Optimal transport via general cost functions

Petr Mokrov

Moscow Institute of Physics and Technology

2020

Plan of presentation

- Task formulation
- Applications
- Progress so far
- References

Task formulation

Let $\mathbb{P}$ and $\mathbb{Q}$ be probability distributions on $\mathbb{R}^n$, $c : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+$ is cost function (e.g. $c(x, y) = \|x - y\|_p^p$)

Monge's optimal transportation problem

$$\int_{\mathcal{X}} c(x, T(x)) d\mathbb{P}(x) \to \min_T$$

over all measurable maps $T : \forall A \in \mathcal{B}(\mathbb{R}^n) : \mu_{\mathbb{Q}}[A] = \mu_{\mathbb{P}}[T^{-1}(A)]$

Kantorovich's optimal transportation problem

$$\int\limits_{\mathbb{R}^n \times \mathbb{R}^n} c(\mathrm{x}, \mathrm{y}) d\pi(\mathrm{x}, \mathrm{y}) \to \min_{\pi \in \prod(\mathbb{P}, \mathbb{Q})}$$

Dual Kantorovich's optimal transportation problem

$$\int\limits_{\mathbb{R}^n} \phi(x) d\mathbb{P}(x) + \int_{\mathbb{R}^n} \psi(x) d\mathbb{Q}(x) \to \max_{\phi(x) + \psi(y) \leq c(x,y)}$$

C. Villani: Topics in optimal transportation

Task formulation

To solve the dual problem we introduce c-**transform**:

$$\phi^c(y) = \min_{x \in \mathbb{R}^n} (c(x, y) - \phi(x))$$

To construct ϕ^c from ϕ we consider argument of c-transform:

$$h_\phi(y) = \arg\min_{x \in \mathbb{R}^n} (c(x, y) - \phi(x)) \Rightarrow$$
$$\Rightarrow \phi^c(y) = c(T_\phi(y), y) - \phi(T_\phi(y))$$

We can model $\phi(x)$ and $h_\phi(y)$ as NNs and consider the following min-max optimization:

$$\int_{\mathbb{R}^n} \phi(x) d\mathbb{P}(x) + \int_{\mathbb{R}^n} c(h(y), y) - \phi(h(y)) d\mathbb{Q}(y) \to \max_{\phi} \min_{h}$$

The optimal transportation map T can be found from optimal potentials ϕ^* and ϕ^{c*}

Applications

- Latent space optimal transport
- Image-to-image style transfer
- Domain adaptation
- Image super resolution

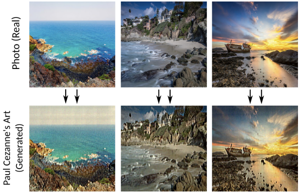


Figure: Style transfer

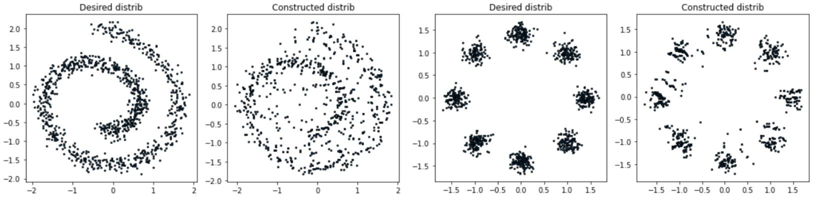
Figure: Color transfer

Korotin et. al.: Wasserstein-2 Generative Networks

Progress so far

Optimal transport mapping via ICNNs

- article link: https://arxiv.org/pdf/1908.10962.pdf
- The authors solves Optimal Transportation problem for quadratic cost function $c(x, y) = \|x - y\|_2^2$
- The authors exploits ICNNs, which represent convex functions
- Our reproduction link: https://github.com/PetrMokrov/OTGeneralization
- Our reproduction images:



Progress so far

Three-Player Wasserstein GAN via Amortised Duality

- article link: https://www.ijcai.org/Proceedings/2019/0305.pdf
- The authors generalize Wasserstein GAN by considering arbitrary cost function and introducing c-transform as additional optimization problem
- 3 - player min-max-min optimization problem
- Our reproduction images:

References

- **OT via ICNN**: Optimal transport mapping via input convex neural networks
  https://arxiv.org/abs/1908.10962
- **ICNN**: Input convex neural networks
  https://arxiv.org/abs/1609.07152
- **W-2 Generative Networks** Wasserstein-2 Generative Networks
  https://arxiv.org/abs/1909.13082
- **3 Player Wasserstein GAN** Three-Player Wasserstein GAN via Amortised Duality
  https://www.ijcai.org/Proceedings/2019/0305.pdf
- **Topics on OT**: Topics on Optimal Transportation by C. Villani
  amazon