Macroeconomic Theory

Assignment III

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ECO 7938 - MACROECONOMIC THEORY II

Problem Set 3

Instructions: This problem set is due on April 8th before the beginning of class. Make sure that you are submitting on Canvas clear and ordered answers, and to submit your code so I can replicate your work. Failure to meet these requirements will translate in a 30 point deduction from your final grade. You may work in groups of 2, but you need to submit individual answers and explicitly mention the members of your group.

Transitional Dynamics

An economy is populated by a continuum of atomistic and identical households that maximize their lifetime utility given by

$$U = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left(\theta \ln c_t + (1-\theta)\ln(1-l_t)\right)$$

Households own capital and rent it to firms at a rate r_t. They earn a wage w_t for every hour worked, and they face a labor income tax τ. They accumulate capital according to the law of motion

$$k_{t+1} = i_t + (1-\delta)k_t$$

Finally, assume that households receive a lump-sum transfer T_t from the government.
The economy is also composed by a continuum of identical firms that use capital and labor to produce a final good according to the production function

$$y_t = k_t^{\alpha} l_t^{1-\alpha}$$

Lastly, assume that the government rebates back to households all revenues raised from the labor income tax.

(a) (20 points) Assume that $\delta = 0.025$, $\alpha = 0.3$, $\tau = 0.1$, and a net return of capital equal to 1%. Find the steady state of the model, as well as the total labor income revenues raised in the steady state. . Note that to find it, you will need to calibrate β and θ. For θ, you can set it so that the steady state amount of worked hours is equal to 1/3. Show your work.

(b) (20 points) Using the parameters that you found in part (a), plot steady state total labor income revenues for different values of τ.[1] What do you see? Are labor income revenues always increasing in the labor income tax rate? Why?

(c) (20 points) Assume now that the government announces that the labor income tax rate will be raised from 10% to 20%, effective immediately. Find the new steady state of the model and compare equilibrium allocations. What do you see? where is welfare larger?

(d) (20 points) Compute the transitional dynamics of the model. This is, find how equilibrium allocations along transition path from the initial to the new steady state. Interpret your results.

(e) (20 points) Suppose that the government is evaluating a policy that instead of raising the labor income tax rate from 10% to 20% imposes a new tax on capital income $r_t k_t$. Assume that the tax rate is τ^k. If the government wants to raise the same amount of revenues as with the 20% labor income tax rate, what would the capital tax rate have to be in order to do so? Compare the steady state allocations from this alternative to the one with the 20% labor income tax. Which alternative would you prefer? Carefully justify your answer.

[1] Hint: you need to re-solve the steady state for every value of the labor income tax. You can generate a grid of, for example, 50 values ranging from $\tau = 0.01$ to $\tau = 0.9$.

(a) (20 points) Assume that $\delta = 0.025$, $\alpha = 0.3$, $\tau = 0.1$, and a net return of capital equal to 1%. Find the steady state of the model, as well as the total labor income revenues raised in the steady state. . Note that to find it, you will need to calibrate β and θ. For θ, you can set it so that the steady state amount of worked hours is equal to 1/3. Show your work.

HH Problem:

$$\max_{C_t, l_t, K_{t+1}} E_0 \sum_{t=0}^{\infty} \beta^t \left(\theta \ln C_t + (1-\theta)\ln(1-l_t)\right)$$

$$\text{s.t. } C_t + K_{t+1} = W_t l_t (1-\tau) + r_t K_t + K_t(1-\delta) + T_t \quad , \quad T_t = \tau W_t l_t.$$

$$\mathcal{L} = E_0 \sum_{t=0}^{\infty} \beta^t \left[\theta \ln C_t + (1-\theta)\ln(1-l_t) + \lambda_t\left(W_t l_t(1-\tau) + r_t K_t + K_t(1-\delta) + T_t - C_t - K_{t+1}\right)\right]$$

$$\frac{\partial \mathcal{L}}{\partial C_t}: \quad \frac{\theta}{C_t} = \lambda_t \qquad (1.1)$$

$$\frac{\partial \mathcal{L}}{\partial l_t}: \quad \frac{1-\theta}{1-l_t} = \lambda_t W_t (1-\tau) \qquad (1.2)$$

$$\frac{\partial \mathcal{L}}{\partial K_{t+1}}: \quad \beta^t \lambda_{t+1}(1 + r_{t+1} - \delta) = \lambda_t \qquad (1.3)$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_t}: \quad C_t + K_{t+1} = W_t l_t (1-\tau) + r_t K_t + T_t \qquad (1.4)$$

From (1.1), (1.2), $\quad \theta \frac{1}{C_t} W_t (1-\tau) = (1-\theta)\frac{1}{1-l_t}. \qquad (2.1) \quad$ Intra.

From (1.1) (1.3), $\quad \beta^t \frac{1}{C_{t+1}}(1 + r_{t+1} - \delta) = \frac{1}{C_t}. \qquad (2.2), \quad$ Euler

Firm Problem. $\quad \max_{K_t, l_t} K_t^{\alpha} l_t^{1-\alpha} - W_t l_t - r_t K_t$

$$\frac{\partial \mathcal{L}}{\partial l_t}: (1-\alpha) K_t^{\alpha} l_t^{-\alpha} = W_t \qquad (2.3)$$

$$\frac{\partial \mathcal{L}}{\partial K_t}: \alpha K_t^{\alpha-1} l_t^{1-\alpha} = r_t \qquad (2.4)$$

R.C.

$$\begin{cases} C_t + K_{t+1} = \cancel{W_t l_t}(1-\tau) + \cancel{r_t K_t} + K_t(1-\delta) + \cancel{T_t} \quad \tau_t W_t l_t. \\ W_t l_t + r_t K_t = K_t^{\alpha} l_t^{1-\alpha} \end{cases}$$

$$\Rightarrow C_t + K_{t+1} - K_t(1-\delta) = K_t^{\alpha} l_t^{1-\alpha}$$

$$(C_t + i_t = y_t)$$

$$\begin{cases} \frac{\theta}{C_t} W_t (1-\tau) = \frac{1-\theta}{1-l_t} \\ \beta \frac{1}{C_{t+1}}(1 + r_{t+1} - \delta) = \frac{1}{C_t} \\ (1-\alpha) K_t^{\alpha} l_t^{-\alpha} = W_t \\ \alpha K_t^{\alpha-1} l_t^{1-\alpha} = r_t \\ C_t + K_{t+1} - K_t(1-\delta) = K_t^{\alpha} l_t^{1-\alpha} \end{cases} \xRightarrow{\text{in S.S}} \begin{cases} \frac{\theta}{C} W(1-\tau) = \frac{1-\theta}{1-l} \\ \beta(1 + r - \delta) = 1 \\ (1-\alpha) K^{\alpha} l^{-\alpha} = W \\ \alpha K^{\alpha-1} l^{1-\alpha} = r \\ C + \delta K = K^{\alpha} l^{1-\alpha}. \end{cases}$$

since $\delta=0.025$. $\alpha=0.3$ $\tau=0.1$ $l=\frac{1}{3}$. $r-\delta=0.01 \to r=0.035$.

$$\begin{cases} \frac{\theta}{c}w(0.9)=\frac{1-\theta}{\frac{2}{3}} \\ 1.01\beta=1 \\ 0.7K^{0.3}\left(\frac{1}{3}\right)^{-0.3}=w. \\ 0.3K^{-0.7}\left(\frac{1}{3}\right)^{0.7}=0.035 \\ c+0.025K=K^{0.3}\left(\frac{1}{3}\right)^{0.7} \end{cases}$$

unknown: θ, c, w. β K

$$\Longrightarrow \begin{cases} \beta \approx 0.990 \\ w \approx 1.758 \\ K \approx 7.175 \\ c \approx 0.658 \\ \theta \approx 0.384 \end{cases}$$

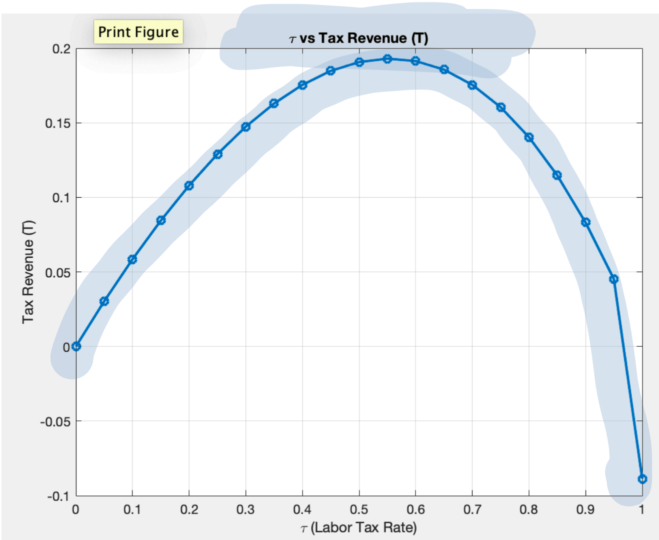
(b) (20 points) Using the parameters that you found in part (a), plot steady state total labor income revenues for different values of τ.[1] What do you see? Are labor income revenues always increasing in the labor income tax rate? Why?

known $\tau=(0,1,20)$

$\delta=0.025$. $\alpha=0.3$ $\beta=0.99$ $\theta=0.384$

unknown: l, r, w, k, c.

please check my coding using matlab: HW4cm-cal.m.

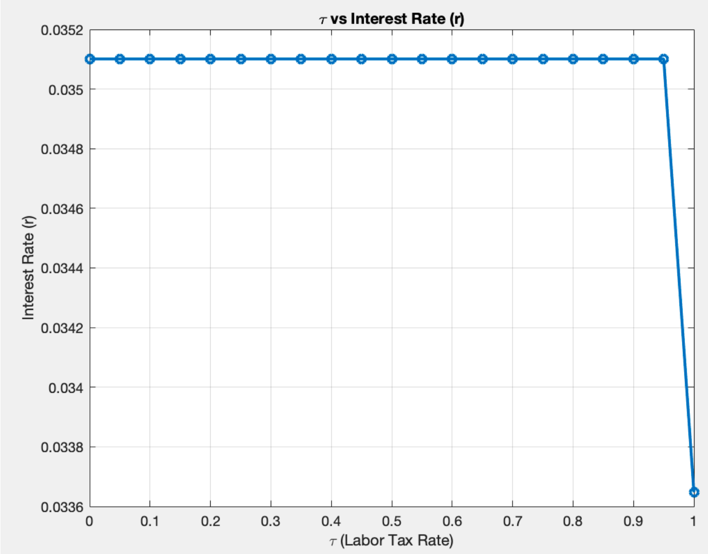


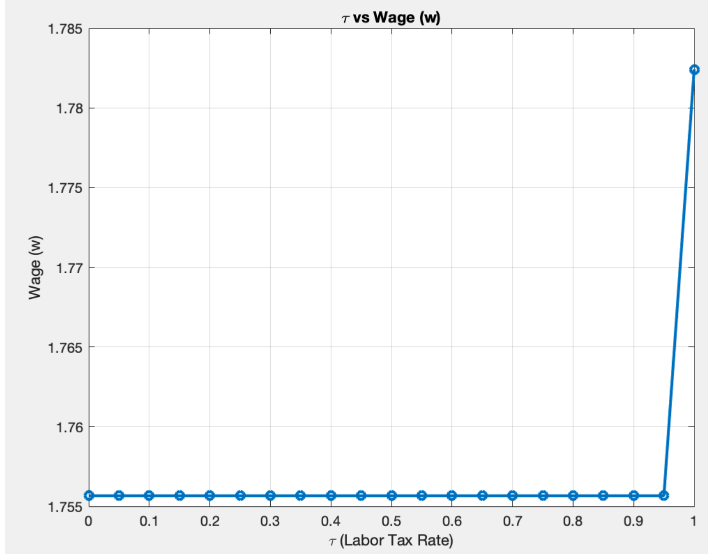
Total labor income revenue is increasing first, and decreases.

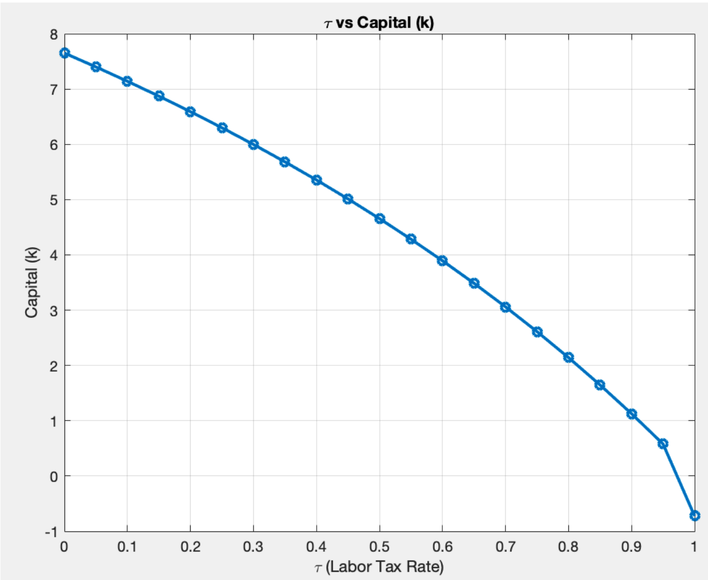
No.

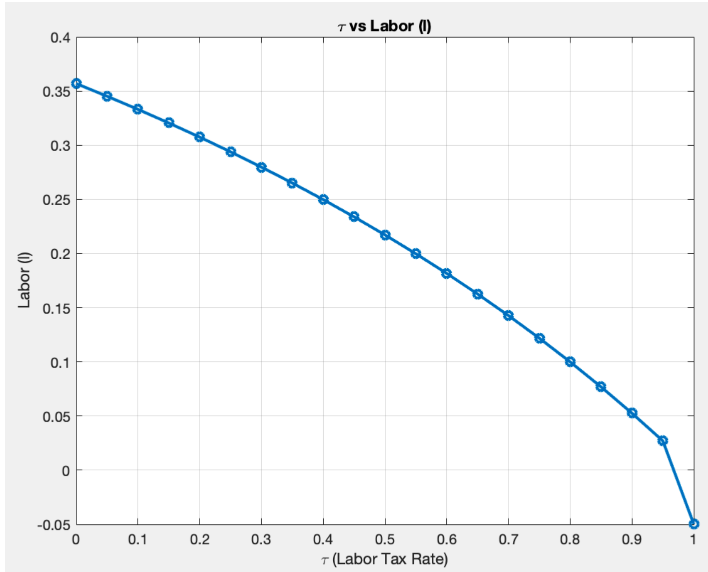
Since after $\tau \approx 0.55$, people are not willing to work since high tax.

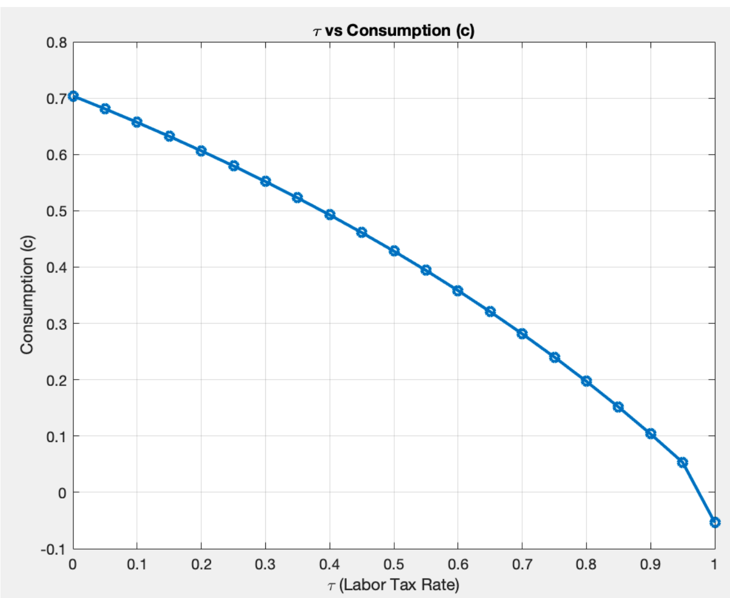
The rest of plots those are not necessary to be here, but I think they are helpful.











(c) (20 points) Assume now that the government announces that the labor income tax rate will be raised from 10% to 20%, effective immediately. Find the new steady state of the model and compare equilibrium allocations. What do you see? where is welfare larger?

Please check my coding using matlab:

mn: math.m

$\tau = 10\%$ $\quad\quad\quad\quad$ $\tau = 20\%$

$$\begin{cases} c = 0.657 \\ l = 0.333 \\ k = 7.14 \\ w = 1.756 \\ r = 0.035 \\ T = 0.059 \end{cases} \quad \begin{cases} c = 0.606 & \downarrow \\ l = 0.308 & \downarrow \\ k = 6.591 & \downarrow \\ w = 1.756 & \rightarrow \\ r = 0.035 & \rightarrow \\ T = 0.108 & \uparrow \end{cases}$$

consumption, labor, captical decrease.

Welfare Analysis.

Defin $U = \sum \beta^t U(c_t, l_t)$ as lifetime utility.

If in the S.S forever.

$$U_0 = \sum \beta^t U(c^{ss}, l^{ss}) \text{ is lifetime utility at the initial steady state.}$$

$$= U(c^{ss}, l^{ss}) \sum \beta^t$$

$$= \frac{U(c^{ss}, l^{ss})}{1-\beta} = \frac{(\theta \ln c + (1-\theta) \ln(1-l))}{1-\beta}$$

since the economy get the stead state very soon, so we use U_0 to do the welfare analysis.

so we need to compare $U_0|_{\tau=10\%}$ and $U_0|_{\tau=20\%}$

we can ignore $1-\beta$, since β is a constant in both case.

$$(1-\beta) U_0|_{\tau=10\%} = 0.384 \cdot \ln(0.657) + (1-0.384) \cdot \ln(\tfrac{2}{3}) \approx -0.4111$$

$$(1-\beta) U_0|_{\tau=20\%} = 0.384 \cdot \ln(0.606) + (1-0.384) \cdot \ln(1-0.308) \approx -0.4191$$

Thus when $\tau = 10\%$ (tax rate is 10%) ~~at~~, welfare is bigger

(d) (20 points) Compute the transitional dynamics of the model. This is, find how equilibrium allocations along transition path from the initial to the new steady state. Interpret your results.

$$\frac{\theta}{C_t} W_t (1-\tau_t) = \frac{1-\theta}{1-l_t} \quad (3.1)$$

$$\beta^t \frac{1}{C_{t+1}} (1+r_{t+1}-\delta) = \frac{1}{C_t} \quad (3.2)$$

$$(1-\alpha) K_t^{\alpha} l_t^{-\alpha} = W_t \quad (3.3)$$

$$\alpha K_t^{\alpha-1} l_t^{1-\alpha} = r_t \quad (3.4)$$

$$C_t + K_{t+1} - K_t(1-\delta) = K_t^{\alpha} l_t^{1-\alpha} \quad (3.5)$$

We can get Euler:

$$\beta \cdot \frac{1}{K_{t+1}^{\alpha} l_{t+1}^{1-\alpha} - K_{t+2} + (1-\delta)K_{t+1}} \cdot \left[1+\alpha K_{t+1}^{\alpha-1} l_{t+1}^{1-\alpha} - \delta\right] = \frac{1}{K_t^{\alpha} l_t^{1-\alpha} - K_{t+1} + (1-\delta)K_t}$$

Time iteration method

Step 1: set $T=30$. (check after and might be changed)

Step 2: Guess the sequence of $\hat{K}_t$ as a linear decreasing function. (since from (c), we know $k(ss)$ in $\tau=20\%$ is lower)

Step 3. Use the Euler equation to iteratively solve for the updated path of the $\check{K}_t$ (updated).

For example, $T=1$, ☆ $\tau=0.1$ for $T>1$ $\tau=0.2$

① Use K_1 (the initial stedy state capital) and $\hat{K}_3$ to slove for $\hat{K}_2$,

② Then use $\check{K}_2$ and $\hat{K}_4$ to solve for $\check{K}_3$,

③ Then use $\check{K}_3$ and $\hat{K}_5$ to solve for $\check{K}_4$ and so on.

(however, the Euler of only K_t is hard to get, we use l_t involved)

Step 4: This give a new sequence of $\check{K}_t$,

Check whether $\|\check{K}_t - \hat{K}_t\| < \varepsilon$,

if yes, stoop the loop, go to step5

if no, update $\hat{k}_t \xleftarrow{\text{use}} \tilde{k}_t$, and repeat Step 3, 4

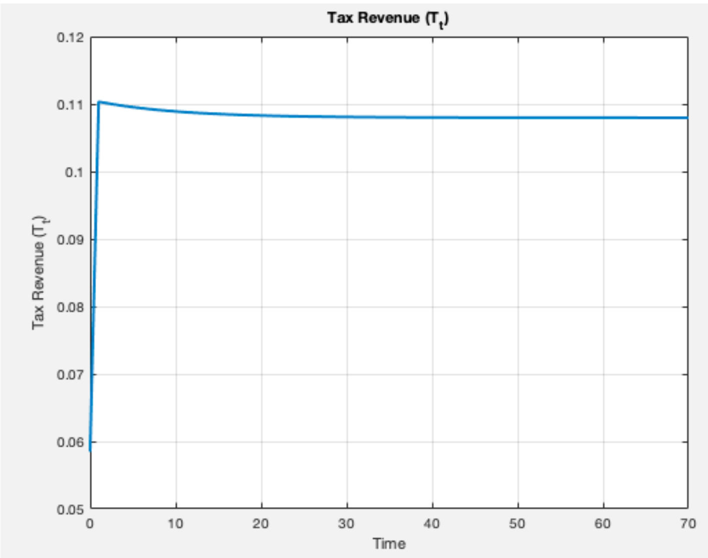
Step 5: when the loop ends, plot the paths of k_t,

and l_t, w_t, r_t, c_t using equations above and parameters

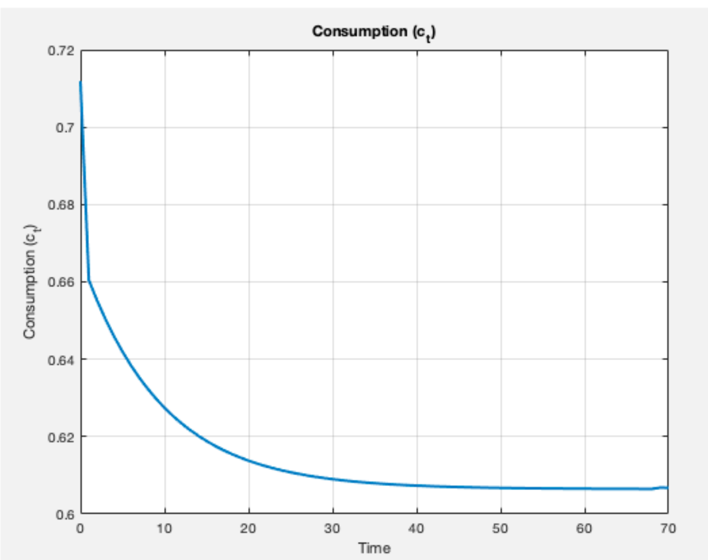
of $\delta = 0.025$, $\alpha = 0.3$ $\beta = 0.99$. $\theta = 0.384$.

Please check my MATLAB code runing:

HW3d.



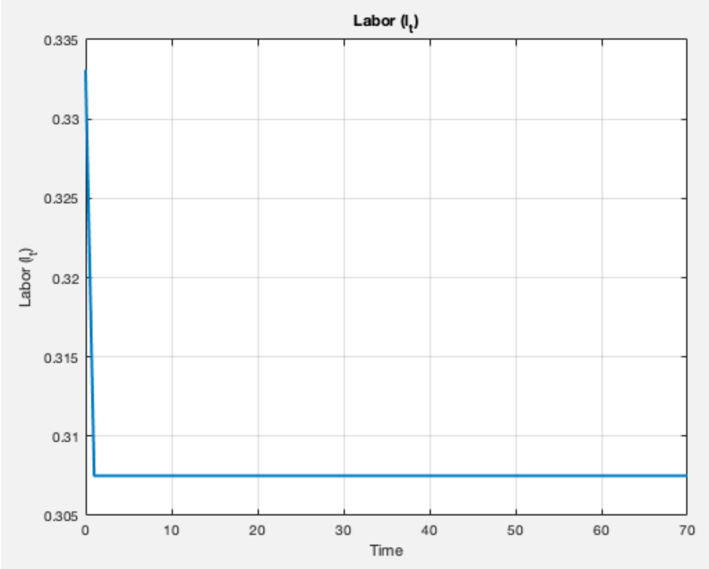
since government increased tax rate from 10% to 20% immediately, There's a total tax jump from 0.059. Afterward, since high tax, people are less willing to work, less labor, less tax, but still higher than the initial.



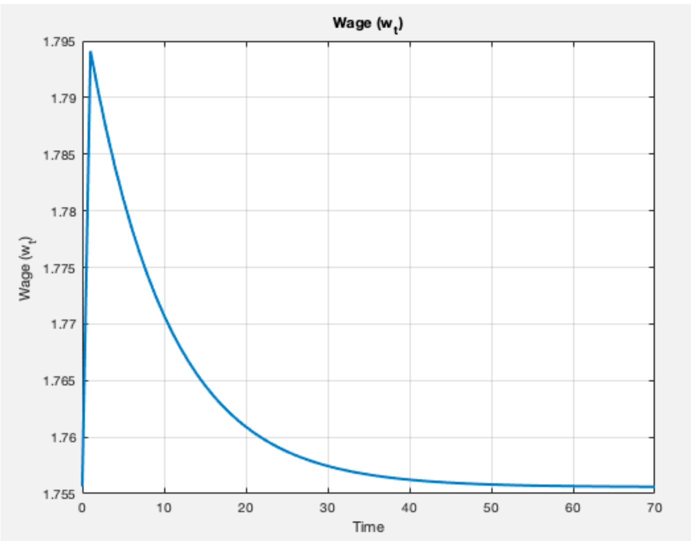
higher income tax

lower disposable income

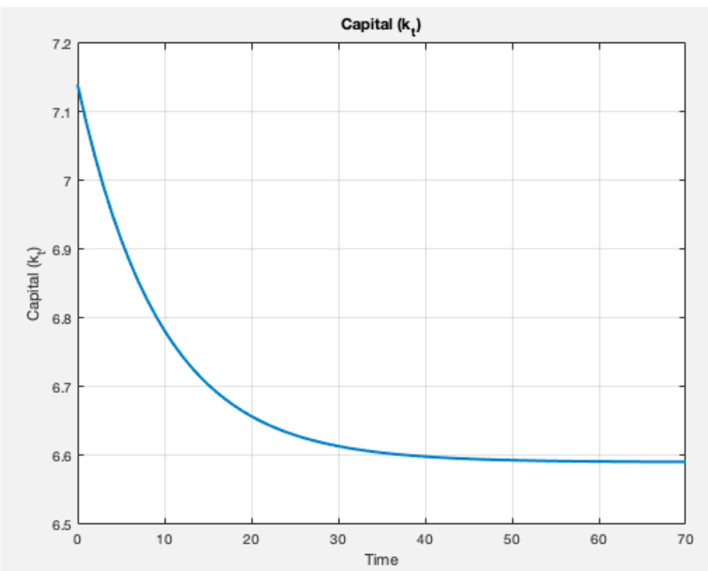
people consume less



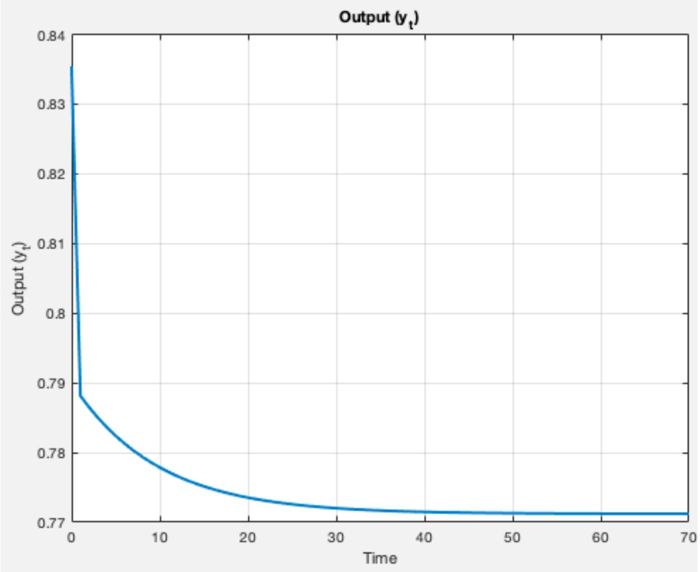
Since high tax, people are less willing to work, less labor.



Since high tax, people are less willing to work, the firm still hold the initial capital storage, they need labor, so they have to hire with higher wage, and gradually drop as costing storage of initial capital

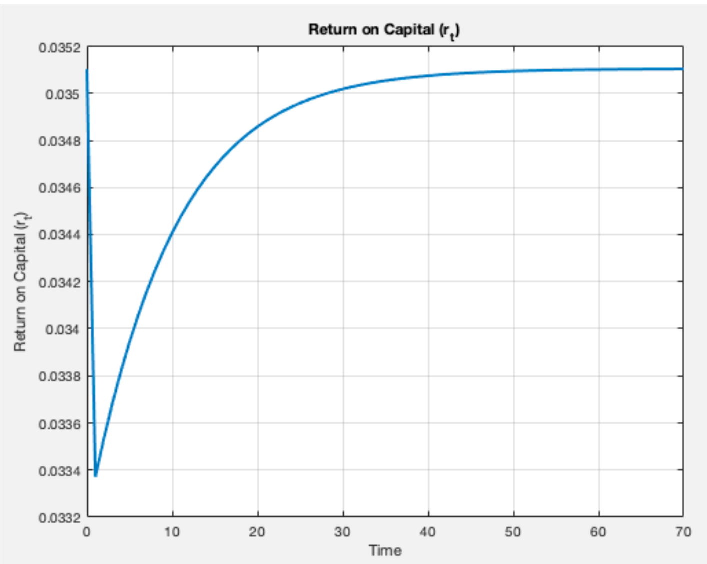


The capital cannot be costed immediately in addition, people are less willing to work, so the firm cost the high storage of capital slowly.



less labor, less output

even though the initial capital level.



initial drop since marginal return of capital immediately drops. (since less labor)
Afterword, the conomy adjust to the new tax rate, the capital and labd stablize. As a result, r_t gradually recovers.

(e) (20 points) Suppose that the government is evaluating a policy that instead of raising the labor income tax rate from 10% to 20% imposes a new tax on capital income $r_t k_t$. Assume that the tax rate is τ^k. If the government wants to raise the same amount of revenues as with the 20% labor income tax rate, what would the capital tax rate have to be in order to do so? Compare the steady state allocations from this alternative to the one with the 20% labor income tax. Which alternative would you prefer? Carefully justify your answer.

From $T_t = 20\% \, l_t W_t \longrightarrow T_t = \tau_t^k r_t k_t$

CASE 1 CASE 2.

since when $\tau = 20\%$

in s.s. $T^l = \tau w l = 0.2 \times 1.756 \times 0.308 \approx 0.108.$

we need τ^k satisfy:

$$T^k = \tau^k r K = 0.108.$$

Known: $\alpha = 0.3$ $\delta = 0.025$ $\beta = 0.99$ $\theta = 0.384$

unknown: ① τ^k.

② $\{c, l, k, w, r\}$.

From Firm. $(1-\alpha) k_t^{\alpha} l_t^{-\alpha} = W_t$

$$\alpha k_t^{\alpha-1} l_t^{1-\alpha} = r_t$$

$$\max_{C_t, l_t, k_{t+1}} E_0 \sum_{t=0}^{\infty} \beta^t \left(\theta \ln C_t + (1-\theta) \ln(1-l_t)\right)$$

s.t. $C_t + k_{t+1} = W_t l_t + (1 + r_t(1-\tau^k) - \delta) k_t + T_t^k$, $T_t^k = \tau r_t k_t$

$$\mathcal{L} = E_0 \sum_{t=0}^{\infty} \beta^t \left[\theta \ln C_t + (1-\theta) \ln(1-l_t) + \lambda\left(W_t l_t + (1 + r_t(1-\tau^k) - \delta) k_t + T_t - C_t - k_{t+1}\right)\right]$$

C_t, l_t, k_{t+1}

$\frac{\partial \mathcal{L}}{\partial C_t}$: $\frac{\theta}{C_t} = \lambda_t$

$\frac{\partial \mathcal{L}}{\partial l_t}$: $\frac{1-\theta}{1-l_t} = \lambda_t W_t$

$\frac{\partial \mathcal{L}}{\partial k_{t+1}}$: $\beta \lambda_{t+1} \left(1 + r_t(1-\tau^k) - \delta\right) = \lambda_t.$

$$\left\{ \begin{array}{l} \frac{1-\theta}{1-l_t} = \frac{\theta}{C_t} W_t. \quad (4.3) \\ \beta \frac{1}{C_{t+1}} \left(1 + r_t(1-\tau^k) - \delta\right) = \frac{1}{C_t}. \quad (4.4) \end{array} \right.$$

B.R. $C_t + k_{t+1} = \cancel{w_t l_t} + (1 + \cancel{r_t(1-\tau^k)} - \delta)k_t + \cancel{\tau^k r_t k_t}$.

$\cancel{r_t k_t} + \cancel{w_t l_t} = k_t^{\alpha} l_t^{1-\alpha}$

$$C_t + k_{t+1} = k_t^{\alpha} l_t^{1-\alpha} + (1-\delta)k_t$$

$$\Rightarrow k_t^{\alpha} l_t^{1-\alpha} = C_t + k_{t+1} - (1-\delta)k_t.$$

$$\begin{cases} \frac{1-\theta}{1-l_t} = \frac{\theta}{C_t} w_t. & (4.1) \\ \beta \frac{1}{C_{t+1}} (1 + r_t(1-\tau^k) - \delta) = \frac{1}{C_t}. & (4.2) \\ (1-\alpha) k_t^{\alpha} l_t^{-\alpha} = w_t & (4.3) \\ \alpha k_t^{\alpha-1} l_t^{1-\alpha} = r_t & (4.4) \\ k_t^{\alpha} l_t^{1-\alpha} = C_t + k_{t+1} - (1-\delta)k_t. & (4.5) \end{cases} \xrightarrow{\text{in. S.S}} \begin{cases} \frac{1-\theta}{1-l} = \frac{\theta}{c} w. \\ \beta[1 + r(1-\tau^k) - \delta] = 1 \\ (1-\alpha) k^{\alpha} l^{-\alpha} = w \\ \alpha k^{\alpha-1} l^{1-\alpha} = r \\ k^{\alpha} l^{1-\alpha} = c + \delta k. \\ \tau^k \cdot r \cdot k = 0.108 \end{cases}$$

since $\alpha = 0.3$ $\delta = 0.025$ $\beta = 0.99$ $\theta = 0.384$, unknown: l, c, w, r, τ^k k

with MATLAB:
run_ss_capital_tax.m

$$\begin{cases} c = 0.4874 \\ l = 0.3219 \\ k = 1.6979 \\ w = 1.1529 \\ r = 0.0957 \\ \tau^k = 0.6572. \end{cases}$$

for welfare analysis:

$$(1-\beta) \mho_0 |_{\tau^k = 0.6572} = \theta \ln c + (1-\theta) \ln(1-l)$$

$$= 0.384 \times \ln 0.4874 + (1 - 0.384) \times \ln(1 - 0.3219)$$

$$= -0.5152.$$

since $(1-\beta) \mho_0 |_{\tau = 20\%} = -0.4191$ $>$ $(1-\beta) \mho_0 |_{\tau^k = 0.6572} = -0.5152.$

$\Rightarrow$ $\mathcal{V}_0|_{\tau=20\%} > \mathcal{V}_0|_{\tau^k=0.6572}$

Thus. welfare of taking labor tax is higher. I prefer this.

① since, in captical taking model, people consume less and work more, since our utility function is $U = E_0 \sum_{t=0}^{\infty} \beta^t (\theta \ln C_t + (1-\theta)\ln(1-l_t))$, showing utility in increasing in consumption and disutility in laboring. So, labor tax model's welfare is lower.

②. Taking labor making people invest less, then the firm can only hire lower labor due to limited investment. As a result, people earn less. consume less, ending up with lower welfare level.