Macroeconomic Theory

Assignment IV

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ECO 7938 - MACROECONOMIC THEORY II

Problem Set 3

Instructions: This problem set is due on April 22nd before the beginning of class. Make sure that you are submitting on Canvas clear and ordered answers, and to submit your code so I can replicate your work. Failure to meet these requirements will translate in a 30 point deduction from your final grade. You may work in groups of 2, but you need to submit individual answers and explicitly mention the members of your group.

Nonlinearities and Financial Crises

An economy is populated by a continuum of atomistic and identical households that maximize their lifetime utility given by

$$U = \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma}}{1-\sigma}$$

where c_t is a composite good, which is aggregated from tradable and nontradable goods according to the following function

$$c_t = \left[\omega (c_t^T)^{-\eta} + (1-\omega)(c_t^N)^{-\eta}\right]^{-\frac{1}{\eta}}$$

Every period households receive a tradable and a nontradable endowment y_t^T and y_t^N, respectively, and can hold one-period bonds b_{t+1} issued at a world interest rate r^* (borrowing implies negative b_{t+1}), denominated in units of the tradable goods. Funds raised by issuing bonds b_{t+1} are then $\frac{b_{t+1}}{1+r^*}$.

Households face a borrowing constraint of the form

$$\frac{b_{t+1}}{1+r^*} \geq -\kappa \left(y_t^T + p_t^N y_t^N\right),$$

where p_t^N is the relative price of nontradable goods in terms of tradables (the price of tradables has been normalized to 1).

(a) (20 points) Characterize the competitive equilibrium of this economy.

(b) (20 points) Assume that $\eta = 0.205$, $\omega = 0.3$, $\beta = 0.96$, $\beta(1 + r^*) = 1$, $\kappa = 0.3$, $\sigma = 2$, $y_t^N = 1$ and $y_t^T = 1$. Lastly, assume that the borrowing constraint is marginally binding for every t. Find the unconstrained allocations and prices of this economy (c_t^T, b_{t+1}, p_t^N).

(c) (20 points) Suppose now that there's a wealth-neutral shock in period 0 that lowers the tradable endowment by 5%. Find the new period-0 equilibrium allocations and prices. What do you observe? Is the decline in consumption and prices proportional to the decline in endowment? Why?

(d) (20 points) Repeat the same exercise of part (c) but now for a range of declines of the tradable endowment (start at 5% and go all the way to 20%). Is the decline in consumption and prices amplified in a monotonic way? why?

(e) (20 points) How do your answers to part (c) vary if we had lower values of κ? You can try with values as low as $\kappa = 0.1$ and go up to $\kappa = 0.3$. What do you observe? Is the decline in prices and consumption monotonic with respect to the value of κ? Provide intuition for your results.

(a) (20 points) Characterize the competitive equilibrium of this economy.

Representative HH's Problem:

$$\max_{C_t^T,\, C_t^N,\, b_{t+1}} U = \sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\sigma}}{1-\sigma} \qquad \text{where } c_t = \left[\omega (c_t^T)^{-\eta} + (1-\omega)(c_t^N)^{-\eta}\right]^{-\frac{1}{\eta}}$$

s.t. ① $1 \cdot C_t^T + P_t^N C_t^N + \frac{b_{t+1}}{1+r^*} = y_t^T + P_t^N y_t^N + b_t$

normalized prices

② $\frac{b_{t+1}}{1+r^*} \geq -\kappa (y_t^T + P_t^N y_t^N)$

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \left[\underbrace{\frac{C_t^{1-\sigma}}{1-\sigma}}_{U_t} + \lambda_t \left(y_t^T + P_t^N y_t^N + b_t - C_t^T - P_t^N C_t^N - \frac{b_{t+1}}{1+r^*} \right) + \mu_t \left(\frac{b_{t+1}}{1+r^*} + \kappa (y_t^T + P_t^N y_t^N) \right) \right]$$

FOCs:

C_t^T: $\frac{\partial U_t}{\partial C_t} \cdot \frac{\partial C_t}{\partial C_t^T} - \lambda_t = 0$

$$\cancel{\frac{1}{1-\sigma}} \cancel{(1-\sigma)} \cdot C_t^{-\sigma} \cdot \cancel{-\frac{1}{\eta}} \left[\omega (C_t^T)^{-\eta} + (1-\omega)(C_t^N)^{-\eta} \right]^{-\frac{1}{\eta}-1} \cdot \cancel{(-\eta)}\, \omega\, C_t^{T\,-\eta-1} = \lambda_t.$$

$$C_t^{-\sigma} \cdot \left[\omega (C_t^T)^{-\eta} + (1-\omega) C_t^{N\,-\eta} \right]^{-\frac{1}{\eta}-1} \cdot \omega\, C_t^{T\,-\eta-1} = \lambda_t. \qquad ①$$

C_t^N: $\frac{\partial U_t}{\partial C_t} \cdot \frac{\partial C_t}{\partial C_t^N} - \lambda_t P_t^N = 0$

$$\lambda_t P_t^N = C_t^{-\sigma} \cdot \left[\omega (C_t^T)^{-\eta} + (1-\omega)(C_t^N)^{-\eta} \right]^{-\frac{1}{\eta}-1} (1-\omega)(C_t^N)^{-\eta-1} \qquad ②$$

b_{t+1}: $\cancel{\beta^t} \lambda_t \left(-\frac{1}{1+r^*}\right) + \cancel{\beta^t} \mu_t \left(\frac{1}{1+r^*}\right) + \beta^{\cancel{t+1}} \lambda_{t+1} = 0.$

$$-\lambda_t + \mu_t + \beta \lambda_{t+1} (1+r^*) = 0 \qquad ③$$

$$\lambda_t = \mu_t + \beta \lambda_{t+1} (1+r^*)$$

①, ②
$$\frac{\lambda_t P_t^N}{\lambda_t} = \frac{\cancel{C_t^{-\sigma}} \cdot \cancel{\left[\omega (C_t^T)^{-\eta} + (1-\omega)(C_t^N)^{-\eta} \right]^{-\frac{1}{\eta}-1}} (1-\omega)(C_t^N)^{-\eta-1}}{\cancel{C_t^{-\sigma}} \cdot \cancel{\left[\omega (C_t^T)^{-\eta} + (1-\omega) C_t^{N\,-\eta} \right]^{-\frac{1}{\eta}-1}} \cdot \omega\, C_t^{T\,-\eta-1}}$$

$$P_t^N = \frac{1-\omega}{\omega} \left(\frac{C_t^N}{C_t^T} \right)^{-\eta-1}$$

$$= \frac{1-\omega}{\omega} \left(\frac{C_t^T}{C_t^N} \right)^{1+\eta} \qquad ④$$

①, ③, $\underline{c_t^{-\sigma}\cdot\left[\omega(c_t^T)^{-\eta}+(1-\omega)(c_t^N)^{-\eta}\right]^{-\frac{1}{\eta}-1}\cdot\omega\, c_t^{T-\eta-1}} = \mu_t + \beta(1+r^*)\,\underline{c_{t+1}^{-\sigma}\left[\omega \cdots\cdots (t+1)\right]}$

$= \frac{\partial U_t}{\partial c_t^T} \xrightarrow{\text{let}} U_T(t)$ $\qquad = \frac{\partial U_{t+1}}{c_{t+1}^T} \xrightarrow{\text{let}} U_T(t+1)$

$$U_T(t) = \mu_t + \beta(1+r^*)U_T(t+1) \qquad ⑥$$

Resource Constrain

Nontradable: $c_t^N = y_t^N$ ⑦

trade tradable: $c_t^T + \frac{b_{t+1}}{1+r^*} = y_t^T + b_t.$ ⑧

HH $\begin{cases} ⑤ \\ ⑥ \end{cases}$

R.C. $\begin{cases} ⑦ \\ ⑧ \end{cases}$ characterize the competitive eq* of the economy.

(b) (20 points) Assume that $\eta = 0.205$, $\omega = 0.3$, $\beta = 0.96$, $\beta(1+r^*) = 1$, $\kappa = 0.3$, $\sigma = 2$, $y_t^N = 1$ and $y_t^T = 1$. Lastly, assume that the borrowing constraint is marginally binding for every t. Find the unconstrained allocations and prices of this economy (c_t^T, b_{t+1}, p_t^N).

$1+r^* = \frac{1}{\beta} = \frac{1}{0.96}.$ $\quad 1+r^* \approx 1.042.$

$c_t^N = y_t^N = 1.$

since binding for every t, $\mu_t = 0.$

$$\begin{cases} p_t^N = \frac{1-\omega}{\omega}\left(\frac{c_t^T}{c_t^N}\right)^{1+\eta} & (2.1) \\ U_T(t) = \mu_t + \beta(1+r^*)U_T(t+1) & (2.2) \\ c_t^N = y_t^N & (2.3) \\ c_t^T + \frac{b_{t+1}}{1+r^*} = y_t^T + b_t. & (2.4) \\ \frac{b_{t+1}}{1+r^*} = -\kappa(y_t^T + p_t^N y_t^N). & (2.5) \end{cases}$$

(2.2): $U_T(t) = 0 + 1\,U_T(t+1)$

$\Rightarrow U_T(t) = U_T(t+1)$

since $U_T(t)$ depends only on two goods.

$c_t^T = c_{t+1}^T = \bar{c}^T$, $c_t^N = c_{t+1}^N = \bar{c}^N$

(2.1). $P_t^N = \frac{1-0.3}{0.3}\left(\frac{\bar{C}^T}{1}\right)^{1.205} = \frac{7}{3}\bar{C}^{T\,1.205}$. (C)

(2.5) $\frac{b_{t+1}}{1.024} = -0.3(1+P_t^N - 1)$ since

$b_{t+1} = -0.3\cdot 1.024\left(1+\frac{7}{3}\bar{C}^{T\,1.205}\right)$ (B)

(2.4). when $t=0$

$\bar{C}^T + \frac{b_1}{1+r^*} = 1 + b_0$

$b_0 = \bar{C}^T + \frac{b_1}{1+r^*} - 1$

$= \bar{C}^T + \frac{\bar{C}^T + \frac{b_2}{1+r^*} - 1}{1+r^*} - 1$

$= \vdots$

$= \bar{C}^T + \frac{\bar{C}^T}{1+r^*} + \frac{\bar{C}^T}{(1+r^*)^2} + \cdots \frac{\bar{C}^T}{(1+r^*)^n} + \frac{b_n}{(1+r^*)^n} + \cdots +$

$b_0 = (\bar{C}^T - 1)\sum_{t=0}^{\infty}\frac{1}{(1+r^*)^t} = (\bar{C}^T - 1)\sum_{t=0}^{\infty}\beta^t = \frac{\bar{C}^T - 1}{1-\beta}$.

$\bar{C}^T = (1-\beta)b_0 + 1$ (A).

$\bar{C}^T = 0.04 b_0 + 1$

$\therefore C_0^T = 0.04 b_0 + 1$

(2.1), (2.5), (A),

$b_0 = 0.04 b_0 + 1 + \frac{b_1}{1.042} - 1$

$b_0 \approx -1.0064$.

$b_1 = -1.0064 = b_{t+1}$

by (A): $C_0^T = 0.9597$.

(C): $P_0^N = 2.2206$.

(c) (20 points) Suppose now that there's a wealth-neutral shock in period 0 that lowers the tradable endowment by 5%. ① Find the new period-0 equilibrium allocations and prices. ② What do you observe? ③ Is the decline in consumption and prices proportional to the decline in endowment? Why?

①

Step 1, Get BB. Curve (Maximize → Euler → P_t^N function)

$P_t^N \sim C_t^T$

(2.1) $$P_t^N = \frac{1-\omega}{\omega}\left(\frac{C_t^T}{C_t^N}\right)^{\eta+1}$$

Since $C_t^N = 1$

$$P_t^N = \frac{1-\omega}{\omega} {C_t^T}^{\eta+1}$$ BB

Get MM Curve (Borrowing constraint + tradable R.C.) $C_t^T \sim y_t^T + P_t^N + b_t$

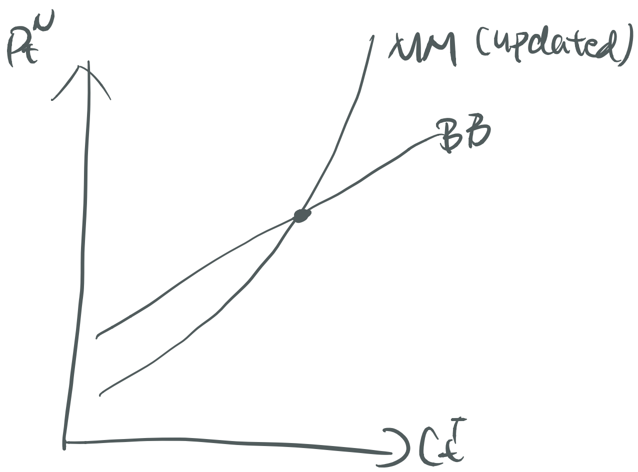
$$\begin{cases} \frac{b_{t+1}}{1+r^*} = -\kappa(y_t^T + P_t^N) \\ C_t^T + \frac{b_{t+1}}{1+r^*} = y_t^T + b_t \end{cases}$$ (2.2)

$\Rightarrow$ $C_t^T - \kappa(y_t^T + P_t^N) = y_t^T + b_t$

$C_t^T = y_t^T + b_t + \kappa(y_t^T + P_t^N)$

$= (1+\kappa)y_t^T + \kappa P_t^N + b_t$

t=0: $C_0^T = (1+\kappa)y_0^T + \kappa P_0^N + b_0$ MM



Step 2: ① introduce shock and sub into MM, ② MM + BB ⇒ C_0^T

$y_0^T \downarrow 5\%$ since original $y_0^T = 1 \to y_0^T = 0.95$.

①. $C_0^T = 0.95(1+\kappa) + \kappa P_0^N + b_0$ updated MM

②. $\begin{cases} \text{updated MM} \\ \text{BB.} \end{cases}$ ← $P_t^N = \frac{1-\omega}{\omega} {C_t^T}^{\eta+1}$

when t=0 $P_0^N = \frac{1-\omega}{\omega} {C_0^T}^{\eta+1}$

$$C_0^T = 0.95(1+k_2) + k_2\left[\frac{1-\omega}{\omega} C_0^{T\ \eta+1}\right] + b_0$$

Since, $\omega = 0.3$, $k_2 = 0.3$ $\eta = 0.205$ $b_0 = -1.0064$

$$C_0^T = 0.95(1+0.3) + 0.3\left[\frac{1-0.3}{0.3} C_0^{T\ 0.205+1}\right] - 1.0064.$$

by Matlab:

$C_0^{T'} \approx 0.629.$

Step 3: BB $\Rightarrow P_0^N$

$$P_0^{N'} = \frac{1-\omega}{\omega} C_0^{T\ \eta+1} = \frac{1-0.3}{0.3}[-1.0064]^{1.205} \approx 1.33.$$

②.

Original	Now		
$C_0^T = 0.9597.$	$C_0^{T'} \approx 0.629$	↓ 34%	
$P_0^N = 2.2206.$	$P_0^{N'} \approx 1.33.$	↓, 40%	magnified decline.

③. Not proportional, decline in C_0^T is 34%

and decline in P_0^N is 40%

the decline from C_0^T to P_0^N is magnified/amplified

by Borrowing constrain + tradable R.C), where we set MM curve.

$$\begin{cases} \frac{b_{t+1}}{1+r^*} = -k_2(y_t^T + P_t^N) \\ C_t^T + \frac{b_{t+1}}{1+r^*} = y_t^T + b_t \end{cases}$$

Ⓐ since P_t^N ↓ borrowing constrain ↓. borrowing ↓

Ⓑ b_{t+1} ↓. since y_t^T is fixed. C_t^T ↓.

by BB curve

$$P_0^N = \frac{1-\omega}{\omega}(C_0^T)^{\eta+1}$$

Ⓒ C_t^T ↓. P_0^N ↓.

However, it's not finished. back to MM & BB

$$\begin{cases} \dfrac{b_{t+1}}{1+r^*} = -\kappa_t (y_t^T + p_t^N) & \text{(A)} \\ c_t^T + \dfrac{b_{t+1}}{1+r^*} = y_t^T + b_t & \text{(B)} \end{cases}$$

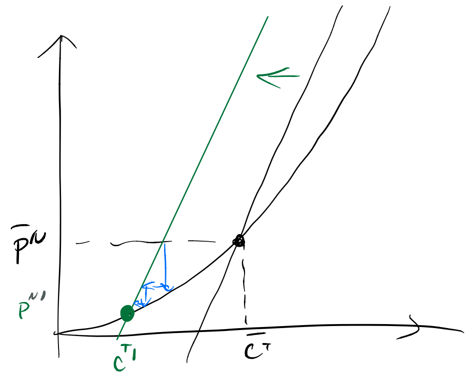
$P_0^N \downarrow$. borrowing constraint $\downarrow$ more.

$b_{t+1} \downarrow$, $C_t^T \downarrow$ more,

$$P_0^N = \frac{1-\omega}{\omega} (C_0^T)^{\eta+1} \quad \text{(C)}$$

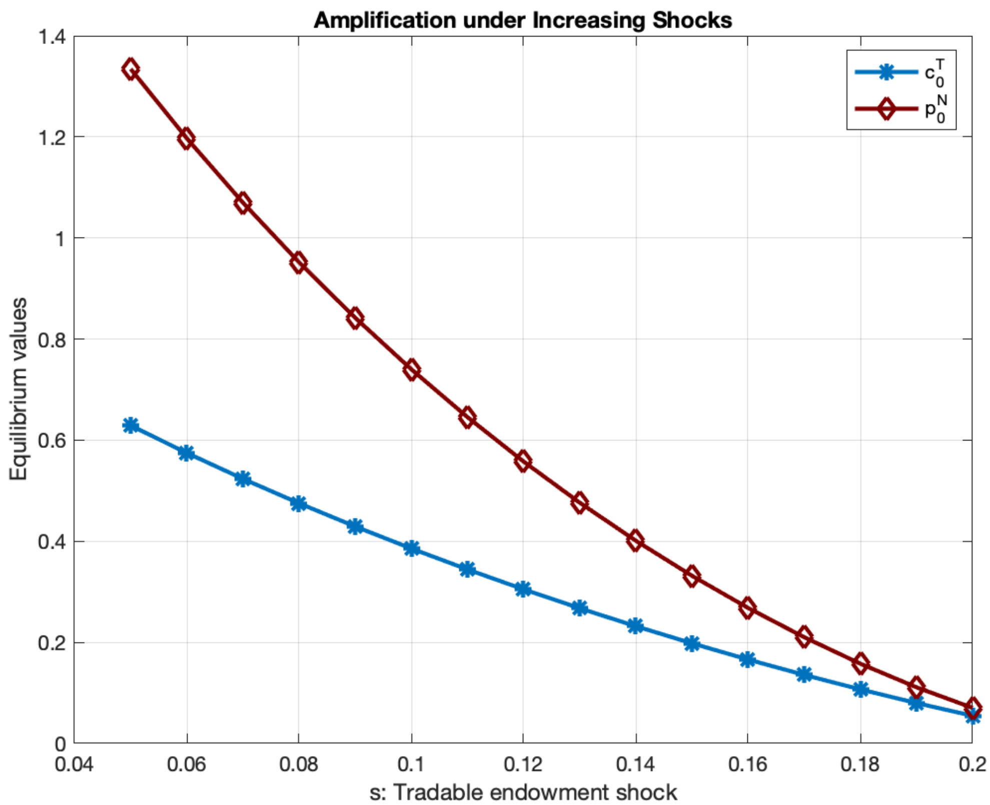
$C_t^T \downarrow$. $P_0^N \downarrow$.

Thus it's the general eq^* amplification.



(d) (20 points) Repeat the same exercise of part (c) but now for a range of declines of the tradable endowment (start at 5% and go all the way to 20%). Is the decline in consumption and prices amplified in a monotonic way? why?

Please run MATLAB code. HWtd.m.



No.

We can see as the tradable income shock s increases from 5% to 20%, both eq^* tradable consumption c_0^T and nontradable price P_0^N decrease.

However, the decline is not proportional.

And both show concave pattern, indicating that the magnitude of decline.

become smaller with large shocks.

Potential Reason: the eq* amplification is strong for shock shocks, when borrowing constraints tighten significantly, but diminishes for large shocks as borrowing capacity has already hit the limit.

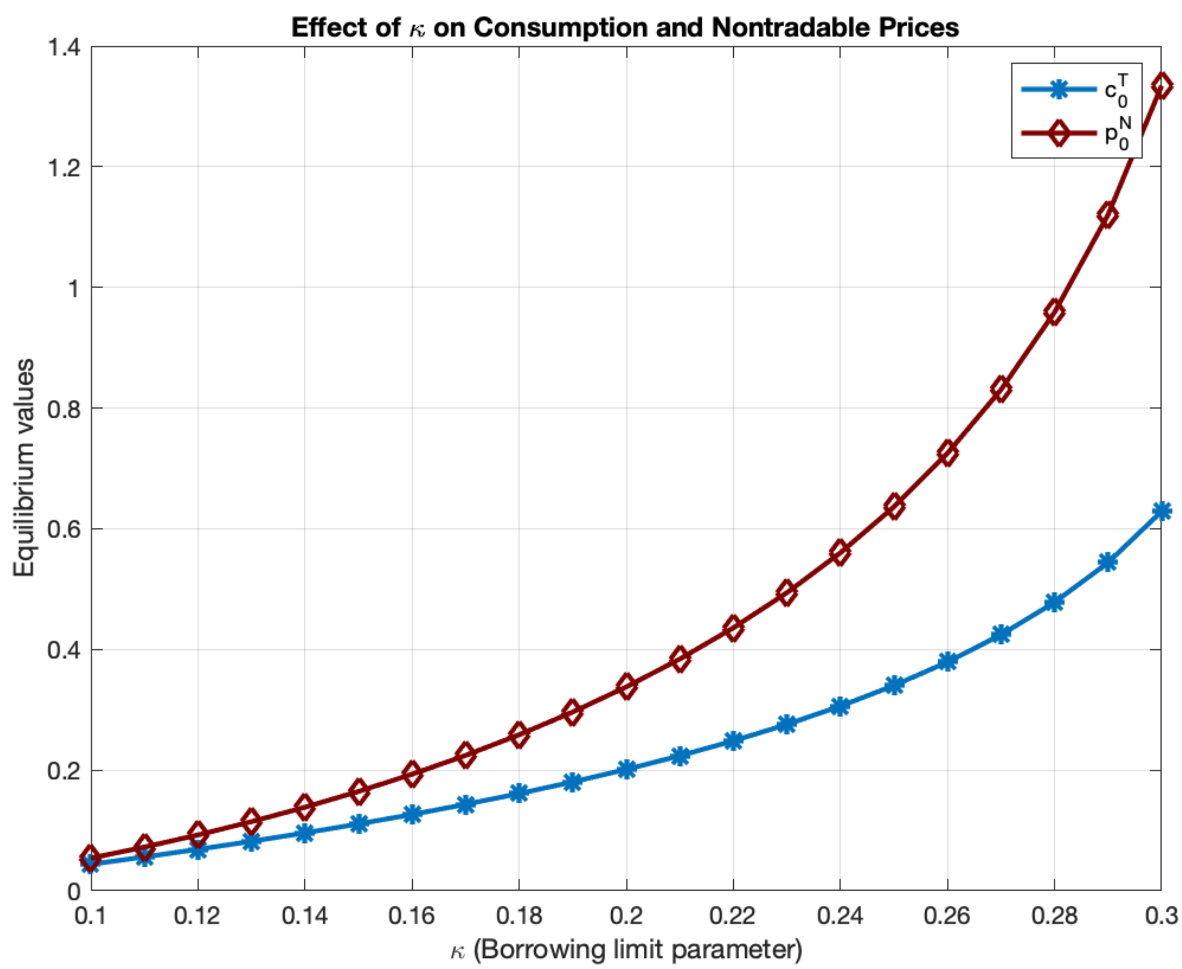
(e) (20 points) How do your answers to part (c) vary if we had lower values of κ? You can try with values as low as $\kappa = 0.1$ and go up to $\kappa = 0.3$. What do you observe? Is the decline in prices and consumption monotonic with respect to the value of κ? Provide intuition for your results.

Original $C_0^T = 0.95(1+\kappa) + \kappa\left[\frac{1-\omega}{\omega} C_0^{T\,\eta+1}\right] + b_0$

Since, $\omega = 0.3$, $\kappa = 0.3$ $\eta = 0.205$ $b_0 = -1.0064$

$p_0^N = \frac{1-\omega}{\omega} C_0^{T\,\eta+1} = \frac{1-0.3}{0.3}[-1.0064]^{1.205} \approx 1.33$

Please run MATLAB code:



we can see from the figure, the eq* value of tradable consumption c_0^T and the relative price of nontradable P_0^N increase monotonically with respect to the borrowing parameter κ, indicating that the amplification effect is stronger with κ is low, and weak as κ increases.

Intuition: a high κ allows HH to borrow more against their endowments, which helps smooth consumption following a negative tradable income shock.

This borrowing capacity mitigates the decline in c_0^T, and due to the CES aggregator, also prevents a sharp drop of P_0^N.