Question 1

Part a

We have a total of 10 topics and need to pick 3 so the amount of possible combinations is $\binom{10}{3} = 120$.

Part b

There are $\binom{10-n}{3}$ combinations of exams such that no studied topics appear.

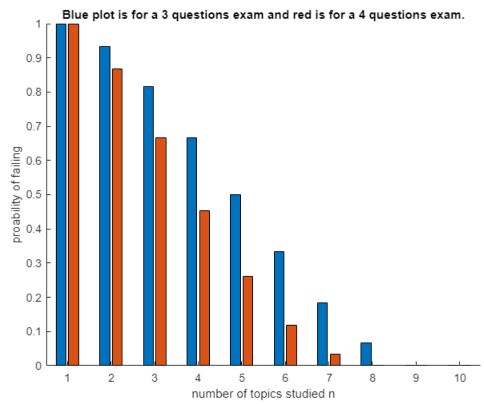
Thus using the value from part a, we have $\frac{\binom{10-n}{3}}{120}$.

Part c

Let's derive an expression for passing the exam and then take its inverse. There are 2 cases to consider. When 2 out of the 3 topics were studied, there are $(10-n) * \binom{n}{2}$ exam combinations. This multiplies the number of combinations for the non studied topic by the combinations of 2 studied topics. Then the scenario that all 3 topics on the exam were studied, there are $\binom{n}{3}$ combinations thus the probability of failing the exam is: $1 - [\frac{(10-n)*\binom{n}{2}+\binom{n}{3}}{\binom{10}{3}}]$. The plot is provided in the following question (part d).

Part d

When an exam is composed of 4 question there are 3 cases to consider in order to obtain an expression for passing the exam. Using the same idea from part c, we sum the combinations for knowing, 2, 3 and 4 of the topics on the exam. we obtain the following expression for failing the exam: $1 - [\frac{\binom{10-n}{2}\binom{n}{2}+(10-n)\binom{n}{3}+\binom{n}{4}}{\binom{10}{4}}]$ Using the code in the Appendix section A, I was able to plot the expressions for both 3 and 4 questions exams. We notice that a 4 questions exam system is easier to pass for any number of studied topics, with the biggest difference occuring when $n = 4$ and $n = 5$.



Part e

The code for the stochastic simulation is provided in the appendix section B.

Part f

The code for the extended simulation is provided in the appendix section C. The expression for a 95% confidence interval using the central limit theorem is $[\mu - 1.96\sigma, \mu + 1.96\sigma]$ Here we have $\mu = E[X_i]$. So we only need to find and expression for σ. We have $var(\frac{1}{n}\sum_{i=1}^{N} Y_i) = \frac{1}{n^2}\sum_{i=1}^{N} var(Y_i) = \frac{var(Y_i)}{n} = \sigma^2$. Finally, $\sigma = \sqrt{\frac{E[X_i]*(1-E[X_i)}{N}} = \sqrt{\frac{0.1497}{N}}$. So our expression for a 95% confidence interval using CLT is $[E[X_i] - 1.96\sqrt{\frac{0.1497}{N}}, E[X_i] + 1.96\sqrt{\frac{0.1497}{N}}]$ Using $N = 1000$ and $N = 10000$ we then get $[0.79272, 0.84068]_{N=1000}$ and $[0.80912, 0.82428]_{N=10000}$

Part g

I will use chebyshev's inequality with 95% confidence and solve for n in order to find how many times I should run my simulation. Let's choose $\epsilon = 0.1$, we have $E[X_i] = 0.8167$ and $var(X_i) = 0.1497$ from previous questions. Using these values in the inequality we obtain: $P(|Y - E[X_i| \geq 0.1) \leq \frac{0.1497}{n(0.1)^2} \Leftrightarrow \frac{0.1497}{n(0.1)^2} \leq 0.05 \Leftrightarrow n \geq 299$. So I will run my simulation 300 times. Running with $N = 1000$ we find that the emprical mean falls 94.3% of the times within the interval. With $N = 10000$ it is 96.3%. We find that for both values of N, we get a percentage to be expected since we built the confidence intervals with 95% accuracy and the obtained percentages are very close.

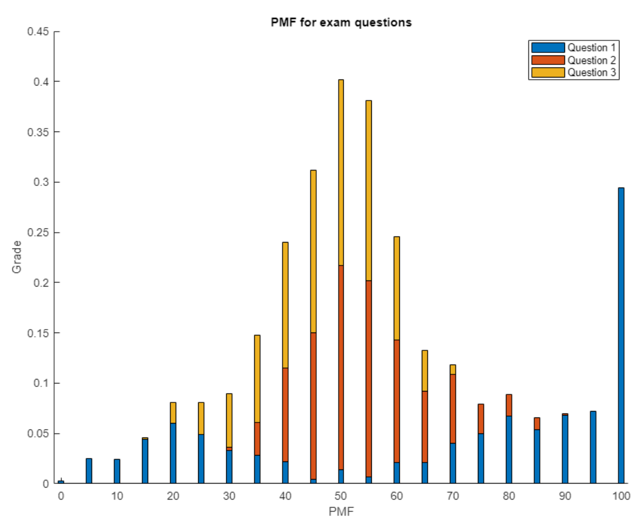
Part h

If students know that topics from the previous exam are more likely to appear, then overall students might choose to prioritise studying those topics. However this also only applies to 3 topics and thus some students may also still decide to study potentially easier topics instead of a certain topic from the previous year. So in general a system like this would affect overall how the class studies but it would not get rid of students deciding to only study different topics either. Thus I believe it makes sense to use the same weights for drawing the exam questions as for drawing the topics a student decides to study in my simulation. Using the simulation, there is no surprise there, as topics are more predictable, with the empirical mean for passing getting to 0.5 near 4.5 topics studied, and gets near 0.8 when 6 topics are studied. In the case that exam questions are less likely to appear if they were in the previous exam then with students using the same approach as previously outlined, except this time being less likely to study those topics, we also notice an increase overall in means for passing for all amounts of topics studied however the difference is not as big as the previous approach. Finally, if questions are excpeted to be more predictable based on the previous exam when they are in fact chosen at random will negatively affect the mean of passing students for all amounts of studied topics. This is due to the fact that students will prioritise a subset of topics which are infact as likely to be drawn as any other topic.

Question 2

Part a

Using code provided in the appendix section E I was able to produce the following plot.



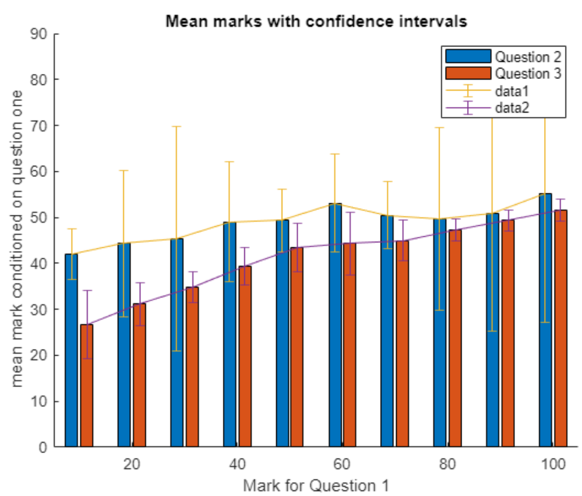
We notice that question 1 is very spread out and uneven in probability. We can assume that it was very easy to obtain a grade of 100% given that a student studied that topic. Question 2 seems to be following the student distribution as well as question 3 however we notice higher grades for question meaning that it is probably easier. Question 3 seem to be harder as we notice a significant probability for grades below 40%. This approach can be good to use as if we notice low values for the PMF then that exam is likely to be hard. On the other hand, it could lead to false positives to as students as a group are likely to study similar topics, we could notice a low PMF for a question that was in fact easy but the students did not study for.

Part b

The code for this question is added to the appendix section F. I decided to use binnings with a range of 10% as it allowed for a cleaner display of means and variances while keeping accuracy (for instance making the ranges too wide would loose too much accuracy and thus would make it harder to draw conclusions).

Part c

The code for this question is also in the appendix section F as it was integrated with part b's code. I decided to use CLT confidence intervals as it is easy to obtain means and variances using matlab.



In question 2, we notice that the means for all categories is always above 40. We also notice that students who did poorly in question 1 go similar grades in question 2 as students who graded high in question 1. We also notice that students with high scores for question 1 obtained very varied scores in question 2, we notice this thanks to the confidence intervals being very big, thus it seems question 2 harder question to obtain high grades in than question 1, however it seems to be relatively easy to obtain a score of 40%. In question 3, we notice a more usual distribution with students who did poorly in question 1 getting lower grades than those who scored well in question 1. However we also notice once again that those who obtained high grades in question 1 obtained a grade near 50% for question 3 with very small confidence intervals meaning that this question is very likely to be hard.

Part d

Using the mark obtained in question 1 as an input to do model and marks for the following questions as training data, we can use linear regression to predict marks for a student. We would use the hypothesis $h_\theta(x) = \theta_0 + \theta_1 x_1 + \theta_2 x_2$. Where We would then measure the accuracy of our model using the following cost function $J(\theta) = \frac{1}{m}\sum_{i=1}^{m}(h_\theta(x^i) - y^i)^2$. Finally, to get the most accurate predictions, we want to find values for our parameters such that they minimize the outputs of our loss function. One way to do this is by using gradient descent. Thus we can use the following algorithm to find such values. Repeat the following 2 steps: Step 1 - for j=0 to n: $tmp_j = \theta_j - \frac{2\alpha}{m}\sum_{i=1}^{m}(h_\theta(x^{(i)}) - y^{(i)})x_j^{(i)}$. Step 2 - for j=0 to n: $\theta_j = tmp_j$ According to the graph from part c, it seems that question 3 follows a linear regression. Question 2 however is not as clear, it seems to follow a linear pattern however the confidence interval towards to end are too wide to be able to make decisions on which model might be better suited, perhaps a cubic model could work better here. The type of randomness assumed in linear regression is applicable for exam marks as many different factors outside of topics studied affect a grade, things such as fatigue, performance, stress or even lecturer grading for example.

Part e

Since we know that marks X_{ij} are normally distributed (ie $X_{ij} \sim N(S_i - D_j, \sigma^2)$). Then the probability density function is $\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}(\frac{x-(S_i-D_j)}{\sigma})^2}$. Finally, given that all student marks are independent we can use this PDF to obtain a log-likelihood function. $\prod_{i=1}^{N}\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-S_i-D_j)^2}{2\sigma^2}} = (\sigma^2 2\pi)^{-N/2}e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{N}(S_i-D_j)^2}$

Part f

We are looking to estimate values for parameters S_i, D_j such that they maximise the above log-likelihood function. We can use gradient ascent, the algorithm is the same as part d however we will be using the negative loss function such that we find a maximum. Thus the algorithm here is also in 2 steps. Step 1 - for j=0 to n: $tmp_j = \theta_j - \frac{2\alpha}{m}\sum_{i=1}^{m}(h_\theta(x^{(i)}) + y^{(i)})x_j^{(i)}$. Step 2 - for j=0 to n: $\theta_j = tmp_j$ Note that in this case we have 2 unknown parameters θ, these are S_i and D_j.

Appendix

Section A - Question 1d

```
y4qs = zeros([1 10]);
y3qs = zeros([1 10]);
% for all values of n
for n = 1:10
    % probability of failing for a 3 questions exam
    y3qs(n) = 1 - (((10-n) *
        (customNChooseK(n,2))/customNChooseK(10,3))+
        (customNChooseK(n,3)/customNChooseK(10,3)));
    % probability of failing for a 4 questions exam
    y4qs(n) = 1 - (((10-n)*(customNChooseK(n,3)) /
        customNChooseK(10,4))+ (customNChooseK(n,4) /
        customNChooseK(10,4)) + (customNChooseK(10-n,2) *
        (customNChooseK(n,2)/customNChooseK(10,4))));
end

figure
hold on

xAxis = 1:1:10;
yAxis = [y3qs; y4qs];
xlabel('number of topics studied n')
ylabel('proability of failing')
bar(xAxis, yAxis);
title('Blue plot is for a 3 questions
    exam and red is for a 4 questions exam.')

function [m] = customNChooseK(n,k)
    if(n < k)
        m = 0;
    else
        m = nchoosek(n,k);
    end
end
```

Section B - Question 1e

```
function [X] = simulation3Questions(n)
    % number of topics
    topics=1:10;
    % draw for exam questions
    questions = randperm(numel(topics),3);
    % draw for studied subjects
    studiedSubjects = randperm(numel(topics),n);

    % get common topics
```

```
    pos=intersect(questions, studiedSubjects);
    % return 1 if pass or 0 if fail
    if(length(pos) >= 2)
        X = 1;
    else
        X =0;
    end
end
```

Section C - Question 1f

Extended stochastic simulation.

```
function [X] = simulationEmpiricalMean(n, N)
    passCount = 0;
    % run N times keeping count of amount of pass
    for i = 1:N
        % number of topics
        topics=1:10;
        % draw exam questions
        questions = randperm(numel(topics),3);
        % draw studied subjects
        studiedSubjects = randperm(numel(topics),n);

        % get common topics
        common=intersect(questions, studiedSubjects);
        if(length(common) >= 2)
            passCount = passCount + 1;
        end
    end
    % return empirical mean
    X = passCount / N;
end
```

Section D - Question 1g

Code for running the extended stochastic simulation N times and displaying the % of times that the simulated empirical mean falls within a confidence interval.

```
% count for times mean falls within interval
count = 0;
% 1000 or 10000 depending on simulation to run
N = 1000;
% Amount of times to run simulation
size = 300;
% Values of the confidence interval (here for N = 1000)
```

```
lo = 0.79272;
hi = 0.84068;

% Run simulation size times and increment count if
% the mean falls within the CI
for i = 1:size
    % function from appendix section C
    % returns the empirical mean.
    simMean = simulationEmpiricalMean(7, N);
    if(simMean >= lo && simMean <= hi)
        count = count + 1;
    end
end
% Display % of means falling within confidence interval
disp(count/size * 100);
```

Section E - Question 2a

Code for running obtaining the PMF for all 3 questions from the data set and plot the results.

```
data = load('data.txt');
nstudents = length(data(:,1));

countsQ1=zeros(1, 101);
countsQ2=zeros(1, 101);
countsQ3=zeros(1, 101);

gradesQ1 = data(:,1);
gradesQ2 = data(:,2);
gradesQ3 = data(:,3);

% Go through all students keeping counts
% For all grades
for i=1:nstudents
    countsQ1(gradesQ1(i) + 1) =
        countsQ1(gradesQ1(i)+ 1) + 1;
    countsQ2(gradesQ2(i) + 1) =
        countsQ2(gradesQ2(i)+ 1) + 1;
    countsQ3(gradesQ3(i) + 1) =
        countsQ3(gradesQ3(i)+ 1) + 1;
end
figure
hold on

combined = [countsQ1/nstudents;
```

```
    countsQ2/nstudents; countsQ3/nstudents];
bar(0:100,combined, 1,'stacked');
xlabel('PMF');
ylabel('Grade');
legend(['Question 1'; 'Question 2'; 'Question 3']);
title('PMF for exam questions');
bar(xAxis, yAxis);
```

Section F - Question 2b and 2c

Code for calculating means, variances and confidence intervals for question 2 and 3 conditioned on question 1.

```
% Q2 b
m = 1000;
data = load('data.txt');
means = 1:10;
vars = 1:10;
cis =[];

% go through all 10 categories
for i = 1:10
    index=1;
    studentsInCat = zeros([1 1]);
    % go through all students and
    % add to category if grade from q1 is in
    % the current category
    for j = 1:m
        grade = data(j, 1);
        if (grade < i*10) && grade >= (i*10)-10
            studentsInCat(index,1) = j;
            index = index + 1;
        end
    end

    % get list of grades from all
    % students who fall in the category
    % and their means
    nbStudents = numel(studentsInCat);
    index = 1;
    % array to contain all grades of
    % all students for q2 and 3 from the
    % current category
    grades = 1:2*nbStudents;
    for cQuestion = 2:3
        gradeSum = 0;
```

```
        for cStudent = 1:nbStudents
            grade = data(studentsInCat(cStudent)
                ,cQuestion);
            grades(index) = grade;
            index = index + 1;
            gradeSum = gradeSum + grade;
        end

        % keep variance and mean for
        % current question / bracket
        vars(i,cQuestion-1) = var(grades);
        means(i,cQuestion-1) = gradeSum/nbStudents;

        % 2c - Compute confidence interval
        % Using CLT
        stdDev = sqrt(vars(i,cQuestion-1))
            /sqrt(nbStudents);
        cis(i,cQuestion-1,1) = means(i)-(2*stdDev);
        cis(i,cQuestion-1,2) = means(i)+(2*stdDev);
    end
end

figure
hold on

% plot means for both questions & all 10 categories
xaxis = 10:10:100;
yaxis = [reshape(means(:,1),1,10);
    reshape(means(:,2),1,10)];
xlabel('Mark for Question 1');
ylabel(' mean mark conditioned on question one');
title('Mean marks with confidence intervals');
bar(xaxis, yaxis);

errq1 = ones(10,1);
errq2 = ones(10,1);

% get difference between high and lower bound as error
for i = 1:10
    errq1(i) = cis(i,1,2) - cis(i,1,1);
    errq2(i) = cis(i,2,2) - cis(i,2,1);
end

% plot error bars
legend({'Question 2','Question 3'})
errorbar(xaxis - 1.5,reshape(means(:,1),1,10), errq1);
```

```
errorbar(xaxis + 1.5,reshape(means(:,2),1,10), errq2);
```