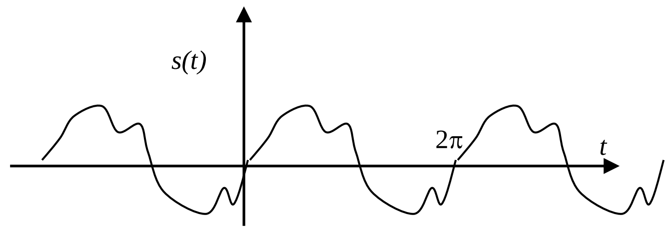
FOURIER SERIES REPRESENTATION OF SIGNALS

Any periodic time-domain signal can be represented (or modelled or approximated) as a d.c. component plus a sum of sine and cosine functions.
Consider the periodic signal *s(t)* shown right and for convenience let its period be 2π seconds.



The ***Fourier series representation*** of this signal can be expressed as:

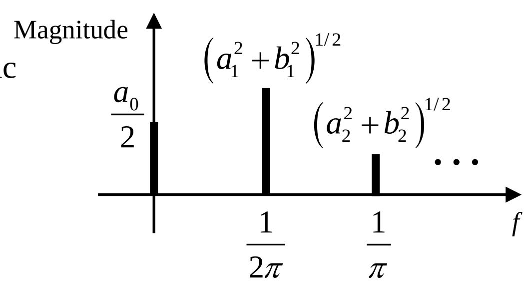
$$s(t) = \frac{a_0}{2} + a_1 \cos t + b_1 \sin t + a_2 \cos 2t + b_2 \sin 2t + \cdots + a_n \cos nt + b_n \sin nt + \cdots \qquad (1)$$

where a_n and b_n are constants and are called the ***Fourier coefficients*** of the function (or signal) *s(t)*.

Note that $\frac{a_0}{2}$ represents the DC component of *s(t)*. The combined terms $a_1 \cos t + b_1 \sin t$ represent the fundamental sinusoidal component or first harmonic. $a_2 \cos 2t + b_2 \sin 2t$ represents the second harmonic etc.

Ex. Show mathematically and by a function m-file demo that $a_1 \cos t + b_1 \sin t$ can represent a sinusoid of period 2π and arbitrary amplitude and phase.

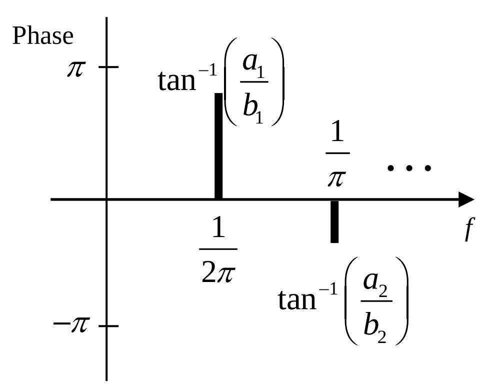
So if we can determine a_n (for $n = 0, 1, 2, \cdots \infty$) and b_n (for $n = 1, 2, \cdots \infty$) then we can model our signal exactly. In practice we can generally get an adequate approximation of our signal with a finite number of harmonics.



The useful thing about the Fourier series representation of a periodic signal is that it enables us to represent a signal in the frequency-domain as shown right. But first we need a way to determine the Fourier series coefficients a_n and b_n.

If we multiply both sides of equation (1) above by $\cos mt$ and integrate from -π to +π we get:

$$\int_{-\pi}^{\pi} s(t) \cos mt\,dt = \frac{a_0}{2} \int_{-\pi}^{\pi} \cos mt\,dt + a_1 \int_{-\pi}^{\pi} \cos t \cos mt\,dt + b_1 \int_{-\pi}^{\pi} \sin t \cos mt\,dt + \cdots$$



Now $\int_{-\pi}^{\pi} \cos nt \cos mt\,dt = \begin{cases} 0 & \text{for} \quad m \neq n \\ \pi & \text{for} \quad m = n \end{cases}$ (prove this)

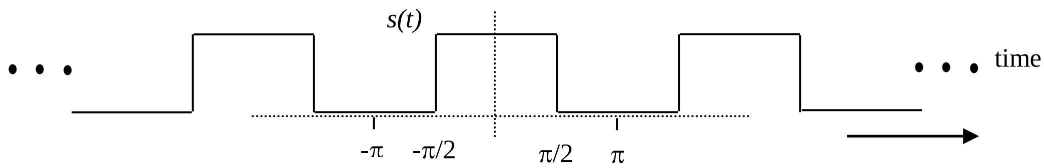
And $\int_{-\pi}^{\pi} \sin nt \cos mt\,dt = 0$ for all integers *m, n* (prove this)

Therefore $a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} s(t) \cos nt\,dt$ for $n = 0, 1, 2, \cdots \infty$

Similarly, if we multiply both sides of equation (1) by $\sin mt$ and use:

$\int_{-\pi}^{\pi} \sin nt \sin mt\,dt = \begin{cases} 0 & \text{for} \quad m \neq n \\ \pi & \text{for} \quad m = n \end{cases}$ we get $b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} s(t) \sin nt\,dt$ for $n = 1, 2, \cdots \infty$

Example.



$$a_0 = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} 1dt = 1 \qquad a_n = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} 1\cos ntdt = \frac{1}{n\pi}\left[\sin nt\right]_{-\pi/2}^{\pi/2} = \frac{1}{n\pi}\left[\sin\frac{n\pi}{2} - \sin\left(\frac{-n\pi}{2}\right)\right] = \frac{2\sin\left(\frac{n\pi}{2}\right)}{n\pi}$$

$$= \frac{\sin\left(\frac{n\pi}{2}\right)}{\left(\frac{n\pi}{2}\right)}$$

Therefore: $a_n = \begin{cases} \dfrac{-2}{n\pi} & \text{for} \quad n = 3, 7, 11, \cdots \\ \dfrac{2}{n\pi} & \text{for} \quad n = 1, 5, 9, \cdots \\ 0 & \text{for} \quad n = 2, 4, 6, \cdots \end{cases}$ $\qquad b_n = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} 1\sin ntdt = \frac{1}{n\pi}\left[-\cos nt\right]_{-\pi/2}^{\pi/2} = 0$

Therefore: $s(t) = \frac{1}{2} + \frac{2}{\pi}\cos t - \frac{2}{3\pi}\cos 3t + \frac{2}{5\pi}\cos 5t - \frac{2}{7\pi}\cos 7t + \cdots =$

$$\frac{1}{2} - \frac{2}{\pi}\sum_{n=1}^{\infty}(-1)^n \frac{\cos(2n-1)t}{2n-1}$$

Eg. Write an m-file to sum and plot the above harmonics one at a time for t = -2π to 2π

Sol.

```
% script m-file to show that a square wave of frequency 1/2pi
% is made up of a sum of cosine functions of the form:
% {sin(n*pi/2)/(n*pi)}*cos(2*pi*n*f0*t)

t = -2*pi:.01:2*pi;  % a 4π second time vector
s = 0.5*ones(size(t));  % this is the first Fourier coefficient
plot(t,s)
pause

for n = 1:100, % to see more Fourier terms added, increase the upper limit
   s = s+(2/pi)*((-1)^(n))*cos((2*n-1)*t)/(2*n-1);
   plot(t,s)
   pause
   n
end;
```