Contents

1 Introduction **1**

2 The Biogeochemical Model **4**

3 Model Equations **13**

4 Analysis of the Differential Equation System **15**
4.1 Units . . . 15
4.2 General Estimations . . . 16
4.3 The NPZ Model as Initial Value Problem . . . 18
4.4 Mass Conservation . . . 18
4.5 Continuity and Differentiability . . . 20
4.6 Existence and Uniqueness of Solutions . . . 23
4.7 Positivity of Solutions . . . 25
4.8 Continuously Dependency of Solutions . . . 28
4.9 Differential Dependency of Solutions . . . 29

5 Equilibria **31**
5.1 Model LLM: No quadratic loss term . . . 35
5.2 Model QQM: With quadratic loss terms for P and Z . . . 47
5.3 Model LQM: With optional term for Z . . . 55
5.4 Model QLM: With optional term for P . . . 57

6 Summary and Outlook **62**
6.1 Summary . . . 62
6.2 Outlook . . . 63

7 Bibliography **65**

8 Appendix **67**
8.1 Tables: Variables and Parameters . . . 67

1 Introduction

The oceans are characteristic for Earth´s surface. Not only do they cover the majority of the globe, they also enclose many biotopes. The ocean´s physics, chemistry and biology, changings in currents, temperature, salinity, acidity, or biota, have a huge impact on human life, via climate changes, sea level rise, changes within the food chain, or on the amount of CO_2 within the atmosphere. Major changings, as declining pH-value, attenuation of the thermohaline circulation, or extinction of certain plankton species, would influence life on earth substantially.

Plankton - ocean organisms drifting in the water - plays an important role within the cycle of CO_2 by consuming carbon, and is an irreplaceable part of the marine food chain. There are many plankton species of different sizes in the ocean. Phytoplankton (e.g. cyanobacteria or diatoms) lives by building biomass from light energy (photosynthesis) and is dependent on nutrients like nitrate and phosphate. Herbivore plankton, e.g. ciliates, rotifers, and copepods, feed on plankton. Nitrogen is a primary production (biomass production) determining factor, so it limits the amount of the carbon which is consumed. Knowledge on the amount of nitrogen, carbon etc. found in ocean water, and on the plankton population (including plankton blooms etc.) is important to biological, ecological, chemical and other subjects. Furthermore, plankton itself is a basic food resource for marine animals some of which again are important protein sources for human nourishment. Understanding plankton life cycles is one of the major aims of ocean science (for more details on plankton and on the biology and geography of the oceans see [Ott]).

For modelling natural processes concerning some kind of growths and losses, mathematicians often use differential equations or systems of differential equations. Lotka Volterra models (as described in [Pruess] in example (e) on page 9) are examples for such modelling with two variables. They describe the occurence of one prey species with the first variable and one predator species with the second variable. The Lotka Volterra model is a system of two differential equations. Each equation represents the changing in the population in one of the species, dependent on the occurence of the other one.

The plankton life cycle can be modelled with differential equation systems either with three or four variables: the amount of nutrients (N), the population of phytoplankton (P), the population of zooplankton (Z or H or h), and in some models the amount of detritus (D, particulate organic material, e.g. from dead and decomposed plankton cells). Parameters are used to describe the influence of some natural processes, like daylength or the efficiency of biomass production.

For deeper investigation of plankton life in the ocean more advanced models are

needed, for example to understand or predict plankton blooms or attenuation of certain plankton species.

The degree of extension or simplification of a model depends on the aim of the research. In some cases, like investigating blooms and attenuation of plankton species, it might be necessary to introduce an extra variable Q to allow the scientist to examine the interaction of two different classes of phytoplankton, represented by P and Q, and their interaction with herbivore zooplankton H (NPQH model, [Evans]). Sometimes for investigation of equilibria and stability a reduced model with two variables P and Z is more appropriate ([Evans, Parslow]).

An extended model was developed from a Volterra Lotka model by Evans and Parslow [Evans, Parslow]. Their NPH model characterizes interdependencies between nutrition N, phytoplankton P, acting as prey, and herbivore zooplankton H, which represents the predator here.

Biogeochemical models as NPZ or NPZD models can be zero or one dimensional models. The NPZ model was developed from the Evans and Parslow NPH model [Evans, Parslow] and introduced in the dissertation *Data-assimilation studies of marine, nitrogen based, ecosystem models in the North Atlantic Ocean* (written by M. Schartau, Kiel 2001, [Schartau]). It is a zero dimensional model: the concentrations of nutrients and plankton are calculated as a mean value within the *mixed layer*. The mixed layer is the upper layer of the ocean with contact to the atmosphere; it belongs to the euphotic zone where photosynthesis is possible. The same paper presents a one dimensional model as well: the NPZD model allows variations within the vertical axe of the mixed layer.

Differential equation systems can be used to investigate the general behaviour of the variables by a theoretical mathematical analysis. Finding equilibria by fixing the derivatives at zero and then solving the system, as it has been done with Schartau's one dimensional NPZD model in [Heinle], is an example for this kind of analysis. Furthermore, models with differential equations are used for numerical approaches. They are discretized, implemented, and computed with certain initial values and parameters in order to calculate or forecast natural processes (weather, plankton blooms...). The two models described in [Schartau] are analyzed this way in his dissertation. The mathematical equations have been discretized and implemented. A great part of the work refers to the optimization of parameters: the author studies methods for parameter choice that guarantee minimized deviations between model output and the results of actual measurement results from the ocean. The author used a three dimensional ocean circulation model with measured data from the *Bermuda Atlantic Time-series Study* (BATS) and other experiments to determine boundary conditions for the parameters in his models.

More information on the measuring data sources (BATS, NABE and OWS INDIA) can be found at [JGOFS].

In this diploma thesis, the zero dimensional NPZ model equation system from [Schartau] is analyzed with a theoretical approach. At first the model equations and the parameters are introduced. Then some information on units and mass conservation is given. Existence and uniqueness of solutions, positivity and dependance of solutions are considered then. The topic of equilibria is more extensive and is put into a separate section for a better overview. For equilibria some numerical experiments are done. The code is written in MATLAB (Student Version MATLAB 7.10.0 R 2010 a). The figures with the results are shown in section 5, programs and result tables are made available on github:

<https://github.com/PetraFuhs/Analysis_marine_ecosystem_model> .

The results may be helpful to interpret results from model implementations, for choosing parameters, or developing improved models. The used methods can easily be applied to other model equation systems for a theoretical analysis. Nevertheless, all results are of a theoretical character and cannot give information on actual occurrence of plankton in certain parts of the ocean.

2 The Biogeochemical Model

In this section the NPZ model [Schartau] is introduced. The model is a system of three nonlinear ordinary differential equations. It consists of three variables N, P, and Z and a number of parameters (for the equation system see 3.0.1). It can be used in different versions either with linear plankton loss functions, quadratic plankton loss functions, or linear loss function for one of the plankton variables and quadratic loss function for the other one. The parameters used in the model are described below in the parameter list. The model is applicable for the uppermost water layer in the ocean (more information on this layer follows within this section). The depth of this layer is controlled by parameter M. The parameter M (see below) is varied during a system run within implementations, and hence not constant in time unless we consider a certain point in time. Additionally, this parameter causes the model to be not continuous. This issue is discussed in the parameter list. All other parameters are considered constant (in some cases simplified into average value constants).

At first a figure with a sketch from [Schartau] (Figure 2.0.1) is depicted to show the idea of the model. In the figure the variables are the big red coloured letters, all parameters that are used in all model versions are smaller and black, and the optional parameters are green symbols. Below the figure, the variables, followed by the parameters, are explained, completed by the explanation of some auxiliary functions. All important variables, parameters and auxiliary terms with units and domains are centralized in tables 1 to 3 in the appendix (8). The model equations are introduced in the following section.

The figure from [Schartau], Figure 2.0.1, illustrates the idea of the model and the meaning of variables and parameters. Below the parameters are explained in detail.

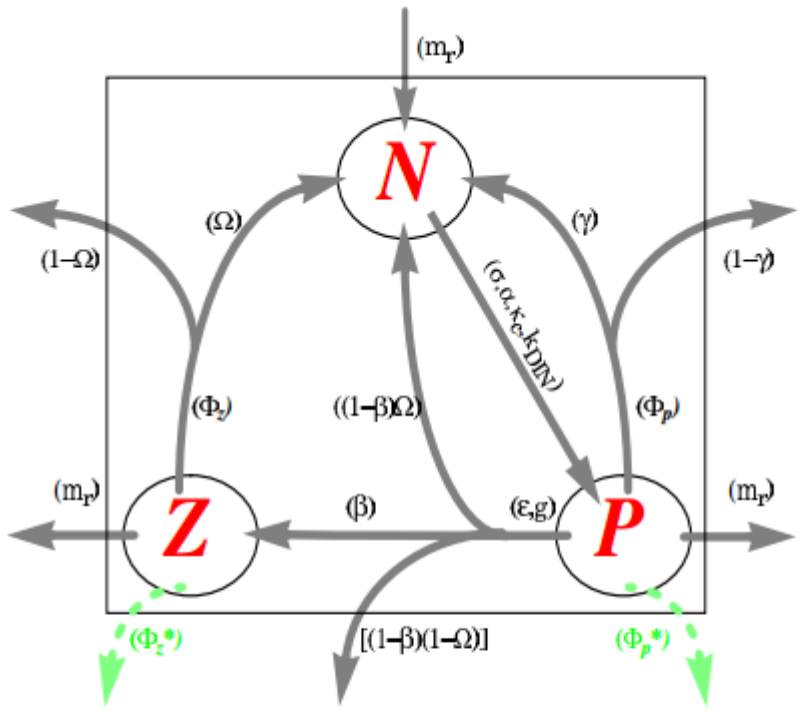

Figure 2.1: Structure of the nitrogen-based ecosystem model with three state variables, phytoplankton (P), herbivorous zooplankton (Z) and dissolved inorganic nitrogen (N). The arrows that point outside of the mean square box indicate a loss of mass out of the mixed layer, yielding the non-conservative property of the model. The greek letters (in brackets) are the parameters that are involved in the processes represented by the arrows.

Figure 2.0.1: NPZ model (from [Schartau]), with original caption

The figure illustrates gains and losses of the three variables and mutual interdependencies. The model has no mass conservation (some of the arrows point out of the picture or lead into the picture from the outside). The figure shows gains are mainly caused by currents, while losses within the model are caused by currents and by remineralisation and sinking out of the layer. The mass conservation issue is discussed in section 4.4. The not explained parameters in the figure on the arrow from N to P are only used for calculating resp. estimating other parameters and are not mentioned here in detail.

As mentioned before, the model consists of the three variables N, P and Z. Basically the domains of all variables are $\mathbb{R}$. Due to the fact that the variables represent concentrations which always are nonnegative, in all applications only

values within $[0, \infty)$ are expected. Therefore the model should provide positivity, i.e. the feature to obtain nonnegative results in all cases of nonnegative initial values in all three variables. The positivity of this model (unless the parameter β is extended) is proven in section 4.7. Furthermore all denominators of fractions should be unequal zero; for this reason the domain of N is constrained:

Let k_{min} be the lowest possible value of the (later introduced) nonnegative parameter k, then $N \in (-k_{min}, \infty)$. The fraction u that makes such a constraint preferable is given in the list of variables in the next section of this chapter.

The model immanent parameters are taken from [Schartau], as are estimations of the considered ranges. The upper bounds are subject to changing in [Schartau]. For example in the data assimilation process (the process of feeding actual measured values into the computing program in order to improve the quality of the results) he finds two useful sets of parameters for the NZP model with quadratic loss terms that include parameter values of $\mu, g, \varepsilon, \beta$ greater than their given upper bound. The lower bounds still hold, especially all parameters are greater than zero. In case of parameters that theoretically represent a fraction and therefore should range in the interval $(0, 1]$, and are fixed at a value greater than 1 for numerical reasons, this leads to a discrepancy. The reason for effects like this is that the data assimilation process of the parameters has shown that a compromise in the bounds of the parameters can lead to a better match of measured vs. calculated values. Nevertheless, in case of β this theoretically could cause the model to loose its positivity. This issue is addressed in section 4.7. Other parameters' extended upper bounds do not have serious impact on the theoretical analysis.

The variables are as follows:

- $N \in (-k_{min}, \infty)$, $[N] = \frac{mmolN}{m^3}$. This variable stands for nutrients, based on dissolved inorganic nitrogen. In the model the variable N is used within the growth function $u = \frac{N}{N+k}$ (k is a positive constant parameter; u and k will be explained later). To avoid a zero in the denominator the domain of N is constrained. Let k_{min} be the smallest possible parameter k ($k_{min} > 0$) and let $\mathbb{R}_+$ be the set of all nonnegative real values. Then using $N \in (-k_{min}, \infty)$ will result in a continuous differentiable function u, the fraction has no singularity. In section 4.7 the NPZ model is proven to have the positivity feature, i.e. all initial values $(N, P, Z)^T \in \mathbb{R}^3_+$ obtain nonnegative results, therefore this constraint does not affect the applications of the model.

  The variable N can be transferred into the variable $\Delta N = N_D - N$ without impact on the basic mathematical features of the model. This might be convenient if a zero is desired explicitely in all variables; e.g. if the trivial

equilibrium (see section 5) $(N_D, 0, 0)^T$ would be easier to handle if all values are zero, it is replaced by the equivalent $(\Delta N, 0, 0)^T$ with $N = N_D \Rightarrow \Delta N = N_D - N = 0$. Both versions can be used similarly. For reasons of clarity and comprehensibility this thesis waives the term ΔN.

- $P \in \mathbb{R}$, $[P] = \frac{mmolN}{m^3}$. This variable refers to the concentration of phytoplankton (biomass).
- $Z \in \mathbb{R}$, $[Z] = \frac{mmolN}{m^3}$: This variable refers to herbivorous zooplankton concentration (biomass). Some authors use H or h instead of Z.

Following parameters are relevant in this thesis:

- $M \in [1, 400]$, $[M] = m$, is the layer depth of the mixed layer. In [Schartau] this parameter is used to control the changing of the seasons within the implementation. The value is determined in daily or steps with the help of temperature values from a three dimensional ocean model. The mixed layer is shallow in summer and deep in winter; the value varies between 1 m and 200 m. For numeric experiments the paper allows depths up to 400 m, so $M \in [1, 400]$. These boundaries should not be extended. If the NPZ system models the processes within the layer, its depth should not draw close to zero. And too great depths lead into the dysphotic or even aphotic zone without photosynthesis taking place; all assumptions on phytoplankton growth are not valid there.

  Due to the discretization the parameter itself is introduced as a step function in [Schartau], hence it is not continuous and not differentiable, and above all, it is not constant. Nevertheless, the actual depth of the mixed layer is assumed to be continuous differentiable due to the fact that the modelled natural processes, as changings in temperature or daylength, are smooth. Furthermore in this thesis the model gets analyzed at (arbitrary) certain time points, as it can be seen in section 5. If the step function M is stated as continuous from below (without loss of generality), then for every fixed $t \in \mathbb{R}_+$ an unique value for M is provided. This way t can be treated as another parameter, and so the model can be considered as autonomous system.

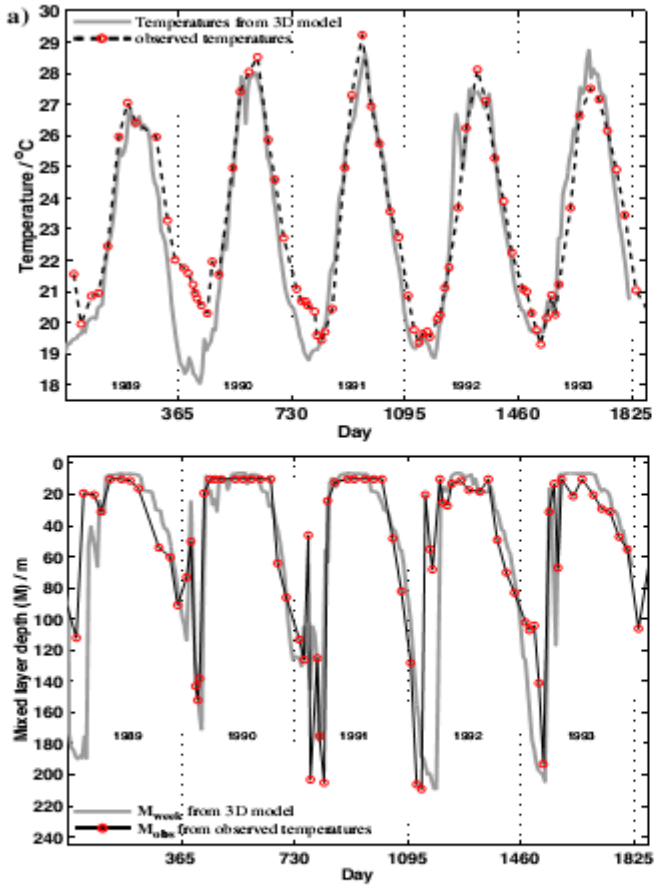

Figure 2.2: TOP: Weekly mean temperatures near Bermuda of the upper ocean layer from a general circulation model of the North Atlantic, under realistic forcing taken from the ECMWF reanalysis and the observed temperatures from averaged CTD profiles (BATS: biweekly-monthly measurements). BOTTOM: The resulting mixed layer depths M_{obs} and M_{week} when a 0.2°C temperature criterion is applied.

Figure 2.0.2: Temperature and mixed layer depth, Bermuda, from 1989 to 1993 (from [Schartau]) , with original caption

Figure 2.0.2 ([Schartau], fig.2.2) gives an impression on the actual behavior of M within a period of five years.

- $N_D \in [1, 20]$, $[N_D] = \frac{mmolN}{m^3}$ is a parameter named deep dissolved nitrogen concentration. It refers to the concentration of nitrogen in the layer underneath the mixed layer.

  The parameter N_D refers to the concentration of nutrients in the deeper layer.

According to [Schartau], section 2.2., authors prefer two different approaches for N_D: fixing it as a constant or calculating it with the help of information on chemical topics regarding NO_2 and NO_3. Here (as in [Schartau]) the more common constant version is chosen.

This parameter is used within the summand F_N in the equation for N. Usually in the ocean the inequalitiy $N_D \geq N$ holds. The model does not guarantee this feature; a value for N_D is chosen before N is calculated. Nevertheless, within the numerical analysis of the equilibria with three different values of N_D, three different parameter sets and 400 values for M the case $N > N_D$ did not occur once; the model results for N were equal or less N_D in all tested cases. On the other hand for the positivity (section 4.7) $N_D \geq 0$ is sufficient, $N_D > N$ is not necessary. Hence the used requirement $N_D \geq 1 \frac{mmolN}{m^3}$ is sufficient. The lower bound should not be lowered as extremely low values of nutrients below the mixed layer are not intended in the model. The upper bound is an estimation anyway and might be extended; higher values mathematically lead to more nutrient accumulation in the mixed layer.

To estimate the upper bound of the domain and for numeric experiments in section 5 real data is used, at depth of scarce below $400m$ (BATS Station $1.5 - 2 \frac{mmolN}{m^3}$; NABE Station $8 \frac{mmol}{m^3}$; OWS INDIA $16 \frac{mmolN}{m^3}$; source for this values: Schartau, personal communication).

- $\mu \in [0.100, 3.000]$, $[\mu] = \frac{1}{d}$, is a daily averaged light limited phytoplankton growth rate parameter. It is used as a factor for the phytoplankton growth function (see u). It is daylight dependent and therefore time dependent. The lower bound should be kept greater zero; in case of $\mu = 0$ the phytoplankton growth function u vanishes; the model loses its aim. Violating the upper bound does not any harm and is done in one of the used parameter sets. In [Schartau] μ is replaced by a constant value for finding useful parameter sets. The same constant is used here. The growth rate is influenced by the duration of daylight and is calculated by an approximation (for details, see [Schartau], section 2.2.1.).

- $k \in [0.100, 0.700]$, $[k] = \frac{mmolN}{m^3}$, is the half-saturation constant of N uptake rate, necessary for the phytoplankton growth function u. The bounds are justified by biological reasons. A value of $k = 0$ would fix the phytoplankton growth function at 1 which is not desired. If other changings in the boundaries can be made is a biological question and cannot be discussed here.

- $m_r \in [0.010, 3.000]$, $[m_r] = \frac{m}{d}$, stands for the turbulent mixing rate for gain and loss of nutrients and plankton in the mixed layer due to water currents. This parameter appears within the term $\frac{m_r}{M}$ in the differential equations of all three variables N, P and Z. The lower bound is fixed to avoid $m_r = 0$ which would lead to a loss of the F_N term and changings in the prefactors of P and Z. For positivity (section 4.7) the requirement $m_r > 0$ can be loosened to $m_r \geq 0$.

- $g \in [0.100, 1.000]$, $[g] = \frac{1}{d}$. This parameter refers to the maximum grazing rate and is necessary to calculate G, a function that models herbivore grazing, and hence appears as a factor for Z in two of the differential equations (see below). More details on the behavior of G is found in section 4.2.

- $\varepsilon \in [0.100, 3.000]$, $[\varepsilon] = \frac{m^6}{(mmolN)^2 d}$. This constant parameter is the prey capture rate and as well is necessary to calculate G, see below.
  Note that this parameter is the same letter as the ε mathematicians use for small neighbourhoods. ε will be used in both senses in this thesis; this will not cause confusion because of the particular context. More details on G are found in section 4.2.

- $\Omega \in [0.010, 1.000]$, $[\Omega] = 1$, is a remineralization parameter, modelling fractions of cells that are remineralized into nutrients within the mixed layer in three cases: remineralization of zooplankton, remineralization of cells during the grazing process, resulting in an increase of N, and remineralization of cells during the grazing process, resulting in a loss of material out of the mixed layer. Ω needs to be greater zero. Otherwise the influence of zooplankton in the nutrient equation could not be modelled. As a fraction naturally Ω is limited to 1. This value is exceeded in one of the parameter sets [Schartau] uses; the impact on positivity is described in section 4.7, all other features are not influenced.

- $\gamma \in [0.010, 1.000]$, $[\gamma] = 1$, is a remineralization parameter, modelling the fraction of phytoplankton cells that are remineralized, so that the nutrients that emerge from this process either lead to an increase of N within the mixed layer or are flown out of the mixed layer. As other fractions, this parameter should range in $(0, 1]$; the boundaries of γ are not extended in any numerical experiments. A value of $\gamma = 0$ would change the behavior of the nutrient equation.

- $\beta \in [0.100, 1.000]$, $[\beta] = 1$, is the constant zooplankton assimilation efficiency

parameter (efficiency of building molecules into biomass). This parameter must not be equal zero. Otherwise the model would not not allow zooplankton growth on phytoplankton grazing; the material would be added to the nutrients instead. This would be a huge change in the model behavior and applicability. A value of $\beta > 1$ occurs in a parameter set and is discussed in section 4.7.

- $\Phi_P \in [0.010, 0.100]$, $[\Phi_P] = \frac{1}{d}$, is a constant parameter for linear losses of phytoplankton. Together with the parameter γ it refers to increasing of N due to remineralization and to loss of mass out of the layer. This loss parameter is used in all model versions including versions with quadratic loss terms.

- $\Phi_Z \in [0.010, 0.100]$, $[\Phi_Z] = \frac{1}{d}$, is a constant parameter for linear losses of zooplankton. Together with the parameter Ω it refers to increasing of N due to remineralization and to loss of mass out of the layer. It is referred to as Φ_h or Φ_H by some authors. This loss parameter is used in all model versions including versions with quadratic loss terms.

- $\Phi_P^* \in [0.010, 1.000]$, $[\Phi_P^*] = \frac{m^3}{mmolNd}$, is an optional parameter and is used to model enhanced phytoplankton mortality. It can be used to obtain model versions with (constant) quadratic phytoplankton loss rates, e.g. due to enhanced sinking out of the layer or enhanced feeding. It is used additionally to Φ_P.

- $\Phi_Z^* \in [0.010, 1.000]$, $[\Phi_Z^*] = \frac{m^3}{mmolNd}$, is an optional parameter and is used to model enhanced zooplankton mortality. As the parameter Φ_P^* for quadratic losses of P, it can be used to obtain model versions with (constant) quadratic zooplankton loss rates. It is used additionally to Φ_Z.

Now the terms u, G and F_N that are used in the model equations are explained.

- $G = \frac{g\varepsilon P^2}{g+\varepsilon P^2}$, $[G] = \frac{1}{d}$, is a zooplankton growth function, concerning all three equations of the model. It depends on the concentration of phytoplankton and the grazing rate and prey capture rate parameters. Greater values of G will increase the concentration of nutrients and zooplankton and decrease the concentration of phytoplankton. The boundaries of the value of G are estimated in section 4.2, where also a figure of the behavior can be found. It is also analyzed there if it is possible to allow the used parameters to be equal zero. Biologists classify G as a Holling type 3 function, see figure 18 in [Maynard Smith].

- $u = \frac{N}{k+N}$, $[u] = 1$, is a growth function for phytoplankton. As for G the estimation and behavior are given in section 4.2. Great values of u decrease nutrient and increase phytoplankton concentration.

- $F_N = \frac{m_r}{M}(N_D - N)$, $[F_N] = \frac{mmolN}{m^3 d}$, is a function that describes the increase in the concentration of N within the mixed layer caused by entrainment of nutrients from the layer below. In [Schartau] the term is $\frac{m_r + w_e}{M}$ with w_e being a parameter on entrainment velocity. The parameter $w_e > 0$ was removed due to recent experience with the model (source: Schartau, personal communication). The removal of w_e does not influence the mathematical properties of the model and could be easily added again if necessary. As before, estimation and behavior are laid out in section 4.2.

Variables, parameters and terms are put together in tables 1 to 3 in the appendix in section 8.

3 Model Equations

In this section, the NPZ model differential equation system is introduced. The model is a nonautonomous system of three nonlinear ordinary differential equations of first order with three variables and with rational terms, but without singularities.

The variables and parameters have been explained in the section before. The model is displayed in the common way first. After that it is shown in a more detailed way in order to make the time dependencies within the model visible. A short discussion of the auxiliary terms u, G and F_N will be done in the following section.

As mentioned in the first section, the model is developed by M. Schartau (see [Schartau], Appendix, Equations A1, A2, A3) from the NPH models according to [Evans, Parslow].

Let $\mathbb{R}_+$ be the set of nonnegative real values including zero, then the differential equation system of the model is as follows:

$$
\begin{aligned}
&f : (-k_{min}, \infty) \times \mathbb{R}^2 \times \mathbb{R}_+ \to \mathbb{R}^3, \\
&f_1(N,P,Z,t) = \tfrac{dN}{dt} = [-\mu u + \gamma \Phi_P]P + [(1-\beta)G + \Phi_Z]\Omega Z + F_N \\
&f_2(N,P,Z,t) = \tfrac{dP}{dt} = [\mu u - \Phi_P - \tfrac{m_r}{M}]P - GZ - \Phi_P^* P^2 \\
&f_3(N,P,Z,t) = \tfrac{dZ}{dt} = [\beta G - \Phi_Z - \tfrac{m_r}{M}]Z - \Phi_Z^* Z^2.
\end{aligned}
\tag{3.0.1}
$$

The last terms in the second and third equation, $\Phi_P^* P^2$ and $\Phi_Z^* Z^2$, are optional to allow model versions with linear and quadratic loss terms.

Basically some parameters are time dependent themselves or functions of parameters, others are constant. The time dependent parameter μ can be replaced with an average constant value, which is done here. N_D is treated as constant here in spite of the fact that some authors prefer a more chemical approach which leads to a time dependent function $N_D(t)$; this issue has been mentioned in section 2. The parameter M is a control parameter for the seasons, it is described in section 2, too. For analyses at certain times it can be treated constant.

To visualize the structure of the auxiliary terms u, G, F_N, and the dependencies that could be important if the system was to be discretized and implemented, the model equations are printed again, considering all dependencies and compound terms:

$$\frac{dN}{dt} = [-\mu(t)\frac{N(t)}{k+N(t)} + \gamma\Phi_P]P(t) + [(1-\beta)\frac{g\varepsilon P(t)^2}{g+\varepsilon P(t)^2} + \Phi_Z]\Omega Z(t) + \frac{m_r}{M(t)}(N_D(t) - N(t))$$
$$\frac{dP}{dt} = [\mu(t)\frac{N(t)}{k+N(t)} - \Phi_P - \frac{m_r}{M(t)}]P(t) - \frac{g\varepsilon P(t)^2}{g+\varepsilon P(t)^2}Z(t) - \Phi_P^* P(t)^2$$
$$\frac{dZ}{dt} = [\beta\frac{g\varepsilon P(t)^2}{g+\varepsilon P(t)^2} - \Phi_Z - \frac{m_r}{M(t)}]Z(t) - \Phi_Z^* Z(t)^2.$$

The denominators of all fractions are greater than zero:

- $M > 0$ by definition
- $k + N > 0$ because $k \geq k_{min} > 0$ and $N > -k_{min}$
- $g + \varepsilon P^2 > 0$ because $(g > 0 \wedge \varepsilon > 0 \wedge P > 0)$

The typical behavior of the fractions G and u is presented in section 4.2, together with a figure of F_N, u and G there .

4 Analysis of the Differential Equation System

This and the following section (section 5) are related to the analysis of the NPZ equation system. In this section at first the all variables and parameters are transformed into dimensionless quantities. After some general estimations on the auxiliary terms u, G and F_N and the introduction of the NPZ model as an initial value problem, it is shown that the system does not provide mass conservation. It is proven that the system is continuous and local (not global) Lipschitz continuous, and the local existence of a unique solution is concluded. After that a view on the positivity of solutions (in case of nonnegative initial values) is taken and requirements to the parameters for positivity are discussed. This section ends with a consideration on continuous and differentiable dependency.

The following sections' topics are the equilibria of the four possible model versions. It includes the results of the MATLAB programs (calculating equilibria with certain parameter sets and their stability).

4.1 Units

All values of variables, parameters and compound or auxiliary terms are listed in section 2 and in the tables 1 to 3 (Appendix, section 8). With the following transformation from dimensionful to dimensionless variables and parameters the proofs in section 5 will be easier to handle.
For every variable and parameter a dimensionless variable resp. parameter is defined, using the following method (applied from [Slawig], section 2.4.): Let A be the dimensionful variable or parameter with $[A] = a$. Then the dimensionless variable or parameter $\widetilde{A}$ is defined to match the expression $A = \widetilde{A}a$, therefore it is defined $\widetilde{A} := \frac{A}{a}$, and is substituted into the given equations. This is done with all variables and parameters. If for example $A = \frac{B}{C}$ with some B, C, $[B] = b$, $[C] = c$, the resulting equation would be $\widetilde{A}a = \frac{\widetilde{B}b}{\widetilde{C}c}$. From the equation $A = \frac{B}{C}$ for the unit a follows $a = \frac{b}{c}$. Altogether, the units cancel each other now, and this obtains the equation $\widetilde{A} = \frac{\widetilde{B}}{\widetilde{C}}$. So this method leads to the basic equation again, now with dimensionless variables and parameters. The tilde can be skipped then; all equations are written the same way as before, but now all parameters and variables can be treated as dimensionless.

Define

- $\widetilde{N} := N\frac{m^3}{mmolN}$; $\widetilde{P} := P\frac{m^3}{mmolN}$; $\widetilde{Z} := Z\frac{m^3}{mmolN}$,
- $\widetilde{M} := M\frac{1}{m}$; $\widetilde{N_D} := N_D\frac{m^3}{mmolN}$,

- $\widetilde{\mu} := \mu d; \quad \widetilde{k} := k\frac{m^3}{mmolN}; \quad \widetilde{m_r} := m_r\frac{d}{m},$
- $\widetilde{g} := gd; \quad \widetilde{\varepsilon} := \varepsilon\frac{(mmolN)^2 d}{m^6},$
- $\widetilde{\Omega} := \Omega \quad \widetilde{\gamma} := \gamma; \quad \widetilde{\beta} := \beta$
- $\widetilde{\Phi_P} := \Phi_P d; \quad \widetilde{\Phi_Z} := \Phi_Z d,$
- $\widetilde{\Phi_P^*} := \Phi_P^*\frac{mmolNd}{m^3}; \quad \widetilde{\Phi_Z^*} := \Phi_Z^*\frac{mmolNd}{m^3}.$

Now these new dimensionless variables and parameters are substituted into the differential equations. Due to the definition of the variables and parameters with a tilde all units are cancelled. The system is exactly the same as before, but now with all variables and parameters dimensionless. The tilde will be omitted from here on.

4.2 General Estimations

For a better overview on the model and its fractions it is convenient to estimate boundaries for $u = \frac{N}{k+N}$ and $G = \frac{g\varepsilon P^2}{g+\varepsilon P^2}$ in case of nonnegative variables N and P. Additionally, the term boundaries for $F_N = \frac{m_r}{M}(N_D - N)$ are estimated.
First, the term $u = \frac{N}{k+N}$ is considered.
Lower bound:
With $N \geq 0$ and $k \in [0.100, 0.700]$ it follows $\frac{N}{k+N} \geq 0$.
Upper bound:

$$\begin{aligned}\frac{N}{k+N} &= \frac{k+N-k}{k+N} \\ &= 1 - \frac{k}{k+N}, \ \lim_{N\to\infty}\frac{k}{k+N} = 0.\end{aligned}$$

In summary it follows $\frac{N}{k+N} \in [0, 1)$.
Second, the boundaries of term $G = \frac{g\varepsilon P^2}{g+\varepsilon P^2}$ are estimated:
With $P \geq 0, g \in [0.100, 1.000], \varepsilon \in [0.100, 3.000]$, the followings boundaries are found:
Lower bound:

$$\frac{g\varepsilon P^2}{g+\varepsilon P^2} \geq \frac{0}{g} = 0,$$

Upper bound:

$$\begin{aligned}\frac{g\varepsilon P^2}{g+\varepsilon P^2} &= \frac{g^2+g\varepsilon P^2-g^2}{g+\varepsilon P^2} = \frac{g^2+g\varepsilon P^2}{g+\varepsilon P^2} - \frac{g^2}{g+\varepsilon P^2} \\ &= \frac{g(g+\varepsilon P^2)}{g+\varepsilon P^2} - \frac{g^2}{g+\varepsilon P^2} = g - \frac{g^2}{g+\varepsilon P^2}, \ \lim_{P\to\infty}\frac{g^2}{g+\varepsilon P^2} = 0\end{aligned}$$

In summary it follows $G \in [0, g)$.
If for some reason the lower bound of the parameter ε is extended to 0, in case of $\varepsilon = 0$ it holds $G = 0$ for all values of P.
If the lower bound of parameter g is set equal 0, the definition:
$P = 0 \Rightarrow G := 0$ holds.
This way an extension of the domain of g down to 0 is possible without developing problems with undefined terms.
Now the term $F_N = \frac{m_r}{M}(N_D - N)$ is estimated. The boundaries of N_D are an estimation; the greatest value used is $16\frac{mmolN}{m^3}$, so the estimated upper bound is $20\frac{mmolN}{m^3}$.
As long as the implementation of the model provides $N_D \geq N$, the following estimation holds:
$N_D - N \in [0, 20]$. The fraction $\frac{m_r}{M}$ ranges between 2.5×10^{-5} and 3.
$\Rightarrow 0 \times 2.5 \times 10^{-5} \leq F_N \leq 20 \times 3 \Rightarrow 0 \leq F_N \leq 60$.
Due to the fact that $N > N_D$ could occur, the model does not guarantee this boundaries, the Matlab programs contain comparison of the calculated F_N with this boundaries and display a notification if $N > N_D$. In the calculations with the used parameter sets (see section 5) all results have been within the boundaries.

The figure (4.2.3) shows a typical behavior of u respectively F_N in dependence of N, and of G in dependence of P. The values are exemplarily calculated with the model version with two quadratic loss terms. The used values are $k = 0.636, g = 0.982, \varepsilon = 3.473$ and $m_r = 0.107$ from parameter set II (parameter sets in table 4, section 8), and $N_D = 8$. The variables N and P are the results of calculating the equilibrium $QQM2$ (section 5) for mixed layer depths from 1 to 400.

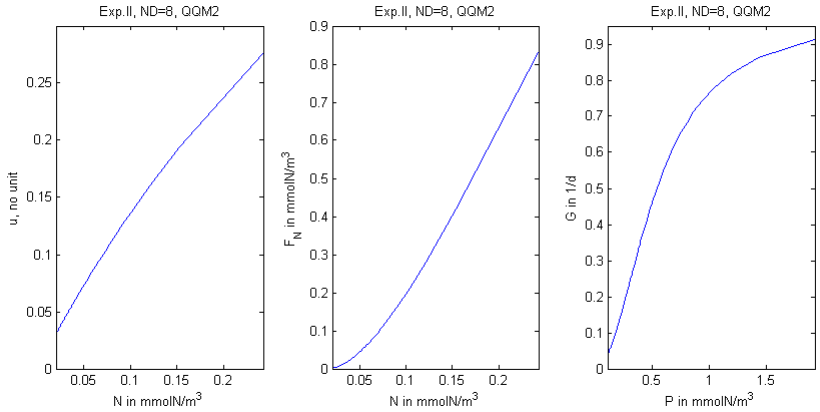

Figure 4.2.3: Behavior of u, F_N, G (from left to right); Exp.II, $N_D = 8$, Eq.QQM2

4.3 The NPZ Model as Initial Value Problem

The NPZ model as used here is a continuous function. This will be explained in the next section. It can be considered as an initial value problem (IVP) as described in [Pruess][1], equation 2.1.

Definition 4.3.1. ***Initial Value Problem***
Let $U \subset \mathbb{R}^{n+1}$ *open,* $f : U \to \mathbb{R}^n$ *continuous and* $(t_0, x_0) \in U$*. Then*
$\dot{x} = f(t, x)$,
$x(t_0) = x_0$
is called an initial value problem (IVP) .

This can be applied to equation (3.0.1):

Definition 4.3.2. ***NPZ Model as Initial Value Problem (IVP)***
$U = (-k_{min}, \infty) \times \mathbb{R}^3 \in \mathbb{R}^4$ *open;*
$f : U \to \mathbb{R}^3$,
$f(x, t) = (f_1(x, t), f_2(x, t), f_3(x, t))^T$,
$x(t_0) = x_0 = (N_0, P_0, Z_0)^T$*: initial values* N_0, P_0, Z_0 *at starting time* t_0
is called the NPZ initial value problem.

4.4 Mass Conservation

The NPZ model does not provide mass conservation. If a model provides mass conservation the sum of the right sides of the equations should add up to zero. This is not the case here, as the following proof shows:

Proposition 4.4.1. ***No Mass Conservation of NPZ Model***
The NPZ model 3.0.1 does not provide mass conservation.

Proof:

The right sides of the three model equations are added. In models with mass conservation this sum is equal zero.

[1] In [Pruess] this open subset is called G. To avoid the name of the subset getting mixed up with the growth function G this subset is renamed into U here.

$$
\begin{aligned}
&[-\mu u + \gamma\Phi_P]P + [(1-\beta)G + \Phi_Z]\Omega Z + F_N \\
&+[\mu u - \Phi_P - \tfrac{m_r}{M}]P - GZ - \Phi_P^* P^2 \\
&+[\beta G - \Phi_Z - \tfrac{m_r}{M}]Z - \Phi_Z^* Z^2 \\
&= [-\mu u + \gamma\Phi_P + \mu u - \Phi_P - \tfrac{m_r}{M}]P + [(1-\beta)\Omega G + \Phi_Z\Omega - G + \beta G - \Phi_Z - \tfrac{m_r}{M}]Z \\
&+F_N - \Phi_P^* P^2 - \Phi_Z^* Z^2 \\
&= [(\gamma-1)\Phi_P - \tfrac{m_r}{M}]P + [(1-\beta)\Omega G - (1-\beta)G + \Phi_Z(\Omega-1) - \tfrac{m_r}{M}]Z \\
&+F_N - \Phi_P^* P^2 - \Phi_Z^* Z^2 \\
&= [(\gamma-1)\Phi_P - \tfrac{m_r}{M}]P + [(1-\beta)(\Omega-1)G + \Phi_Z(\Omega-1) - \tfrac{m_r}{M}]Z + F_N \\
&-\Phi_P^* P^2 - \Phi_Z^* Z^2.
\end{aligned}
$$

Assuming that the model provides mass conservation, equation 4.4.2 holds:

$$
\begin{aligned}
&[(\gamma-1)\Phi_P - \tfrac{m_r}{M}]P + [(1-\beta)(\Omega-1)G + \Phi_Z(\Omega-1) - \tfrac{m_r}{M}]Z + F_N \\
&-\Phi_P^* P^2 - \Phi_Z^* Z^2 = 0.
\end{aligned}
\tag{4.4.2}
$$

This has to be considered in four versions: without optional terms; with optional term $\Phi_P^* P^2$, with optional term $\Phi_Z^* Z^2$ and with both optional terms. As the sum needs to be zero for all variables, it follows: as Φ_P^* and Φ_Z^* are unequal zero, this is a contradiction. This model versions do not provide mass conservation. If no optional terms are used, with 4.4.2 it follows

$(\gamma-1)\Phi_P = \frac{m_r}{M}$
and $(1-\beta)(\Omega-1)G + \Phi_Z(\Omega-1) = \frac{m_r}{M}$
and $F_N = \frac{m_r}{M}(N_D - N) = 0$.

If this is valid for all variables and layer depths it follows $m_r = 0$. This is a contradiction to $m_r \in [0.010, 3.000]$ (see 2). If now the requirement $m_r > 0$ is loosened and m_r is set equal zero, it follows with the prefactor of P in equation 4.4.2:

$(\gamma-1)\Phi_P = 0$. This is a contradiction to the requirement $\gamma > 0$ and $\Phi_P > 0$. Therefore the model version without quadratic loss terms, too, does not provide mass conservation.

□

As long as the prefactors of the variables P and Z in 4.4.2 (and the prefactors of the optional term variables) are negative or at most zero, together with the fact that all solutions are nonnegative for nonnegative initial values (which is proven in section 4.7.1), the sum of concentrations would not increase to infinity within this differential equation, and solutions would not blow up. The prefactor of P, $[(\gamma - 1)\Phi_P - \frac{m_r}{M}$, is negative if the domains of γ and m_r are not varied (and no such variation is done in this thesis or in [Schartau]). The prefactor of Z requires $(\beta \leq 1 \wedge \Omega \leq 1)$ or $(\beta > 1 \wedge \Omega > 1)$ to remain negative. Albeit the term $F_N = \frac{m_r}{M}(N_D - N)$ usually is greater zero and hence is a source for nutrients in the model. Although the prefactor of N itself, $-\frac{m_r}{M}$, is negative, in summary the term F_N in normal ocean conditions adds mass to the system, and the case $N > N_D$ only occurs as an exception (this would lead to a notification in the MATLAB programs). Therefore no conclusion on the question of blow ups is possible at this point; more investigation is necessary.

As the sum of the equations is not zero, the allover sum of mass within the system is not constant. The figure 2.0.1 shows this fact, too. For nutrients, phytoplankton and zooplankton arrows protrude out of the system margins: it gains matter via N, controlled by parameter m_r (turbulent mixing rate, this includes matter gains via the parameter N_D) and loses matter via P and Z, controlled by m_r and by β (zooplankton assimilation efficiency) and γ and Ω (remineralization parameters). If a model with mass conservation is necessary, e.g. the NPZD model in [Schartau] can be used.

4.5 Continuity and Differentiability

This section discusses the continuity and differentiabilty feature of the NPZ equation system. The continuity is explained, differentiability and local Lipschitz continuity are proven, and it is shown that the system is not global Lipschitz continuous.

At first continuity is proven.

Proposition 4.5.1. ***Continuity of NPZ Model***
The NPZ model 3.0.1 is continuous with respect to all three variables.

Proof:
The NPZ model, considered as a system of three component functions in three variables N, P, Z, is a combination of polynomials and rational functions. As mentioned before, the rational functions do not have any poles (i.e. no points of discontinuity), as long as N is limited to $(-k_{min}, \infty)$. According to the corollary on

page 14 of [Koenigsberger], rational functions (understood as functions in finite-dimensional normed spaces) are continuous within their domains. As the NPZ model is a function in a finite-dimensional Euclidean space, this corollary applies here. Due to the fact that no zeros occur in the denominators, no additionally boundaries of the domains are necessary here to gain continuity. Therefore the model is continuous with respect to all three variables.

□

Further analysis of the IVP 4.3.2 requires local Lipschitz continuity as a precondition. First, the definition of local Lipschitz continuity is given; then the feature for NPZ model is proven by applying a lemma from [Koenigsberger].
The definition of Lipschitz continuity is cited from Definition 2.1.1. in [Pruess]

Definition 4.5.2. ***Local and Global Lipschitz Continuity***

Given is the IVP (4.3.2).

1. $f : U \to \mathbb{R}^n$ *is local Lipschitz continuous with respect to* x*, if for every* $(t_1, x_1) \in U$ *a closed ball* $\bar{B}_r(x_1)$ *and an* $\alpha > 0$ *with* $[t_1-\alpha, t_1+\alpha]\times\bar{B}_r(x_1) \subset U$ *and a constant* $L = L(t_1, x_1) > 0$ *exist, in a way that*

$$|f(t,x) - f(t,\widetilde{x})| \leq L(t_1,x_1)|x-\widetilde{x}|, \; if \; |t-t_1| \leq \alpha, \; and \; x, \; \widetilde{x} \in \bar{B}_r(x_1).$$

2. f *is global Lipschitz continuous with respect to* x*, if the constant* L *is independent of* $(t_1, x_1) \in U$*, i.e.*

$$|f(t,x) - f(t,\widetilde{x})| \leq L|x-\widetilde{x}|, \; if \; (t,x),(t,\widetilde{x}) \in U.$$

With the following lemma, shown in [Koenigsberger], page 138, it can be concluded that the NPZ equation system is local Lipschitz continuous:

Lemma 4.5.3. ***NPZ Continuously Partial Differentiable, Local Lipschitz***

Let $U \in \mathbb{R} \times \mathbb{R}^n$ *be open.*

$f : U \to \mathbb{R}^n, (t,x) \mapsto f(t,x)$ *with* $t \in \mathbb{R}, x \in \mathbb{R}^n$.

Let f *be partially differentiable with respect to* $(x_1, ..., x_n)$ *for every* $(t,x) \in U$*, and let all partial derivatives be continuous in* U*. Then* f *is local Lipschitz con-*

tinuous with respect to x.

With $U = (0, \infty) \times (-k_{min}, \infty) \times \mathbb{R}^2$ open, this lemma can be applied on the NPZ model, and leads to the following proposition:

Proposition 4.5.4. ***Differentiability in NPZ System***
The NPZ model 3.0.1 is continuously partially differentiable and local Lipschitz continuous.

Proof
As polynomials and pole free rationals are partially differentiable everywhere within the domain, all three components f_1, f_2, f_3, as compositions of polynomials and pole free rationals, are partially differentiable. Therefore $f = (f_1, f_2, f_3)^T : U \to \mathbb{R}^3$ is partially differentiable. The partial derivatives are polynomials and rationals again; the Jacobian matrix is

$$\boldsymbol{Jac}(N,P,Z)^T = \begin{pmatrix} \left\{-\mu\frac{k}{(k+N)^2}P - \frac{m_r}{M}\right\} & \left\{(-\mu\frac{N}{k+N} + \gamma\Phi_P) + (1-\beta)\frac{2g^2\varepsilon P}{(g+\varepsilon P^2)^2}\Omega Z\right\} & \left\{((1-\beta)\frac{g\varepsilon P^2}{g+\varepsilon P^2} + \Phi_Z)\Omega\right\} \\ \left\{\mu\frac{k}{(k+N)^2}P\right\} & \left\{\mu\frac{N}{k+N} - \Phi_P - \frac{m_r}{M} - \frac{2g^2\varepsilon P}{(g+\varepsilon P^2)^2}Z - 2\Phi_P^* P\right\} & \left\{-\frac{g\varepsilon P^2}{g+\varepsilon P^2}\right\} \\ \{0\} & \left\{\beta\frac{2g^2\varepsilon P}{(g+\varepsilon P^2)^2}Z\right\} & \left\{\beta\frac{g\varepsilon P^2}{g+\varepsilon P^2} - \Phi_Z - \frac{m_r}{M} - 2\Phi_Z^* Z\right\} \end{pmatrix}$$

The following consideration leads to the conclusion that all denominators within Jacobian matrix entries are unequal zero: All denominators either are the same as in the model itself ($k + N > 0; g + \varepsilon P^2 > 0$), and already have been proven to be unequal zero, or they consist of squares of the same positive terms ($(k + N)^2 > 0; (g + \varepsilon P^2)^2 > 0$). Squares of terms that are not equal zero are unequal zero themselves (within the fields of real and complex numbers). Hence all partial derivatives are compositions of polynomials and pole free rationals, therefore all partial derivatives are continuous in U. It can be concluded that the function is continuously partially differentiable (and therefore continuously differentiable). Lemma 4.5.3 can be applied; local Lipschitz continuity is proven.

□

Now it is shown that the NPZ model is not globally Lipschitz continuous.

Proposition 4.5.5. ***NPZ not Globally Lipschitz***
The NPZ model is not globally Lipschitz continuous.

Proof:

This proposition can be proven with Lemma V on page 113 in [Walter], and a contraposition:

Lemma 4.5.6. ***Bounded Partial Derivatives, Global Lipschitz Continuity***

Let $f : U \to \mathbb{R}^n$ be partial differentiable, and let U be convex. If all partial derivatives are continuous and bounded, then f is global Lipschitz continuous.

U is a Cartesian product of convex sets and therefore convex itself, and partial differentiability has been explained before. Hence the requirements for Lemma 4.5.6 are fulfilled. But not all partial derivatives are bounded, as can be seen from the Jacobian matrix entry in row 3, column 2:
$\frac{\partial f_3}{\partial P} = \beta \frac{2g^2 \varepsilon P}{(g+\varepsilon P^2)^2} Z.$
This term consists of three factors: β, a fraction as function of variable P, and the variable Z. The second factor (the fraction) is analysed at first. The first two derivations of the factor $\widehat{f} := \frac{2g^2\varepsilon P}{(g+\varepsilon P^2)^2}$ are:

$\widehat{f}'(P) = \frac{2\varepsilon g^3 - 6\varepsilon^2 g^2 P^2}{(g+\varepsilon P^2)^3}$

$\widehat{f}''(P) = \frac{24\varepsilon^2 g^2 P(\varepsilon P^2 - g)}{(g+\varepsilon P^2)^4}$

With $\widehat{f}'(P) = 0 \Rightarrow P = \sqrt{\frac{g}{3\varepsilon}}$ it follows

$\Rightarrow \widehat{f}''(\sqrt{\frac{g}{3\varepsilon}}) = \frac{24\varepsilon^2 g^2 \sqrt{\frac{g}{3\varepsilon}}(\varepsilon \frac{g}{3\varepsilon} - g)}{(g+\varepsilon\frac{g}{3\varepsilon})^4} = \frac{24\varepsilon^2 g^2 \sqrt{\frac{g}{3\varepsilon}}(-\frac{2}{3}g)}{(\frac{4}{3}g)^4} < 0.$

As a consequence, the considered factor has a maximum at $P = \sqrt{\frac{g}{3\varepsilon}}$. The value of the factor in this maximum is greater zero. Substituted into the partial derivative, the matrix entry can be written as a constant > 0 (with $\beta > 0$) multiplied with Z and is not bounded.

Now, if Lemma 4.5.6 says that in cases like this, if all partial derivatives are bounded the function is global Lipschitz continuous, then by contraposition it can be concluded that if at least one partial derivative is not bounded, the function is not global Lipschitz continuous. □

In summary, the model is differentiable and local Lipschitz continuous, but the Lipschitz continuity does not hold globally.

4.6 Existence and Uniqueness of Solutions

In the last section it has been proven that f is local Lipschitz continuous in x.

Now, considering the IVP 4.3.2 within this section, by Uniqueness Theorem, it is concluded that the initial value problem locally has not more than one solution. The Picard Lindelöf Theorem and the Extension Theorem (all in [Pruess]) provide the existence of a local solution (which is unique in this case). In systems with global Lipschitz continuity a global unique solution exists, albeit this is not the case in the NPZ equation system. Due to the fact that global Lipschitz continuity could not be proven, it is not possible to guarantee a global unique solution.

Now the theorems are given, and then applied to show the existence of a unique local solution of the IVP 4.3.2.

The Uniqueness Theorem is number 2.1.3. in [Pruess]:

Theorem 4.6.1. ***Uniqueness Theorem for IVP 4.3.1***
Let $U \subset \mathbb{R}^{n+1}$ be open and $f : U \to \mathbb{R}^n$, f continuous, be local Lipschitz continuous with respect to x. Then the IVP 4.3.1 has at most one solution.

Now this theorem is applied, together with the earlier proven local Lipschitz continuity, and leads to

Corollary 4.6.2. *The IVP 4.3.2 locally has at most one solution.*

Proof:
Application of theorem 4.6.1 .

□

For existence of a local solution the Picard Lindelöf (2.2.2. in [Pruess]) theorem is used.

Theorem 4.6.3. ***Picard Lindelöf Theorem***
Let $U \subset \mathbb{R}^{n+1}$ be open, $f : U \to \mathbb{R}^n$ continuous and lokal Lipschitz continuous with respect to x. Then a $\delta > 0$ and a unique, continuously differentiable function $x : J_\delta \to \mathbb{R}^n$ with $J_\delta := [t_0 - \delta, t_0 + \delta]$ exists, with $(t, x(t)) \in U$ for every $t \in J_\delta$, and $x = x(t)$ is a solution of the IVP 4.3.1 within the interval J_δ.

With U open and f continuous and local Lipschitz continuous the model meets all requirements for Picard Lindelöf theorem. This obtains the corollary

Corollary 4.6.4. *The IVP 4.3.2 locally has got a solution.*

Proof:
Application of theorem 4.6.3.

□

With help of the Extension Theorem (2.3.2., [Pruess]) this can be refined.

Theorem 4.6.5. ***Extension Theorem***
Let $f : U \to \mathbb{R}^n$ be continuous and local Lipschitz continuous with respect to x. Then the solution for the IVP 4.3.1 exists on the maximal interval (t_-, t_+) with $t_-(t_0, x_0) =: t_- < t_0 < t_+ := t_+(t_0, x_0)$. The terminal point to the right t_+ is characterized by the following alternatives:

1. *$t_+ = \infty : x(t)$ is a global solution to the right.*
2. *$t_+ < \infty$ and $\underline{\lim}_{t \to t_+} dist((t, x(t)), \partial U) = 0$: the solution $x(t)$ gets arbitrarily close to the boundary of U.*
3. *$t_+ < \infty$ and $\underline{\lim}_{t \to t_+} dist((t, x(t)), \partial U) > 0, \lim_{t \to t_+} |x(t)| = \infty$: the solution blows up to infinity.*

The extension theorem, with $f : U \to \mathbb{R}^n$ being continuous and local Lipschitz continuous, can be applied here and provides three alternatives for the behaviour of the local solution to the right. A blow up cannot be excluded unless more detailed information on the term F_N is available (section 4.4).

Altogether these considerations obtain the result that the NPZ model provides a unique local solution.

4.7 Positivity of Solutions

4.7.1 General Positivity Considerations

All variables of the NPZ model represent concentrations. Hence in nature all variables will be nonnegative at all times ("'negative concentrations"' do not exist). Therefore the model, understood as IVP (see 4.3.2), should have the following feature (as it is explained in [Pruess] on page 71):

Definition 4.7.1. ***Positivity of Solutions***
Let $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ be continuous and local Lipschitz continuous with respect to x. Consider the IVP 4.3.1 with solution $x(t)$. If the initial value of the IVP, x_0, is nonnegative in all components, that is $x_{0_j} \geq 0, j = 1, ..., n$, it is implied that $x_j(t) \geq 0$ for all $j = 1, ..., n$, with $t \geq 0, x(t) = [x_1(t), x_2(t), ..., x_n(t)]^T \in \mathbb{R}^n$. This feature is called the positivity of the solution $x(t)$.

In this section a definition on quasi positive functions is given ([Pruess], page 71). The model is proven a quasi positive function, and with this fact, the conclusion is gained that positivity of solution is a feature of the NPZ model (unless

$\beta > 1$): if for a given time t_0 it holds $N(t_0) \geq 0, P(t_0) \geq 0, Z(t_0) \geq 0$ and $\beta \leq 1$, all three variables are nonnegative for every $t > t_0$. This is done with the help of theorem 4.2.2. from [Pruess].
The next definition is from [Pruess], page 71.

Definition 4.7.2. ***Quasi positive function***

Let $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be continuous and local Lipschitz in x, and let $x(t)$ be the solution of the IVP 4.3.1.

Let $x \in \mathbb{R}^n$ and $x_j \geq 0, j = 1, .., n$.

The function f is called quasi positive if for every $k \in 1, .., n, t \geq t_0$ and $x_k = 0$ it holds $f_k(t, x) \geq 0$.

Theorem 4.2.2., [Pruess], reads

Theorem 4.7.3. ***Positivity***

Let $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be continuous and lokal Lipschitz in x, and let f be quasi positive. If $x(t)$ is the solution of the IVP, and if $x_{0_j} \geq 0$ for all $j \in 1, .., n$, it follows that $x_j \geq 0$ for all $t \in [t_0, t_+)$, the maximum existence intervall of the solution to the right.

4.7.2 Positivity in general NPZ Model

Now for the NPZ model the positivity theorem is stated and proven.

Theorem 4.7.4. ***Positivity of NPZ Model***

Consider the IVP 4.3.2, and let $N_0 \geq 0, P_0 \geq 0, Z_0 \geq 0$ be the initial value of the IVP 4.3.2 at time t_0. Let $\beta \leq 1$. Let $N_D \geq 0$, $m_r \geq 0$. If $x(t) = (N(t), P(t), Z(t))^T$ is the solution of the IVP 4.3.2 at $t \geq t_0$ then $(N(t) \geq 0) \wedge (P(t) \geq 0) \wedge (Z(t) \geq 0)$.

Proof:

According to section 4.5 the model function is continuous and local Lipschitz continuous with respect to $x = (N, P, Z)^T$. Because of theorem 4.7.1 it is sufficient to prove $x(t)$ quasi positive.

Let $N_0 \geq 0, P_0 \geq 0, Z_0 \geq 0$.
Let $P = 0$. Then the corresponding equation from system 3.0.1 reads

$$f_2(N, 0, Z) = [\mu \frac{N}{k + N} - \Phi_P - \frac{m_r}{M}]P - \frac{g\varepsilon P^2}{g + \varepsilon P^2} Z - \Phi_P^* P^2 = 0.$$

This result holds for model versions with and without quadratic loss term.
Now setting $Z = 0$ (instead of $P = 0$) with the corresponding equation it follows

$$f_3(N, P, 0) = [\beta \frac{g\varepsilon P^2}{g + \varepsilon P^2} - \Phi_Z - \frac{m_r}{M}]Z - \Phi_Z^* Z^2 = 0.$$

This result also holds for versions with and without quadratic loss term. Now, $N = 0$ is considered:

$$\begin{aligned} f_1(0, P, Z) &= [-\mu \tfrac{N}{k+N} + \gamma\Phi_P]P + [(1-\beta)\tfrac{g\varepsilon P^2}{g+\varepsilon P^2} + \Phi_Z]\Omega Z + \tfrac{m_r}{M}(N_D - N) \\ &= [\gamma\Phi_P]P + [(1-\beta)\tfrac{g\varepsilon P^2}{g+\varepsilon P^2} + \Phi_Z]\Omega Z + \tfrac{m_r}{M}N_D. \end{aligned}$$

Now $\beta \leq 1$ is assumed, although certain numerical experiments in [Schartau] allow $\beta > 1$. If this assumption is violated, further considerations have to be made to guarantee the positivity (see section 4.7.3).
Applying this two assumptions and the already shown nonnegative P and Z, this equation is nonnegative.
Therefore, in general the model provides the positivity feature.

□

Some considerations on the situation of extended parameter β are stated in the next section.

4.7.3 Positivity in NPZ Model with Extended Domain of Parameter β

In some of the numeric experiments in [Schartau] some parameters are slightly greater than their initial boundaries (as it is mentioned in section 2). If $\beta > 1$, this would result in
$(1-\beta)\frac{g\varepsilon P^2}{g+\varepsilon P^2} < 0$, and if
$|(1-\beta)\frac{g\varepsilon P^2}{g+\varepsilon P^2}| > \Phi_Z$, even
$[(1-\beta)\frac{g\varepsilon P^2}{g+\varepsilon P^2} + \Phi_Z]\Omega Z < 0$ would occur.

In this case positivity of the model would require

$$[\gamma\Phi_P]P + \frac{m_r}{M}N_D \geq \|[(1-\beta)\frac{g\varepsilon P^2}{g+\varepsilon P^2} + \Phi_Z]\Omega Z\|. \tag{4.7.3}$$

This requirement might be violated if high concentrations of Z, compared to N_D and P, occur. Within this thesis, in section 5 the parameter sets for experiment II and III are used, with $\beta = 1.003$ (exp.II) respective $\beta = 1.045$ (exp.III). Actually in the calculations (regarding the QQM equilibria, section 5.2.1) the requirement 4.7.3 is fulfilled; the positivity is preserved. Nevertheless, a closer look to this situ-

ation is necessary if the model is implemented and run with this kind of extended parameter sets.

4.8 Continuously Dependency of Solutions

In this section the definition of continuously dependent solutions is introduced, and the solutions of our IVP are proven continuously dependent.
At first the necessary definitions and theorem from [Pruess] are given, then the NPZ model is analyzed on this issue.
[Pruess], page 67, has this definition for a graph:

Definition 4.8.1. ***Graph of a Solution*** *Let $x(t)$ be the solution of the IVP 4.3.1 on its maximum existence intervall (t_-, t_+). Let $J = [a, b] \subset (t_-, t_+)$ be a compact intervall with $t_0 \in (a, b)$. Then $graph_J(x) := (t, x(t)) : t \in J \subset U$.*

The definition of continuously dependent solutions is cited from 4.1.1.([Pruess]).

Definition 4.8.2. ***Continuously Dependent Solution***

The given solution $x(t)$ is continuously dependent on (t_0, x_0, f), if for every compact intervall $J \subset (t_-, t_+)$ there exists a compact neighbourhood $K \subset U$ of $graph_J(x)$, so that
for every $\varepsilon > 0$ there exists a $\delta > 0$, so that the solution $y(t)$ of the IVP $\dot{y} = g(t, y), y(\tau_0) = y_0$, exists for all $t \in [a, b]$ and fulfills the inequality

$$|x(t) - y(t)| \leq \varepsilon, \forall t \in [a, b],$$

if $g : U \to \mathbb{R}^n$ is continuous, local Lipschitz continuous with respect to x, and

$$|\tau_0 - t_0| \leq \delta, |x_0 - y_0| \leq \delta, \sup_{(s,z)\in K} |f(s, z) - g(s, z)| \leq \delta.$$

This feature guarantees that continuous variations of the initial value do not cause stepwise changes of the solutions and is a very useful feature to describe natural smooth processes in a realistic way.

The following theorem ([Pruess], 4.1.2.) provides continuously dependency for the given IVP:

Theorem 4.8.3. ***Continuously Dependent Solution***

Let $U \subset \mathbb{R}^{n+1}$ be open with $(t_0, x_0) \in U, f : U \to \mathbb{R}^n$ be continuous and local Lipschitz continuous with respect to x. Then the solution $x(t)$ of the IVP continuously depends on (t_0, x_0, f).

With local Lipschitz continuity of the NPZ model it follows:

Corollary 4.8.4. *Solutions of the IVP 4.3.2 are continuously dependent on the initial values.*

Proof:

The corollary directly is obtained from theorem 4.8.3.

□

4.9 Differential Dependency of Solutions

This section is a refinement of the conclusions in the previous section. With a theorem in [Timmann] (pages 22 and 23) it can be shown that the solutions of the IVP 4.3.2 are differentiably dependent on the initial values. [Timmann] concludes on page 24 that this result can be broadened on the parameters.

The consideration starts with a definition of the characteristic function of a system ([Timmann], page 23).

Definition 4.9.1. ***Characteristic Function of the Equation System***

Consider the equation system $\dot{x} = f(t, x)$ with $U \subset \mathbb{R}^{n+1}$ open, $f : U \to \mathbb{R}^n$, and let f be continuously differentiable with respect to the variables $x_1, ..., x_n$ (and hence local Lipschitz continuous with respect to x).

Let $\phi : I \to \mathbb{R}^n$ with $\phi(t_0) = x_0$ be the solution of the IVP within the compact interval I.

Then the characteristic function $\overline{\Phi}(t, t_0, x_0)$ of the system is defined as follows: $\overline{\Phi}(t, t_0, x_0) := \phi(t)$.

The theorem on page 23 in [Timmann] does not use any requirements above the assumptions of definition 4.9.1.

Theorem 4.9.2. ***Differentiability of Characteristic Function***

The characteristic function ϕ, as defined in 4.9.1, is continuously differentiable within its domain, hence continuously partial differentiable with respect to all variables.

This means that a differentiable variation in the initial value causes differentiable changes in the solution.

According to the considerations in [Timmann], page 24 (as an explanation, not fixed into a theorem), the conclusion in 4.9.2 can be broadened on differentiable dependence on the parameters. The IVP is transferred in a way as the following general example shows.

Assume the IVP is depending on m parameters $p_1, ..., p_m \in \mathbb{R}$:

$\dot{x} = f(t, x, p_1, ..., p_m); x(t_0) = t_0$

is transferred into the equivalent system

$\dot{x} = f(t, x, z_1, ..., z_m); x(t_0) = x_0$

$\dot{z_1} = 0; z_1(t_0) = p_1$, ...,

$\dot{z_m} = 0; z_m(t_0) = p_m$

Now it can be stated that the solutions depend on the parameters the same way as on the initial values. This leads to the conclusion that the NPZ models solutions are differentiable dependent on the initial value and the parameters. This might be helpful, because continuous or differential variations on initial values or parameters can be done without changing the solution in a not differentiable way or even cause points of incontinuity.

5 Equilibria

This section is related to the analysis if the NPZ model supports equilibria. Those equilibria that can be determined with reasonable effort are calculated with some given parameter combinations with MATLAB and their stability is tested (see below).

An equilibrium is a condition of the system where at a point of time the concentration values of nutrients, phytoplankton and zooplankton remain in their present value. Hence in this condition the derivations of all three variables are zero.

To find equilibria, the system needs to be solved with all derivatives being fixed at zero.

$\frac{dN_v}{dt} = 0, \frac{dP_v}{dt} = 0, \frac{dZ_v}{dt} = 0$ provides

$$0 = \frac{dN}{dt} = [-\mu u + \gamma\Phi_P]P + [(1-\beta)G + \Phi_Z]\Omega Z + F_N$$
$$0 = \frac{dP}{dt} = [\mu u - \Phi_P - \frac{m_r}{M}]P - GZ(-\Phi_P^* P^2)$$
$$0 = \frac{dZ}{dt} = [\beta G - \Phi_Z - \frac{m_r}{M}]Z(-\Phi_Z^* Z^2).$$

During the analysis of the equilibria the system equations are considered at a certain point of time; all parameters are constant, including M, μ and N_D. It has to be kept in mind that within the model the changes of the seasons are controlled by parameter M. Changings in temperature, currents, daylight etc. can influence the plankton growth and loss, and M is modelling such changes by adjusting the mixed layer depth. Greater changes in M will occur at least every few months, and this leads to changings in the results of concentrations. Therefore the results in this chapter are only valid within a short period of time without changings in M.

Due to the fact that the equilibria refer to concentration values, negative or complex solutions are neglected.

For all results a numerical analysis of the model equations is calculated for three different constant values of N_D (1.5, 8 and 16, see section 2), and with parameter sets that have been found useful in [Schartau] (ibd. table 2.4.), and for mixed layer depths from 1 to 400 (with an $1m$ grid).

One parameter set is used for the model without quadratic loss terms (experiment I) and two sets for the model with both quadratic loss terms (experiment II and experiment III). The parameter values are found in table 1. For a better overview the units of the parameters are added although in calculation dimensionless values are used (see section 4.1).

Table 1: Parameter Sets

Parameter	Unit	Experiment		
		I	II	III
μ	$\frac{1}{d}$	2.362	1.679	3.775
Φ_P	$\frac{1}{d}$	0.016	0.033	0.017
k	$\frac{mmolN}{m^3}$	0.350	0.636	0.560
g	$\frac{1}{d}$	0.743	0.982	1.560
ε	$\frac{m^6}{mmolN^2d}$	3.088	3.473	3.740
Φ_Z	$\frac{1}{d}$	0.067	0.010	0.011
β	1	1;0.852	1.003	1.045
m_r	$\frac{m}{d}$	0.643	0.107	1.270
Ω	1	1;0.348	0.990	1.006
γ	1	1;0.973	0.990	0.870
Φ_P^*	$\frac{m^3}{mmolNd}$	–	0.168	0.122
Φ_Z^*	$\frac{m^3}{mmolNd}$	–	0.257	0.098

It has to be noted that parameter set III exceeds the upper bounds of the parameter domains in parameter $\mu, g, \varepsilon, \beta$, and Ω. In case of β and Ω this influences the positivity feature (see section 4.7) and theoretically causes a discrepancy as these parameters should constitute a ratio between 0 and 1. On the other hand this situation did not cause any numerical issues. The other upper bounds are not rigorous.

The equilibria are analyzed by the method of linearized stability. To introduce the concept of linearized stability a definition(5.1.1.) and a theorem (5.4.1.) from [Pruess] are applied.

The corresponding definition ([Pruess], 5.1.1.) reads

Definition 5.0.3. ***Stability of Equilibria***

Let $U \subset \mathbb{R}^n$ open, $\mathbb{R} \times U \to \mathbb{R}^n$ continuous and local Lipschitz continuous with respect to x. The IVP is $\dot{x} = f(t, x)$;
$x(t_0) = x_0$
with $t_0 \in \mathbb{R}$ and $x_0 \in U$.

Let $f(t,0) = 0$ and $x(t,x_0), t \geq 0$ be the solution of the IVP.

1. *The solution $x_* = 0$ is called stable, if for every $\varepsilon > 0$ there exists a $\delta > 0$ with $\overline{B_\delta(0)} \subset U$, the solution $x(t,x_0)$ corresponding to $x_0 \in \overline{B_\delta(0)} \subset U, \forall t \geq t_0$ and*

$$|x(t,x_0)| \geq \varepsilon, \forall |x_0| \leq \delta, t \geq t_0$$

   holds.

2. *$x_* = 0$ is called unstable, if x_* is not stable.*

3. *$x_* = 0$ is called attractive if there exists a $\delta_0 > 0$ so with $\overline{B_{\delta_0}(0)} \subset U$, the solution $x(t,x_0)$ corresponding to $x_0 \in \overline{B_{\delta_0}(0)} \forall t \geq t_0$ exists, and*

$$\lim_{t\to\infty} \|x(t,x_0)\| = 0, \forall x_0 \in \overline{B_{\delta_0}(0)}$$

   .

4. *$x_* = 0$ is called asymptotically stable, if $x_* = 0$ is stable and attractive.*

For stability analysis the IVP is considered as an autonomous system (as equilibria are calculated with a fixed t), and theorem 5.4.1.from [Pruess] is used:

Theorem 5.0.4. ***Stability Features of Equilibria***
Let $\dot{x} = f(x)$ be an autonomous differential equation system with $f : \mathbb{R}^n \to \mathbb{R}^n$ continuously differentiable.
Let x_* be an equilibrium (i.e. $f(x_*) = 0$).
Define $A := f(x_*)$. Then following statements hold:

1. *If $\Re\lambda_j < 0$ for all eigenvalues λ_j of A, the equilibrium x_* is asymptotially stable regarding the given IVP.*

2. *If an eigenvalue $\lambda_j, \Re\lambda_j > 0$ exists, the equilibrium x_* is unstable regarding the given IVP.*

Remark
In case of $\Re\lambda_j = 0$ theorem 5.0.4 is not sufficient to obtain a result on stability.
Due to the optional quadratic loss terms in the model the system has four versions. The labeling is adopted from a similar classification of the NPZD model that also works with optional terms, used in [Heinle]:

1. Model LLM: Both optional quadratic loss terms are neglected, linear loss terms for phytoplankton and zooplankton

2. Model QQM: Both optional terms are considered greater than zero

3. Model LQM: The optional quadratic loss term for phytoplankton is neglected; the quadratic loss term for zooplankton is considered to be greater than zero

4. Model QLM: The quadratic loss term for phytoplankton is considered to be greater than zero, the optional quadratic loss term for zooplankton is neglected.

The last two variations are considered for reasons of completeness; the parameter sets in [Schartau] do not provide a version with only one quadratic loss term; nevertheless in such cases the parameter sets for QQM variation is used and one of the quadratic loss terms is replaced by zero. The LQM system also is analyzed for NPZD model in [Heinle]; QLM is not considered there either.

The analysis is done in the order given above. For every model variation the model equations are presented again, with the matching number of optional terms, for a better overwiev. Then all those equilibria that can be found with reasonable effort are outlined and proven. All those points (exception: trivial solution $(N_D, 0, 0)^T$) are calculated with the given values for N_D, parameter set I (LLM) resp. parameter sets II and III (QQM, LQM, and QLM), all of which for all mixed layer depths from 1 to $400m$ (in steps of $1m$). The programming is done with MATLAB Student Version V 7.10.0 (R2010a). The variables are calculated and it is confirmed that assumptions for certain equilibria fulfilled. The results are shown in plots, some of them scatterplots; or scatterplot on interpolation line, if convenient. Within all MATLAB calculation roots of third degree polynomials and of eigenvalues of the Jacobian matrix are being calculated by the standard MATLAB algorithms *eig* and *roots* . Within the proof a theoretical approach is mentioned to calculate the number of real roots of a third degree polynomial (based on the Cardano approach, compare [Doerrie]; this formula is used in MATLAB, too, but the calculation of the up to three roots is done by MATLAB algorithms. All terms that exceed the given boundaries lead to a notification during program run (but not necessarily to a program abortion). Additionally it is confirmed that derivations are almost zero: the deviations from zero in this values in all our MATLAB analyses usually about the size of roughly $\pm 10^{-13}$ or much smaller in all model versions but the QLM version. The deviation in QLM is much greater, up to a

size of 0.1, especially with shallow mixed layer depths - this issue is going to be discussed in section 5.4.1. In all cases the stability is tested and the results given: for each calculation a stability value of -1 is assigned to cases with the result 'asymptotically stable', $+1$ is assigned to the result 'unstable', and 0 stands for not determined. These stability results[2] are shown in scatter plots visualizing the dependency of the stability from the layer depth.
The results for concentrations in cases of equilibria also are displayed in figures. The programs and an example of the result tables (made by the programs) are stored in tables on github (<https://github.com/PetraFuhs/Analysis_marine_ecosystem_model>).

In all programs the parameter values are stored in vectors at the top of the program to enable the user to change values easily (all necessary units and domains are laid out in the appendix tables). The acceptable deviations from zero (both for eigenvalues and derivations) are fixed this way, too. The code section for figure making is found at the bottom of the programs and can be adjusted according to the aim of the program use easily.

The programs are:

- LLM1.m: tool for testing the stability of the trivial solution $(N_D, 0, 0)$ (for all model versions) with all layer depths
- LLM2.m: calculation and stability test for equilibrium number $LLM2$.
- LLM3.m: calculation and stability test for equilibria numbers $LLM3$ and $LLM4$.
- QQM.m: calculation and stability test for equilibria $QQM2, QQM3$ and $QQM4$.
- QLM.m: calculation and stability test for equilibria $QLM2$ and $QLM3$

The LQM cases are similar to cases in other model versions, no program has been made for LQM version equilibria.

5.1 Model LLM: No quadratic loss term

The equation system [3] without optional terms at equilibria is

[2] The lines in stability scatterplots look broadly due to the fact that 400 values are sketched
[3] Remark: the equation numbers within the equilibria section are not consecutive with the other sections

$$
\begin{aligned}
0 &= [-\mu\tfrac{N}{k+N} + \gamma\Phi_P]P + [(1-\beta)\tfrac{g\varepsilon P^2}{g+\varepsilon P^2} + \Phi_Z]\Omega Z + \tfrac{m_r}{M}(N_D - N) \quad (I_{LLM}) \\
0 &= [\mu\tfrac{N}{k+N} - \Phi_P - \tfrac{m_r}{M}]P - \tfrac{g\varepsilon P^2}{g+\varepsilon P^2}Z \quad (II_{LLM}) \\
0 &= [\beta\tfrac{g\varepsilon P^2}{g+\varepsilon P^2} - \Phi_Z - \tfrac{m_r}{M}]Z \quad (III_{LLM}).
\end{aligned}
\tag{5.1.4}
$$

There are up to four equilibria (N, P, Z) in the LLM system. The trivial equilibrium $LLM_1 = (N_D, 0, 0)$ (also valid for all other model versions), with all plankton being eradicated and the nutrient value limited to N_D in the mixed layer, is independent from the choice of parameters and always existing. This equilibrium is unstable over all depths and all considered values of N_D. Figure 5.1.4 shows the results of the stability tests as a scatter plot. The equilibrium is instable at all depths and with all tested values of N_D.

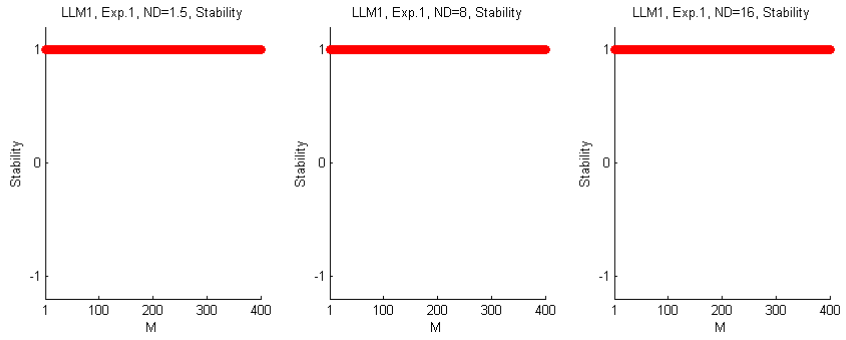

Figure 5.1.4: Stability, Equilibrium $LLM1$, $N_D = 1.5; 8; 16$ from left to right

The second point, $LLM2$, can only exist if two certain assumptions are fulfilled (see below), and if zooplankton has become extinct in the layer. The third and fourth point, $LLM3$ and $LLM4$, exist if a third rsp. fourth assumption ($LLM4$) are satisfied. At equilibria $LLM3$ and $LLM4$ both phytoplankton and zooplankton exist together. These points are roots of an quadratic equation; hence both, one or none of them might be nonnegative. This is why in model LLM two, three or four equilibria can occur, depending on the chosen parameters. The stated points are critical points; if these points provide equilibria (nonnegative real values), depends on the chosen parameter sets. The results are shown with each of the proof sections.

Proposition 5.1.1. *The system LLM provides up to four equilibria:*

1. $LLM_1 = (N_D, 0, 0)$

   $(N = N_D,$
   $P = 0,$
   $Z = 0)$

2. *If assumption 1:* $\mu M - \Phi_P M - m_r \neq 0$

   and assumption 2 : $\gamma\Phi_P M - \Phi_P M - m_r \neq 0$ *are satisfied, a critical point LLM2 exists:*

   $LLM_2 = (k\frac{\Phi_P M + m_r}{\mu M - \Phi_P M - m_r}, \frac{m_r}{\gamma\Phi_P M - \Phi_P M - m_r}(-N_D + k\frac{\Phi_P M + m_r}{\mu M - \Phi_P M - m_r}), 0)$

   $(N = k\frac{\Phi_P M + m_r}{\mu M - \Phi_P M - m_r},\ P = \frac{m_r}{\gamma\Phi_P M - \Phi_P M - m_r}(-N_D + k\frac{\Phi_P M + m_r}{\mu M - \Phi_P M - m_r}),$
   $Z = 0)$

3. *With the auxiliary terms*
   $G_1 = (\frac{\Phi_Z M + m_r}{\beta M})$
   $\xi_1 = (\frac{gG_1}{\varepsilon(g - G_1)})$
   $\xi_2 = ([(1-\beta)G_1 + \Phi_Z]\Omega\frac{\sqrt{\xi_1}}{G_1})$
   $a_1 = (\frac{M}{m_r}[\sqrt{\xi_1}(\mu - \gamma\Phi_P) - \xi_2(\mu - \Phi_P - \frac{m_r}{M}) + \frac{m_r}{M}(k - N_D)])$, *and*
   $b_1 = (\frac{kM}{m_r}(-\gamma\Phi_P\sqrt{\xi_1} + \xi_2(\Phi_P + \frac{m_r}{M})) - kN_D)$
   and the assumptions:
   assumption 3: $g\beta M > \Phi_Z M + m_r$,
   assumption 4: $\frac{1}{4}a_1^2 \geq b_1$, *and*
   assumption 5: $[(a_1 < 0) \vee (a_1 \geq 0 \wedge b_1 < 0)]$
   the third possible equilibrium is

   $LLM_3 = (-\frac{a_1}{2} + \frac{1}{2}\sqrt{a_1^2 - 4b_1}, \sqrt{\xi_1}, \frac{\sqrt{\xi_1}}{G_1}[\mu\frac{-\frac{a_1}{2} + \frac{1}{2}\sqrt{a_1^2 - 4b_1}}{k - \frac{a_1}{2} + \frac{1}{2}\sqrt{a_1^2 - 4b_1}} - \Phi_P - \frac{m_r}{M}])$

   $(N = -\frac{a_1}{2} + \frac{1}{2}\sqrt{a_1^2 - 4b_1},$
   $P = \sqrt{\xi_1},$
   $Z = \frac{\sqrt{\xi_1}}{G_1}[\mu\frac{-\frac{a_1}{2} + \frac{1}{2}\sqrt{a_1^2 - 4b_1}}{k - \frac{a_1}{2} + \frac{1}{2}\sqrt{a_1^2 - 4b_1}} - \Phi_P - \frac{m_r}{M}])$

4. *With the auxiliary terms* $G_1, \xi_1, \xi_2, a_1, b_1)$ *as for* LLM_3 *and assumption 6:*
   $(g\beta M > \Phi_Z M + m_r) \wedge (b_1 \leq 0) \wedge (a_1 \leq 0)$
   $LLM_4 = (-\frac{a_1}{2} - \frac{1}{2}\sqrt{a_1^2 - 4b_1}, \sqrt{\xi_1}, \frac{\beta M}{\Phi_Z M + m_r}[\mu\frac{N}{k+N} - \Phi_P - \frac{m_r}{M}]\sqrt{\frac{g(\Phi_Z M + m_r)}{\varepsilon(g\beta M - \Phi_Z M - m_r)}})$

$(N = -\frac{a_1}{2} - \frac{1}{2}\sqrt{a_1^2 - 4b_1},$
$P = \sqrt{\xi_1},$
$Z = \frac{\beta M}{\Phi_Z M + m_r}[\mu \frac{-\frac{a_1}{2} - \frac{1}{2}\sqrt{a_1^2 - 4b_1}}{k - \frac{a_1}{2} - \frac{1}{2}\sqrt{a_1^2 - 4b_1}} - \Phi_P - \frac{m_r}{M}]\sqrt{\frac{g(\Phi_Z M + m_r)}{\varepsilon(g\beta M - \Phi_Z M - m_r)}})$

Proof:

For a good overview on the system solving steps the equation system is noted again:

$$0 = [-\mu \tfrac{N}{k+N} + \gamma \Phi_P]P + [(1-\beta)\tfrac{g\varepsilon P^2}{g+\varepsilon P^2} + \Phi_Z]\Omega Z + \tfrac{m_r}{M}(N_D - N) \quad (I_{LLM})$$
$$0 = [\mu \tfrac{N}{k+N} - \Phi_P - \tfrac{m_r}{M}]P - \tfrac{g\varepsilon P^2}{g+\varepsilon P^2} Z \quad (II_{LLM})$$
$$0 = [\beta \tfrac{g\varepsilon P^2}{g+\varepsilon P^2} - \Phi_Z - \tfrac{m_r}{M}]Z \quad (III_{LLM}).$$

From equation III_{LLM} the proof is proceeded in two parts: in the first part with $Z = 0$, in the second one with $Z \neq 0$.

First part: $Z = 0 \quad 0 = [-\mu \frac{N}{k+N} + \gamma \Phi_P]P + \frac{m_r}{M}(N_D - N) \quad (I_{LLM,Z=0})$

$$0 = [\mu \tfrac{N}{k+N} - \Phi_P - \tfrac{m_r}{M}]P \quad (II_{LLM,Z=0})$$

Equation $II_{LLM,Z=0}$ provides two cases:
Case $(Z = 0 \wedge P = 0)$ and case $(Z = 0 \wedge P \neq 0)$.

Case $(Z = 0 \wedge P = 0)$

In this case equation $I_{LLM,Z=0}$ shows
$0 = \frac{m_r}{M}(N_D - N)$
With $m_r > 0$, and $M > 0$ this only holds for
$N = N_D$. This leads to equilibrium LLM1.

Case $Z = 0 \wedge P \neq 0$

$$0 = [-\mu \tfrac{N}{k+N} + \gamma \Phi_P]P + \tfrac{m_r}{M}(N_D - N) \quad (I_{LLM,Z=0})$$

$$0 = [\mu \tfrac{N}{k+N} - \Phi_P - \tfrac{m_r}{M}]P \quad (II_{LLM,Z=0})$$

Equation $II_{LLM,Z=0}$ shows with $P \neq 0$

$\mu\frac{N}{k+N} - \Phi_P - \frac{m_r}{M} = 0$

$\Leftrightarrow \mu\frac{N}{k+N} = \Phi_P + \frac{m_r}{M} = \frac{\Phi_P M + m_r}{M}$

and, as $\mu > 0$:
$\Leftrightarrow \frac{N}{k+N} = \frac{\Phi_P M + m_r}{\mu M}$ (1)

$\Leftrightarrow N = k\frac{\Phi_P M + m_r}{\mu M} + N\frac{\Phi_P M + m_r}{\mu M}$

$\Leftrightarrow N(1 - \frac{\Phi_P M + m_r}{\mu M}) = k\frac{\Phi_P M + m_r}{\mu M}$

$\Leftrightarrow N\frac{\mu M - \Phi_P M - m_r}{\mu M} = k\frac{\Phi_P M + m_r}{\mu M}$

Provided $\mu M - \Phi_P M - m_r \neq 0$ this leads to
$\Leftrightarrow N = k\frac{\Phi_P M + m_r}{\mu M - \Phi_P M - m_r}$ (2)

Now with (1) and (2) equation $I_{LLM,Z=0}$ supplies
$0 = [-\frac{\Phi_P M + m_r}{M} + \gamma\Phi_P]P + \frac{m_r}{M}(N_D - k\frac{\Phi_P M + m_r}{\mu M - \Phi_P M - m_r})$

$\Leftrightarrow \frac{\gamma\Phi_P M - \Phi_P M - m_r}{M}P = \frac{m_r}{M}[-N_D + k\frac{\Phi_P M + m_r}{\mu M - \Phi_P M - m_r}]$

Now it is assumed
$\gamma\Phi_P M - \Phi_P M - m_r \neq 0.$

Hence
$\Leftrightarrow P = \frac{m_r}{\gamma\Phi_P M - \Phi_P M - m_r}(-N_D + k\frac{\Phi_P M + m_r}{\mu M - \Phi_P M - m_r}).$

So, with the mentioned assumptions, equilibrium $LLM2$ is obtained. The calculation with $N_D = 1.5, 8, 16$, parameter set I and for $M = 1$ to 400 with LLM2.m the following results are gained:

The behavior of assumptions 1 and 2 are exactly the same for all values of N_D; the behavior of the assumptions calculated with $N_D = 1.5$ is plotted and the other two are skipped. The second plot is a detail to assumption 1 to make the fact that the value is unequal zero at all values of M visible (figure 5.1.5).

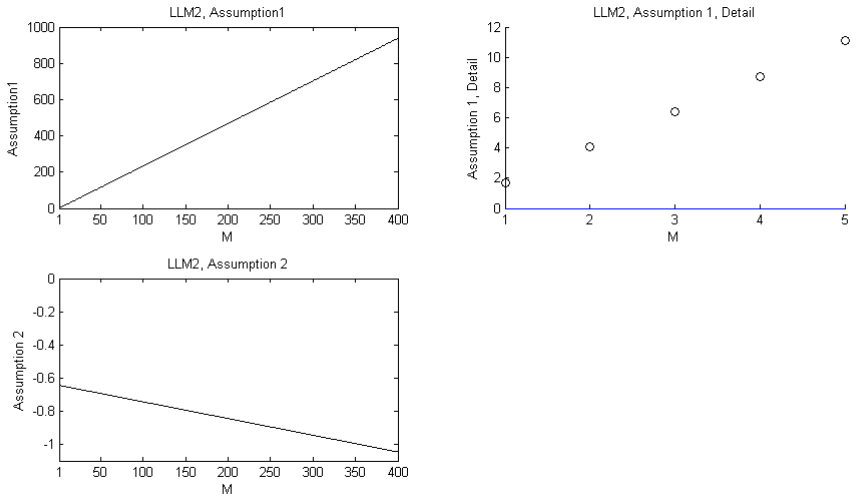

Figure 5.1.5: Assumptions Eq.$LLM2$ (1st row:Assump. 1; 2nd row: Assum.2)

Figure 5.1.6 depicts the results of LLM2 (in this case $Z = 0$ for all values of M, hence this variable is not shown in the plots). Changing of the parameter N_D leads to variations in the results of P and leaves the values of N unaltered in this equilibrium. The results for P are shown in blue for $N_D = 1.5$, red for $N_D = 8$, and green for $N_D = 16$. The last two subplots show P in dependency of N.

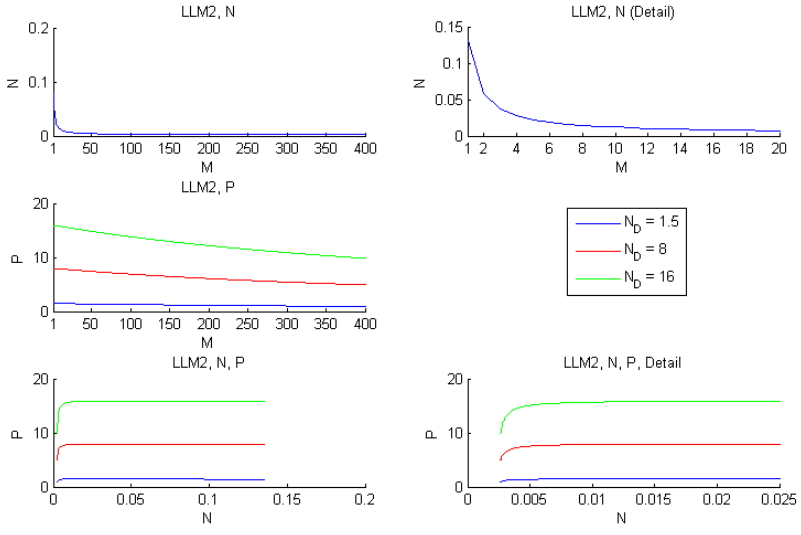

Figure 5.1.6: Results Eq.$LLM2$ (1st row: N; 2nd row: P, Legend; 3rd row: P on N), right: Details

The equilibria are unstable at all depths exept for $M = 1m$ (see figure 5.1.7).

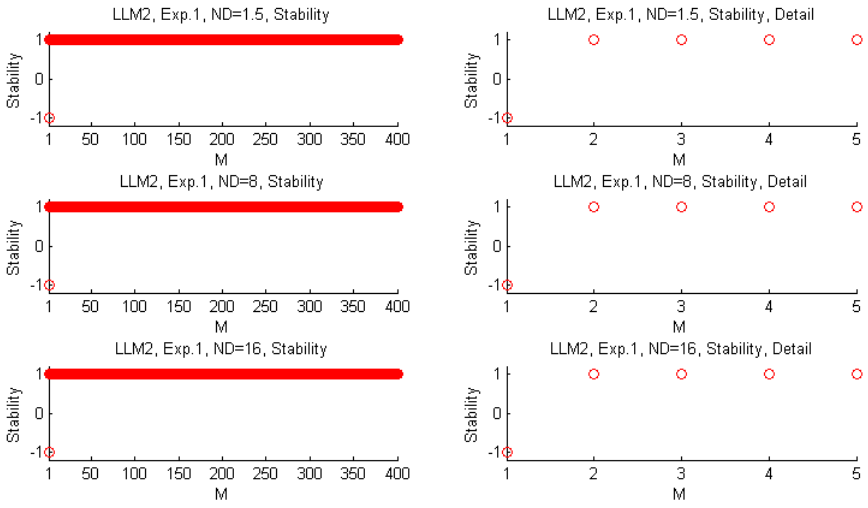

Figure 5.1.7: Stability, Equilibrium $LLM2$ (1st row: $N_D = 1.5$ and 1.5 detailed; 2nd row: $N_D = 8$ and 8 detailed; 3rd row: $N_D = 16$ and 16 detailed)

Second part: $Z \neq 0$ Again the proof starts with the initial equation system:
$0 = [-\mu\frac{N}{k+N} + \gamma\Phi_P]P + [(1-\beta)\frac{g\varepsilon P^2}{g+\varepsilon P^2} + \Phi_Z]\Omega Z + \frac{m_r}{M}(N_D - N) \quad (I_{LLM})$

$$0 = [\mu\frac{N}{k+N} - \Phi_P - \frac{m_r}{M}]P - \frac{g\varepsilon P^2}{g+\varepsilon P^2}Z \quad (II_{LLM})$$

$$0 = [\beta\frac{g\varepsilon P^2}{g+\varepsilon P^2} - \Phi_Z - \frac{m_r}{M}]Z \quad (III_{LLM}).$$

From Equation III_{LLM} with $Z \neq 0$ it holds

$$0 = \beta\frac{g\varepsilon P^2}{g+\varepsilon P^2} - \Phi_Z - \frac{m_r}{M}$$

and, as $\beta > 0$:

$$\Leftrightarrow \frac{g\varepsilon P^2}{g+\varepsilon P^2} = \underbrace{\frac{\Phi_Z M + m_r}{\beta M}}_{=:\, G_1} \quad (3)$$

Remark: G_1 equals G in this calculation (hence $G_1 > 0$), the index illustrates that this expression is only dependant on parameters and not variables. This can be done as long as one considers equilibria without changings in P).

$\Leftrightarrow g\varepsilon P^2 = gG_1 + \varepsilon P^2 G_1$

$\Leftrightarrow \varepsilon P^2(g - G_1) = gG_1$

and, as $\varepsilon > 0$

$\Leftrightarrow P^2(g - G_1) = \frac{g}{\varepsilon} G_1$

With $G = G1$ and $G < g$ it is known $g - G_1 \neq 0$

$$\Leftrightarrow P^2 = \underbrace{\frac{gG_1}{\varepsilon(g - G_1)}}_{=:\ \xi_1} \quad (4)$$

$\Rightarrow P = \sqrt{\xi_1} \vee P = -\sqrt{\xi_1}$.

With P being a real, nonnegative value, $\sqrt{\xi_1}$ it can be concluded $\xi_1 \geq 0$.
For the same reason the solution $P = -\sqrt{\xi_1}$ is neglected. Hence
$P = \sqrt{\xi_1}$ (5).

$\Rightarrow \frac{gG_1}{\varepsilon(g - G_1)} \geq 0$.

This information leads to assumption 6 (by substituting back G_1):

$\frac{g(\Phi_Z M + m_r)}{\varepsilon(g\beta M - \Phi_Z M - m_r)} \geq 0$.

Because of the parameter domains $g > 0, \varepsilon > 0, \Phi_Z > 0, m_r > 0, M > 0, \beta > 0$

and the necessary condition
$g - G_1 \neq 0 \Rightarrow g\beta M - \Phi_Z M - m_r \neq 0$

the conclusion is
$g\beta M > \Phi_Z M + m_r$.

Now with a solution for P, Z can be found: Substitution of (3) and (5) into equation II_{LLM} leads to
$0 = [\mu \frac{N}{k+N} - \Phi_P - \frac{m_r}{M}]\sqrt{\xi_1} - G_1 Z$

$\Leftrightarrow G_1 Z = [\mu\frac{N}{k+N} - \Phi_P - \frac{m_r}{M}]\sqrt{\xi_1}$

$\Leftrightarrow Z = \frac{\sqrt{\xi_1}}{G_1}[\mu\frac{N}{k+N} - \Phi_P - \frac{m_r}{M}]$ (6) (observe $G_1 > 0$)

With equation I_{LLM},

$$0 = [-\mu\tfrac{N}{k+N}+\gamma\Phi_P]\sqrt{\xi_1}+\underbrace{[(1-\beta)G_1+\Phi_Z]\Omega\frac{\sqrt{\xi_1}}{G_1}}_{=:\ \xi_2}[\mu\tfrac{N}{k+N}-\Phi_P-\tfrac{m_r}{M}]+\tfrac{m_r}{M}(N_D-N)$$

$$\Leftrightarrow 0 = [-\mu\tfrac{N}{k+N} + \gamma\Phi_P]\sqrt{\xi_1} + \xi_2[\mu\tfrac{N}{k+N} - \Phi_P - \tfrac{m_r}{M}] + \tfrac{m_r}{M}(N_D - N)$$

$$\Leftrightarrow 0 = [-\mu N + k\gamma\Phi_P + N\gamma\Phi_P]\sqrt{\xi_1} + \xi_2[\mu N - k\Phi_P - N\Phi_P - k\tfrac{m_r}{M} - N\tfrac{m_r}{M}] + k\tfrac{m_r}{M}N_D + N\tfrac{m_r}{M}N_D - k\tfrac{m_r}{M}N - N\tfrac{m_r}{M}N$$

$$\Rightarrow 0 = N^2[-\tfrac{m_r}{M}] + N[\sqrt{\xi_1}(-\mu+\gamma\Phi_P) + \xi_2(\mu - \Phi_P - \tfrac{m_r}{M}) + \tfrac{m_r}{M}(N_D - k)] + k(\gamma\Phi_P\sqrt{\xi_1} - \xi_2\Phi_P - \xi_2\tfrac{m_r}{M} + \tfrac{m_r}{M}N_D)$$

can be received.

As $m_r \neq 0$, it is obtained

$$0 = N^2 + N\underbrace{\frac{M}{m_r}[\sqrt{\xi_1}(\mu-\gamma\Phi_P) - \xi_2(\mu - \Phi_P - \frac{m_r}{M}) + \frac{m_r}{M}(k - N_D)]}_{=:\ a_1} + \underbrace{\frac{kM}{m_r}(-\gamma\Phi_P\sqrt{\xi_1} + \xi_2(\Phi_P + \frac{m_r}{M})) - kN_D}_{=:\ b_1}$$

Hence for the computed P and Z two possible results for N are found, which are the two roots of

$N^2 + a_1 N + b_1 = 0$.

The roots of this equation are

$N_1 = -\frac{a_1}{2} + \frac{1}{2}\sqrt{a_1^2 - 4b_1}$ and

$N_2 = -\frac{a_1}{2} - \frac{1}{2}\sqrt{a_1^2 - 4b_1}$.

For analysis it is assumed $a_1^2 \geq 4b_1$ because all solutions for N are real. In addition, N_1 and N_2 must be non-negative as an equilibrium of the model. Now the proof is proceeded with a case-by-case analysis:

Case N_1 $\quad N_1 = -\frac{a_1}{2} + \frac{1}{2}\sqrt{a_1^2 - 4b_1}$
$\Rightarrow -\frac{a_1}{2} + \frac{1}{2}\sqrt{a_1^2 - 4b_1} \geq 0$
$\Rightarrow \frac{1}{2}\sqrt{a_1^2 - 4b_1} \geq \frac{a_1}{2}$
$\Rightarrow \sqrt{a_1^2 - 4b_1} \geq a_1$

If $a_1 < 0$ all values of b_1 with $b_1 \leq \frac{1}{4}a_1^2$ meet the requirement.
If $a_1 \geq 0$, $b_1 \leq 0$ is required.

Case N_2

In this case $b_1 \leq \frac{1}{4}a_1^2$ is necessary as well, and
$N_2 = -\frac{a_1}{2} - \frac{1}{2}\sqrt{a_1^2 - 4b_1}$
$\Rightarrow -\frac{a_1}{2} - \frac{1}{2}\sqrt{a_1^2 - 4b_1} \geq 0$
$\Rightarrow \frac{1}{2}\sqrt{a_1^2 - 4b_1} \leq -\frac{a_1}{2}$.
If $a_1 < 0$ all values of b_1 with $0 \leq b_1 \leq \frac{1}{4}a_1^2$ meet the requirement.
If $a_1 = 0$ the solution holds for $b_1 = 0$.
With $a_1 > 0$ no useful root is found here.

In summary, two possible values of N for equilibria can be found. Now substituting N_1 respectively N_2 into equation (6) leads to Z_1 resp. Z_2:

$$Z_1 = \frac{\sqrt{\xi_1}}{G_1}[\mu\frac{N_1}{k+N_1} - \Phi_P - \frac{m_r}{M}]$$

$$Z_2 = \frac{\sqrt{\xi_1}}{G_1}[\mu\frac{N_2}{k+N_2} - \Phi_P - \frac{m_r}{M}]$$

This results in the equilibria $LLM_3 = (N_1, P, Z_1)$ and $LLM_4 = (N_2, P, Z_2)$.

□

The calculations and plots for both $LLM3$ and $LLM4$ with the usual values for N_D and M and with parameter set I are done with the program LLM3.m (Results: figure 5.1.8).

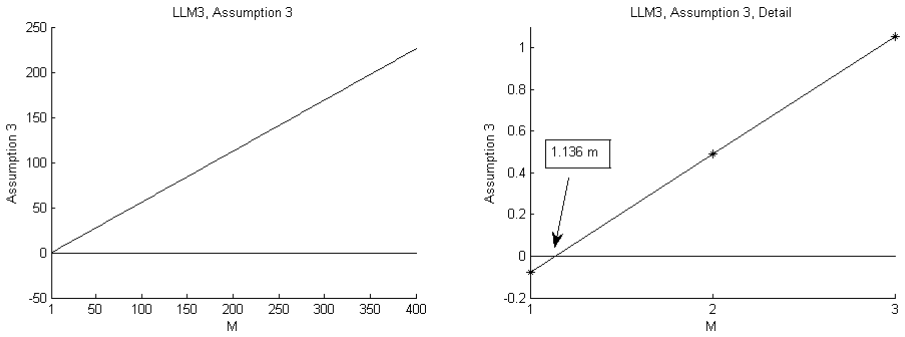

Figure 5.1.8: Assumption3, Equilibrium $LLM3$ (2nd Plot: Detail, Scatterplot on Interpolation Plot)

With the used parameters an equilibrium for $LLM3$ for all $M \geq 2$ is obtained. In case of $M = 1$ assumption 3 (which results in the same values with all three variations of N_D) is not fulfilled. The MATLAB results show that assumption 3 is not valid at a depth of $M = 1$, but for all other depths (with parameter set I). Applying the parameters from set I into the assumption and solving the unequality for M obtains that the assumption is fulfilled for all $M > 1.136m$. This, in case of $M \leq 1.136$ (exp.I) leads to negative values for G and the solution is rejected (for LLM3 see figures 5.1.9 and 5.1.10). In the case of $LLM4$ the root of N is negative; hence there is no equilibrium $LLM4$ with the given parameter set.

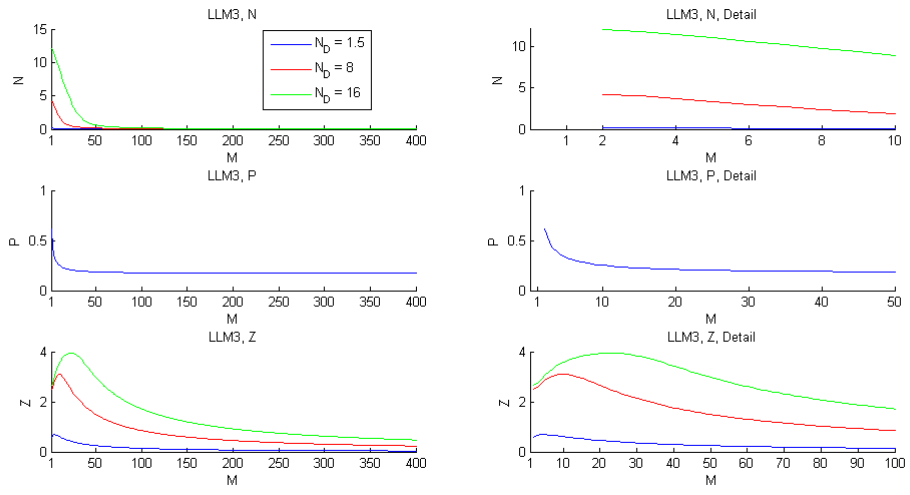

Figure 5.1.9: $LLM3$, Results, Comparing N_D Values; Left: All Depths, Right: Detail; 1st Row: N, 2nd Row: P, 3rd Row: Z

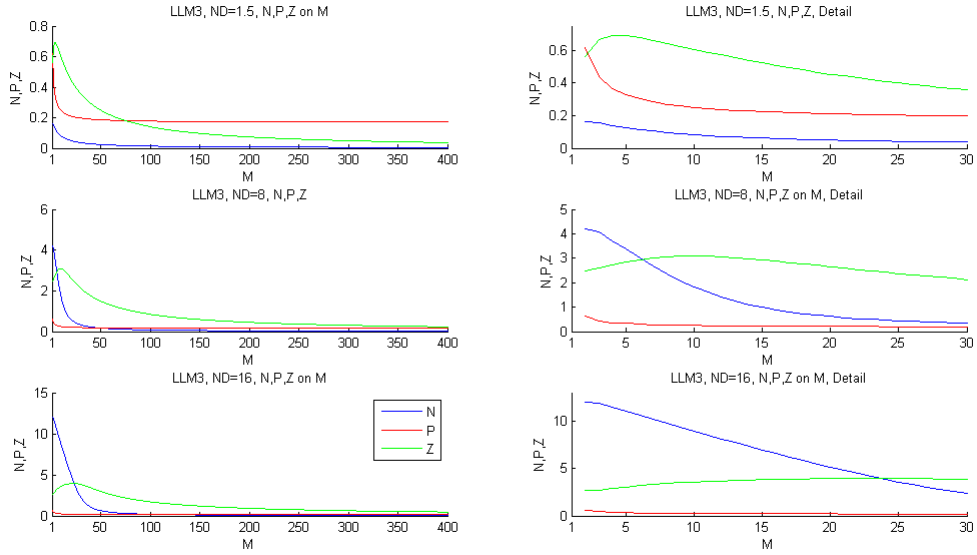

Figure 5.1.10: *LLM3*, Results, Comparing N,P, and Z; Left: All Depths, Right: Details; 1st Row: $N_D = 1.5$, 2nd Row: $N_D = 8$, 3rd Row: $N_D = 16$

The behavior of P is equal for all values of N_D. The first collection of plots shows each variable with respect to M; the second collection includes pictures for all variables together (calculated with a certain value of N_D).

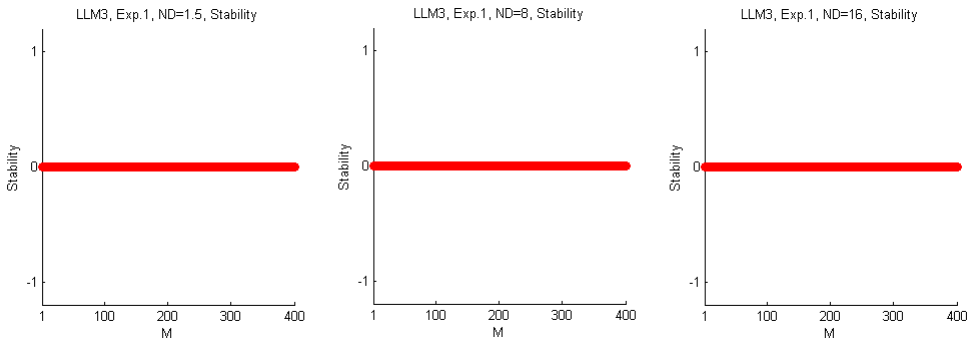

Figure 5.1.11: Stability LLM3, $N_D = 1.5, 8, 16$ (from left to right)

On the stability no conclusion can be done with the eigenvalues of the Jacobian matrix (figure 5.1.11), even if the program accepts the eigenvalue as zero only with a deviation of $10^{-}12$. In this case, to determine the stability of the equilibria, a

different approach would be necessary.

5.2 Model QQM: With quadratic loss terms for P and Z

For the QQM model both quadratic loss terms apply. Hence the model equations look as follows:

$$0 = [-\mu\tfrac{N}{k+N} + \gamma\Phi_P]P + [(1-\beta)\tfrac{g\varepsilon P^2}{g+\varepsilon P^2} + \Phi_Z]\Omega Z + \tfrac{m_r}{M}(N_D - N) \quad (I_{QQM})$$

$$0 = [\mu\tfrac{N}{k+N} - \Phi_P - \tfrac{m_r}{M}]P - \tfrac{g\varepsilon P^2}{g+\varepsilon P^2}Z - \Phi_P^* P^2 \quad (II_{QQM})$$

$$0 = [\beta\tfrac{g\varepsilon P^2}{g+\varepsilon P^2} - \Phi_Z - \tfrac{m_r}{M}]Z - \Phi_Z^* Z^2 \quad (III_{QQM}).$$

The number of Equilibria depends on the parameters. One equilibrium can be found for $P = 0$ and $Z = 0$ that always exists, and it is shown that there are not more than three equilibria for $P \neq 0$ and $Z = 0$ (depending on the chosen parameter sets). The values for N_2, N_3, N_4 are three (not necessarily real and non-negative) roots of a third degree equation. These roots are not given with formulae here. They are calculated with chosen parameter sets with the MATLAB program "'QQM.m"'. Complex and negative roots are sorted out, and with the useful (real nonnegative) results the program calculates P.

In case of $Z \neq 0$ the system cannot be solved analytically and the maximum number of Equilibria cannot be determined due to high degrees of rational functions.

The number of equilibria of the QQM system with $Z = 0$ is proven, and figures of the results, with the usual three values of N_D, all of this with parameter set II, are shown. After that with parameter set III parts of result tables and figures regarding QQM_2. With these parameters QQM_3 does not exist. The equilibrium QQM_4 does not exist with both of the used parameter sets.

Proposition 5.2.1. ***QQM Equilibria***

The QQM system with $Z = 0$ has one equilibrium $QQM_1 = (N_D, 0, 0)$ and up to three equilibria (QQM_2, QQM_3, QQM_4) with $P \neq 0$.

Proof: Equation III_{QQM}
$0 = [\beta \frac{g\varepsilon P^2}{g+\varepsilon P^2} - \Phi_Z - \frac{m_r}{M}]Z - \Phi_Z^* Z^2$
shows two cases:

Case $Z = 0$ and case $Z \neq 0$.

Case $Z = 0$ With $Z = 0$ the equation system transforms into

$$0 = [-\mu \frac{N}{k+N} + \gamma \Phi_P]P + \frac{m_r}{M}(N_D - N) \quad (I_{QQM,Z=0})$$

$$0 = [\mu \frac{N}{k+N} - \Phi_P - \frac{m_r}{M}]P - \Phi_P^* P^2 \quad (II_{QQM,Z=0})$$

From Equation $II_{QQM,Z=0}$ one obtains two subcases:
Case $(Z = 0 \wedge P = 0)$ and case $(Z = 0 \wedge P \neq 0)$.

(Case $Z = 0 \wedge P = 0$**)** $I_{QQM,Z=0}$ supplies
$0 = \frac{m_r}{M}(N_D - N)$
hence, as $\frac{m_r}{M} \neq 0$
$N = N_D$.
This is the equilibrium QQM_1.

Case $Z = 0 \wedge P \neq 0$ $0 = [\mu \frac{N}{k+N} - \Phi_P - \frac{m_r}{M}]P - \Phi_P^* P^2$ $(II_{QQM,Z=0}$)

$$\Leftrightarrow 0 = P(\mu \frac{N}{k+N} - \Phi_P - \frac{m_r}{M} - \Phi_P^* P)$$

and, with $P \neq 0$:

$$\Leftrightarrow 0 = \mu \frac{N}{k+N} - \Phi_P - \frac{m_r}{M} - \Phi_P^* P$$

$$\Leftrightarrow \Phi_P^* P = \mu \frac{N}{k+N} - \Phi_P - \frac{m_r}{M}$$

and, as $\Phi_P^* \neq 0$:

$$\Leftrightarrow \quad \mathrm{P} = \quad \frac{1}{\Phi_P^*}[\mu \frac{N}{k+N} - \Phi_P - \frac{m_r}{M}] \quad (7)$$

Substitution into $I_{QQM,Z=0}$ leads to

$$\begin{aligned}
\Leftrightarrow \quad 0 = \quad & [-\mu\tfrac{N}{k+N} + \gamma\Phi_P]\tfrac{1}{\Phi_P^*}[\mu\tfrac{N}{k+N} - \Phi_P - \tfrac{m_r}{M}] + \tfrac{m_r}{M}(N_D - N) \\
\Leftrightarrow \quad 0 = \quad & \tfrac{1}{\Phi_P^*}[-\mu^2\tfrac{N^2}{(k+N)^2} + \mu\tfrac{N}{k+N}\Phi_P + \mu\tfrac{N}{k+N}\tfrac{m_r}{M} + \gamma\Phi_P\mu\tfrac{N}{k+N} - \gamma\Phi_P^2 - \gamma\Phi_P\tfrac{m_r}{M}] + \tfrac{m_r}{M}N_D - \\
& \tfrac{m_r}{M}N
\end{aligned}$$

Multiplication with $(k+N)^2$ obtains

$$\begin{aligned}
\Leftrightarrow \quad 0 = \quad & \tfrac{1}{\Phi_P^*}[-\mu^2N^2 + \mu N\Phi_P(k+N) + \mu N\tfrac{m_r}{M}(k+N) + \gamma\Phi_P\mu N(k+N) - \gamma\Phi_P^2(k+ \\
& N)^2 - \gamma\Phi_P\tfrac{m_r}{M}(k+N)^2] + \tfrac{m_r}{M}N_D(k+N)^2 - \tfrac{m_r}{M}N(k+N)^2 \\
\Leftrightarrow \quad 0 = \quad & \tfrac{1}{\Phi_P^*}[-\mu^2N^2 + \mu N\Phi_P k + \mu N^2\Phi_P + \mu N\tfrac{m_r}{M}k + \mu N^2\tfrac{m_r}{M} + \gamma\Phi_P\mu Nk + \gamma\Phi_P\mu N^2 - \\
& \gamma\Phi_P^2k^2 - 2\gamma\Phi_P^2kN - \gamma\Phi_P^2N^2 - \gamma\Phi_P\tfrac{m_r}{M}k^2 - 2\gamma\Phi_P\tfrac{m_r}{M}kN - \gamma\Phi_P\tfrac{m_r}{M}N^2] + \\
& \tfrac{m_r}{M}N_Dk^2 + 2\tfrac{m_r}{M}N_DkN + \tfrac{m_r}{M}N_DN^2 - \tfrac{m_r}{M}Nk^2 - 2\tfrac{m_r}{M}N^2k - \tfrac{m_r}{M}N^3 \\
\Leftrightarrow \quad 0 = \quad & N^3[-\tfrac{m_r}{M}] + N^2[\tfrac{1}{\Phi_P^*}(-\mu^2 + \mu\Phi_P + \mu\tfrac{m_r}{M} + \gamma\Phi_P\mu - \gamma\Phi_P^2 - \gamma\Phi_P\tfrac{m_r}{M}) + \tfrac{m_r}{M}N_D - \\
& 2\tfrac{m_r}{M}k] + N[\tfrac{1}{\Phi_P^*}(\mu\Phi_Pk + \mu\tfrac{m_r}{M}k + \gamma\Phi_P\mu k - 2\gamma\Phi_P^2k - 2\gamma\Phi_P\tfrac{m_r}{M}k) + 2\tfrac{m_r}{M}N_Dk - \\
& \tfrac{m_r}{M}k^2] + \tfrac{1}{\Phi_P^*}(-\gamma\Phi_P^2k^2 - \gamma\Phi_P\tfrac{m_r}{M}k^2) + \tfrac{m_r}{M}N_Dk^2 \\
\Leftrightarrow \quad 0 = \quad & N^3[-\tfrac{m_r}{M}] + N^2[\tfrac{1}{\Phi_P^*}(\mu(-\mu + \Phi_P + \tfrac{m_r}{M} + \gamma\Phi_P) + \gamma\Phi_P(-\Phi_P - \tfrac{m_r}{M})) + \tfrac{m_r}{M}(N_D - \\
& 2k)] + N[\tfrac{k}{\Phi_P^*}[\Phi_P(\mu + \gamma\mu - 2\gamma\Phi_P - 2\gamma\tfrac{m_r}{M}) + \mu\tfrac{m_r}{M}] + \tfrac{km_r}{M}(2N_D - k)] + \\
& \tfrac{\gamma\Phi_Pk^2}{\Phi_P^*}(-\Phi_P - \tfrac{m_r}{M}) + \tfrac{m_r}{M}N_Dk^2
\end{aligned}$$

and, as $m_r \neq 0$

$$\begin{aligned}
\Leftrightarrow \quad 0 = \quad & N^3 \\
& + N^2 \underbrace{(-\frac{M}{m_r})[\frac{1}{\Phi_P^*}(\mu(-\mu + \Phi_P + \frac{m_r}{M} + \gamma\Phi_P) + \gamma\Phi_P(-\Phi_P - \frac{m_r}{M})) + \frac{m_r}{M}(N_D - 2k)]}_{=:a_2} \\
& + N \underbrace{(-\frac{M}{m_r})[\frac{k}{\Phi_P^*}[\Phi_P(\mu + \gamma\mu - 2\gamma\Phi_P - 2\gamma\frac{m_r}{M}) + \mu\frac{m_r}{M}] + \frac{km_r}{M}(2N_D - k)]}_{=:b_2} \\
& + \underbrace{\frac{\gamma\Phi_Pk^2M}{\Phi_P^*m_r}[\Phi_P + \frac{m_r}{M}] - N_Dk^2}_{=:c_2}
\end{aligned}$$

So it can be simplified

$0 = N^3 + a_2N^2 + b_2N + c_2$ with

$a_{QQM} := (-\frac{M}{m_r})[\frac{1}{\Phi_P^*}(\mu(-\mu + \Phi_P + \frac{m_r}{M} + \gamma\Phi_P) + \gamma\Phi_P(-\Phi_P - \frac{m_r}{M})) + \frac{m_r}{M}(N_D - 2k)]$

$b_{QQM} := (-\frac{M}{m_r})[\frac{k}{\Phi_P^*}[\Phi_P(\mu + \gamma\mu - 2\gamma\Phi_P - 2\gamma\frac{m_r}{M}) + \mu\frac{m_r}{M}] + \frac{km_r}{M}(2N_D - k)]$

$c_{QQM} := \frac{\gamma\Phi_Pk^2M}{\Phi_P^*m_r}[\Phi_P + \frac{m_r}{M}] - N_Dk^2.$

This third degree polynomial can support up to three zeros (roots). According to [Doerrie] the discriminant

$\theta = 18a_2b_2c_2 + a_2^2b_2^2 - 27c_2^2 - 4b_2^3 - 4a_2^3c_2$
can be used for a conclusion on the number of roots:

- $\theta > 0$ obtains 3 different real roots
- $\theta = 0$ obtains 1 double and one single real root (or one triple root)
- $\theta < 0$ obtains one real root and one pair of conjugated complex roots.

So, for chosen parameters, θ can be calculated to decide if in this case N supports $3, 2$ or 1 real root.

Now for chosen parameters and the assumption $Z = 0$ one can proceed as shown in [Dörrie] [4]. With chosen parameters, this supports up to three results of N: N_{QQM2}, N_{QQM3} and N_{QQM4}. As MATLAB is used for the calculation the [Dörrie] method for the roots is not used. MATLAB provides a numerical method for the roots of our third degree polynom, using the eigenvalues of the companion matrix (the results have to be seen as an approximation). With this approach roots that actually lead to equilibria can be found.

Substitution of the results of N into (7) will result in up to three Equilibria
$QQM_2 = (N_{QQM2}, P_{QQM2}, 0)$,
$QQM_3 = (N_{QQM3}, P_{QQM3}, 0)$,
$QQM_4 = (N_{QQM4}, P_{QQM4}, 0)$.

For the QQM Model (two quadratic loss terms) Schartau finds two useful sets of parameters in his experiments (experiment II and III). With this parameter sets the roots of N are calculated with MATLAB QQM.m. With the parameter set "'experiment II"' (see table 1) and the same three values for N_D as before provides the following results:

[4] Sometimes for the calculation of zeros it might be necessary to use complex numbers inside the calculation even if the results are real numbers. For further information see [Dörrie]

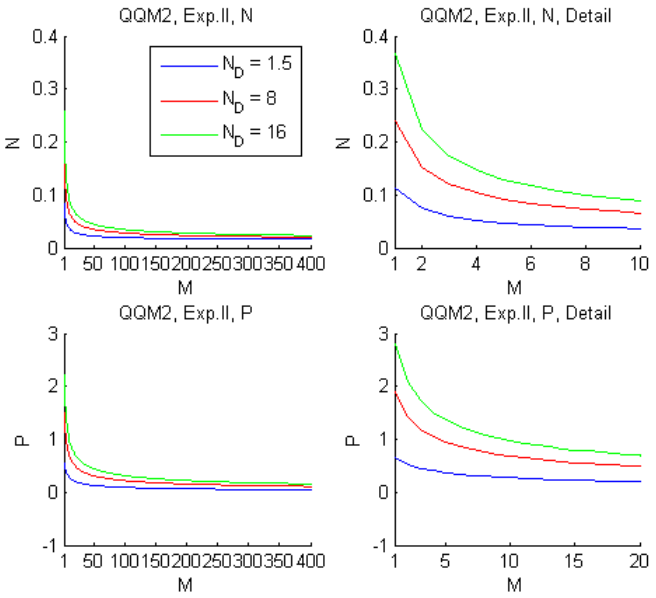

Figure 5.2.12: Equilibrium QQM2, Parameter Set II. Left: All Depths, Right: Details. First Row: N, Second Row: P

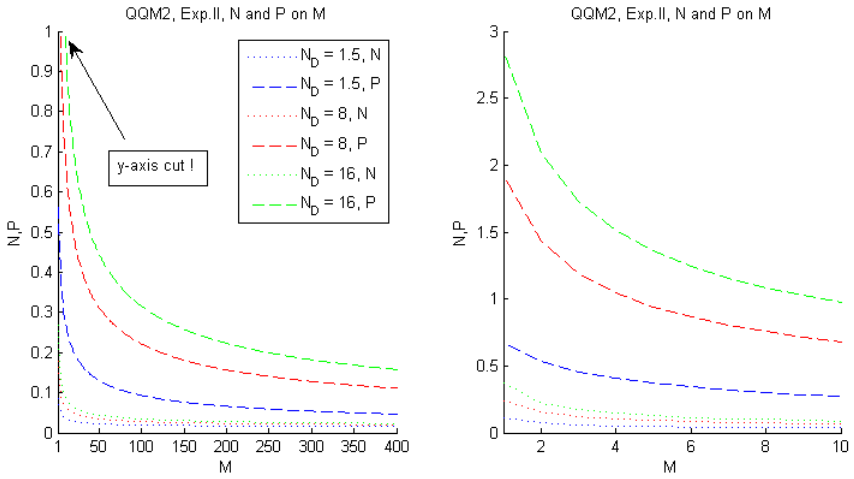

Figure 5.2.13: Equilibrium QQM2, Parameter Set II. Left: Comparison N (dotted), P (dashed), $N_D = 1.5, 8, 16$, y-axis cut for better overview

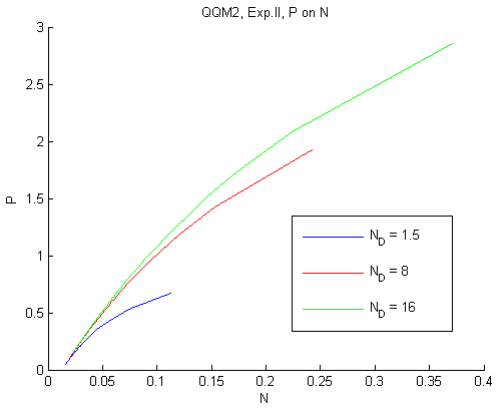

Figure 5.2.14: Equilibrium QQM2, Parameter Set II. P dependent on N

The equilibrium $QQM2$ exists for all mixed layer depths and is shown in figures 5.2.12, 5.2.13, 5.2.14. As it can be seen in figure ***, the equilibrium is unstable for layer depths from $1m$ to $291m$; in $292m$ the stability cannot be determined, and for depths from $293m$ on the equilibrium is stable, see figure 5.2.15.

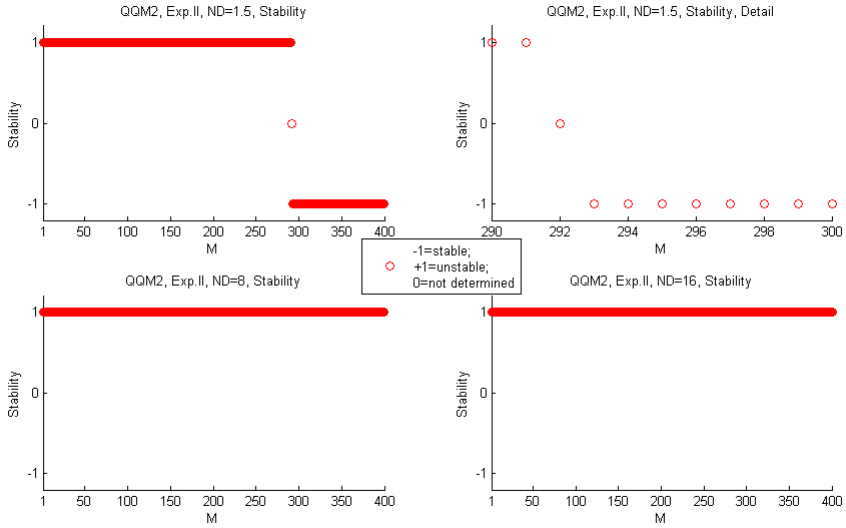

Figure 5.2.15: Equilibrium QQM2, Parameter Set II, Stability. 1st Row: $N_D = 1.5$, with Detail. 2nd Row: $N_D = 8, N_D = 16$

The transition between stable and unstable equilibria within two meters of layer depth only holds if any deviation from zero of more than 10^{-5} is considered unequal zero. If this value is made greater there will be a range of layer depths around

$292m$ where the stability cannot be determined.

For Equilibrium QQM_3 sufficient values for N if the mixed layer is deep enough. With $N_D = 1.5$ the $QQM3$ equilibrium is found at $22m$ and deeper, with $N_D = 8$ it exists at $131m$ or deeper, and with $N_D = 16, M = 264m$ is the shallowest depth with an equilibrium $QQM3$. In all these cases this roots of N produce negative values for P. Therefore the solution for QQM_3 is rejected. For $QQM4$ no sufficient roots of N occur with this parameter set.

Applying the experiment III parameter set, the system only provides the equilibrium $QQM2$ as well. With this parameter set the third degree polynomial supports only one useful (real and nonnegative) root. The following three figures show the features of the QQM_2 results with parameter set III and the stability results.

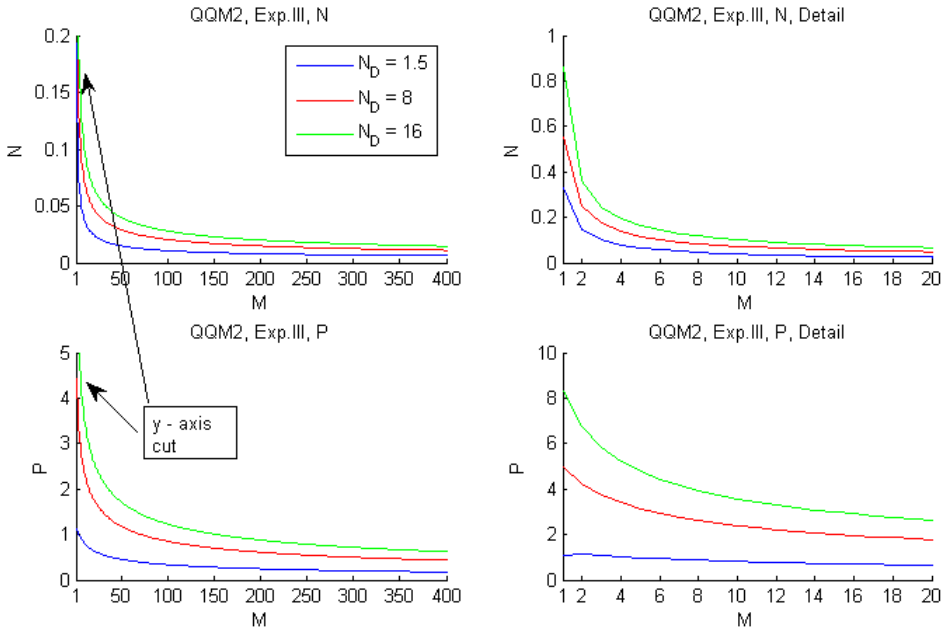

Figure 5.2.16: QQM2, Exp.III, 1st row: N, 2nd: P; right: Details

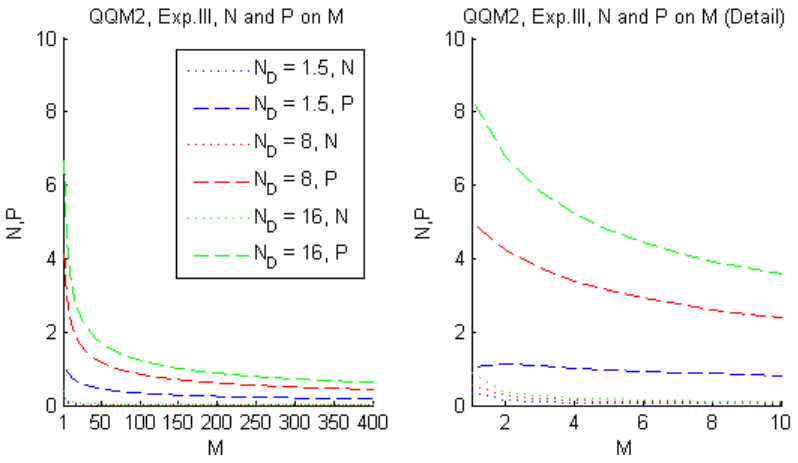

Figure 5.2.17: QQM2, Exp.III,N (dotted),P(dashed), different N_D.

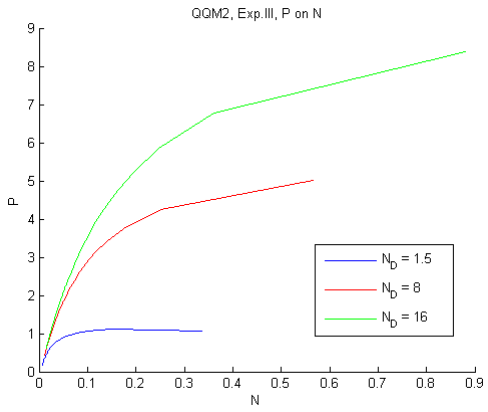

Figure 5.2.18: QQM, Exp.III, P dependent on N

As it can be seen in figure 5.4.24, all results are unstable with one exception; in case of $N_D = 1.5$ and a layer depth 1 m the equilibrium is asymptotically stable.

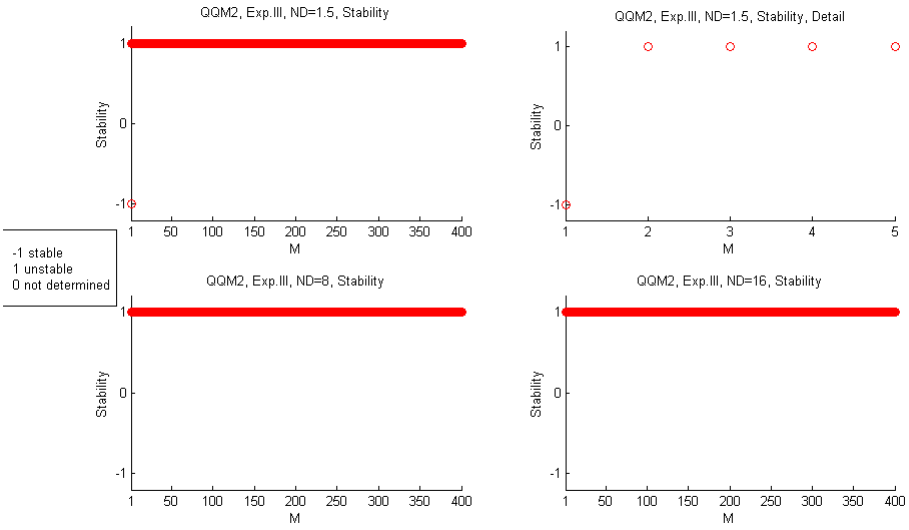

Figure 5.2.19: QQM2, Exp.III, Stability

(Case $Z \neq 0 \wedge P \neq 0$) Equation III_{QQM} can be written as

$$0 = [\beta \frac{g\varepsilon P^2}{g+\varepsilon P^2} - \Phi_Z - \frac{m_r}{M} - \Phi_Z^* Z]Z$$

With $Z \neq 0$, it can be concluded

$$0 = \beta \frac{g\varepsilon P^2}{g+\varepsilon P^2} - \Phi_Z - \frac{m_r}{M} - \Phi_Z^* Z$$
$$\Rightarrow \Phi_Z^* Z = \beta \frac{g\varepsilon P^2}{g+\varepsilon P^2} - \Phi_Z - \frac{m_r}{M}$$

and, as $\Phi_Z^* \neq 0$:

$$Z = \frac{1}{\Phi_Z^*}(\beta \frac{g\varepsilon P^2}{g+\varepsilon P^2} - \Phi_Z - \frac{m_r}{M})$$

Substitution of this term for Z into I_{QQM} or II_{QQM} leads to higher degree polynomials with terms like $(\frac{g\varepsilon P^2}{g+\varepsilon P^2})^2$ and cannot be solved analytically. To find equilibria in this case, a parameter set should be chosen, and the task should be approached with numerical methods.

□

5.3 Model LQM: With optional term for Z

This model version with linear phytoplankton loss term and quadratic loss term can be traced back to LLM for extincted zooplankton and to QQM with persisting

zooplankton.

Proposition 5.3.1. ***LQM Equilibria***

Regarding equilibria, the model LQM behaves like the model LLM if $Z = 0$*. If* $Z \neq 0$*, its equilibria equal the model QQM with* $Z \neq 0$*.*

Proof:

The model equations in this model version are

$$0 = [-\mu\tfrac{N}{k+N} + \gamma\Phi_P]P + [(1-\beta)\tfrac{g\varepsilon P^2}{g+\varepsilon P^2} + \Phi_Z]\Omega Z + \tfrac{m_r}{M}(N_D - N) \quad (I_{LQM})$$

$$0 = [\mu\tfrac{N}{k+N} - \Phi_P - \tfrac{m_r}{M}]P - \tfrac{g\varepsilon P^2}{g+\varepsilon P^2}Z \quad (II_{LQM})$$

$$0 = [\beta\tfrac{g\varepsilon P^2}{g+\varepsilon P^2} - \Phi_Z - \tfrac{m_r}{M}]Z - \Phi_Z^* Z^2. \quad (III_{LQM})$$

Case $Z = 0, P = 0$ obtains $LQM_1 = LLM_1 = (N_D, 0, 0)$.

Case $Z = 0, P \neq 0$ leads to the system

$$0 = [-\mu\tfrac{N}{k+N} + \gamma\Phi_P]P + \tfrac{m_r}{M}(N_D - N) \quad (I_{LQM,Z=0})$$

$$0 = [\mu\tfrac{N}{k+N} - \Phi_P - \tfrac{m_r}{M}]P \quad (II_{LQM,Z=0})$$

These equations are the same as in the LLM case with $Z = 0$ and obtain the same results.

Case $Z \neq 0$:

The equation

$$0 = [\beta\tfrac{g\varepsilon P^2}{g+\varepsilon P^2} - \Phi_Z - \tfrac{m_r}{M}]Z - \Phi_Z^* Z^2. \quad (III_{LQM})$$

equals the third model equation in model QQM. Solving for Z and substitution into the other equations leads to unsolvable polynomials the same way it does in QQM; a numerical approach should be done here as well.

□

5.4 Model QLM: With optional term for P

The model with optional quadratic loss term for phytoplankton, and linear loss term for zooplankton, is

$0 = [-\mu\frac{N}{k+N} + \gamma\Phi_P]P + [(1-\beta)\frac{g\varepsilon P^2}{g+\varepsilon P^2} + \Phi_Z]\Omega Z + \frac{m_r}{M}(N_D - N) \quad (I_{QLM})$

$0 = [\mu\frac{N}{k+N} - \Phi_P - \frac{m_r}{M}]P - \frac{g\varepsilon P^2}{g+\varepsilon P^2}Z - \Phi_P^* P^2 \quad (II_{QLM})$

$0 = [\beta\frac{g\varepsilon P^2}{g+\varepsilon P^2} - \Phi_Z - \frac{m_r}{M}]Z \quad (III_{QLM}).$

Proposition 5.4.1. ***QLM Equilibria***

If $Z = 0$, the model QLM has the same equilibria like the model QQM with $Z = 0$. If $Z \neq 0$ the model QLM provides up to two equilibria.

Proof:

Case $Z = 0$

As the QLM models' difference to the QQM model is limited to the third equation (which, with $Z = 0$, disappears here) the system provides the same equilibria like in the $QQM, Z = 0$ model. This is the same situation as in $QQM, P \neq 0, Z = 0$ and obtains the same results.

Case $Z \neq 0$

Let $Z \neq 0$. Then from equation III_{QLM} follows:

$\beta G - \Phi_Z - \frac{m_r}{M} = 0$

This equation equals the case $LLM, Z \neq 0$, and hence it leads to

$G = G_1$ and $P = \sqrt{\xi_1}$

with

$G_1 = \frac{\Phi_Z M + m_r}{\beta M}$ and

$\xi_1 = \frac{gG_1}{\varepsilon(g-G_1)}$

This can be applied to equation II_{QLM} and provides

$$0 = [\mu\tfrac{N}{k+N} - \Phi_P - \tfrac{m_r}{M}]\sqrt{\xi_1} - G_1 Z - \Phi_P^* \xi_1$$

$$\Rightarrow 0 = \mu\tfrac{N}{k+N}\sqrt{\xi_1} - \Phi_P\sqrt{\xi_1} - \tfrac{m_r}{M}\sqrt{\xi_1} - G_1 Z - \Phi_P^*\xi_1, \text{ and with } G_1 \neq 0\text{:}$$

$$\Rightarrow Z = \tfrac{\mu}{G_1}\tfrac{N}{k+N}\sqrt{\xi_1} - \underbrace{\frac{\Phi_P}{G_1}\sqrt{\xi_1} - \frac{m_r}{MG_1}\sqrt{\xi_1} - \frac{\Phi_P^*}{G_1}\xi_1}_{=:\xi_3}$$

Substituted into I_{QLM} this obtains

$$0 = [-\mu\tfrac{N}{k+N} + \gamma\Phi_P]\sqrt{\xi_1} + [(1-\beta)G_1 + \Phi_Z]\Omega[\tfrac{\mu}{G_1}\tfrac{N}{k+N}\sqrt{\xi_1} - \xi_3] + \tfrac{m_r}{M}(N_D - N)$$

$$\Leftrightarrow 0 = \tfrac{N}{k+N}\underbrace{(-\mu\sqrt{\xi_1} + (1-\beta)\Omega\mu\sqrt{\xi_1} + \Phi_Z\Omega\frac{\mu}{G_1}\sqrt{\xi_1}}_{=:xi_4} - \tfrac{m_r}{M}N$$

$$+\underbrace{\gamma\Phi_P\sqrt{\xi_1} - [(1-\beta)G_1 + \Phi_Z]\Omega\xi_3 + \frac{m_r}{M}N_D}_{=:\xi_5}$$

$$\Leftrightarrow 0 = -\tfrac{m_r}{M}N^2 + N(\xi_4 - \tfrac{m_r}{M}k + \xi_5) + \xi_5 k$$

$$\Leftrightarrow 0 = N^2 + N\underbrace{(-\xi_4\frac{M}{m_r} + k - \xi_5\frac{M}{m_r})}_{:=\xi_6} + \underbrace{(-k\xi_5\frac{M}{m_r})}_{:=\xi_7}$$

$$\Leftrightarrow 0 = N^2 + \xi_6 N + \xi_7$$

$$\Rightarrow N = -\tfrac{\xi_6}{2} \pm \sqrt{(\tfrac{\xi_6}{2})^2 - \xi_7}.$$

This polynomial obtains two roots; if one of them or both provide a nonnegative real value depends on the chosen parameters. With parameter sets II and III this is done with the program QLM.m. The results are put into the equation regarding Z to compute values for Z.

□

The results of the MATLAB QLM.m program show that with parameter set II there is one equilibrium each exist if calculated with $N_D = 1.5$ (stable), with $N_D = 8$ from 1 to $20m$ (stability result not determined), and with $N_D = 16$ at all depths (no determined stability, too). The second polynomial root provides an asymptotically stable equilibrium at $N_D = 8$, and the values match the equilibrium based on the first root. The results are shown in the figures below.

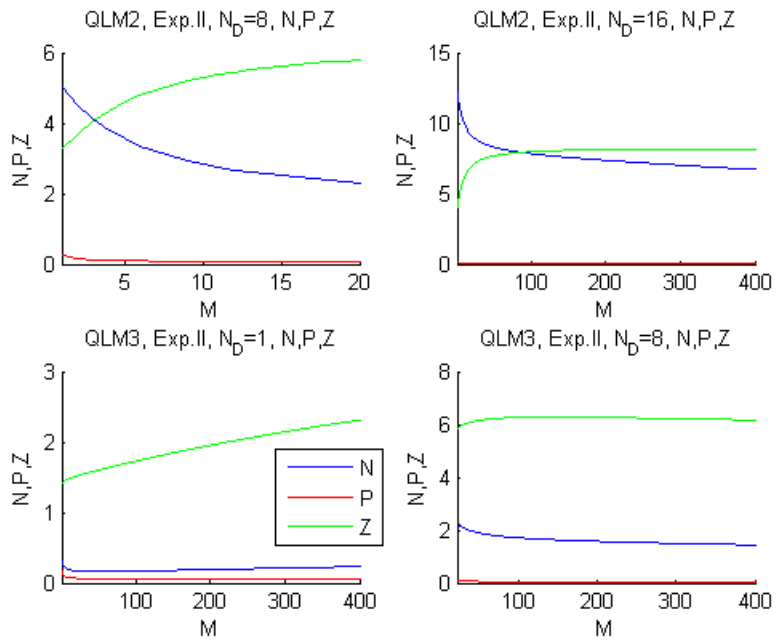

Figure 5.4.20: 1st row: QLM2 ($N_D = 8, 16$) , 2nd row: QLM3, $N_D = 1.5, 8$

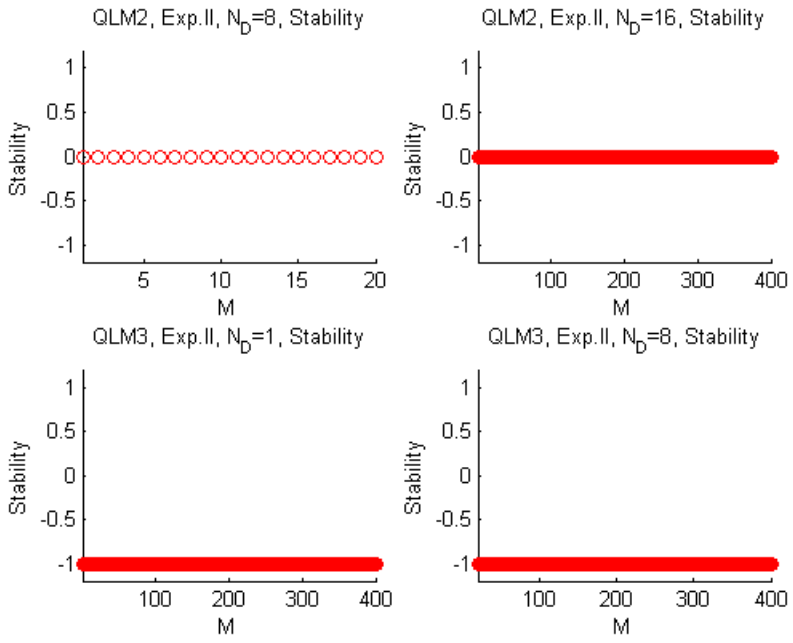

Figure 5.4.21: Stability Results, QLM2

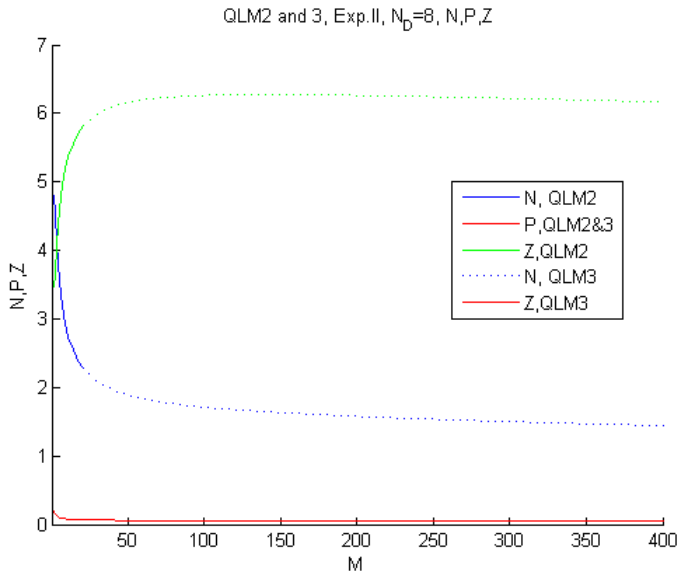

Figure 5.4.22: Results QLM2 and QLM3 (dotted) match

With parameter set III the following equilibria are obtained:
With $N_D = 1.5$ and $N_D = 8$ in a depth of $65m$ and deeper: an asymptotically stable equilibrium each, and with $N_D = 16$ the system provides with the first root an equilibrium with undetermined stability in 65 to $224m$, and in the same way as with parameter set II a matching one in $225m$ to $400m$, in this case stable.
The results are shown in the following figures. Note the very low values of P.

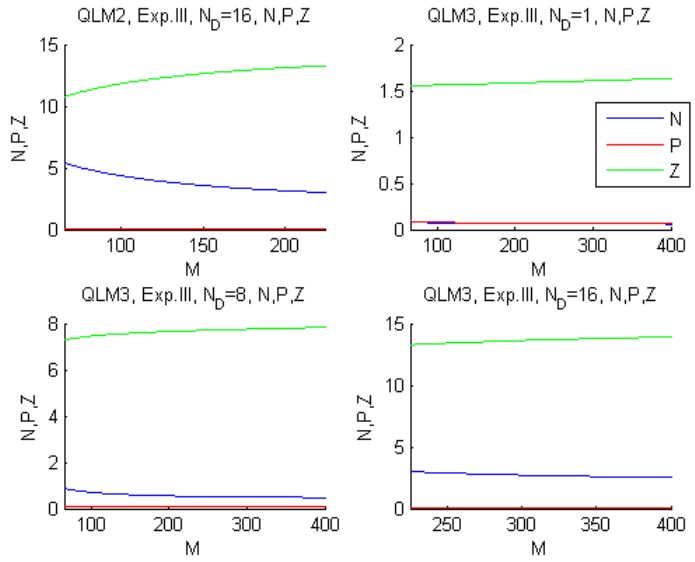

Figure 5.4.23: Results QLM with Parameter Set III

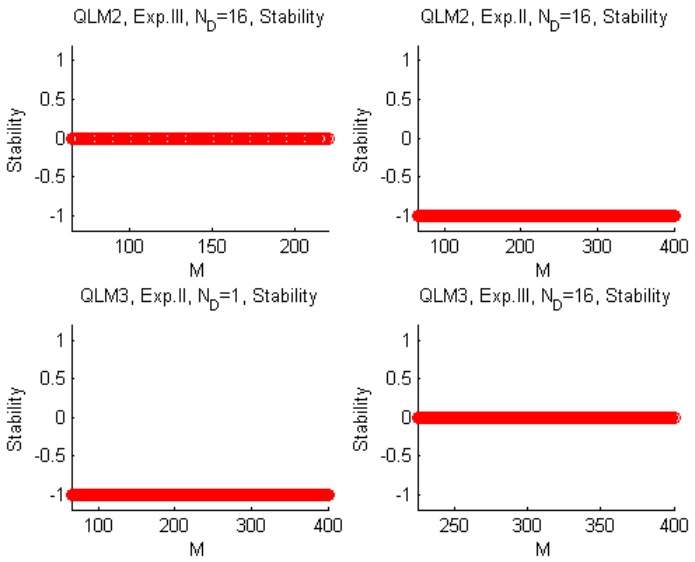

Figure 5.4.24: Stability Results QLM with Parameter Set III

The program QLM obtains very high deviations from zero within the results of $\frac{dP}{dt}$. This might be caused by the fact that during the proof of the equilibria several interim results are fixed in order to avoid extremely long equations. The interim results themselves are combined to more interim results and so on, up to interim result ξ_6 and ξ_7 (compare proof 5.4.1). Rounded values are multiplied with

rounded values, and combined with parameter M (which has got a huge range of values). In summary this might lead to very imprecise results in the MATLAB calculation. Furthermore there might be an error in the QLM proof or in the program QLM. Anyway the model QLM itself might not be very useful (it is not considered in a detailed way within the mentioned literature).

6 Summary and Outlook

6.1 Summary

In summary, the here explained and considered zero dimensional NPZ model, based on the models of Evans and Parslow [Evans,Evans, Parslow], provides helpful features for analysing plankton life cycles. A number of parameters enable the scientist to adjust the equations on his aim of research, e.g. the region of the ocean, season or analyzed pankton species (growth functions, loss functions etc. are parameter dependent). After a detailed explanation of the model variables and parameters, the continuity and differentiability has been explored. As a local Lipschitz continuous system the NPZ model guarantees a local unique solution, positivity, and continuous dependency of solutions. Above that the solutions depend on initial values including parameters in a differentiable way. A global unique solution could not be proven as this requires global Lipschitz continuity. Additionally, the mathematical model does not provide mass conservation.

Those equilibria that can be gained from polynomials of at most third degree are lined out and can be calculated with some basic MATLAB programs included into this thesis. Figures of all results that might be interesting have been plotted; for special research issues more figures can be created because the programs store all results during the program run in tables, including those who have not been plotted in this thesis.

Some potential equilibrium points could not be found theoretically due to the high degrees of the rational functions. More advanced numerical research with fixed parameter sets might enlighten this topic. During the stability analysis it turned out that with some of the points no conclusion could be made (because the Jacobian matrix exhibits an eigenvalue with a real part equal zero). An analysis with Ljapunow function approach, as described in [Pruess], section 5.5., could obtain more knowledge on this issue. The MATLAB programs can be used for analysis of the system with varied parameters. Parameter sets (for description of the features of certain situations of phytoplankton and zooplankton in the ocean) and allowed variations from zero are stored at the top of the program codes and

can be changed easily. Albeit optimizing parameter sets with respect to measured values from ocean experiments is a much more advanced task and currently subject of research, especially if data assimilation is used (as can be seen in [Schartau]). For this reason no numerical experiments with parameter variations above those parameter sets given in [Schartau] have been carried out.

The model can be used in different versions with linear or quadratic plankton loss functions; the choice of the loss function does have an impact on the equilibria but not on any of the other model features.

The model is useful for issues that request analysis of the mixed layer as a whole. If variations of the nutrient and plankton concentrations within the vertical axe in the mixed layer are needed, the one dimensional NPZD model might be a more suitable choice.

6.2 Outlook

Some ideas on further analysis of the NPZ model have already been mentioned in the last section (applying numerical methods on the remaining equilibria, applying Ljapunov theory).

As the NPZ model does not provide mass conservation, it requests an enhancement, e.g. by adding the mass conservation while introducing extra terms for detritus and adding an extra equation for derivation of detritus to the system, as a kind of a zero dimensional NPZD model. On the other hand the mass conservational one dimensional NPZD model already exists, is introduced in [Schartau] together with a parameter optimization consideration, and all equilibria that can be found with reasonable effort are considered in [Heinle]; hence an enhancement of the NPZ model might not be necessary at all. All methods that are used in this thesis can be applied to other systems of (ordinary) differential equations.
This thesis does not deal with the task of periodic solutions. A periodic solution is a solution $f(t)$ of an IVP with the feature $f(t) = f(t+T)$ for all $t \in \mathbb{R}_+$, with $T > 0$ being the period. Differential equation systems that describe plankton cycles in the ocean might have periodic features, as the plankton itself is influenced by annual periodic events as the seasons. If poisonous algae blooms occur, these cycles sometimes even are reported in mainstream media. Understanding annual cycles in ocean biota is an interesting topic and subject of science, although the situation in nature can require complicated modeling: plankton life cycles may be influenced of several processes with periods of different lengths (one, two or more years) that interfere each other. Certain plankton species can exist parallel and

influence each other with alternating dominance. ENSO circulation might influence the cycles (see [Evans], [Evans,Evans, Parslow] etc.). Knowing about periodic features in plankton models would be preferable. Nevertheless the usual methods do not apply with the NPZ model:

The Floquet theory (as described in [Timmann], page 96) is a method for linear differential equations and does not apply with the nonlinear NPZ system. The Bendixson criterion ([Timmann], page 64) applies for systems with a domain $U \subset \mathbb{R}^2$. The domain of the system in this thesis is three dimensional.

For further studies the reduced model suggested in [Evans, Evans, Parslow], consisting of only two variables P and Z, might be helpful. The growth function is replaced by a sinus and the dimension is reduced because the nutrient variable and equation are skipped. This way Evans and Parslow find a pseudo-equilibrium cycle that might be useful to describe plankton bloom cycles. As the reduced model of Evans and Parslow is two-dimensional an attempt to apply the Bendixson criterion might be promising.

Furthermore [Gajewski] (section VI, § 2) offers a further approach. This method is working within a suitable Hilbert space, which again is its own dual space. For using this method, the system is understood as a nonlinear operator, and this operator is investigated on a number of features, among them coercivity. With appropriate functional analytical skills, this approach might lead to more knowledge on the question of periodical solutions of the NPZ model.

7 Bibliography

References

[Doerrie] Heinrich Dörrie. *Kubische und biquadratische Gleichungen* München, Leibniz Verlag, 1948

[Evans, Parslow] Geoffrey T.Evans & John S. Parslow. *A Model of Annual Plankton Cycles* Biological Oceanography, 3:3, 327-347, 1985 http://dx.doi.org/10.1080/019655

[Evans] Geoffrey T. Evans. *A framework for discussing seasonal succession and coexistence of phytoplankton species* Limnology and Oceanography, 33(5), 1988, 1027-1036

[Franks] Peter J.S. Franks. *NPZ Models of Plankton Dynamics: Their Construction, Coupling to Physics, and Application*, Journal of Oceanography, Vo. 58, pp.379 to 387, 2002

[Gajewski] Herbert Gajewski, Konrad Gröger, Klaus Zacharias. *Nichtlineare Operatorgleichungen und Operatordifferentialgleichungen* Akademie-Verlag Berlin, 1974

[Heinle] Anna Heinle, Thomas Slawig. *Internal dynamics of NPZD type ecosystem models*, Ecological Modelling 254(2013)33-42, 2013

[Koenigsberger] Konrad Königsberger. *Analysis 2, 5., korrigierte Auflage* Springer Verlag, 2003

[Ott] Jörg Ott. *Meereskunde : Einführung in die Geographie und Biologie der Ozeane, 2.Auflage* Stuttgart, Ulmer, 1996

[Pruess] Jan W. Prüss & Mathias Wilke. *Gewöhnliche Differentialgleichungen und dynamische Systeme*, Basel, Birkhäuser, 2010

References

[Schartau] Markus Schartau. *Data-assimilation studies of marine, nitrogen based, ecosystem models in the North Atlantic Ocean*, CAU Kiel, Dissertation, 2001

[Slawig] Thomas Slawig. *Klimamodelle und Klimasimulationen* Christian-Albrechts-Universität zu Kiel, Institut für Informatik, Exzellenzcluster *The Future Ocean* , Vorlesungsscript Wintersemester 2011/12, Stand 3.2.2012

[Maynard Smith] , John Maynard Smith. *Models in Ecology* Cambridge University Press, 1974

[Timmann] Steffen Timmann. *Repetitorium der gewöhnlichen Differentialgleichungen* Binomi Verlag, Springe, 2005

[Walter] Wolfgang Walter. *Gewöhnliche Differentialgleichungen* Springer Verlag, 2000

[JGOFS] *Joint Global Ocean Flux Study* (JGOFS) , see *http://usjgofs.whoi.edu/research/index.html* , retrieved Nov.20,2014, 10:36 UTC .

8 Appendix

8.1 Tables: Variables and Parameters

The tables are organized as follows:

1. Table 1: Variables
2. Table 2: Parameters
3. Table 3: Auxiliary Terms
4. Table 4: Experimental Parameter Sets

8.1.1 Variables

Variables	Domain	Unit
N	$(-k_m in, \infty)$	$\frac{mmolN}{m^3}$
P	$\mathbb{R}$	$\frac{mmolN}{m^3}$
Z	$\mathbb{R}$	$\frac{mmolN}{m^3}$

8.1.2 Parameters

Parameters	Domain	Unit
M	$[1, 400]$	m
N_D	$[1, 20]$	$\frac{mmolN}{m^3}$
μ	$[0.100, 3.000]$	$\frac{1}{d}$
N_D	$[1, 20]$	$\frac{mmolN}{m^3}$
g	$[0.100, 1.000]$	$\frac{1}{d}$
m_r	$[0.010, 3.000]$	$\frac{m}{d}$
k	$[0.100, 0.700]$	$\frac{mmolN}{m^3}$
Φ_P	$[0.010, 0.100]$	$\frac{1}{d}$
Φ_Z	$[0.010, 0.100]$	$\frac{1}{d}$
Ω	$[0.010, 1.000]$	1
Φ_P^*	[0.010,1.000]	$\frac{m^3}{mmolNd}$
Φ_Z^*	[0.010,1.000]	$\frac{m^3}{mmolNd}$
β	$[0.100, 1.000]$	1
γ	$[0.010, 1.000]$	1
ε	$[0.100, 3.000]$	$\frac{m^6}{(mmolN)^2d}$

8.1.3 Auxiliary terms

Auxiliary terms	Domain	Unit
F_N	$\mathbb{R}_+ \times \mathbb{R}_+ \times [1, 400]$	$\frac{mmol}{m^3d}$
G	$\mathbb{R}_{\geq 0}$	$\frac{1}{d}$
u	$\mathbb{R}_{\geq 0}$	1

8.1.4 Experimental Parameter Sets

Parameter	Unit	Experiment		
		I	II	III
μ	$\frac{1}{d}$	2.362	1.679	3.775
Φ_P	$\frac{1}{d}$	0.016	0.033	0.017
k	$\frac{mmolN}{m^3}$	0.350	0.636	0.560
g	$\frac{1}{d}$	0.743	0.982	1.560
ε	$\frac{m^6}{mmolN^2d}$	3.088	3.473	3.740
Φ_Z	$\frac{1}{d}$	0.067	0.010	0.011
β	1	1;0.852	1.003	1.045
m_r	$\frac{m}{d}$	0.643	0.107	1.270
Ω	1	1;0.348	0.990	1.006
γ	1	1;0.973	0.990	0.870
Φ_P^*	$\frac{m^3}{mmolNd}$	–	0.168	0.122
Φ_Z^*	$\frac{m^3}{mmolNd}$	–	0.257	0.098

Erklärung

Hiermit erkläre ich, dass die vorliegende Arbeit selbständig verfasst worden ist und keine anderen als die angegebenen Quellen und Hilfsmittel verwendet worden sind.

Kiel, den 25.11.2014

(Petra Fuhs)