THE FOURIER TRANSFORM

The Fourier Transform of a time-domain signal can be thought of as a generalization of the Fourier Series representation of the signal. It is defined as:

$$H(j\omega) = \int_{-\infty}^{\infty} h(t) e^{-j\omega t} dt$$

Where $h(t)$ is the time-domain signal. Note the similarity between the above definition and the equations for determining the Fourier series coefficients a_n and b_n. For a particular value of frequency, say ω_1, the value of the Fourier transform, $H(j\omega_1)$, will represent the sinusoidal component of $h(t)$ which has frequency equal to ω_1. Note also that $H(j\omega_1)$ represents this sinusoidal component as a complex number and thereby represents the amplitude and phase of the component.

A sufficient condition for the Fourier transform to exist is that $\int_{-\infty}^{\infty} |h(t)| dt < \infty$. If this condition is satisfied then we can always evaluate the Fourier integral. Mathematically the integral is said to converge.

Given a system's impulse response, the Fourier transform allows us to determine its frequency response.

Eg. An exponential pulse is defined by $h(t) = \begin{cases} e^{-\alpha t} & \text{for} \quad t \geq 0 \\ 0 & \text{for} \quad t < 0 \end{cases}$. Find its Fourier transform.

Solution: $H(j\omega) = \int_{0}^{\infty} e^{-\alpha t} e^{-j\omega t} dt = \int_{0}^{\infty} e^{-(\alpha + j\omega)t} dt = \left| \frac{e^{-(\alpha+j\omega)t}}{-(\alpha + j\omega)} \right|_{0}^{\infty} = \frac{1}{\alpha + j\omega}$

Note that because $h(t) = 0$ for $t < 0$, we can set the lower limit of the Fourier integral to zero.

The inverse Fourier transform is defined as: $h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(j\omega) e^{j\omega t} d\omega$

The reason for the $\frac{1}{2\pi}$ scaling factor is that the frequency variable ω is expressed in radians/sec as opposed to Hertz which is the true reciprocal of time measured in seconds. The Fourier transform pair can also be expressed as:

$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt$ and $x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi ft} df$

Eg. The impulse response of a circuit is approximately equal to an exponentially decaying pulse given by:

$$h(t)=\begin{cases}5\times10^3 e^{-t/\tau} & \text{for} \quad t\geq 0\\ 0 & \text{for} \quad t<0\end{cases}$$

The time-constant τ of the circuit is 1 ms. Use the Fourier transform to determine the response of the circuit to an input sinewave of amplitude 3V and frequency 200Hz.

Solution:

Using the last example above, we can say that the Fourier transform of an impulse response of the form $Ae^{-\alpha t}$ is given by: $\dfrac{A}{\alpha + j\omega}$

So in this case, A = 5 x 10^3 and α = $1/\tau$ = 1000 rad/s which means that the frequency response of the circuit is:

$$H(j\omega)=\frac{5000}{1000+j\omega}$$

At a frequency of 200 Hz, $\omega = 2\pi \times 200 = 400\pi$ rad/s so that

$$H(j\omega)\Big|_{\omega=400\pi}=\frac{5000}{1000+j400\pi} \quad =\frac{5}{1+j1.26}$$

Therefore, the magnitude response at 200 Hz is equal to $\left|\dfrac{5}{1+j1.26}\right| \;=\; \dfrac{5}{\sqrt{1+(1.26)^2}}$ = 3.1

And the phase response at 200 Hz is equal to $\angle\left(\dfrac{5}{1+j1.26}\right)$ = - $\tan^{-1}(1.26)$ = - 0.9 rad or -52^{o}

So the output of the circuit in response to an input sinewave of amplitude 3V and frequency 200Hz is equal to a sinewave of amplitude 3 x 3.1 = 9.3V with a phase lag relative to the input sinewave of 52 ^{o}

Ex. What delay in milliseconds does this phase lag correspond to?

Note that the frequency of the output sinewave is equal to that of the input sinewave i.e. 200 Hz.

Ex. Write an m-file to calculate and plot the magnitude and phase response of the above circuit.