \dots, \quad \text{for } x \in \mathbf{R}$$

and

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \dots, \quad \text{for } -1 < x < 1$$

It follows that

$$\begin{aligned} e^x + \frac{1}{1-x} &= (1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \dots) \\ &\quad +(1 + x + x^2 + x^3 + x^4 + \dots) \\ &= (1+1) + (1+1)x + (\frac{1}{2!} + 1)x^2 \\ &\quad +(\frac{1}{3!} + 1)x^3 + (\frac{1}{4!} + 1)x^4 + \dots \\ &= 2 + 2x + \frac{3}{2}x^2 + \frac{7}{6}x^3 + \frac{25}{24}x^4 + \dots, \end{aligned}$$

at least for $-1 < x < 1$. The series above is therefore the Taylor series about 0 for the function $f(x) = e^x + \frac{1}{(1-x)}$; it is valid for $-1 < x < 1$.

Adding, subtracting and multiplying Taylor series II

- Any two Taylor series with the same centre can be added or subtracted term by term in a similar manner.
- The resulting Taylor series is valid for all values of x for which both original Taylor series are valid. It may also be valid for further values of x.

Adding, subtracting and multiplying Taylor series III

Find the Taylor series about 0 for the function

$$\ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x),$$

and determine a range of validity for this series.

Using the Taylor series for $\ln(1+x)$ and the series for $\ln(1-x)$ we found above, we have

$$\begin{aligned} \ln(1+x) - \ln(1-x) &= (x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 - \ldots) - (-x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4 - \frac{1}{5} \\ &= 2x + \frac{2}{3}x^3 + \frac{2}{5}x^5 + \ldots; \end{aligned}$$

that is

$$\ln\left(\frac{1+x}{1-x}\right) = 2x + \frac{2}{3}x^3 + \frac{2}{5}x^5 + \ldots.$$

The Taylor series for $\ln(1+x)$ and $\ln(1-x)$ are each valid for $-1 < x < 1$, so the series derived here is also valid for $-1 < x < 1$.

Comment

Adding, subtracting and multiplying Taylor series IV

It can be shown that every positive real number t can be expressed in the form $t = \frac{(1+x)}{(1-x)}$ for some x in the range $-1 < x < 1$. Thus the Taylor series in this activity can be used to find an approximation for $\ln t$ for any t in the domain $(0, \infty)$ of $\ln$.

Adding, subtracting and multiplying Taylor series V

In contrast, the series for $\ln(1+x)$ can be used to find approximations for $\ln t$ only for $0 < t \leqslant 2$, since these are the only values of t that can be expressed in the form $t = 1 + x$ for some x in the range $-1 < x \leqslant 1$. For both series, the further x is from 0, the more terms of the series have to be evaluated in order to obtain the desired accuracy.

This equation can be rearranged as $x = 1 - \frac{2}{(t+1)}$, which gives a suitable x for any given positive t. For example, if $t = 3$, then $x = \frac{1}{2}$.

Adding, subtracting and multiplying Taylor series VI

You have seen that Taylor series can be added and subtracted. A Taylor series can also be multiplied term by term by a non-zero constant. The resulting series is valid for every value of x for which the original Taylor series is valid. For example, we can multiply the Taylor series for e^x by 3 to deduce that

$$3e^x = 3 + 3x + \frac{3}{2!}x^2 + \frac{3}{3!}x^3 + \frac{3}{4!}x^4 + \dots, \quad \text{for } x \in \mathbf{R}$$

(Recall that $e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \dots$.)

Note we will later see that this is like multiplying two Taylor series together i.e. multiplying the Taylor series for the function $f(x) = 3$ by that for the function $g(x) = e^x$.

Adding, subtracting and multiplying Taylor series VII

(a) Use the binomial series to find the Taylor series about 0 for $\dfrac{1}{(1+x)^2}$, and a range of validity for this series.

(b) By using the fact that

$$\frac{1}{(3+x)^2} = \frac{1}{3^2} \times \frac{1}{(1+\frac{x}{3})^2},$$

find the Taylor series about 0 for $\dfrac{1}{(3+x)^2}$, and determine a range of validity for this series.

(a)

Taking $\alpha = -2$ in the binomial series gives

$$\begin{aligned} \frac{1}{(1+x)^2} &= 1 - 2x + \frac{(-2)(-3)}{2!}x^2 + \frac{(-2)(-3)(-4)}{3!}x^3 + \ldots \\ &= 1 - 2x + 3x^2 - 4x^3 + \ldots, \end{aligned}$$

for $-1 < x < 1$.

Adding, subtracting and multiplying Taylor series VIII

(b)

Using the result of part (a) with x replaced by $\frac{x}{3}$, we obtain

$$
\begin{aligned}
\frac{1}{(3+x)^2} &= \frac{1}{3^2} \times \frac{1}{(1+\frac{x}{3})^2} \\
&= \frac{1}{3^2} \times (1 - 2(\frac{x}{3}) + \frac{(-2)(-3)}{2!}(\frac{x}{3})^2 + \frac{(-2)(-3)(-4)}{3!}(\frac{x}{3})^3 + \dots) \\
&= \frac{1}{3^2} \times (1 - 2(\frac{x}{3}) + 3(\frac{x}{3})^2 - 4(\frac{x}{3})^3 + \dots), \\
&= \frac{1}{3^2} \times (1 - \frac{2}{3}x + \frac{3}{3^2}x^2 - \frac{4}{3^3}x^3 + \dots), \\
&= \frac{1}{3^2}m, -\frac{2}{3^3}x + \frac{3}{3^4}x^2 - \frac{4}{3^5}x^3 + \dots),
\end{aligned}
$$

This series is valid for $-1 < \frac{x}{3} < 1$; that is, for $-3 < x < 3$.

Comment

The Taylor series about 0 for any function of the form $(c+x)^\alpha$ can be deduced from the series for $(1+x)^\alpha$ in a similar way; that is, by first expressing $(c+x)^\alpha$ as $c^\alpha(1+\frac{x}{c})^\alpha$.

Adding, subtracting and multiplying Taylor series IX

You may have come across the functions $\sinh$ and $\cosh$ in your study of mathematics or you may have noticed their presence on your calculator. These are called hyperbolic functions; they can be defined in terms of the exponential function by the equations

$$\sinh x = \frac{1}{2}(e^x - e^{-x})$$

and

$$\cosh x = \frac{1}{2}(e^x + e^{-x})$$

We can use these equations, together with the Taylor series for e^x, to find the Taylor series for $\sinh x$ and $\cosh x$, as you will see shortly. The graphs of $\sinh$ and $\cosh$ are shown in Figure 4.1.

(There are various pronunciations in use for the functions $\sinh$ and $\cosh$; the most common ones are "shine" or "sine-sh" for $\sinh$ and simply "cosh" for $\cosh$.)

There are other hyperbolic functions, such as $\tanh$, which is defined by $\tanh x = \frac{(\sinh x)}{(\cosh x)}$, and usually pronounced "tansh".

Adding, subtracting and multiplying Taylor series X

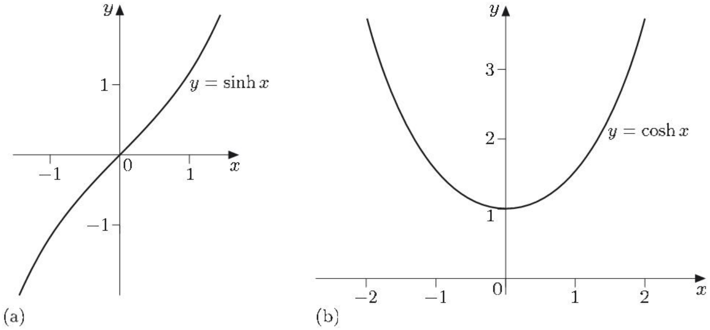

Figure 4.1 Functions sinh and cosh

Adding, subtracting and multiplying Taylor series XI

Although you may not expect it from either the definitions or the graphs of $\sinh$ and $\cosh$, these functions have many properties analogous to those of the trigonometric functions $\sin$ and $\cos$:

- $\cosh$, like $\cos$, is an even function, and $\sinh$, like $\sin$, is an odd function.
- The derived function of $\sinh$ is $\cosh$ and $\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$, for all $x, y \in \mathbf{R}$. These two properties are directly analogous to properties of $\sin$ and $\cos$.

Some of the properties of $\sinh$ and $\cosh$ are similar, but not directly analogous, to those of $\sin$ and $\cos$:

- You have met the identity $\cos^2 x + \sin^2 = 1$; the corresponding identity for the hyperbolic functions is $\cosh^2 x - \sinh^2 x = 1$.

These properties of $\sinh$ and $\cosh$ can be verified by using the given expressions for $\sinh$ and $\cosh$ in terms of e.

Adding, subtracting and multiplying Taylor series XII

- The identity $\cosh^2 x - \sinh^2 x = 1$ allows $\cosh$ and $\sinh$ to be used to define parametric equations for the hyperbola in standard form. This is why such functions are called hyperbolic.

- The next example involves using the techniques of adding and multiplying by a constant, together with the technique of substitution, to find the Taylor series about 0 for the function $\cosh$.

Adding, subtracting and multiplying Taylor series XIII

Find the Taylor series about 0 for the function $f(x) = \cosh x$, and determine a range of validity for this series.

We use the formula $\cosh x = \frac{1}{2}(e^x + e^{-x})$. Now

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!} + \dots, \quad \text{for } x \in \mathbf{R}$$

On replacing x by $-x$, we obtain

$$e^{-x} = 1 - x + \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \frac{1}{4!} - \dots, \quad \text{for } x \in \mathbf{R}$$

Therefore

$$\frac{1}{2}(e^x + e^{-x}) = \frac{1}{2}((1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!} + \dots) + (1 - x + \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \frac{1}{4!} - \dots)),$$

for $x \in \mathbf{R}$; that is

$$\cosh x = 1 + \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \dots, \quad \text{for } x \in \mathbf{R}$$

Adding, subtracting and multiplying Taylor series XIV

In the next activity you are asked to use a similar method to find the Taylor series about 0 for the function $\sinh$.

Activity to do at home:

Find the Taylor series about 0 for the function $f(x) = \sinh x$, and determine a range of validity for this series.

Adding, subtracting and multiplying Taylor series XV

$$
\begin{aligned}
\sinh x &= x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \frac{1}{7!}x^7 + \frac{1}{9!}x^9 + \ldots, \text{ for } x \in \mathbb{R} \\
\cosh x &= 1 + \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \frac{1}{6!}x^6 + \frac{1}{8!}x^8 + \ldots, \text{ for } x \in \mathbb{R}
\end{aligned}
$$

Notice that the Taylor series about 0 for $\sinh$ and $\cosh$ are similar to those for the corresponding trigonometric functions. The only difference is that all the coefficients in the series for $\cosh x$ and $\sinh x$ are positive, whereas the coefficients of the series for $\cos x$ and $\sin x$ alternate in sign.

Adding, subtracting and multiplying Taylor series XVI

You have seen that Taylor series can be added, subtracted and multiplied by a non-zero constant.

We can also multiply two Taylor series together. The resulting Taylor series is valid for all values of x for which both original Taylor series are valid. It may also be valid for further values of x.

Adding, subtracting and multiplying Taylor series XVII

- It is easy to write down the Taylor series about 0 of a polynomial

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n,$$

because this is already in the form of the series in the given formula (with $a = 0$, and the coefficients of all terms from that in x^{n+1} onwards equal to 0).

- It follows that the Taylor series about 0 of such a polynomial is the polynomial itself! It is valid for all $x \in \mathbf{R}$.

Adding, subtracting and multiplying Taylor series XVIII

Find the Taylor series about 0 for the function $f(x) = \dfrac{(1-x)}{(1+x)}$, and determine a range of validity for this series.

The Taylor series for $1-x$ is $1-x$.

The Taylor series for $\frac{1}{(1+x)}$ is

$$1 - x + x^2 - x^3 + \dots$$

Therefore

$$\begin{aligned}
\frac{1-x}{1+x} &= (1-x)(1-x+x^2+\dots) \\
&= 1(1-x+x^2+\dots) - x(1-x+x^2+\dots) \\
&= (1-x+x^2+\dots) - (x-x^2+x^3-\dots) \\
&= 1 - 2x + 2x^2 - 2x^3 + \dots.
\end{aligned}$$

Adding, subtracting and multiplying Taylor series XIX

The Taylor series about 0 for $1-x$ and $\frac{1}{(1+x)}$ are valid for $x \in \mathbf{R}$ and for $-1 < x < 1$, respectively. Hence the above Taylor series is valid for $-1 < x < 1$.

n

Find the Taylor series about 0 for each of the following functions. In each case determine a range of validity for the series.

(a) $f(x) = x^2 \sin x$

(b) $f(x) = (1+x)\cos x$

Adding, subtracting and multiplying Taylor series XX

(a)

Using the Taylor series about 0 for $\sin x$, we obtain

$$\begin{aligned} x^2 \sin x &= x^2 \left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \dots \right) \\ &= x^3 - \frac{1}{3!}x^5 + \frac{1}{5!}x^7 - \dots \end{aligned}$$

for $x \in \mathbb{R}$.

Adding, subtracting and multiplying Taylor series XXI

(b)

$$
\begin{aligned}
(1+x)\cos x &= (1+x)\left(1-\frac{1}{2!}x^2+\frac{1}{4!}x^4-\dots\right)\\
&= \left(1-\frac{1}{2!}x^2+\frac{1}{4!}x^4-\dots\right)+x\left(1-\frac{1}{2!}x^2+\frac{1}{4!}x^4-\dots\right)\\
&= \left(1-\frac{1}{2!}x^2+\frac{1}{4!}x^4-\dots\right)+\left(x-\frac{1}{2!}x^3+\frac{1}{4!}x^5-\dots\right)\\
&= 1+x-\frac{1}{2!}x^2-\frac{1}{2!}x^3+\frac{1}{4!}x^4+\frac{1}{4!}x^5-\dots,
\end{aligned}
$$

for $x \in \mathbb{R}$.

Adding, subtracting and multiplying Taylor series XXII

- In each case where we have multiplied together two Taylor series, one of the series had only finitely many non-zero terms (that is, it was a polynomial).

- Multiplying together two Taylor series both of which have infinitely many non-zero terms is usually a difficult task, and you will not be asked to carry out any complete multiplications of this type in this module.

- However, it is possible to multiply together the first few terms of two infinite Taylor series to find the first few terms of the product Taylor series. This is illustrated in the next example.

Adding, subtracting and multiplying Taylor series XXIII

Find the cubic Taylor polynomial about 0 for the function $f(x) = e^x \cos x$.

Using the Taylor series about 0 for e^x and for $\cos x$ from before, we have

$$\begin{aligned}
e^x \cos x &= (1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots)(1 - \frac{x^2}{2!} + \dots) \\
&= 1(1 - \frac{x^2}{2!} + \dots) + x(1 - \frac{x^2}{2!} + \dots) + \frac{x^2}{2!}(1 - \frac{x^2}{2!} + \dots) \\
&\quad + \frac{x^3}{3!}(1 - \frac{x^2}{2!} + \dots) + \dots \\
&= (1 - \frac{1}{2}x^2 + \dots) + (x - \frac{1}{2}x^3 + \dots) + (\frac{1}{2}x^2 - \dots) + (\frac{1}{6}x^3 - \dots) + \dots \\
&= 1 + x - \frac{1}{3}x^3 + \dots.
\end{aligned}$$

Therefore the cubic Taylor polynomial about 0 for $f(x) = e^x \cos x$ is

$$1 + x - \frac{1}{3}x^3.$$

In the solution to the last example at each stage all the terms that could eventually result in terms with power 3 or less were retained, and any terms which would affect only terms with power 4 or more were ignored. A similar method is required in the next activity.

To do at home:
Use the Taylor series about 0 for $\frac{1}{(1+x)}$ and for $\sin x$ to find the cubic Taylor polynomial about 0 for the function

$$f(x) = \frac{\sin x}{1+x}$$

Differentiating and integrating Taylor series I

You have seen that the Taylor series about 0 for the function $f(x) = \sin x$ is

$$x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \dots.$$

Let us consider the effect of differentiating this series term by term, in the way that we would if it had only finitely many terms and was therefore a polynomial. We obtain the series

$$1 - \frac{3}{3!}x^2 + \frac{5}{5!}x^4 - \dots = 1 - \frac{1}{2!}x^{+}\frac{1}{4!}x^4 - \dots.$$

You may recognise this as the Taylor series about 0 for $\cos x$. So by differentiating term by term the Taylor series about 0 for $f(x) = \sin x$, we have obtained the Taylor series about 0 for its derived function $f'(x) = \cos x$.

This observation suggests that:

Term-by-term differentiation of a Taylor series about 0 for a function f yields the Taylor series about 0 for its derivative f'.

We now verify this conjecture.

Differentiating and integrating Taylor series II

Let f be a function that is differentiable infinitely many times at 0, and let $g = f'$. The Taylor series about 0 for f is

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \cdots + \frac{f^{(k)}(0)}{k!}x^k + \ldots.$$

Differentiating this series term by term gives the series

$$\begin{aligned} & 0 + f'(0) + \frac{f''(0)}{2!}2x + \frac{f^{(3)}(0)}{3!}3x^2 + \cdots + \frac{f^{(k)}(0)}{k!}kx^{k-1} + \ldots \\ = \quad & f'(0) + f''(0)x + \frac{f^{(3)}(0)}{2!}x^2 + \cdots + \frac{f^{(k)}(0)}{(k-1)!}x^{k-1} + \ldots \end{aligned}$$

Since $g = f'$ we have $g(0) = f'(0)$, $g'(0) = f''(0)$, $g''(0) = f^{(3)}(0)$, and so on. Therefore the series above can be written as

$$g(0) + g'(0)x + \frac{g''(0)}{2!}x^2 + \cdots + \frac{g^{(k-1)}(0)}{(k-1)!}x^{(k-1)} + \ldots$$

which is the Taylor series about 0 for $g = f'$. (The general term is expressed in terms of $k-1$ instead of k.)

Differentiating and integrating Taylor series III

Taylor series can also be integrated term by term.

If the Taylor series about 0 for a function f is integrated term by term, then the result is the Taylor series about 0 of an antiderivative of f. These properties of Taylor series are summarised below.

Differentiating and integrating Taylor series IV

Let f be a function that is differentiable infinitely many times at 0. If the Taylor series about 0 for $f(x)$ is

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n + \ldots,$$

then the Taylor series for $f'(x)$ is

$$a_1 + 2a_2 x + 3a_3 x^2 + \cdots + \frac{a_n}{n+1} x^n + \ldots,$$

and the Taylor series for any antiderivative of $f(x)$ is

$$c + a_0 x + \frac{a_1}{2} x^2 + \frac{a_2}{3} x^3 + \cdots + \frac{a_n}{n+1} x^{n+1} + \ldots,$$

where c is an arbitrary constant.

Any range of validity for the Taylor series for f which is an open interval is also a range of validity for the Taylor series for f' and for any antiderivative of f.

Differentiating and integrating Taylor series V

Note:

- We have used a_0 for $f(0)$, a_1 for $f'(0)$, a_2 for $\frac{f''(0)}{2!}$, and so on, to simplify the notation.
- The Taylor series for an antiderivative of f may also be valid at one or both of the endpoints of the open interval, even if the Taylor series for f is not valid there.
- The results in the box on the previous slide can be extended to Taylor series with centres other than 0, but we shall not need to use this fact in this module.

Differentiating and integrating Taylor series VI

We have seen that

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - \dots, \quad \text{for } -1 < x < 1.$$

Use this fact to find the Taylor series about 0 for each of the following functions f. In each case state a range of validity for the series.

(a) $f(x) = \dfrac{1}{(1+x)^2}$

(b) $f(x) = \ln(1+x)$

(a)

Differentiating both sides of the given equation yields

$$-\frac{1}{(1+x)^2} = -1 + 2x - 3x^2 + 4x^3 - \dots, \quad \text{for } -1 < x < 1.$$

Multiplying both sides by -1 gives the required Taylor series:

$$\frac{1}{(1+x)^2} = 1 - 2x + 3x^2 - 4x^3 + \dots, \quad \text{for } -1 < x < 1.$$

Differentiating and integrating Taylor series VII

(b)

Integrating both sides of the given equation yields

$$\ln(1+x) = c + x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 - \dots, \quad \text{for } -1 < x < 1.$$

where c is an arbitrary constant. Taking $x = 0$ gives $\ln 1 = c$; hence $c = 0$. Therefore the required Taylor series is

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 - \dots, \quad \text{for } -1 < x < 1.$$

Differentiating and integrating Taylor series VIII

Comments

- An alternative way to find the Taylor series in part (a) is to take $\alpha = -2$ in the binomial series, as was done earlier in this module.
- This is the standard Taylor series for $\ln(1+x)$, as is to be expected. Part (b) shows the connection between this standard series and the series for $\dfrac{1}{(1+x)}$.

Differentiating and integrating Taylor series IX

Verify that term-by-term differentiation of the Taylor series about 0 for the function $f(x) = e^x$ leaves the series unchanged.

The Taylor series about 0 for e^x is

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \dots, \quad \text{for } x \in \mathbf{R}$$

Differentiating this series gives

$$\begin{aligned} & 0 + 1 + \frac{2}{2!}x + \frac{3}{3!}x^2 + \frac{4}{4!}x^3 + \dots \\ = \quad & 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \dots, \end{aligned}$$

which is the same series.

This result corresponds to the fact that the derivative of e^x is e^x.

Differentiating and integrating Taylor series X

In the following we make use of the fact that:

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 - \dots, \quad \text{for } -1 < x < 1.$$

Use integration to deduce the Taylor series about 0 for $\arctan x$ (i.e. $\tan^{-1}$), and state a range of validity for this series.

Integrating both sides of the given equation yields

$$\tan^{-1} x = c + x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \dots,$$

for $-1 < x < 1$, where c is an arbitrary constant. Taking $x = 0$ gives $\tan^{-1} 0 = c$; hence $c = 0$.

Therefore

$$\tan^{-1} x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \dots,$$

for $-1 < x < 1$.

Differentiating and integrating Taylor series XI

In the final activity of this section you are asked to use both substitution and integration to find the first few terms of the Taylor series about 0 for the inverse sine function.

(a) Use the binomial series to find the Taylor series about 0 for the function $\dfrac{1}{\sqrt{1+x}}$, and evaluate the coefficients of the first three terms. State a range of validity for this series.

(b) Hence, by using substitution, find the first three terms of the Taylor series about 0 for the function $\dfrac{1}{\sqrt{1-x^2}}$. Determine a range of validity for this series.

(c) By integrating the series in part (b) term by term, find the first three non-zero terms in the Taylor series about 0 for the function $f(x) = \sin^{-1} x$, and state a range of validity for this series.

Differentiating and integrating Taylor series XII

(a)

Taking $\alpha = -\frac{1}{2}$ in the binomial series gives

$$\begin{aligned} \frac{1}{\sqrt{1+x}} &= 1 - \frac{1}{2}x + \frac{(-\frac{1}{2})(-\frac{3}{2})}{2!}x^2 + \ldots \\ &= 1 - \frac{1}{2}x + \frac{3)}{8}x^2 + \ldots, \end{aligned}$$

for $-1 < x < 1$.

Differentiating and integrating Taylor series XIII

(b)

Replacing x by $-x^2$ in the above series gives

$$\begin{aligned} \frac{1}{\sqrt{1-x^2}} &= 1 - \frac{1}{2}(-x^2) + \frac{3}{8}(-x^2)^2 + \ldots, \\ &= 1 + \frac{1}{2}x^2 + \frac{3}{8}x^4 + \ldots. \end{aligned}$$

This Taylor series is valid for $-1 < (-x^2) < 1$.

The left-hand inequality here is $-1 < -x^2$ which is equivalent to $1 > x^2$; that is $-1 < x < 1$.

The right-hand inequality is $-x^2 < 1$, which is equivalent to $x^2 > -1$ therefore does not place any restriction on x, since the square of any real number is non-negative.

Thus the Taylor series is valid for $-1 < x < 1$.

Differentiating and integrating Taylor series XIV

(c)

Integrating both sides of the above equation gives

$$\int \frac{1}{\sqrt{1-x^2}}\, dx = \int (1 + \frac{1}{2}x^2 + \frac{3}{8}x^4 + \dots)\, dx;$$

$$\sin^{-1} x = c + x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \dots,$$

for $-1 < x < 1$, where c is an arbitrary constant.

Taking $x = 0$ gives $\sin^{-1} 0 = c$; hence $c = 0$.

Therefore

$$\sin^{-1} x = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \dots,$$

for $-1 < x < 1$.