Question 1

Part a

We have a total of 10 topics and need to pick 3 per exam so the amount of possible combinations is $\binom{10}{3} = 120$. This works for unordered topics without replacement.

Part b

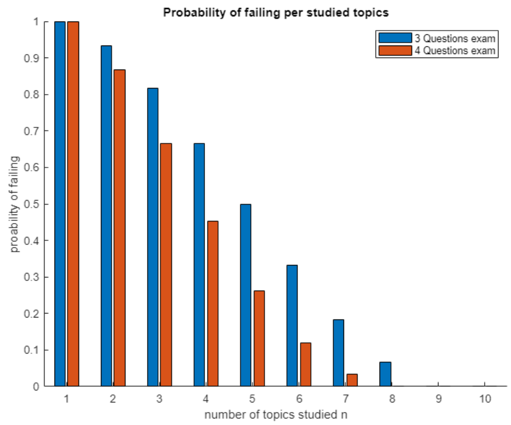
There are $\binom{10-n}{3}$ combinations of exams such that no studied topics appear. Dividing this number with the total combination of exams from part a gives us the probability of failing: $\frac{\binom{10-n}{3}}{120}$.

Part c

It is easier to derive an expression for passing. There are 2 cases to consider. When 2 out of the 3 topics were studied, there are $(10-n)*\binom{n}{2}$ combinations. This multiplies the number of combinations for the non studied topic by the combinations of 2 studied topics. When all 3 topics on the exam were studied, there are $\binom{n}{3}$ combinations thus the probability of failing the exam is: $1 - [\frac{(10-n)*\binom{n}{2}+\binom{n}{3}}{\binom{10}{3}}]$. The plot is provided in the following question (part d).

Part d

When an exam is composed of 4 question there are 3 cases to consider in order to obtain an expression for passing the exam. Using the same idea from part c, we sum the amount of combinations for knowing, 2, 3 and 4 of the topics on the exam: $1 - [\frac{\binom{10-n}{2}\binom{n}{2}+(10-n)\binom{n}{3}+\binom{n}{4}}{\binom{10}{4}}]$. Using the code in the Appendix section A, I obtained the following plot. We notice that a 4 questions exam system is easier to pass for any number of studied topics, with the biggest difference occuring when $n = 4$ and $n = 5$ and very low chance of failing when $n \geq 6$. For a 3 questions exam the probability is as low when $n \geq 8$, making the exam much simpler to pass with a 4 questions system.



Part e

The code for the stochastic simulation is in the appendix section B.

Part f

The code for the extended simulation is in the appendix section C. The expression for a 95% confidence interval using the central limit theorem is $[\mu - 1.96\sigma, \mu + 1.96\sigma]$. Here we have $\mu = E[X_i]$. So we only need to find and expression for σ. $var(\frac{1}{n}\sum_{i=1}^{N} Y_i) = \frac{1}{n^2}\sum_{i=1}^{N} var(Y_i) = \frac{var(Y_i)}{n} = \sigma^2$. Finally, $\sigma = \sqrt{\frac{E[X_i]*(1-E[X_i)}{N}} = \sqrt{\frac{0.1497}{N}}$. So our expression for a 95% confidence interval using CLT is $[E[X_i] - 1.96\sqrt{\frac{0.1497}{N}}, E[X_i] + 1.96\sqrt{\frac{0.1497}{N}}]$ Using $N = 1000$ and $N = 10000$ we then get $[0.79272, 0.84068]_{N=1000}$ and $[0.80912, 0.82428]_{N=10000}$

Part g

I will use Chebyshev's inequality with 95% confidence and solve for n to find how many times I should run my simulation. Let's choose $\epsilon = 0.1$, we have $E[X_i] = 0.8167$ and $var(X_i) = 0.1497$ from previous questions. Then we have: $P(|Y - E[X_i| \geq 0.1) \leq \frac{0.1497}{n(0.1)^2} \Leftrightarrow \frac{0.1497}{n(0.1)^2} \leq 0.05 \Leftrightarrow n \geq 299$. So I will run my simulation 300 times. Running with $N = 1000$ we find that the emprical mean falls 94.3% of the times within the interval. With $N = 10000$ it is 96.3%. We find that both values of N fall very close to 95%. The CLT confidence intervals were built with with 95% accuracy so everything is in order.

Part h

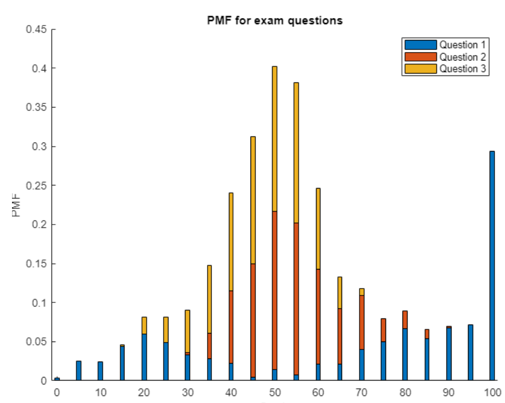
If students know that topics from the previous exam are more likely to appear, then overall students they will likely prioritise studying those topics. However this also only applies to 3 topics and thus some students may also still decide to study potentially easier topics instead of a certain topic from the previous year. So in general a system like this would affect overall how the class studies but it would not get rid of students deciding to only study different topics either. Thus I believe it makes sense to use the same weights for drawing the exam questions as for drawing the topics a student decides to study. Using the simulation, we notice an overall increase in the means for passing, with the empirical mean getting to 0.5 near 4.5 topics studied, and gets near 0.8 when 6 topics are studied. In the case that exam questions are less likely to appear if they were in the previous exam then with students using the same approach as previously outlined, except this time being less likely to study those topics, we also notice an increase overall in means for passing for all amounts of topics studied however the difference is not as big as the previous since this leaves 7 remaining topics that are more likely to be drawn, so the probability for one of these to appear in the exam would be significantly lower than previously where only 3 topics were more likely to be drawn. Finally, if questions are excpected to be more predictable based on the previous exam when they are in fact chosen at random will negatively affect the mean of passing students for all amounts of studied topics. This is due to the fact that students will prioritise a subset of topics which are infact as likely to be drawn as any other topic.

Question 2

Data id: $0.29 : 0.5 - 0.374 : 2 - 0.584 : 2 - 0$

Part a

Using code provided in the appendix E, I was able to produce the following plot. We notice that question 1 is very spread out and uneven in probability. We can assume that it was very easy to obtain a grade of 100% given that a student studied that topic. Question 2 and 3 seem to be following student distribution.



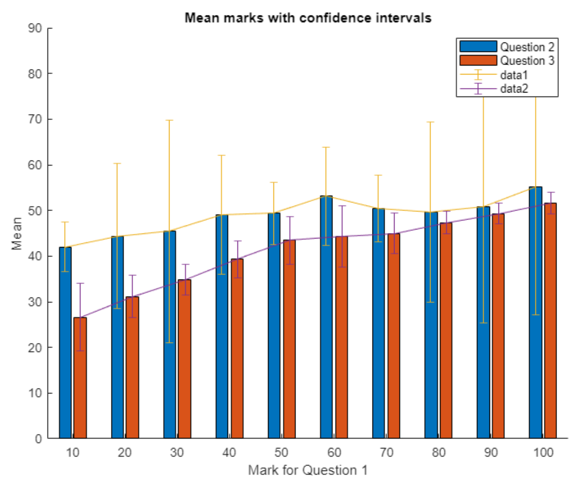
However we notice higher grades for question 2 meaning that it is probably easier. Question 3 seem to be harder as we notice a significant probability for grades below 40%. This approach can be good to use as if an entire class obtains low grades, the PMF will reflect that and that can be a good indicator that an question is likely to be hard. On the other hand, it could lead to false positives as students as a group are likely to study similar topics, we could notice a low PMF for a question that was in fact easy but the students did not study for.

Part b

The code for this question is added to the appendix section F. I decided to use binnings with a range of 10% as it allowed for a cleaner display of means and variances while keeping accuracy (for instance making the ranges too wide would loose too much accuracy and thus would make it harder to draw conclusions).

Part c

The code to obtain the following plot is also in the appendix section F as it was integrated with part b's code. I decided to use CLT 95% confidence intervals.



In question 2, we notice that the means for all categories is always above 40. We also notice that students who did poorly in question 1 go similar grades in question 2 as students who scored relatively high in question 1. Using confidence intervals, we observe that students with high grades for question 1 obtained very varied scores in question 2 as the intervals are very large. Thus it seems question 2 is a harder question to obtain high grades in than question 1, however it seems to be relatively easy to obtain a score of 40%. In question 3, we notice a more usual distribution with students who did poorly in question 1 getting lower grades than those who scored well in question 1. However we also notice once again that those who obtained high grades in question 1 obtained a grade near 50% for question 3 with very small confidence intervals meaning that this question is very likely to be hard to score high in.

Part d

Using the mark obtained in question 1 as an input to our model and marks for the following questions as training data, we can use linear regression to predict marks for a student. We would use the hypothesis $h_\theta(x) = \theta_0 + \theta_1 x_1 + \theta_2 x_2$. We would then measure the accuracy of our model using the following cost function $J(\theta) = \frac{1}{m}\sum_{i=1}^{m}(h_\theta(x^i) - y^i)^2$. Finally, to get the most accurate predictions, we want to find values for our parameters such that they minimize the outputs of our loss function. One way to do this is by using gradient descent. Thus we can use the following algorithm to find such values. Repeat the following 2 steps: Step 1 - for j=0 to n: $tmp_j = \theta_j - \frac{2\alpha}{m}\sum_{i=1}^{m}(h_\theta(x^{(i)}) - y^{(i)})x_j^{(i)}$. Step 2 - for j=0 to n: $\theta_j = tmp_j$. According to the graph from part c, it seems that question 3 follows a linear regression. Question 2 however is not as clear, it seems to follow a linear pattern however the confidence intervals for grades over 70% are too wide to be able to make decisions on which model might be better suited, perhaps a cubic model could work better here. The type of randomness assumed in linear regression is applicable for exam marks as many different factors outside of previously obtained grades affect a grade, things such as fatigue, performance, stress or even lecturer grading for example.

Part e

Since we know that marks X_{ij} are normally distributed ($X_{ij} \sim N(S_i - D_j, \sigma^2)$). Then the probability density function is $\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}(\frac{x-(S_i-D_j)}{\sigma})^2}$. Finally, given that all student marks are independent we can use this PDF to obtain a log-likelihood function. $\prod_{i=1}^{N}\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-S_i-D_j)^2}{2\sigma^2}} = (\sigma^2 2\pi)^{-N/2}e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{N}(S_i-D_j)^2}$.

Part f

We are looking to estimate values for parameters S_i, D_j such that they maximise the above log-likelihood function. We can use gradient ascent, the algorithm is the same as part d however we will be using the negative loss function such as to find a maximum. Thus the algorithm here is also in 2 steps. Step 1 - for j=0 to n: $tmp_j = \theta_j + \frac{2\alpha}{m}\sum_{i=1}^{m}(h_\theta(x^{(i)}) + y^{(i)})x_j^{(i)}$. Step 2 - for j=0 to n: $\theta_j = tmp_j$ Note that in this case we have 2 unknown parameters θ, these are S_i and D_j.

Appendix

Section A - Question 1c and 1d

Code to plot the probability of failing against the number of studied topics for both 3 and 4 questions exam types.

```
y4qs = zeros([1 10]);
```

```
y3qs = zeros([1 10]);
% for all values of n
for n = 1:10
    % probability of failing for a 3 questions exam
    y3qs(n) = 1 - (((10-n) *
        (customNChooseK(n,2))/customNChooseK(10,3))+
        (customNChooseK(n,3)/customNChooseK(10,3)));
    % probability of failing for a 4 questions exam
    y4qs(n) = 1 - (((10-n)*(customNChooseK(n,3)) /
        customNChooseK(10,4))+ (customNChooseK(n,4) /
        customNChooseK(10,4)) + (customNChooseK(10-n,2) *
        (customNChooseK(n,2)/customNChooseK(10,4))));
end

figure
hold on

xAxis = 1:1:10;
yAxis = [y3qs; y4qs];
xlabel('number of topics studied n')
ylabel('proability of failing')
bar(xAxis, yAxis);
legend(['3 Questions exam'; '4 Questions exam']);
title('Probability of failing per studied topics');

function [m] = customNChooseK(n,k)
    if(n < k)
        m = 0;
    else
        m = nchoosek(n,k);
    end
end
```

Section B - Question 1e

Code for a stochastic simulation of the exam setup with three questions. Function takes a number n which designs the amount of studied topics. The simulation picks randomly a set of studied topics and exam topics and returns 1 if the student passes, else 0.

```
function [X] = simulation3Questions(n)
    % number of topics
    topics=1:10;
    % draw for exam questions
    questions = randperm(numel(topics),3);
    % draw for studied subjects
```

```
    studiedSubjects = randperm(numel(topics),n);

    % get common topics
    pos=intersect(questions, studiedSubjects);
    % return 1 if pass or 0 if fail
    if(length(pos) >= 2)
        X = 1;
    else
        X =0;
    end
end
```

Section C - Question 1f

Extended stochastic simulation. Runs the above simulation in Section B N times and returns the empirical mean.

```
function [X] = simulationEmpiricalMean(n, N)
    passCount = 0;
    % run N times keeping count of amount of pass
    for i = 1:N
        % number of topics
        topics=1:10;
        % draw exam questions
        questions = randperm(numel(topics),3);
        % draw studied subjects
        studiedSubjects = randperm(numel(topics),n);

        % get common topics
        common=intersect(questions, studiedSubjects);
        if(length(common) >= 2)
            passCount = passCount + 1;
        end
    end
    % return empirical mean
    X = passCount / N;
end
```

Section D - Question 1g

Code for running the extended stochastic simulation N times and displaying the percentage of times that the simulated empirical mean falls within a confidence interval.

```
% count for times mean falls within interval
```

```
count = 0;
% 1000 or 10000 depending on simulation to run
N = 1000;
% Amount of times to run simulation
size = 300;
% Values of the confidence interval (here for N = 1000)
lo = 0.79272;
hi = 0.84068;

% Run simulation size times and increment count if
% the mean falls within the CI
for i = 1:size
    % function from appendix section C
    % returns the empirical mean.
    simMean = simulationEmpiricalMean(7, N);
    if(simMean >= lo && simMean <= hi)
        count = count + 1;
    end
end
% Display % of means falling within confidence interval
disp(count/size * 100);
```

Section E - Question 2a

Code for running obtaining the PMF for all 3 questions from the data set and plot the results.

```
data = load('data.txt');
nstudents = length(data(:,1));

countsQ1=zeros(1, 101);
countsQ2=zeros(1, 101);
countsQ3=zeros(1, 101);

gradesQ1 = data(:,1);
gradesQ2 = data(:,2);
gradesQ3 = data(:,3);

% Go through all students keeping counts
% For all grades
for i=1:nstudents
    countsQ1(gradesQ1(i) + 1) =
        countsQ1(gradesQ1(i)+ 1) + 1;
    countsQ2(gradesQ2(i) + 1) =
        countsQ2(gradesQ2(i)+ 1) + 1;
    countsQ3(gradesQ3(i) + 1) =
```

```
        countsQ3(gradesQ3(i)+ 1) + 1;
end
figure
hold on

combined = [countsQ1/nstudents;
    countsQ2/nstudents; countsQ3/nstudents];
bar(0:100,combined, 1,'stacked');
xlabel('Grade');
ylabel('PMF');
legend(['Question 1'; 'Question 2'; 'Question 3']);
title('PMF for exam questions');
bar(xAxis, yAxis);
```

Section F - Question 2b and 2c

Code for calculating means, variances and confidence intervals for question 2 and 3 conditioned on question 1.

```
% Q2 b
m = 1000;
data = load('data.txt');
means = 1:10;
vars = 1:10;
cis =[];

% go through all 10 categories
for i = 1:10
    index=1;
    studentsInCat = zeros([1 1]);
    % go through all students and
    % add to category if grade from q1 is in
    % the current category
    for j = 1:m
        grade = data(j, 1);
        if (grade < i*10) && grade >= (i*10)-10
            studentsInCat(index,1) = j;
            index = index + 1;
        end
    end

    % get list of grades from all
    % students who fall in the category
    % and their means
    nbStudents = numel(studentsInCat);
    index = 1;
```

```
    % array to contain all grades of
    % all students for q2 and 3 from the
    % current category
    grades = 1:2*nbStudents;
    for cQuestion = 2:3
        gradeSum = 0;
        for cStudent = 1:nbStudents
            grade = data(studentsInCat(cStudent)
                ,cQuestion);
            grades(index) = grade;
            index = index + 1;
            gradeSum = gradeSum + grade;
        end

        % keep variance and mean for
        % current question / bracket
        vars(i,cQuestion-1) = var(grades);
        means(i,cQuestion-1) = gradeSum/nbStudents;

        % 2c - Compute confidence interval
        % Using CLT
        stdDev = sqrt(vars(i,cQuestion-1))
            /sqrt(nbStudents);
        cis(i,cQuestion-1,1) = means(i)-(2*stdDev);
        cis(i,cQuestion-1,2) = means(i)+(2*stdDev);
    end
end

figure
hold on

% plot means for both questions & all 10 categories
xaxis = 10:10:100;
yaxis = [reshape(means(:,1),1,10);
    reshape(means(:,2),1,10)];
xlabel('Mark for Question 1');
ylabel('Mean');
title('Mean marks with confidence intervals');
bar(xaxis, yaxis);

errq1 = ones(10,1);
errq2 = ones(10,1);

% get difference between high and lower bound as error
for i = 1:10
    errq1(i) = cis(i,1,2) - cis(i,1,1);
```

```
    errq2(i) = cis(i,2,2) - cis(i,2,1);
end

% plot error bars
legend({'Question 2','Question 3'})
errorbar(xaxis - 1.5,reshape(means(:,1),1,10), errq1);
errorbar(xaxis + 1.5,reshape(means(:,2),1,10), errq2);
```