Ivan Franko National University of Lviv

Stallions

Maksym Shcherba, Petro Tarnavskyi, Yarema Stiahar

48th Annual World Championship of the International Collegiate Programming Contest

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Contest (1)

template.hpp

28 lines

```cpp
// hash = 9fd99b

#include <bits/stdc++.h>
using namespace std;

#define FOR(i, a, b) for(int i = (a); i < (b); i++)
#define RFOR(i, a, b) for(int i = (a) - 1; i >= (b); i--)
#define SZ(a) int(a.size())
#define ALL(a) a.begin(), a.end()
#define PB push_back
#define MP make_pair
#define F first
#define S second

typedef long long LL;
typedef vector<int> VI;
typedef vector<LL> VL;
typedef pair<int, int> PII;
typedef pair<LL, LL> PLL;
typedef double db;

int main()
{
  ios::sync_with_stdio(0);
  cin.tie(0);

  return 0;
}
```

compilation.txt

2 lines

```
g++ -O2 -std=c++20 -Wno-unused-result -Wshadow -Wall -o %e %e.cpp
g++ -std=c++20 -Wshadow -Wall -o %e %e.cpp -fsanitize=address -fsanitize=undefined -D_GLIBCXX_DEBUG -g
```

s.sh

6 lines

```bash
for((i = 0; ; i++)) do
  echo $i
  ./gen $i > in
  diff -w <(./a < in) <(./brute < in) || break
  [ $? == 0 ] || break
done
```

hash.sh

1 lines

```bash
cpp -dD -P -fpreprocessed $1 | tr -d '[:space:]'| md5sum |cut -c-6
```

Rules

Reject incorrect solutions from your teammates. Try to find counterexamples.

Discuss implementation and try to simplify the solution.

Avoid getting stuck on the problem.

Regularly discuss how many problems need to be solved and what steps to take, starting from the middle of the contest.

At the end of the contest, try to find a problem with an easy implementation.

Troubleshoot

Pre-submit

F9. Create a few manual test cases. Calculate time and memory complexity. Check the limits. Be careful with overflows, constants, clearing mutitestcases, uninitialized variables.

Wrong answer

F9. Print your solution! Read your code. Check pre-submit. Are you sure your algorithm works? Think about precision errors and hash collisions. Have you understood the problem correctly? Write the brute and the generator.

Runtime error

F9. Print your solution! Read your code. F9 with generator. Memory limit exceeded.

Time limit exceeded

What is the complexity of your algorithm? Are you copying a lot of unnecessary data? (References) Do you have any infinite loops? Use arrays, unordered maps instead of vectors and maps.

Pragmas

- `#pragma GCC optimize ("Ofast")` will make GCC auto-vectorize loops and optimizes floating points better. It is not unexpected to see your floating-point error analysis go to waste.

- `#pragma GCC target ("avx2")` can double performance of vectorized code, but causes crashes on old machines.

- `#pragma GCC optimize("unroll-loops")` enables aggressive loop unrolling, which reduces the number of branches and optimizes parallel computation.

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```

Data Structures (2)

dsu.hpp

f097c2, 31 lines

```
struct DSU
{
  int n;
  VI p, sz;

  DSU(int _n = 0)
  {
    n = _n;
    p.resize(n);
    iota(ALL(p), 0);
    sz.assign(n, 1);
  }
  int find(int v)
  {
    if (v == p[v])
      return v;
    return p[v] = find(p[v]);
  }
  bool unite(int u, int v)
  {
    u = find(u);
    v = find(v);
    if (u == v)
      return false;
    if (sz[u] > sz[v])
      swap(u, v);
    p[u] = v;
    sz[v] += sz[u];
    return true;
  }
};
```

fenwick.hpp

6a7a21, 20 lines

```
struct Fenwick
{
  int n;
  VL t;

  Fenwick(int _n = 0): n(_n), t(n) {}

  void upd(int i, LL x)
  {
    for (; i < n; i |= i + 1)
      t[i] += x;
  }
  LL query(int i)
  {
    LL ans = 0;
    for (; i >= 0; i = (i & (i + 1)) - 1)
      ans += t[i];
    return ans;
  }
};
```

fenwick-lower-bound.hpp

730a06, 17 lines

```
int lowerBound(LL x)
{
  LL sum = 0;
  int i = -1;
  int lg = 31 - __builtin_clz(n);
  while (lg >= 0)
  {
    int j = i + (1 << lg);
    if (j < n && sum + t[j] < x)
    {
      sum += t[j];
      i = j;
    }
    lg--;
  }
  return i + 1;
}
```

Minimum on a Segment

Maintain two Fenwick trees with $n = 2^k$ — one for the original array and the other for the reversed array. If $n > 1$, you can use: `n = 1 << (32 - __builtin_clz(n - 1))`.

When querying for the minimum on the segment, only consider segments $[(i\&(i+1)), i]$ that are completely inside $[l, r]$.

Add on a Segment

Maintain two Fenwick trees: tMult and tAdd.

To add x on the segment $[l, r]$, tMult.upd(l, x), tMult.upd(r, $-x$), tAdd.upd(l, $-x \cdot (l-1)$), tAdd.upd(r, $x \cdot r$).

$r\cdot$ tMult.query(r) + tAdd.query(r) is the sum on $[0, r]$.

min= and sum with Segment Tree

Store in each node: $\max, \text{cntMax}, \text{max2}, \text{sum}$.

In update check l, r conditions and:

- if (val $\geq$ max) return;
- else if (val $>$ max2) update this node;
- else go to left and right

You can do max= and += on segment at the same time. Time: $O(\log n)$. Each extra descent decreases number of diferent elements in segment.

treap.hpp

Description: uncomment in split for explicit key or in merge for implicit priority. Minimum and reverse queries.

5af66f, 144 lines

```
mt19937 rng;

struct Node
{
  int l, r;
  int x, y;
  int cnt, par;
  int rev, mn;

  Node(int value)
  {
    l = r = -1;
    x = value;
    y = rng();
    cnt = 1;
    par = -1;
    rev = 0;
    mn = value;
  }
};

struct Treap
{
  vector<Node> t;

int getCnt(int v)
{
  if (v == -1)
    return 0;
  return t[v].cnt;
}
int getMn(int v)
{
  if (v == -1)
    return INF;
  return t[v].mn;
}
int newNode(int val)
{
  t.PB({val});
  return SZ(t) - 1;
}
void upd(int v)
{
  if (v == -1)
    return;
  // important!
  t[v].cnt = getCnt(t[v].l) +
  getCnt(t[v].r) + 1;

  t[v].mn = min(t[v].x, min(getMn(t[v].l), getMn(t[v].r)));
}
void reverse(int v)
{
  if (v == -1)
    return;
  t[v].rev ^= 1;
}
void push(int v)
{
  if (v == -1 || t[v].rev == 0)
    return;
  reverse(t[v].l);
  reverse(t[v].r);
  swap(t[v].l, t[v].r);
  t[v].rev = 0;
}
PII split(int v, int cnt)
{
  if (v == -1)
    return {-1, -1};
  push(v);
  int left = getCnt(t[v].l);
  PII res;
  // elements a[v].x == val will be in right part
  // if (val <= a[v].x)
  if (cnt <= left)
  {
    if (t[v].l != -1)
      t[t[v].l].par = -1;
    // res = split(a[v].l, val);
    res = split(t[v].l, cnt);
    t[v].l = res.S;
    if (res.S != -1)
      t[res.S].par = v;
    res.S = v;
  }
  else
  {
    if (t[v].r != -1)
      t[t[v].r].par = -1;
    // res = split(a[v].r, val);
    res = split(t[v].r, cnt - left - 1);
    t[v].r = res.F;
```

```
      if (res.F != -1)
        t[res.F].par = v;
      res.F = v;
    }
    upd(v);
    return res;
  }
  int merge(int v, int u)
  {
    if (v == -1) return u;
    if (u == -1) return v;
    int res;
    // if ((int)(rng() % (getCnt(v) + getCnt(u))) < getCnt(v))
    if (t[v].y > t[u].y)
    {
      push(v);
      if (t[v].r != -1)
        t[t[v].r].par = -1;
      res = merge(t[v].r, u);
      t[v].r = res;
      if (res != -1)
        t[res].par = v;
      res = v;
    }
    else
    {
      push(u);
      if(t[u].l != -1)
        t[t[u].l].par = -1;
      res = merge(v, t[u].l);
      t[u].l = res;
      if (res != -1)
        t[res].par = u;
      res = u;
    }
    upd(res);
    return res;
  }
  // returns index of element [0, n)
  int getIdx(int v, int from = -1)
  {
    if (v == -1)
      return 0;
    int x = getIdx(t[v].par, v);
    push(v);
    if (from == -1 || t[v].r == from)
      x += getCnt(t[v].l) + (from != -1);
    return x;
  }
};
```

lct.hpp

Description: Link-Cut Tree. Calculate any path queries. Change `upd` to maintain what you need. Don't use `upd` in `push`:). Calculate non commutative functions in both ways and swap them in push. `cnt` – number of nodes in current splay tree. Don't touch `rev`, `sub`, `vsub`. `v->access()` brings `v` to the top and pushes it; its left subtree will be the path from `v` to the root and its right subtree will be empty. Only then `sub` will be the number of nodes in the connected component of `v` and `vsub` will be the number of nodes under `v`. Change upd to calc sum in subtree of other functions. Use `makeRoot` for arbitrary path queries.
Usage: `FOR (i, 0, n) LCT[i] = new snode(i); link(LCT[u], LCT[v]);`
Time: $\mathcal{O}(\log n)$

788027, 159 lines

```
typedef struct Snode* sn;
struct Snode
{
  sn p, c[2]; // parent, children
  bool rev = false; // subtree reversed or not (internal usage)
  int val, cnt; // value in node, # nodes in splay subtree
  int sub, vsub = 0; // vsub stores sum of virtual children
  Snode(int _val): val(_val)
  {
    p = c[0] = c[1] = 0;
    upd();
  }
  friend int getCnt(sn v)
  {
    return v ? v->cnt : 0;
  }
  friend int getSub(sn v)
  {
    return v ? v->sub : 0;
  }
  void push()
  {
    if (!rev)
      return;
    swap(c[0], c[1]);
    rev = false;
    FOR (i, 0, 2)
      if (c[i])
        c[i]->rev ^= 1;
  }
  void upd()
  {
    FOR (i, 0, 2)
      if (c[i])
        c[i]->push();
    cnt = 1 + getCnt(c[0]) + getCnt(c[1]);
    sub = 1 + getSub(c[0]) + getSub(c[1]) + vsub;
  }
  int dir()
  {
    if (!p) return -2;
    FOR (i, 0, 2)
      if (p->c[i] == this)
        return i;
    // p is path-parent pointer
    // -> not in current splay tree
    return -1;
  }

  // checks if root of current splay tree
  bool isRoot()
  {
    return dir() < 0;
  }
  friend void setLink(sn p, sn v, int d)
  {
    if (v)
      v->p = p;
    if (d >= 0)
      p->c[d] = v;
  }
  void rot()
  {
    assert(!isRoot());
    int d = dir();
    sn pa = p;
    setLink(pa->p, this, pa->dir());
    setLink(pa, c[d ^ 1], d);
    setLink(this, pa, d ^ 1);
    pa->upd();
  }
  void splay()
  {
    while (!isRoot() && !p->isRoot())
    {
      p->p->push();
      p->push();
      push();
      dir() == p->dir() ? p->rot() : rot();
      rot();
    }
    if (!isRoot())
      p->push(), push(), rot();
    push();
    upd();
  }
  // bring this to top of tree, propagate
  void access()
  {
    for (sn v = this, pre = 0; v; v = v->p)
    {
      v->splay();
      if (pre)
        v->vsub -= pre->sub;
      if (v->c[1])
        v->vsub += v->c[1]->sub;
      v->c[1] = pre;
      v->upd();
      pre = v;
    }
    splay();
    assert(!c[1]);
  }
  void makeRoot()
  {
    access();
    rev ^= 1;
    access();
    assert(!c[0] && !c[1]);
  }
  friend sn lca(sn u, sn v)
  {
    if (u == v)
      return u;
    u->access();
    v->access();
    if (!u->p)
      return 0;
    u->splay();
    return u->p ? u->p : u;
  }
  friend bool connected(sn u, sn v)
  {
    return lca(u, v);
  }
  void set(int v)
  {
    access();
    val = v;
    upd();
  }
  friend void link(sn u, sn v)
  {
    assert(!connected(u, v));
    v->makeRoot();
    u->access();
    setLink(v, u, 0);
    v->upd();
  }
  // cut v from it's parent in LCT
  // make sure about root or better use next function
  friend void cut(sn v)
  {
    v->access();
    assert(v->c[0]); // assert if not a root
```

```
    v->c[0]->p = 0;
    v->c[0] = 0;
    v->upd();
  }
  // u, v should be adjacent in tree
  friend void cut(sn u, sn v)
  {
    u->makeRoot();
    v->access();
    assert(v->c[0] == u && !u->c[0] && !u->c[1]);
    cut(v);
  }
};
```

ordered-set.hpp

16 lines

```
#include <ext/pb_ds/assoc_container.hpp>
using namespace __gnu_pbds;
using namespace std;
typedef tree<int, null_type, less<int>, rb_tree_tag,
     tree_order_statistics_node_update> ordered_set;

ordered_set s;
s.insert(47);
// Returns the number of elements less then k
s.order_of_key(k);
// Returns iterator to the k-th element or s.end()
s.find_by_order(k);
// Does not exist
s.count();
// Doesn't trigger RE. Returns 0 if compiled using F8
*s.end();
```

sparse-table.hpp

Description: Sparse table for minimum on the range $[l, r), l < r$. You can push back an element in $O(\text{LOG})$ and query anytime.

3cfa53, 19 lines

```
struct SparseTable
{
  VI t[LOG];

  void push_back(int v)
  {
    int i = SZ(t[0]);
    t[0].PB(v);
    FOR (j, 0, LOG - 1)
      t[j + 1].PB(min(t[j][i], t[j][max(0, i - (1 << j))]));
  }
  // [l, r)
  int query(int l, int r)
  {
    assert(l < r && r <= SZ(t[0]));
    int i = 31 - __builtin_clz(r - l);
    return min(t[i][r - 1], t[i][l + (1 << i) - 1]);
  }
};
```

convex-hull-trick.hpp

Description: add(a, b) adds a straight line $y = ax + b$. getMaxY(p) finds the maximum y at $x = p$.

bb0dd6, 72 lines

```
struct Line
{
  LL a, b, xLast;
  Line() {}
  Line(LL _a, LL _b): a(_a), b(_b) {}
  bool operator<(const Line& l) const
  {
    return MP(a, b) < MP(l.a, l.b);
  }
  bool operator<(int x) const
  {
    return xLast < x;
  }
  __int128 getY(__int128 x) const
  {
    return a * x + b;
  }
  LL intersect(const Line& l) const
  {
    assert(a < l.a);
    LL dA = l.a - a, dB = b - l.b, x = dB / dA;
    if (dB < 0 && dB % dA != 0)
      x--;
    return x;
  }
};

struct ConvexHull: set<Line, less<>>
{
  bool needErase(iterator it, const Line& l)
  {
    LL x = it->xLast;
    if (it->getY(x) > l.getY(x))
      return false;
    if (it == begin())
      return it->a >= l.a;
    x = prev(it)->xLast + 1;
    return it->getY(x) < l.getY(x);
  }
  void add(LL a, LL b)
  {
    Line l(a, b);
    auto it = lower_bound(l);
    if (it != end())
    {
      LL x = it == begin() ? -LINF :
          prev(it)->xLast;
      if ((it == begin()
        || prev(it)->getY(x) >= l.getY(x))
        && it->getY(x + 1) >= l.getY(x + 1))
        return;
    }
    while (it != end() && needErase(it, l))
      it = erase(it);
    while (it != begin() && needErase(prev(it), l))
      erase(prev(it));
    if (it != begin())
    {
      auto itP = prev(it);
      Line itL = *itP;
      itL.xLast = itP->intersect(l);
      erase(itP);
      insert(itL);
    }
    l.xLast = it == end() ? LINF : l.intersect(*it);
    insert(l);
  }
  LL getMaxY(LL p)
  {
    return lower_bound(p)->getY(p);
  }
};
```

Graphs (3)

Shortest paths

bellman-ford-moore.hpp

Description: Computes shortest paths from a single source vertex to all of the other vertices in a weighted directed graph.
Time: $\mathcal{O}(nm)$

2e17fb, 35 lines

```
VL spfa(const vector<vector<pair<int, LL>>>& g, int n, int s)
{
  VL dist(n, LINF);
  dist[s] = 0;
  queue<int> q;
  q.push(s);
  VI inQueue(n);
  inQueue[s] = true;
  VI cnt(n);
  bool negCycle = false;
  while (!q.empty())
  {
    int v = q.front();
    q.pop();
    cnt[v]++;
    negCycle |= cnt[v] > n;
    inQueue[v] = false;
    for (auto [to, w] : g[v])
    {
      LL newDist = dist[v] + w;
      if (newDist < dist[to])
      {
        dist[to] = newDist;
        if (!inQueue[to])
        {
          q.push(to);
          inQueue[to] = true;
        }
      }
    }
    if (negCycle)
      break;
  }
  return dist;
}
```

monge-shortest-path.hpp

Description: Finds shortest paths from the vertex 0 to all vertices in a DAG with n vertices, where the edges weights $c(i, j)$ satisfy the Monge property: $\forall i, j, k, l, \quad 0 \le i < j < k < l < n \implies c(i, l) + c(j, k) \ge c(i, k) + c(j, l)$.
Time: $\mathcal{O}(n \log n)$

79be76, 34 lines

```
template<typename F>
VL mongeShortestPath(int n, const F& cost)
{
  VL dist(n, LINF);
  VI amin(n);
  dist[0] = 0;

  auto update = [&](int i, int k)
  {
    LL nd = dist[k] + cost(k, i);
    if (nd < dist[i])
    {
      dist[i] = nd;
      amin[i] = k;
    }
  };

  function<void(int, int)> solve = [&](int l, int r)
  {
    if (r - l == 1)
      return;
    int m = (l + r) / 2;
```

```
    FOR(k, amin[l], min(m, amin[r] + 1))
      update(m, k);
    solve(l, m);
    FOR(k, l + 1, m + 1)
      update(r, k);
    solve(m, r);
  };

  update(n - 1, 0);
  solve(0, n - 1);
  return dist;
}
```

Decompositions

centroid.hpp

19ecf3, 51 lines

```
VI g[N];
int sz[N];
bool usedC[N];

int dfsSZ(int v, int par)
{
  sz[v] = 1;
  for (auto to : g[v])
  {
    if (to != par && !usedC[to])
      sz[v] += dfsSZ(to, v);
  }
  return sz[v];
}

void build(int u)
{
  dfsSZ(u, -1);
  int szAll = sz[u];
  int pr = u;
  while (true)
  {
    int v = -1;
    for (auto to : g[u])
    {
      if (to == pr || usedC[to])
        continue;
      if (sz[to] * 2 > szAll)
      {
        v = to;
        break;
      }
    }
    if (v == -1)
      break;
    pr = u;
    u = v;
  }
  int cent = u;
  usedC[cent] = true;

  // here calculate f(cent)

  for (auto to : g[cent])
  {
    if (!usedC[to])
    {
      build(to);
    }
  }
}
```

hld.hpp

Description: Run dfsSZ(root, -1, 0) and dfsHLD(root, -1, root) to build the HLD. Each vertex v has an index tin[v]. To update on the path, use the process as defined in get(). The values are stored in the vertices.

40c18a, 67 lines

```
VI g[N];
int sz[N];
int h[N];
int p[N];
int top[N];
int tin[N];
int tout[N];
int t = 0;

void dfsSZ(int v, int par, int hei)
{
  sz[v] = 1;
  h[v] = hei;
  p[v] = par;
  for (auto& to : g[v])
  {
    if (to == par)
      continue;
    dfsSZ(to, v, hei + 1);
    sz[v] += sz[to];
    if (g[v][0] == par || sz[g[v][0]] < sz[to])
      swap(g[v][0], to);
  }
}
void dfsHLD(int v, int par, int tp)
{
  tin[v] = t++;
  top[v] = tp;
  FOR (i, 0, SZ(g[v]))
  {
    int to = g[v][i];
    if (to == par)
      continue;
    if (i == 0)
      dfsHLD(to, v, tp);
    else
      dfsHLD(to, v, to);
  }
  tout[v] = t - 1;
}
LL get(int u, int v)
{
  LL res = 0;
  while(true)
  {
    int tu = top[u];
    int tv = top[v];
    if (tu == tv)
    {
      int t1 = tin[u];
      int t2 = tin[v];
      if (t1 > t2)
        swap(t1, t2);
      // query [t1, t2] both inclusive
      res += query(t1, t2);
      break;
    }
    if (h[tu] < h[tv])
    {
      swap(tu, tv);
      swap(u, v);
    }
    res += query(tin[tu], tin[u]);
    u = p[tu];
  }
  return res;
}
```

biconnected-components.hpp

Description: Colors the edges so that the vertices, connected with the same color are still connected if you delete any vertex.
Time: $\mathcal{O}(m)$

1ac7bb, 117 lines

```
struct Graph
{
  int n, m;
  vector<PII> edges;
  vector<VI> g;

  VI used, par;
  VI tin, low, inComp;
  int t = 0, c = 0;
  VI st;

  // components of vertices
  // a vertex can be in several components
  vector<VI> verticesCol;
  // components of edges
  vector<VI> components;
  // col[i] - component of the i-th edge
  VI col;

  Graph(int _n = 0, int _m = 0): n(_n), m(_m), edges(m), g(n),
  used(n), par(n, -1), tin(n), low(n), inComp(n), col(m, -1) {}

  void addEdge(int a, int b, int i)
  {
    assert(0 <= a && a < n);
    assert(0 <= b && b < n);
    assert(0 <= i && i < m);

    edges[i] = MP(a, b);
    g[a].PB(i);
    g[b].PB(i);
  }

  void addComp()
  {
    unordered_set<int> s;
    s.reserve(7 * SZ(components[c]));
    for (auto e : components[c])
    {
      s.insert(edges[e].F);
      s.insert(edges[e].S);
      inComp[edges[e].F] = true;
      inComp[edges[e].S] = true;
    }
    verticesCol.PB(VI(ALL(s)));
  }

  void dfs(int v, int p = -1)
  {
    used[v] = 1;
    par[v] = p;
    low[v] = tin[v] = t++;
    int cnt = 0;
    for (auto e : g[v])
    {
      int to = edges[e].F;
      if (to == v)
        to = edges[e].S;

      if (p == to) continue;
      if (!used[to])
      {
```

```
        cnt++;
        st.PB(e);
        dfs(to, v);

        low[v] = min(low[v], low[to]);

        if ((par[v] == -1 && cnt > 1) ||
        (par[v] != -1 && low[to] >= tin[v]))
        {
          components.PB({});
          while (st.back() != e)
          {
            components[c].PB(st.back());
            col[st.back()] = c;

            st.pop_back();
          }
          components[c].PB(st.back());
          addComp();
          col[st.back()] = c++;

          st.pop_back();
        }
      }
      else
      {
        low[v] = min(low[v], tin[to]);
        if (tin[to] < tin[v])
          st.PB(e);
      }
    }
  }
  void build()
  {
    FOR (i, 0, n)
    {
      if (used[i]) continue;
      dfs(i, -1);
      if (st.empty()) continue;
      components.PB({});
      while (!st.empty())
      {
        int e = st.back();
        col[e] = c;
        components[c].PB(e);
        st.pop_back();
      }
      addComp();
      c++;
    }
    FOR (i, 0, n)
      if (!inComp[i])
        verticesCol.PB(VI(1, i));
  }
};
```

Hierholzer's algorithm

hierholzer.hpp

Description: Finds an Eulerian path in a directed or undirected graph. g is a graph with n vertices. $g[u]$ is a vector of pairs $(v, \text{edge\_id})$. m is the number of edges in the graph. The vertices are numbered from 0 to $n-1$, and the edges - from 0 to $m-1$. If there is no Eulerian path, returns $\{\{-1\},\{-1\}\}$. Otherwise, returns the path in the form (vertices, edges) with vertices containing $m+1$ elements and edges containing m elements. If you need an Eulerian cycle, check vertices[0] = vertices.back().

50cce1, 101 lines

```
// 528807 for undirected
tuple<bool, int, int> checkDirected(vector<vector<PII>>& g)
{
  int n = SZ(g), v1 = -1, v2 = -1;
  bool bad = false;
  VI degIn(n);
  FOR(u, 0, n)
    for (auto [v, e] : g[u])
      degIn[v]++;
  FOR(u, 0, n)
  {
    bad |= abs(degIn[u] - SZ(g[u])) > 1;
    if (degIn[u] < SZ(g[u]))
    {
      bad |= v2 != -1;
      v2 = u;
    }
    else if (degIn[u] > SZ(g[u]))
    {
      bad |= v1 != -1;
      v1 = u;
    }
  }
  return {bad, v1, v2};
}

/*tuple<bool, int, int> checkUndirected(vector<vector<PII>>& g)
{
  int n = SZ(g), v1 = -1, v2 = -1;
  bool bad = false;
  FOR(u, 0, n)
  {
    if (SZ(g[u]) & 1)
    {
      bad |= v2 != -1;
      if (v1 == -1)
        v1 = u;
      else
        v2 = u;
    }
  }
  return {bad, v1, v2};
}*/

pair<VI, VI> hierholzer(vector<vector<PII>> g, int m)
{
  // checkUndirected if undirected
  auto [bad, v1, v2] = checkDirected(g);
  if (bad)
    return {{-1}, {-1}};
  if (v1 != -1)
  {
    g[v1].PB({v2, m});
    // uncomment if undirected
    //g[v2].PB({v1, m});
    m++;
  }
  deque<PII> d;
  VI used(m);
  int v = 0, k = 0;
  while (m > 0 && g[v].empty())
    v++;
  while (SZ(d) < m)
  {
    while (k < m)
    {
      while (!g[v].empty() && used[g[v].back().S])
        g[v].pop_back();
      if (!g[v].empty())
        break;
      d.push_front(d.back());
      d.pop_back();
      v = d.back().F;
      k++;
    }
    if (k == m)
      return {{-1}, {-1}};
    d.PB(g[v].back());
    used[g[v].back().S] = true;
    g[v].pop_back();
    v = d.back().F;
  }
  while (v1 != -1 && d.back().S != m - 1)
  {
    d.push_front(d.back());
    d.pop_back();
    v = d.back().F;
  }
  VI vertices = {v}, edges;
  for (auto [u, e] : d)
  {
    vertices.PB(u);
    edges.PB(e);
  }
  if (v1 != -1)
  {
    vertices.pop_back();
    edges.pop_back();
  }
  return {vertices, edges};
}
```

Maximum matching

kuhn.hpp

Description: mateFor is -1 or mate. addEdge($[0, L)$, $[0, R)$).
Time: 0.6s for $L, R \le 10^5$, $|E| \le 2 \cdot 10^5$

577d83, 76 lines

```
mt19937 rng;

struct Graph
{
  int szL, szR;
  // edges from the left to the right, 0-indexed
  vector<VI> g;
  VI mateForL, usedL, mateForR;

  Graph(int L = 0, int R = 0): szL(L), szR(R), g(L),
    mateForL(L), usedL(L), mateForR(R) {}

  void addEdge(int from, int to)
  {
    assert(0 <= from && from < szL);
    assert(0 <= to && to < szR);

    g[from].PB(to);
  }

  int iter;
  bool kuhn(int v)
  {
    if (usedL[v] == iter) return false;
    usedL[v] = iter;
    shuffle(ALL(g[v]), rng);
    for(int to : g[v])
    {
      if (mateForR[to] == -1)
      {
        mateForR[to] = v;
        mateForL[v] = to;
        return true;
      }
    }
```

```
    for(int to : g[v])
    {
      if (kuhn(mateForR[to]))
      {
        mateForR[to] = v;
        mateForL[v] = to;
        return true;
      }
    }
    return false;
  }
  int doKuhn()
  {
    fill(ALL(mateForR), -1);
    fill(ALL(mateForL), -1);
    fill(ALL(usedL), -1);

    int res = 0;
    iter = 0;

    while(true)
    {
      iter++;

      bool ok = false;
      FOR(v, 0, szL)
      {
        if (mateForL[v] == -1)
        {
          if (kuhn(v))
          {
            ok = true;
            res++;
          }
        }
      }
      if (!ok) break;
    }
    return res;
  }
};
```

edmonds-blossom.hpp

Description: Finds the maximum matching in a graph.

Time: $\mathcal{O}\left(n^2 m\right)$ a2da82, 125 lines

```
struct Graph
{
  int n;
  vector<VI> g;
  VI label, first, mate;

  Graph(int _n = 0): n(_n), g(n + 1), label(n + 1),
    first(n + 1), mate(n + 1) {}

  void addEdge(int u, int v)
  {
    assert(0 <= u && u < n);
    assert(0 <= v && v < n);
    u++;
    v++;
    g[u].PB(v);
    g[v].PB(u);
  }
  void augmentPath(int v, int w)
  {
    int t = mate[v];
    mate[v] = w;
    if (mate[t] != v)
      return;
    if (label[v] <= n)
    {
      mate[t] = label[v];
      augmentPath(label[v], t);
      return;
    }
    int x = label[v] / (n + 1);
    int y = label[v] % (n + 1);
    augmentPath(x, y);
    augmentPath(y, x);
  }
  int findMaxMatching()
  {
    FOR(i, 0, n + 1)
      assert(mate[i] == 0);
    int mt = 0;
    DSU dsu;
    FOR(u, 1, n + 1)
    {
      if (mate[u] != 0)
        continue;
      fill(ALL(label), -1);
      iota(ALL(first), 0);
      dsu.init(n + 1);
      label[u] = 0;
      dsu.unite(u, 0);
      queue<int> q;
      q.push(u);
      while (!q.empty())
      {
        int x = q.front();
        q.pop();
        for (int y: g[x])
        {
          if (mate[y] == 0 && y != u)
          {
            mate[y] = x;
            augmentPath(x, y);
            while (!q.empty())
              q.pop();
            mt++;
            break;
          }
          if (label[y] < 0)
          {
            int v = mate[y];
            if (label[v] < 0)
            {
              label[v] = x;
              dsu.unite(v, y);
              q.push(v);
            }
          }
          else
          {
            int r = first[dsu.find(x)], s = first[dsu.find(y)];
            if (r == s)
              continue;
            int edgeLabel = (n + 1) * x + y;
            label[r] = label[s] = -edgeLabel;
            int join;
            while (true)
            {
              if (s != 0)
                swap(r, s);
              r = first[dsu.find(label[mate[r]])];
              if (label[r] == -edgeLabel)
              {
                join = r;
                break;
              }
              label[r] = -edgeLabel;
            }
            for (int z: {x, y})
            {
              for (int v = first[dsu.find(z)];
                v != join;
                v = first[dsu.find(label[mate[v]])])
              {
                label[v] = edgeLabel;
                if (dsu.unite(v, join))
                  first[dsu.find(join)] = join;
                q.push(v);
              }
            }
          }
        }
      }
    }
    return mt;
  }
  int getMate(int v)
  {
    assert(0 <= v && v < n);
    v++;
    int u = mate[v];
    assert(u == 0 || mate[u] == v);
    u--;
    return u;
  }
};
```

Tutte matrix

Given an undirected graph $G = (V, E)$, its Tutte matrix is:

$$T_{ij} = \begin{cases} x_{ij} & \text{if } i < j \text{ and } (i,j) \in E \\ -x_{ji} & \text{if } i > j \ (i,j) \in E \\ 0 & \text{otherwise.} \end{cases}$$

$\det(T) \neq 0$ if and only if G has a perfect matching.

Flows

dinic.hpp

Description: Finds the maximum flow in a network.

Time: $\mathcal{O}\left(n^2 m\right)$. If all capacities are less than c, then the complexity of the Dinic is bounded by $\mathcal{O}\left(\min(n^{\frac{2}{3}}, \sqrt{cm}) \cdot cm\right)$.   18ca2a, 87 lines

```
struct Graph
{
  struct Edge
  {
    int from, to;
    LL cap, flow;
  };

  int n;
  vector<Edge> edges;
  vector<VI> g;
  VI d, p;

  Graph(int _n): n(_n), g(n), d(n), p(n) {}
  void addEdge(int from, int to, LL cap)
  {
    assert(0 <= from && from < n);
    assert(0 <= to && to < n);
```

```
    assert(0 <= cap);
    g[from].PB(SZ(edges));
    edges.PB({from, to, cap, 0});
    g[to].PB(SZ(edges));
    edges.PB({to, from, 0, 0});
  }
  int bfs(int s, int t)
  {
    fill(ALL(d), -1);
    d[s] = 0;
    queue<int> q;
    q.push(s);
    while (!q.empty())
    {
      int v = q.front();
      q.pop();
      for (int e : g[v])
      {
        int to = edges[e].to;
        if (edges[e].flow < edges[e].cap && d[to] == -1)
        {
          d[to] = d[v] + 1;
          q.push(to);
        }
      }
    }
    return d[t];
  }
  LL dfs(int v, int t, LL flow)
  {
    if (v == t || flow == 0)
      return flow;
    for (; p[v] < SZ(g[v]); p[v]++)
    {
      int e = g[v][p[v]], to = edges[e].to;
      LL c = edges[e].cap, f = edges[e].flow;
      if (f < c && (to == t || d[to] == d[v] + 1))
      {
        LL push = dfs(to, t, min(flow, c - f));
        if (push > 0)
        {
          edges[e].flow += push;
          edges[e ^ 1].flow -= push;
          return push;
        }
      }
    }
    return 0;
  }
  LL flow(int s, int t)
  {
    assert(0 <= s && s < n);
    assert(0 <= t && t < n);
    assert(s != t);
    LL flow = 0;
    while (bfs(s, t) != -1)
    {
      fill(ALL(p), 0);
      while (true)
      {
        LL f = dfs(s, t, LINF);
        if (f == 0)
          break;
        flow += f;
      }
    }
    return flow;
  }
};
```

successive-shortest-path.hpp

Description: Finds the minimum cost maximum flow in a network. If the network contains negative-cost edges, uncomment `initPotentials`.
Time: $\mathcal{O}(|F| \cdot m \log n)$ without negative-cost edges, and $\mathcal{O}(|F| \cdot m \log n + nm)$ with negative-cost edges.

785709, 103 lines

```
struct Graph
{
  struct Edge
  {
    int from, to;
    int cap, flow;
    LL cost;
  };

  int n;
  vector<Edge> edges;
  vector<VI> g;
  VL pi, d;
  VI pred;

  Graph(int _n = 0): n(_n), g(n), pi(n), d(n), pred(n) {}

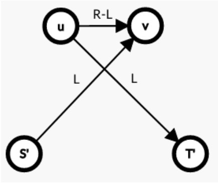
  void addEdge(int from, int to, int cap, LL cost)
  {
    assert(0 <= from && from < n);
    assert(0 <= to && to < n);
    assert(0 <= cap);
    g[from].PB(SZ(edges));
    edges.PB({from, to, cap, 0, cost});
    g[to].PB(SZ(edges));
    edges.PB({to, from, 0, 0, -cost});
  }
  /*void initPotentials(int s)
  {
    vector<vector<pair<int, LL>>> gr(n);
    FOR(v, 0, n)
    {
      for (int e : g[v])
      {
        const Edge& edge = edges[e];
        if (edge.flow < edge.cap)
          gr[v].PB({edge.to, edge.cost});
      }
    }
    pi = spfa(gr, n, s);
  }*/
  pair<int, LL> flow(int s, int t)
  {
    assert(0 <= s && s < n);
    assert(0 <= t && t < n);
    assert(s != t);
    //initPotentials(s);
    int flow = 0;
    LL cost = 0;
    for (int it = 0; ; it++)
    {
      fill(ALL(d), LINF);
      fill(ALL(pred), -1);
      d[s] = 0;
      priority_queue<pair<LL, int>> q;
      q.push({0, s});
      while (!q.empty())
      {
        auto [dv, v] = q.top();
        q.pop();
        if (it > 0 && v == t)
          break;
        if (-dv != d[v])
          continue;
        for (int i : g[v])
        {
          if (edges[i].flow == edges[i].cap)
            continue;
          int to = edges[i].to;
          LL nd = d[v] + edges[i].cost + pi[v] - pi[to];
          if (nd < d[to])
          {
            d[to] = nd;
            pred[to] = i;
            q.push({-nd, to});
          }
        }
      }
      if (d[t] == LINF)
        break;
      int curFlow = INF;
      for (int v = t; v != s;)
      {
        int i = pred[v];
        curFlow = min(curFlow, edges[i].cap - edges[i].flow);
        v = edges[i].from;
      }
      for (int v = t; v != s;)
      {
        int i = pred[v];
        edges[i].flow += curFlow;
        edges[i ^ 1].flow -= curFlow;
        v = edges[i].from;
      }
      flow += curFlow;
      cost += (d[t] + pi[t] - pi[s]) * curFlow;
      FOR(u, 0, n)
        if (it == 0 || d[u] <= d[t])
          pi[u] += d[u] - d[t];
    }
    return {flow, cost};
  }
```

Maximum flow with minimum capacities

On the resulting graph, accumulate maximum flow in the following order:

- from S' to T'
- from S' to T
- from S to T'
- from S to T.

An $S-T$ flow that satisfies the minimum capacities exists if and only if, for all outgoing edges from S' and incoming edges to T', the flow and capacity are equal.



Quadratic supermodular pseudoboolean optimization

$$\sum_i a_i x_i + \sum_i b_i \overline{x_i} + \sum_{i,j} c_{ij} x_i \overline{x_j} \to \min$$

$$c_{ij} x_i x_j = c_{ij} x_i - c_{ij} x_i \overline{x_j}$$

If $a_i \le b_i$, add an edge from S to i of capacity $b_i - a_i$ and add a_i to the answer.

Otherwise, add an edge from i to T of capacity $a_i - b_i$ and add b_i to the answer.

Add an edge from i to j of capacity c_{ij}.

Add the $S - T$ minimum cut to the answer.

Matching tricks

Minimum cut

To find the min-cut, search from vertex S on unsaturated edges. Original edges from used vertices to unused ones are in the min-cut.

Minimum vertex cover

The vertex cover problem is not NP-complete in bipartite graphs. The minimum number of vertices required to cover all **edges** is equal to the size of the maximum matching. To reconstruct the minimum vertex cover, create a directed graph:

- matched edges from the right part to the left part
- unmatched edges from the left part to the right part.

Start traversal from unmatched vertices in the left part. The cover includes vertices from the matching:

- unvisited vertices in the left part
- visited vertices in the right part.

Maximum independent set

The independent set problem is not NP-complete in bipartite graphs. It is the complement of the minimum vertex cover.

Minimum edge cover

A minimum edge cover can be found in **any** graph. The minimum number of edges required to cover all vertices can only be determined in graphs without isolated vertices. By utilizing one edge in the matching, we cover two vertices, while any other vertices are covered using one edge for each.

DAG paths

In a DAG, you can find the minimum number of non-intersecting paths that cover all vertices. Duplicate vertices and create a bipartite graph with edges $u_L \to v_R$. Edges in the matching correspond to edges in the paths.

Dominating set

A dominating set for a graph is a subset D of V such that any vertex is in D, or has a neighbor in D. The dominating set problem is NP-complete **even on bipartite graphs**. It can be found greedily on a tree.

Dominator tree

dominator-tree.hpp

Description: Works for cyclic graphs. par – parent in dfs. p – parent in the DSU. val – vertex with the minimum sdom in dsu. dom – immediate dominator. sdom – semidominator, min vertex with alternate path. bkt – vertices with this sdom. dom[root] = -1. dom[v] = -1 if v is unreachable.

Time: $\mathcal{O}(n)$

e623f1, 106 lines

```
struct Graph
{
  int n;
  vector<VI> g, gr, bkt;
  VI par, used, p, val, sdom, dom, tin;
  int T;
  VI ord;

  Graph(int _n = 0): n(_n), g(n), gr(n), bkt(n), par(n),
    used(n), p(n), val(n), sdom(n), dom(n), tin(n) {}

  void addEdge(int u, int v)
  {
    assert(0 <= u && u < n);
    assert(0 <= v && v < n);
    g[u].PB(v);
    gr[v].PB(u);
  }

  int find(int v)
  {
    if (p[v] == v)
      return v;
    int y = find(p[v]);
    if (p[y] == y)
      return v;
    if (tin[sdom[val[p[v]]]] < tin[sdom[val[v]]])
      val[v] = val[p[v]];
    p[v] = y;
    return y;
  }
  int get(int v)
  {
    find(v);
    // return vertex with min sdom
    return val[v];
  }

  void dfs(int v, int pr)
  {
    tin[v] = T++;
    used[v] = true;
    ord.PB(v);
    par[v] = pr;
    for (auto to : g[v])
    {
      if (!used[to])
        dfs(to, v);
    }
  }

  void build(int s)
  {
    FOR (i, 0, n)
    {
      used[i] = false;
      sdom[i] = i;
      dom[i] = -1;
      p[i] = i;
      val[i] = i;
      bkt[i].clear();
    }
    ord.clear();
    T = 0;

    dfs(s, -1);

    RFOR(i, SZ(ord), 0)
    {
      int v = ord[i];
      for (auto from : gr[v])
      {
        // don't consider unreachable vertices
        if (!used[from])
          continue;
        // find min sdom
        if (tin[sdom[v]] > tin[sdom[get(from)]])
        {
          sdom[v] = sdom[get(from)];
        }
      }
      if (v != s)
        bkt[sdom[v]].PB(v);
      for (auto y : bkt[v])
      {
        int u = get(y);
        // if sdoms equals then this is dom
        // else we will find it later
        if (sdom[y] == sdom[u])
          dom[y] = sdom[y];
        else dom[y] = u;
      }
      // add vertex to dsu
      if (par[v] != -1)
        p[v] = par[v];
    }

    for (auto v : ord)
    {
      if (v == s || dom[v] == -1)
        continue;
      if (dom[v] != sdom[v]) dom[v] = dom[dom[v]];
    }
  }

};
```

Sqrt problems

3-cycles.hpp

Description: Finds all triangles in a graph. Each triangle (v, u, w) increments the cnt.

Time: $\mathcal{O}(m \cdot \sqrt{m})$

406c69, 22 lines

```
int triangles(int n)
{
  vector<VI> ng(n);
  FOR (v, 0, n)
    for (auto u : g[v])
      if (MP(SZ(g[v]), v) < MP(SZ(g[u]), u))
        ng[v].PB(u);
  int cnt = 0;
  VI used(n, 0);
  FOR (v, 0, n)
  {
    for (auto u : ng[v])
      used[u] = 1;
    for (auto u : ng[v])
      for(auto w : ng[u])
        if (used[w])
          cnt++;
    for (auto u : ng[v])
```

```
        used[u] = 0;
    }
    return cnt;
}
```

4-cycles.hpp

Description: Sort d and add breaks to speed up. With breaks works 0.5s for m = $5 \cdot 10^5$.
Time: $\mathcal{O}\left(\sum_{uv \in E} \min(\deg(u), \deg(v))\right) = \mathcal{O}\left(m \cdot \sqrt{m}\right)$

e2c43b, 20 lines

```
LL rect(int n)
{
  LL cnt4 = 0;
  vector<PII> d(n);
  FOR (v, 0, n) d[v] = MP(SZ(g[v]), v);
  VI L(n);
  FOR (v, 0, n)
  {
    for (auto u : g[v])
      if (d[u] < d[v])
        for (auto y : g[u])
          if (d[y] < d[v])
            cnt4 += L[y], L[y]++;
    for (auto u : g[v])
      if (d[u] < d[v])
        for (auto y : g[u])
          L[y] = 0;
  }
  return cnt4;
}
```

Strings (4)

aho-corasick.hpp

a5381b, 64 lines

```
const int AL = 26;

struct Node
{
  int p;
  int c;
  int g[AL];
  int nxt[AL];
  int link;

  Node(int _c, int _p)
  {
    c = _c;
    p = _p;
    fill(g, g + AL, -1);
    fill(nxt, nxt + AL, -1);
    link = -1;
  }
};

struct AC
{
  vector<Node> a;
  AC(): a(1, {-1, -1}) {}

  int addStr(const string& s)
  {
    int v = 0;
    FOR (i, 0, SZ(s))
    {
      // change to [0 AL)
      int c = s[i] - 'a';
      if (a[v].nxt[c] == -1)
      {
        a[v].nxt[c] = SZ(a);
        a.PB(Node(c, v));
      }
      v = a[v].nxt[c];
    }
    return v;
  }
  int go(int v, int c)
  {
    if (a[v].g[c] != -1)
      return a[v].g[c];

    if (a[v].nxt[c] != -1)
      a[v].g[c] = a[v].nxt[c];
    else if (v != 0)
      a[v].g[c] = go(getLink(v), c);
    else
      a[v].g[c] = 0;

    return a[v].g[c];
  }
  int getLink(int v)
  {
    if (a[v].link != -1)
      return a[v].link;
    if (v == 0 || a[v].p == 0)
      return 0;
    return a[v].link = go(getLink(a[v].p), a[v].c);
  }
};
```

suffix-automaton.hpp

b7e92e, 57 lines

```
const int AL = 26;

struct Node
{
  int g[AL];
  int link;
  int len;
  int cnt;
  Node(): link(-1), len(0), cnt(1)
  {
    fill(g, g + AL, -1);
  }
};

struct Automaton
{
  vector<Node> a;
  int head;
  Automaton(): a(1), head(0) {}
  void add(char c)
  {
    // change to [0 AL)
    int ch = c - 'a';
    int nhead = SZ(a);
    a.PB(Node());
    a[nhead].len = a[head].len + 1;
    int cur = head;
    head = nhead;
    while (cur != -1 && a[cur].g[ch] == -1)
    {
      a[cur].g[ch] = head;
      cur = a[cur].link;
    }
    if (cur == -1)
    {
      a[head].link = 0;
      return;
    }
    int p = a[cur].g[ch];
    if (a[p].len == a[cur].len + 1)
    {
      a[head].link = p;
      return;
    }
    int q = SZ(a);
    a.PB(Node());
    a[q] = a[p];
    a[q].cnt = 0;
    a[q].len = a[cur].len + 1;
    a[p].link = a[head].link = q;
    while (cur != -1 && a[cur].g[ch] == p)
    {
      a[cur].g[ch] = q;
      cur = a[cur].link;
    }
  }
};
```

suffix-array.hpp

Description: Cast your string to vector. Don't forget about delimiters. No need to add anything at the end. sa represents permutations of positions if you sort all suffixes.
Time: $\mathcal{O}(n \log n)$

487694, 59 lines

```
void countSort(VI& p, const VI& c)
{
  int n = SZ(p);
  VI cnt(n);
  FOR (i, 0, n)
    cnt[c[i]]++;
  VI pos(n);
  FOR (i, 1, n)
    pos[i] = pos[i - 1] + cnt[i - 1];
  VI p2(n);
  for (auto x : p)
  {
    int i = c[x];
    p2[pos[i]++] = x;
  }
  p = p2;
}

VI suffixArray(VI s)
{
  // strictly smaller than any other element
  s.PB(-1);
  int n = SZ(s);
  VI p(n), c(n);
  iota(ALL(p), 0);
  sort(ALL(p), [&](int i, int j)
  {
    return s[i] < s[j];
  });
  int x = 0;
  c[p[0]] = 0;
  FOR (i, 1, n)
  {
    if (s[p[i]] != s[p[i - 1]])
      x++;
    c[p[i]] = x;
  }
  int k = 0;
  while ((1 << k) < n)
  {
    FOR (i, 0, n)
      p[i] = (p[i] - (1 << k) + n) % n;
```

```
    countSort(p, c);
    VI c2(n);
    PII pr = {c[p[0]], c[(p[0] + (1 << k)) % n]};
    FOR (i, 1, n)
    {
      PII nx = {c[p[i]], c[(p[i] + (1 << k)) % n]};
      c2[p[i]] = c2[p[i - 1]];
      if (pr != nx)
        c2[p[i]]++;
      pr = nx;
    }
    c = c2;
    k++;
  }
  p.erase(p.begin());
  return p;
}
```

lcp.hpp

Description: queryLcp returns the longest common prefix of substrings starting at i and j. 77a30b, 49 lines

```
struct LCP
{
  int n;
  VI s, sa, rnk, lcp;
  SparseTable st;

  LCP(VI _s): n(SZ(_s)), s(_s)
  {
    sa = suffixArray(s);
    rnk.resize(n);
    FOR (i, 0, n)
      rnk[sa[i]] = i;
    lcpArray();
    FOR (i, 0, n - 1)
      st.PB(lcp[i]);
  }

  void lcpArray()
  {
    lcp.resize(n - 1);
    int h = 0;
    FOR (i, 0, n)
    {
      if (h > 0)
        h--;
      if (rnk[i] == 0)
        continue;
      int j = sa[rnk[i] - 1];
      for (; j + h < n && i + h < n; h++)
      {
        if (s[j + h] != s[i + h])
          break;
      }
      lcp[rnk[i] - 1] = h;
    }
  }
  int queryLcp(int i, int j)
  {
    if (i == n || j == n)
      return 0;
    assert(i != j); // return n - i ????
    i = rnk[i];
    j = rnk[j];
    if (i > j)
      swap(i, j);
    // query [i, j)
    return st.query(i, j);
  }
};
```

run-enumerate.hpp

Description: Enumerate all tuples (t, l, r) with t being the minimum period of $s[l, r)$ and $r - l \geq 2 \cdot t$. l and r are maximal. In other words $(t, l-1, r)$ and $(t, l, r+1)$ do not satisfy the previous condition.
The number of runs is $\leq |s|$. Other properties are stated at the end of the function.
Time: $\mathcal{O}(n \log n)$, where $n = |s|$. 3719ad, 62 lines

```
struct Run
{
  int t, l, r;
  bool operator<(const Run& p) const
  {
    return make_tuple(t, l, r) < make_tuple(p.t, p.l, p.r);
  }
  bool operator==(const Run& p) const
  {
    return !(*this < p) && !(p < *this);
  }
};
vector<Run> runEnumerate(VI s)
{
  int n = SZ(s);
  LCP lcp(s); reverse(ALL(s));
  LCP rev(s); reverse(ALL(s));

  vector<Run> runs;
  FOR(inv, 0, 2)
  {
    VI st = {n};
    auto pop = [&](int i)
    {
      int j = st.back();
      int dist = j - i;
      int distPrev = st[SZ(st) - 2] - j;
      int distMn = min(dist, distPrev);

      int len = lcp.queryLcp(i, j);
      if((len >= distMn && dist < distPrev) ||
        (len < distMn && ((s[i + len] < s[j + len]) ^ inv)))
        return true;
      return false;
    };

    RFOR(i, n, 0)
    {
      while(SZ(st) > 1 && pop(i))
        st.pop_back();
      int j = st.back();
      int dist = j - i;
      st.PB(i);

      int x = rev.queryLcp(n - i, n - j);
      int y = lcp.queryLcp(i, j);
      if(x < dist && x + y >= dist)
        runs.PB({dist, i - x, j + y});
    }
  }
  sort(ALL(runs));
  runs.resize(unique(ALL(runs)) - runs.begin());

  //LL sumLen = 0, sumCnt = 0, sum = 0;
  //for(auto [len, l, r] : runs)
  //  sumLen += len, sumCnt += (r - l) / len, sum += r - l;
  //assert(SZ(runs) <= SZ(s));
  //assert(sumLen <= LOG * SZ(s));
  //assert(sumCnt <= 2 * SZ(s));
  //assert(sum <= 2 * LOG * SZ(s));
  return runs;
}
```

suffix-tree.hpp

Description: Ukkonen's algorithm for building a suffix tree. Cast your string to vector. Don't forget about delimiters. `a[v].g[c]` is a transition in format (u, l, r), that goes from v to u and the string spelled out by this transition is the substring $s_{l..r}$. For transitions that go to leaves, $r = \text{INF}$. For the root node which has number 0, `link == -1`. For leaves, `link == -2`. For all other nodes, `link` is maintained explicitly.
Time: $\mathcal{O}(n \log |\Sigma|)$, where Σ is an alphabet f6fdbe, 85 lines

```
struct SuffixTree
{
  struct Transition
  {
    int u, l, r;
  };
  struct Node
  {
    map<int, Transition> g;
    int link;
    Node(): link(-2) {}
  };
  VI s;
  vector<Node> a;
  pair<bool, int> testAndSplit(int v, int l, int r, int c)
  {
    if (v == -1)
      return {true, -1};
    if (l <= r)
    {
      auto [nv, nl, nr] = a[v].g[s[l]];
      if (c == s[nl + r - l + 1])
        return {true, v};
      int newNode = SZ(a);
      a.PB(Node());
      a[v].g[s[l]] = {newNode, nl, nl + r - l};
      a[newNode].g[s[nl + r - l + 1]] = {nv, nl + r - l + 1, nr
          };
      return {false, newNode};
    }
    return {a[v].g.count(c), v};
  }
  PII canonize(int v, int l, int r)
  {
    if (v == -1 && l <= r)
    {
      v = 0;
      l++;
    }
    if (r < l)
      return {v, l};
    Transition cur = a[v].g[s[l]];
    while (cur.r - cur.l <= r - l)
    {
      l += cur.r - cur.l + 1;
      v = cur.u;
      if (l <= r)
        cur = a[v].g[s[l]];
    }
    return {v, l};
  }
  PII update(int v, int l, int r)
  {
    int oldu = 0;
    auto [endPoint, u] = testAndSplit(v, l, r - 1, s[r]);
    while (!endPoint)
    {
      int newNode = SZ(a);
      a.PB(Node());
```

```
      a[u].g[s[r]] = {newNode, r, INF};
      if (oldu != 0)
        a[oldu].link = u;
      oldu = u;
      tie(v, l) = canonize(a[v].link, l, r - 1);
      tie(endPoint, u) = testAndSplit(v, l, r - 1, s[r]);
    }
    if (oldu != 0)
      a[oldu].link = v;
    return {v, l};
  }
  SuffixTree(const VI& _s)
  {
    s = _s;
    // Add the symbol that was not present in 's'
    s.PB(-1);
    a.reserve(2 * SZ(s));
    a = {Node()};
    a[0].link = -1;
    int v = 0, l = 0;
    FOR(i, 0, SZ(s))
    {
      tie(v, l) = update(v, l, i);
      tie(v, l) = canonize(v, l, i);
    }
  }
};
```

z.hpp

e27ac7, 23 lines

```
VI zFunction(const string& s)
{
  int n = SZ(s);
  VI z(n);

  int l = 0;
  int r = 0;
  FOR (i, 1, n)
  {
    z[i] = 0;
    if (i <= r)
      z[i] = min(r - i + 1, z[i - l]);

    while(i + z[i] < n && s[i + z[i]] == s[z[i]])
      z[i]++;
    if(i + z[i] - 1 > r)
    {
      r = i + z[i] - 1;
      l = i;
    }
  }
  return z;
}
```

prefix.hpp

500608, 16 lines

```
VI prefixFunction(const string& s)
{
  int n = SZ(s);
  VI p(n);
  p[0] = 0;
  FOR (i, 1, n)
  {
    int j = p[i - 1];
    while(j != 0 && s[i] != s[j])
      j = p[j - 1];

    if (s[i] == s[j]) j++;
    p[i] = j;
  }
  return p;
}
```

minimal-cyclic-shift.hpp

Description: $s_{shift}, s_{shift+1}, \ldots$ is lexicographically smallest cyclic shift. If more than one answer it finds the minimum value of $shift$.
Time: $\mathcal{O}(n)$ time and memory complexity.

f9f0f3, 29 lines

```
int minimalCyclicShift(VI s)
{
  int n = SZ(s);
  s.resize(2 * n);
  FOR(i, 0, n)
    s[n + i] = s[i];

  int shift = 0;
  VI f(2 * n);
  FOR(i, 1, 2 * n)
  {
    int j = f[i - 1 - shift];
    while(j > 0 && s[shift + j] != s[i])
    {
      if(s[shift + j] > s[i])
        shift = i - j;
      j = f[j - 1];
    }
    if(j == 0 && s[shift] != s[i])
    {
      if(s[shift] > s[i])
        shift = i;
    }
    else
      j++;
    f[i - shift] = j;
  }
  return shift;
}
```

manacher.hpp

Description: $s[i - d0_i, i + d0_i - 1]$, $s[i - d1_i + 1, i + d1_i - 1]$ are palindromes.

e14107, 25 lines

```
vector<VI> manacher(const string& s)
{
  int n = SZ(s);
  vector<VI> d(2);
  FOR (t, 0, 2)
  {
    d[t].resize(n);
    int l = -1;
    int r = -1;
    FOR (i, 0, n)
    {
      if (i <= r)
        d[t][i] = min(r - i + 1, d[t][l + (r - i) + 1 - t]);
      while (i + d[t][i] < n  && i + t - d[t][i] - 1 >= 0
        && s[i + d[t][i]] == s[i + t - d[t][i] - 1])
              d[t][i]++;
      if (i + d[t][i] - t > r)
      {
        r = i + d[t][i] - 1;
        l = i - d[t][i] + t;
      }
    }
  }
  return d;
}
```

palindromic-tree.hpp

d71249, 54 lines

```
const int AL = 26;
struct Node
{
  int to[AL];
  int link;
  int len;
  Node(int _link, int _len)
  {
    fill(to, to + AL, -1);
    link = _link;
    len = _len;
  }
};
struct PalTree
{
  string s;
  vector<Node> a;
  int last;

  PalTree(string t = ""): s(t), a({{-1, -1}, {0, 0}}), last(1)
      {}
  void add(int idx)
  {
    // change to [0, AL)
    int ch = s[idx] - 'a';

    int cur = last;
    while (cur != -1)
    {
      int pos = idx - a[cur].len - 1;
      if (pos >= 0 && s[pos] == s[idx])
        break;
      cur = a[cur].link;
    }
    if (a[cur].to[ch] == -1)
    {
      a[cur].to[ch] = SZ(a);
      int link = a[cur].link;
      while (link != -1)
      {
        int pos = idx - a[link].len - 1;
        if (pos >= 0 && s[pos] == s[idx])
          break;
        link = a[link].link;
      }
      if (link == -1)
        link = 1;
      else
        link = a[link].to[ch];
      a.PB(Node(link, a[cur].len + 2));
    }
    last = a[cur].to[ch];
  }
};
```

Geometry (5)

point.hpp

ff2d7c, 91 lines

```
struct Pt
{
  db x, y;
  Pt operator+(const Pt& p) const
  {
    return {x + p.x, y + p.y};
  }
  Pt operator-(const Pt& p) const
  {
    return {x - p.x, y - p.y};
```

```
  }
  Pt operator*(db d) const
  {
    return {x * d, y * d};
  }
  Pt operator/(db d) const
  {
    return {x / d, y / d};
  }
};
db sq(const Pt& p)
{
  return p.x * p.x + p.y * p.y;
}
db abs(const Pt& p)
{
  return sqrt(sq(p));
}
int sgn(db x)
{
  return (EPS < x) - (x < -EPS);
}
// Returns ‘p‘ rotated counter-clockwise by ‘a‘
Pt rot(const Pt& p, db a)
{
  db co = cos(a), si = sin(a);
  return {p.x * co - p.y * si,
    p.x * si + p.y * co};
}
// Returns ‘p‘ rotated counter-clockwise by 90 degrees
Pt perp(const Pt& p)
{
  return {-p.y, p.x};
}
db dot(const Pt& p, const Pt& q)
{
  return p.x * q.x + p.y * q.y;
}
// Returns the angle between ‘p‘ and ‘q‘ in [0, pi]
db angle(const Pt& p, const Pt& q)
{
  return acos(clamp(dot(p, q) / abs(p) /
    abs(q), (db)-1.0, (db)1.0));
}
db cross(const Pt& p, const Pt& q)
{
  return p.x * q.y - p.y * q.x;
}
// Positive if R is on the left side of PQ,
// negative on the right side,
// and zero if R is on the line containing PQ
db orient(const Pt& p, const Pt& q, const Pt& r)
{
  return cross(q - p, r - p) / abs(q - p);
}
// Checks if argument of ‘p‘ is in [-pi, 0)
bool half(const Pt& p)
{
  assert(sgn(p.x) != 0 || sgn(p.y) != 0);
  return sgn(p.y) == -1 ||
    (sgn(p.y) == 0 && sgn(p.x) == -1);
}
void polarSortAround(const Pt& o, vector<Pt>& v)
{
  sort(ALL(v), [o](Pt p, Pt q)
  {
    p = p - o;
    q = q - o;
    bool hp = half(p), hq = half(q);
    if (hp != hq)
      return hp < hq;
    int s = sgn(cross(p, q));
    if (s != 0)
      return s == 1;
    return sq(p) < sq(q);
  });
}
ostream& operator<<(ostream& os, const Pt& p)
{
  return os << "(" << p.x << "," << p.y << ")";
}
```

line.hpp

83c9af, 50 lines

```
struct Line
{
  // Equation of the line is dot(n, p) + c = 0
  Pt n;
  db c;
  Line (const Pt& _n, db _c): n(_n), c(_c) {}
  // n is the normal vector to the left of PQ
  Line(const Pt& p, const Pt& q):
    n(perp(q - p)), c(-dot(n, p)) {}
  // The "positive side": dot(n, p) + c > 0
  // The "negative side": dot(n, p) + c < 0
  db side(const Pt& p) const
  {
    return dot(n, p) + c;
  }
  db dist(const Pt& p) const
  {
    return abs(side(p)) / abs(n);
  }
  db sqDist(const Pt& p) const
  {
    return side(p) * side(p) / (db)sq(n);
  }
  Line perpThrough(const Pt& p) const
  {
    return {p, p + n};
  }
  bool cmpProj(const Pt& p, const Pt& q) const
  {
    return sgn(cross(p, n) - cross(q, n)) < 0;
  }
  Pt proj(const Pt& p) const
  {
    return p - n * side(p) / sq(n);
  }
  Pt reflect(const Pt& p) const
  {
    return p - n * 2 * side(p) / sq(n);
  }
};
bool parallel(const Line& l1, const Line& l2)
{
  return sgn(cross(l1.n, l2.n)) == 0;
}
Pt inter(const Line& l1, const Line& l2)
{
  db d = cross(l1.n, l2.n);
  assert(sgn(d) != 0);
  return perp(l2.n * l1.c - l1.n * l2.c) / d;
}
```

segment.hpp

687634, 39 lines

```
// Checks if ‘p‘ is in the disk (the region in a plane
// bounded by a circle) of diameter [ab]
bool inDisk(const Pt& a, const Pt& b, const Pt& p)
{
  return sgn(dot(a - p, b - p)) <= 0;
}
// Checks if ‘p‘ lies on segment [ab]
bool onSegment(const Pt& a, const Pt& b, const Pt& p)
{
  return sgn(orient(a, b, p)) == 0 && inDisk(a, b, p);
}
// Checks if the segments [ab] and [cd] intersect
// properly (their intersection is one point
// which is not an endpoint of either segment)
bool properInter(const Pt& a, const Pt& b, const Pt& c, const
    Pt& d)
{
  db oa = orient(c, d, a);
  db ob = orient(c, d, b);
  db oc = orient(a, b, c);
  db od = orient(a, b, d);
  return sgn(oa) * sgn(ob) == -1 && sgn(oc) * sgn(od) == -1;
}
// Returns the distance between [ab] and ‘p‘
db segPt(const Pt& a, const Pt& b, const Pt& p)
{
  Line l(a, b);
  assert(sgn(sq(l.n)) != 0);
  if (l.cmpProj(a, p) && l.cmpProj(p, b))
    return l.dist(p);
  return min(abs(p - a), abs(p - b));
}
// Returns the distance between [ab] and [cd]
db segSeg(const Pt& a, const Pt& b, const Pt& c, const Pt& d)
{
  if (properInter(a, b, c, d))
    return 0;
  return min({segPt(a, b, c), segPt(a, b, d),
      segPt(c, d, a), segPt(c, d, b)});
}
```

polygon.hpp

251907, 67 lines

```
bool isConvex(const vector<Pt>& v)
{
  bool hasPos = false, hasNeg = false;
  int n = SZ(v);
  FOR(i, 0, n)
  {
    int s = sgn(orient(v[i], v[(i + 1) % n], v[(i + 2) % n]));
    hasPos |= s > 0;
    hasNeg |= s < 0;
  }
  return !(hasPos && hasNeg);
}
db areaTriangle(const Pt& a, const Pt& b, const Pt& c)
{
  return abs(cross(b - a, c - a)) / 2.0;
}
db areaPolygon(const vector<Pt>& v)
{
  db area = 0.0;
  int n = SZ(v);
  FOR(i, 0, n)
    area += cross(v[i], v[(i + 1) % n]);
  return abs(area) / 2.0;
}
// Checks if point ‘a‘ is inside the convex
// polygon ‘v‘. Returns true if on the boundary.
// ‘v‘ must not contain duplicated vertices.
// Time: O(log n)
```

```
bool inConvexPolygon(const vector<Pt>& v, const Pt& a)
{
  assert(SZ(v) >= 2);
  if (SZ(v) == 2)
    return onSegment(v[0], v[1], a);
  if (sgn(orient(v.back(), v[0], a)) < 0
    || sgn(orient(v[0], v[1], a)) < 0)
    return false;
  int i = lower_bound(v.begin() + 2, v.end(), a,
  [&](const Pt& p, const Pt& q)
  {
    return sgn(orient(v[0], p, q)) > 0;
  }) - v.begin();
  return sgn(orient(v[i - 1], v[i], a)) >= 0;
}
bool above(const Pt& a, const Pt& p)
{
  return sgn(p.y - a.y) >= 0;
}
bool crossesRay(const Pt& a, const Pt& p,
  const Pt& q)
{
  return sgn((above(a, q) - above(a, p))
    * orient(a, p, q)) == 1;
}
// Checks if point 'a' is inside the polygon
// If 'strict', false when 'a' is on the boundary
bool inPolygon(const vector<Pt>& v, const Pt& a, bool strict =
    true)
{
  int numCrossings = 0;
  int n = SZ(v);
  FOR(i, 0, n)
  {
    if (onSegment(v[i], v[(i + 1) % n], a))
      return !strict;
    numCrossings += crossesRay(a, v[i], v[(i + 1) % n]);
  }
  return numCrossings & 1;
}
```

convex-hull.hpp

4efeb1, 27 lines

```
vector<Pt> convexHull(vector<Pt> v)
{
  if (SZ(v) <= 1)
    return v;
  sort(ALL(v), [](const Pt& p, const Pt& q)
  {
    int dx = sgn(p.x - q.x);
    if (dx != 0)
      return dx < 0;
    return sgn(p.y - q.y) < 0;
  });
  vector<Pt> lower, upper;
  for (const Pt& p : v)
  {
    while (SZ(lower) > 1 &&
      sgn(orient(lower[SZ(lower) - 2], lower.back(), p)) <= 0)
      lower.pop_back();
    while (SZ(upper) > 1 &&
      sgn(orient(upper[SZ(upper) - 2], upper.back(), p)) >= 0)
      upper.pop_back();
    lower.PB(p);
    upper.PB(p);
  }
  reverse(ALL(upper));
  lower.insert(lower.end(), next(upper.begin()), prev(upper.end
     ()));
  return lower;
}
```

tangents-to-convex-polygon.hpp

Description: Returns the indices of tangent points from p. p must be strictly outside the polygon.

e35dbb, 38 lines

```
PII tangentsToConvexPolygon(const vector<Pt>& v, const Pt& p)
{
  int n = SZ(v), i = 0;
  if (n == 2)
    return {0, 1};
  while (sgn(orient(p, v[i], v[(i + 1) % n]))
    * sgn(orient(p, v[i], v[(i + n - 1) % n])) > 0)
    i++;
  int s1 = 1, s2 = -1;
  if (sgn(orient(p, v[i], v[(i + 1) % n])) == s1
    || sgn(orient(p, v[i], v[(i + n - 1) % n])) == s2)
    swap(s1, s2);
  PII res;
  int l = i, r = i + n - 1;
  while (r - l > 1)
  {
    int m = (l + r) / 2;
    if (sgn(orient(p, v[i], v[m % n])) != s1
      && sgn(orient(p, v[m % n], v[(m + 1) % n])) != s1)
      l = m;
    else
      r = m;
  }
  res.F = r % n;
  l = i;
  r = i + n - 1;
  while (r - l > 1)
  {
    int m = (l + r) / 2;
    if (sgn(orient(p, v[i], v[m % n])) == s2
      || sgn(orient(p, v[m % n], v[(m + 1) % n])) != s2)
      l = m;
    else
      r = m;
  }
  res.S = r % n;
  return res;
}
```

minkowski-sum.hpp

Description: Returns the Minkowski sum of two convex polygons.

6feb84, 40 lines

```
vector<Pt> minkowskiSum(const vector<Pt>& v1, const vector<Pt>&
     v2)
{
  if (v1.empty() || v2.empty())
    return {};
  if (SZ(v1) == 1 && SZ(v2) == 1)
    return {v1[0] + v2[0]};
  auto comp = [](const Pt& p, const Pt& q)
  {
    return sgn(p.x - q.x) < 0
      || (sgn(p.x - q.x) == 0
      && sgn(p.y - q.y) < 0);
  };
  int i1 = min_element(ALL(v1), comp) - v1.begin();
  int i2 = min_element(ALL(v2), comp) - v2.begin();
  vector<Pt> res;
  int n1 = SZ(v1), n2 = SZ(v2),
    j1 = 0, j2 = 0;
  while (j1 < n1 || j2 < n2)
  {
    const Pt& p1 = v1[(i1 + j1) % n1];
    const Pt& q1 = v1[(i1 + j1 + 1) % n1];
    const Pt& p2 = v2[(i2 + j2) % n2];
    const Pt& q2 = v2[(i2 + j2 + 1) % n2];
    if (SZ(res) >= 2 && onSegment(res[SZ(res) - 2], p1 + p2,
        res.back()))
      res.pop_back();
    res.PB(p1 + p2);
    int s = sgn(cross(q1 - p1, q2 - p2));
    if (j1 < n1 && (j2 == n2 || s > 0
      || (s == 0 && (SZ(res) < 2
      || sgn(dot(res.back()
      - res[SZ(res) - 2],
      q1 + p2 - res.back())) > 0))))
      j1++;
    else
      j2++;
  }
  if (SZ(res) > 2 && onSegment(res[SZ(res) - 2], res[0], res.
      back()))
    res.pop_back();
  return res;
}
```

ear-clipping.hpp

Description: Finds an arbitrary triangulation of a simple polygon with no three collinear vertices.

3b64b4, 55 lines

```
vector<tuple<int, int, int>> earClipping(const vector<Pt>& v)
{
  int n = SZ(v);
  vector<tuple<int, int, int>> res;
  VI indices(n), ear(n), reflex(n);
  iota(ALL(indices), 0);
  auto updReflexStatus = [&](int i)
  {
    int sz = SZ(indices),
      pos = find(ALL(indices), i) - indices.begin();
    int iPrev = indices[(pos + sz - 1) % sz],
      iNext = indices[(pos + 1) % sz];
    reflex[i] = orient(v[iPrev], v[i], v[iNext]) < 0;
  };
  auto updEarStatus = [&](int i)
  {
    if (reflex[i])
    {
      ear[i] = 0;
      return;
    }
    int sz = SZ(indices),
      pos = find(ALL(indices), i) - indices.begin();
    int iPrev = indices[(pos + sz - 1) % sz],
      iNext = indices[(pos + 1) % sz];
    ear[i] = 1;
    for (int j : indices)
    {
      if (j != iPrev && j != i && j != iNext && reflex[j]
        && inConvexPolygon({v[iPrev], v[i], v[iNext]}, v[j]))
      {
        ear[i] = 0;
        break;
      }
    }
  };
  FOR(i, 0, n)
    updReflexStatus(i);
  FOR(i, 0, n)
    updEarStatus(i);
  RFOR(sz, n + 1, 3)
  {
    int i = 0;
```

```
    while (!ear[indices[i]])
      i++;
    int iPrev = indices[(i + sz - 1) % sz], iNext = indices[(i
        + 1) % sz];
    res.PB({iPrev, indices[i], iNext});
    indices.erase(indices.begin() + i);
    updReflexStatus(iPrev);
    updReflexStatus(iNext);
    updEarStatus(iPrev);
    updEarStatus(iNext);
  }
  return res;
}
```

halfplane-intersection.hpp

Description: Returns the counter-clockwise ordered vertices of the half-plane intersection. Returns empty if the intersection is empty. Adds a bounding box to ensure a finite area.

5c6d01, 47 lines

```
vector<Pt> hplaneInter(vector<Line> lines)
{
  const db C = 1e9;
  lines.PB({{-C, C}, {-C, -C}});
  lines.PB({{-C, -C}, {C, -C}});
  lines.PB({{C, -C}, {C, C}});
  lines.PB({{C, C}, {-C, C}});
  sort(ALL(lines), [](const Line& l1, const Line& l2)
  {
    bool h1 = half(l1.n), h2 = half(l2.n);
    if (h1 != h2)
      return h1 < h2;
    int p = sgn(cross(l1.n, l2.n));
    if (p != 0)
      return p > 0;
    return sgn(l1.c / abs(l1.n) - l2.c / abs(l2.n)) < 0;
  });
  lines.erase(unique(ALL(lines), parallel), lines.end());
  deque<pair<Line, Pt>> d;
  for (const Line& l : lines)
  {
    while (SZ(d) > 1 && sgn(l.side((d.end() - 1)->S)) < 0)
      d.pop_back();
    while (SZ(d) > 1 && sgn(l.side((d.begin() + 1)->S)) < 0)
      d.pop_front();
    if (!d.empty() && sgn(cross(d.back().F.n, l.n)) <= 0)
      return {};
    if (SZ(d) < 2 || sgn(d.front().F.side(inter(l, d.back().F))
        ) >= 0)
    {
      Pt p;
      if (!d.empty())
      {
        p = inter(l, d.back().F);
        if (!parallel(l, d.front().F))
          d.front().S = inter(l, d.front().F);
      }
      d.PB({l, p});
    }
  }
  vector<Pt> res;
  for (auto [l, p] : d)
  {
    if (res.empty() || sgn(sq(p - res.back())) > 0)
      res.PB(p);
  }
  return res;
}
```

circle.hpp

ab2e8c, 42 lines

```
// Returns the circumcenter of triangle abc.
// The circumcircle of a triangle is a circle that passes
    through all three vertices.
Pt circumCenter(const Pt& a, Pt b, Pt c)
{
  b = b - a;
  c = c - a;
  assert(sgn(cross(b, c)) != 0);
  return a + perp(b * sq(c) - c * sq(b)) / cross(b, c) / 2;
}
// Returns circle-line intersection points
vector<Pt> circleLine(const Pt& o, db r, const Line& l)
{
  db h2 = r * r - l.sqDist(o);
  if (sgn(h2) == -1)
    return {};
  Pt p = l.proj(o);
  if (sgn(h2) == 0)
    return {p};
  Pt h = perp(l.n) * sqrt(h2) / abs(l.n);
  return {p - h, p + h};
}
// Returns circle-circle intersection points
vector<Pt> circleCircle(const Pt& o1, db r1, const Pt& o2, db
    r2)
{
  Pt d = o2 - o1;
  db d2 = sq(d);
  if (sgn(d2) == 0)
  {
    // assuming the circles don't coincide
    assert(sgn(r2 - r1) != 0);
    return {};
  }
  db pd = (d2 + r1 * r1 - r2 * r2) / 2;
  db h2 = r1 * r1 - pd * pd / d2;
  if (sgn(h2) == -1)
    return {};
  Pt p = o1 + d * pd / d2;
  if (sgn(h2) == 0)
    return {p};
  Pt h = perp(d) * sqrt(h2 / d2);
  return {p - h, p + h};
}
```

tangents.hpp

Description: Finds common tangents (outer or inner) to two circles. If there are two tangents, returns the pairs of tangency points on each circle (p_1, p_2). If there is one tangent, the circles are tangent to each other at some point p, res contains p four times, and the tangent line can be found as `line(o1, p).perpThrough(p)`. The same code can be used to find the tangent to a circle through a point by setting r_2 to 0 (in which case `inner` doesn't matter).

e50bdb, 20 lines

```
vector<pair<Pt, Pt>> tangents(const Pt& o1, db r1,
  const Pt& o2, db r2, bool inner)
{
  if (inner)
    r2 = -r2;
  Pt d = o2 - o1;
  db dr = r1 - r2, d2 = sq(d), h2 = d2 - dr * dr;
  if (sgn(d2) == 0 || sgn(h2) < 0)
  {
    assert(sgn(h2) != 0);
    return {};
  }
  vector<pair<Pt, Pt>> res;
  for (db sign : {-1, 1})
  {
    Pt v = (d * dr + perp(d) * sqrt(h2) * sign) / d2;
    res.PB({o1 + v * r1, o2 + v * r2});
  }
  return res;
}
```

welzl.hpp

Description: Returns the smallest enclosing circle of points in v

Time: $\mathcal{O}(n)$ (expected)

e33f59, 36 lines

```
pair<Pt, db> welzl(vector<Pt> v)
{
  int n = SZ(v), k = 0, idxes[2];
  mt19937 rng;
  shuffle(ALL(v), rng);
  Pt c = v[0];
  db r = 0;
  while (true)
  {
    FOR(i, k, n)
    {
      if (sgn(abs(v[i] - c) - r) > 0)
      {
        swap(v[i], v[k]);
        if (k == 0)
          c = v[0];
        else if (k == 1)
          c = (v[0] + v[1]) / 2;
        else
          c = circumCenter(v[0], v[1],  v[2]);
        r = abs(v[0] - c);
        if (k < i)
        {
          if (k < 2)
            idxes[k++] = i;
          shuffle(v.begin() + k, v.begin() + i + 1, rng);
          break;
        }
      }
      while (k > 0 && idxes[k - 1] == i)
        k--;
      if (i == n - 1)
        return {c, r};
    }
  }
}
```

closest-pair.hpp

Description: Returns the distance between the closest points

Time: $\mathcal{O}(n \log n)$

8696b6, 23 lines

```
db closestPair(vector<Pt> v)
{
  sort(ALL(v), [](const Pt& p, const Pt& q)
  {
    return sgn(p.x - q.x) < 0;
  });
  set<pair<db, db>> s;
  int n = SZ(v), ptr = 0;
  db h = 1e18;
  FOR(i, 0, n)
  {
    for (auto it = s.lower_bound(MP(v[i].y - h, v[i].x));
      it != s.end() && sgn(it->F - (v[i].y + h)) <= 0; it++)
    {
      Pt q = {it->S, it->F};
      h = min(h, abs(v[i] - q));
    }
    for (; sgn(v[ptr].x - (v[i].x - h)) <= 0; ptr++)
      s.erase({v[ptr].y, v[ptr].x});
    s.insert({v[i].y, v[i].x});
```

```
	}
	return h;
}
```

planar-graph.hpp

Description: Finds faces in a planar graph. Use `addVertex()` and `addEdge()` for initializing the graph and `addQueryPoint()` for initializing the queries. After initialization, call `findFaces()` before using other functions. `getIncidentFaces(i)` returns the pair of faces (u, v) (possibly $u = v$) such that the i-th edge lies on the boundary of these faces. `getFaceOfQueryPoint(i)` returns the face where the i-th query point lies.

629940, 169 lines

```
namespace PlanarGraph
{
struct IndexedPt
{
	Pt p;
	int index;
	bool operator<(const IndexedPt& q) const
	{
		return p.x < q.p.x;
	}
};
struct Edge
{
	// cross(vertices[j].p - vertices[i].p, l.n) > 0
	int i, j;
	Line l;
};
vector<IndexedPt> vertices, queryPoints;
vector<Edge> edges;
struct Comparator
{
	using is_transparent = void;
	static IndexedPt vertex;
	db getY(const Line& l) const
	{
		return -(l.n.x * vertex.p.x + l.c) / l.n.y;
	}
	bool operator()(int i, int j) const
	{
		auto [u1, v1, l1] = edges[i];
		auto [u2, v2, l2] = edges[j];
		if (u1 == vertex.index && u2 == vertex.index)
			return sgn(cross(l1.n, l2.n)) > 0;
		if (v1 == vertex.index && v2 == vertex.index)
			return sgn(cross(l1.n, l2.n)) < 0;
		int dy = sgn(getY(l1) - getY(l2));
		assert(dy != 0);
		return dy < 0;
	}
	bool operator()(int i, const Pt& p) const
	{
		int dy = sgn(getY(edges[i].l) - p.y);
		assert(dy != 0);
		return dy < 0;
	}
} comparator;
IndexedPt Comparator::vertex;
DSU dsu;
VI upperFace, queryAns;

void addVertex(const Pt& p)
{
	vertices.PB({p, SZ(vertices)});
}
void addEdge(int i, int j, const Line& l)
{
	assert(0 <= i && i < SZ(vertices));
	assert(0 <= j && j < SZ(vertices));
	assert(i != j);
	assert(vertices[i].index == i);
	assert(vertices[j].index == j);
	edges.PB({i, j, l});
}
void addEdge(int i, int j)
{
	addEdge(i, j, {vertices[i].p, vertices[j].p});
}
void addQueryPoint(const Pt& p)
{
	queryPoints.PB({p, SZ(queryPoints)});
}
void findFaces()
{
	int n = SZ(vertices), m = SZ(edges);
	const db ROT_ANGLE = 4;
	for (auto& p : vertices)
		p.p = rot(p.p, ROT_ANGLE);
	for (auto& p : queryPoints)
		p.p = rot(p.p, ROT_ANGLE);
	vector<VI> edgesL(n), edgesR(n);
	FOR(k, 0, m)
	{
		auto& [i, j, l] = edges[k];
		l.n = rot(l.n, ROT_ANGLE);
		if (vertices[i].p.x > vertices[j].p.x)
		{
			swap(i, j);
			l.n = l.n * (-1);
			l.c *= -1;
		}
		edgesL[j].PB(k);
		edgesR[i].PB(k);
	}
	sort(ALL(vertices));
	sort(ALL(queryPoints));
	// when choosing INF, remember that we rotate the plane
	addVertex({-INF, INF});
	addVertex({INF, INF});
	addEdge(n, n + 1);
	dsu.init(m + 1);
	set<int, Comparator> s;
	s.insert(m);
	upperFace.resize(m);
	int ptr = 0;
	queryAns.resize(SZ(queryPoints));
	for (const IndexedPt& vertex : vertices)
	{
		int i = vertex.index;
		while (ptr < SZ(queryPoints)
			&& (i >= n || queryPoints[ptr] < vertex))
		{
			const auto& [pt, j] = queryPoints[ptr++];
			Comparator::vertex = {pt, -1};
			queryAns[j] = *s.lower_bound(pt);
		}
		if (i >= n)
			break;
		Comparator::vertex = vertex;
		int upper = -1, lower = -1;
		if (!edgesL[i].empty())
		{
			sort(ALL(edgesL[i]), comparator);
			auto it = s.lower_bound(edgesL[i][0]);
			lower = edgesL[i][0];
			for (int e : edgesL[i])
			{
				assert(*it == e);
				assert(next(it) != s.end());
				upperFace[e] = *next(it);
				it = s.erase(it);
			}
			assert(it != s.end());
			upper = *it;
		}
		if (!edgesR[i].empty())
		{
			sort(ALL(edgesR[i]), comparator);
			if (upper == -1)
			{
				upper = *s.lower_bound(edgesR[i][0]);
			}
			int prv = -1;
			for (int e : edgesR[i])
			{
				s.insert(e);
				if (prv != -1)
				{
					upperFace[prv] = e;
				}
				prv = e;
			}
			upperFace[edgesR[i].back()] = upper;
			dsu.unite(edgesL[i].empty() ? upper : lower, edgesR[i
				][0]);
		}
		else if (lower != -1 && upper != -1)
		{
			dsu.unite(upper, lower);
		}
	}
}
PII getIncidentFaces(int i)
{
	return {dsu.find(i), dsu.find(upperFace[i])};
}
int getFaceOfQueryPoint(int i)
{
	return dsu.find(queryAns[i]);
}
};
```

Mathematics (6)

Number-theoretic algorithms

modular-arithmetics.hpp

03ba2e, 67 lines

```
const int mod = 998244353;

int add(int a, int b)
{
	return a + b < mod ? a + b : a + b - mod;
}

void updAdd(int& a, int b)
{
	a += b;
	if (a >= mod)
		a -= mod;
}

int sub(int a, int b)
{
	return a - b >= 0 ? a - b : a - b + mod;
}
```

```
void updSub(int& a, int b)
{
  a -= b;
  if (a < 0)
    a += mod;
}

int mult(int a, int b)
{
  return (LL)a * b % mod;
}

int binpow(int a, LL n)
{
  int res = 1;
  while (n)
  {
    if (n & 1)
      res = mult(res, a);
    a = mult(a, a);
    n /= 2;
  }
  return res;
}

int inv[N], fact[N], ifact[N];

void init()
{
  inv[1] = 1;
  FOR(i, 2, N)
  {
    inv[i] = mult(mod - mod / i, inv[mod % i]);
  }
  fact[0] = ifact[0] = 1;
  FOR(i, 1, N)
  {
    fact[i] = mult(fact[i - 1], i);
    ifact[i] = mult(ifact[i - 1], inv[i]);
  }
}

int C(int n, int k)
{
  if (k < 0 || k > n)
    return 0;
  return mult(fact[n], mult(ifact[n - k], ifact[k]));
}
```

gcd.hpp

Description: $ax + by = d$, $\gcd(a, b) = |d| \to (d, x, y)$.
Minimizes $|x| + |y|$. And minimizes $|x - y|$ for $a > 0$, $b > 0$.

261b5c, 16 lines

```
tuple<LL, LL, LL> gcdExt(LL a, LL b)
{
  LL x1 = 1, y1 = 0;
  LL x2 = 0, y2 = 1;
  while (b)
  {
    LL k = a / b;
    x1 -= k * x2;
    y1 -= k * y2;
    a %= b;
    swap(a, b);
    swap(x1, x2);
    swap(y1, y2);
  }
  return {a, x1, y1};
}
```

fast-chinese.hpp

Description: $x\%p_i = m_i$, $\text{lcm}(p_i) \le 10^{18}$, $0 \le x < \text{lcm}(p_i) \to x$ or -1.
Time: $\mathcal{O}(n \log(\text{lcm}(p_i)))$

3c13b2, 24 lines

```
LL fastChinese(vector<LL> m, vector<LL> p)
{
  assert(SZ(m) == SZ(p));
  LL aa = p[0];
  LL bb = m[0];
  FOR(i, 1, SZ(m))
  {
    LL b = (m[i] - bb % p[i] + p[i]) % p[i];
    LL a = aa % p[i];
    LL c = p[i];

    auto [d, x, y] = gcdExt(a, c);
    if(b % d != 0)
      return -1;
    a /= d;
    b /= d;
    c /= d;
    b = (b * (__int128)x % c + c) % c;

    bb = aa * b + bb;
    aa = aa * c;
  }
  return bb;
}
```

miller-rabin.hpp

Description: To speed up change candidates to at least 4 random values `rng() % (n - 3) + 2`. Use `__int128` in `mult`.
Time: $\mathcal{O}(|\text{candidates}| \cdot \log n)$

394bc8, 33 lines

```
VI candidates = {2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 47};
bool millerRabin(LL n)
{
  if (n == 1)
    return false;
  if (n == 2 || n == 3)
    return true;
  LL d = n - 1;
  int s = __builtin_ctzll(d);
  d >>= s;

  for (LL b : candidates)
  {
    if (b >= n)
      break;
    b = binpow(b, d, n);
    if (b == 1)
      continue;
    bool ok = false;
    FOR (i, 0, s)
    {
      if (b + 1 == n)
      {
        ok = true;
        break;
      }
      b = mult(b, b, n);
    }
    if (!ok)
      return false;
  }
  return true;
}
```

pollard.hpp

Description: Uses the Miller-Rabin test. `rho` finds a divisor of n. Use `__int128` in `mult`.
Time: $\mathcal{O}\left(n^{1/4} \cdot \log n\right)$.

95237d, 62 lines

```
LL f(LL x, LL c, LL n)
{
  return add(mult(x, x, n), c, n);
}

LL rho(LL n)
{
  const int iter = 47 * pow(n, 0.25);
  while (true)
  {
    LL x0 = rng() % n;
    LL c = rng() % n;
    LL x = x0;
    LL y = x0;
    LL g = 1;
    FOR (i, 0, iter)
    {
      x = f(x, c, n);
      y = f(y, c, n);
      y = f(y, c, n);
      g = gcd(abs(x - y), n);
      if (g != 1)
        break;
    }
    if (g > 1 && g < n)
      return g;
  }
}
VI primes = {2, 3, 5, 7, 11, 13, 17, 19, 23};

VL factorize(LL n)
{
  VL ans;

  for (auto p : primes)
  {
    while (n % p == 0)
    {
      ans.PB(p);
      n /= p;
    }
  }
  queue<LL> q;
  q.push(n);

  while (!q.empty())
  {
    LL x = q.front();
    q.pop();
    if (x == 1)
      continue;
    if (millerRabin(x))
      ans.PB(x);
    else
    {
      LL y = rho(x);
      q.push(y);
      q.push(x / y);
    }
  }
  return ans;
}
```

floor-sum.hpp

Description: Computes $\sum_{i=0}^{n-1} \left\lfloor \frac{a \cdot i + b}{m} \right\rfloor$.

Time: $\mathcal{O}(\log m)$.

7e1c68, 16 lines

```
LL floorSum(LL n, LL m, LL a, LL b)
{
  LL ans = 0;
  while (true)
  {
    ans += (a / m) * n * (n - 1) / 2 + (b / m) * n;
    a %= m;
    b %= m;
    if (a == 0)
      return ans;
    LL k = (a * (n - 1) + b) / m;
    b = a * n - m * k + b;
    n = k;
    swap(a, m);
  }
}
```

min-mod-linear.hpp

Description: Finds $\min\{(ax+b) \mod m \mid 0 \le x < n\}$.

Time: $\mathcal{O}(\log m)$.

874428, 14 lines

```
int minModLinear(LL n, LL m, LL a, LL b)
{
  LL res = m;
  while (n > 0)
  {
    a %= m;
    b = (b % m + m) % m;
    res = min(res, b);
    n = (a * (n - 1) + b) / m;
    b -= m * n;
    swap(a, m);
  }
  return res;
}
```

mod-inequality.hpp

Description: Finds the smallest $x \ge 0$ such that $(ax+b) \mod m \ge c$. Returns -1, if the solution does not exist.

Time: $\mathcal{O}(\log m)$.

e842a1, 15 lines

```
int modInequality(LL m, LL a, LL b, LL c)
{
  a %= m;
  b %= m;
  if (b >= c)
    return 0;
  if (a == 0)
    return -1;
  if (c + a < m)
    return (c - b + a - 1) / a;
  int k = modInequality(a, m, c - b - 1, c + a - m);
  if (k == -1)
    return -1;
  return (k * m + c - b + a - 1) / a;
}
```

Matrices

gaussian.hpp

Description: Solves the system $Ax = b$. Returns (v, w) such that every solution x can be represented as $v + c_1 w_1 + c_2 w_2 + \cdots + c_k w_k$, where v is arbitrary solution, c_i are scalars and w is basis. If there is no solution, returns an empty pair. If the solution is unique, then w is empty.

Time: $\mathcal{O}(nm \min(n, m))$

54e83c, 66 lines

```
pair<VI, vector<VI>> solveLinearSystem(vector<VI> a, VI b)
{
  int n = SZ(a), m = SZ(a[0]);
  assert(SZ(b) == n);
  FOR(i, 0, n)
  {
    assert(SZ(a[i]) == m);
    a[i].PB(b[i]);
  }
  int p = 0;
  VI pivots;
  FOR(j, 0, m)
  {
    // with doubles, abs(a[p][j]) -> max
    if (a[p][j] == 0)
    {
      int l = -1;
      FOR(i, p, n)
        if (a[i][j] != 0)
          l = i;
      if (l == -1)
        continue;
      swap(a[p], a[l]);
    }
    int inv = binpow(a[p][j], mod - 2);
    FOR(i, p + 1, n)
    {
      int c = mult(a[i][j], inv);
      FOR(k, j, m + 1)
        updSub(a[i][k], mult(c, a[p][k]));
    }
    pivots.PB(j);
    p++;
    if (p == n)
      break;
  }
  FOR(i, p, n)
    if (a[i].back() != 0)
      return {};
  VI v(m);
  RFOR(i, p, 0)
  {
    int j = pivots[i];
    v[j] = a[i].back();
    FOR(k, j + 1, m)
      updSub(v[j], mult(a[i][k], v[k]));
    v[j] = mult(v[j], binpow(a[i][j], mod - 2));
  }
  vector<VI> w;
  FOR(q, 0, m)
  {
    if (find(ALL(pivots), q) != pivots.end())
      continue;
    VI d(m);
    d[q] = 1;
    RFOR(i, p, 0)
    {
      int j = pivots[i];
      FOR(k, j + 1, m)
        updSub(d[j], mult(a[i][k], d[k]));
      d[j] = mult(d[j], binpow(a[i][j], mod - 2));
    }
    w.PB(d);
  }
  return {v, w};
}
```

hungarian.hpp

Description: Finds a maximum matching that has the minimum weight in a weighted bipartite graph.

Time: $\mathcal{O}\left(n^2 m\right)$

d0e6fa, 63 lines

```
LL hungarian(const vector<VL>& a)
{
  int n = SZ(a), m = SZ(a[0]);
  assert(n <= m);
  VL u(n + 1), v(m + 1);
  VI p(m + 1, n), way(m + 1);
  FOR(i, 0, n)
  {
    p[m] = i;
    int j0 = m;
    VL minv(m + 1, LINF);
    VI used(m + 1);
    while (p[j0] != n)
    {
      used[j0] = true;
      int i0 = p[j0], j1 = -1;
      LL delta = LINF;
      FOR(j, 0, m)
      {
        if (!used[j])
        {
          LL cur = a[i0][j] - u[i0] - v[j];
          if (cur < minv[j])
          {
            minv[j] = cur;
            way[j] = j0;
          }
          if (minv[j] < delta)
          {
            delta = minv[j];
            j1 = j;
          }
        }
      }
      assert(j1 != -1);
      FOR(j, 0, m + 1)
      {
        if (used[j])
        {
          u[p[j]] += delta;
          v[j] -= delta;
        }
        else
          minv[j] -= delta;
      }
      j0 = j1;
    }
    while (j0 != m)
    {
      int j1 = way[j0];
      p[j0] = p[j1];
      j0 = j1;
    }
  }
  VI ans(n + 1);
  FOR(j, 0, m)
    ans[p[j]] = j;
  LL res = 0;
  FOR(i, 0, n)
    res += a[i][ans[i]];
  assert(res == -v[m]);
  return res;
}
```

simplex.hpp

Description: $c^T x \to \max$, $Ax \le b$, $x \ge 0$.

03c648, 142 lines

```
typedef vector<db> VD;
```

```
struct Simplex
{
  void pivot(int l, int e)
  {
    assert(0 <= l && l < m);
    assert(0 <= e && e < n);
    assert(abs(a[l][e]) > EPS);
    b[l] /= a[l][e];
    FOR(j, 0, n)
      if (j != e)
        a[l][j] /= a[l][e];
    a[l][e] = 1 / a[l][e];
    FOR(i, 0, m)
    {
      if (i != l)
      {
        b[i] -= a[i][e] * b[l];
        FOR(j, 0, n)
          if (j != e)
            a[i][j] -= a[i][e] * a[l][j];
        a[i][e] *= -a[l][e];
      }
    }
    v += c[e] * b[l];
    FOR(j, 0, n)
      if (j != e)
        c[j] -= c[e] * a[l][j];
    c[e] *= -a[l][e];
    swap(nonBasic[e], basic[l]);
  }
  void findOptimal()
  {
    VD delta(m);
    while (true)
    {
      int e = -1;
      FOR(j, 0, n)
        if (c[j] > EPS && (e == -1 || nonBasic[j] < nonBasic[e
            ]))
          e = j;
      if (e == -1)
        break;
      FOR(i, 0, m)
        delta[i] = a[i][e] > EPS ? b[i] / a[i][e] : LINF;
      int l = min_element(ALL(delta)) - delta.begin();
      if (delta[l] == LINF)
      {
        // unbounded
        assert(false);
      }
      pivot(l, e);
    }
  }
  void initializeSimplex(const vector<VD>& _a, const VD& _b,
      const VD& _c)
  {
    m = SZ(_b);
    n = SZ(_c);
    nonBasic.resize(n);
    iota(ALL(nonBasic), 0);
    basic.resize(m);
    iota(ALL(basic), n);
    a = _a;
    b = _b;
    c = _c;
    v = 0;
    int k = min_element(ALL(b)) - b.begin();
    if (b[k] > -EPS)
      return;
    nonBasic.PB(n);
    iota(ALL(basic), n + 1);
    FOR(i, 0, m)
      a[i].PB(-1);
    c.assign(n, 0);
    c.PB(-1);
    n++;
    pivot(k, n - 1);
    findOptimal();
    if (v < -EPS)
    {
      // infeasible
      assert(false);
    }
    int l = find(ALL(basic), n - 1) - basic.begin();
    if (l != m)
    {
      int e = -1;
      while (abs(a[l][e]) < EPS)
        e++;
      pivot(l, e);
    }
    n--;
    int p = find(ALL(nonBasic), n) - nonBasic.begin();
    assert(p < n + 1);
    nonBasic.erase(nonBasic.begin() + p);
    FOR(i, 0, m)
      a[i].erase(a[i].begin() + p);
    c.assign(n, 0);
    FOR(j, 0, n)
    {
      if (nonBasic[j] < n)
        c[j] = _c[nonBasic[j]];
      else
        nonBasic[j]--;
    }
    FOR(i, 0, m)
    {
      if (basic[i] < n)
      {
        v += _c[basic[i]] * b[i];
        FOR(j, 0, n)
          c[j] -= _c[basic[i]] * a[i][j];
      }
      else
        basic[i]--;
    }
  }
  pair<VD, db> simplex(const vector<VD>& _a, const VD& _b,
      const VD& _c)
  {
    initializeSimplex(_a, _b, _c);
    assert(SZ(a) == m);
    FOR(i, 0, m)
      assert(SZ(a[i]) == n);
    assert(SZ(b) == m);
    assert(SZ(c) == n);
    assert(SZ(nonBasic) == n);
    assert(SZ(basic) == m);
    findOptimal();
    VD x(n);
    FOR(i, 0, m)
      if (basic[i] < n)
        x[basic[i]] = b[i];
    return {x, v};
  }
private:
  int m, n;
  VI nonBasic, basic;
  vector<VD> a;
  VD b;
  VD c;
  db v;
};
```

Convolutions

conv-xor.hpp

Description: $c_k = \sum_{i \oplus j = k} a_i b_j$.

b80d13, 24 lines

```
void convXor(VI& a, int k)
{
  FOR(i, 0, k)
    FOR(j, 0, 1 << k)
      if((j & (1 << i)) == 0)
      {
        int u = a[j];
        int v = a[j + (1 << i)];
        a[j] = add(u, v);
        a[j + (1 << i)] = sub(u, v);
      }
}
VI multXor(VI a, VI b, int k)
{
  convXor(a, k);
  convXor(b, k);
  FOR(i, 0, 1 << k)
    a[i] = mult(a[i], b[i]);
  convXor(a, k);
  int d = inv(1 << k);
  FOR(i, 0, 1 << k)
    a[i] = mult(a[i], d);
  return a;
}
```

conv-or.hpp

Description: $c_k = \sum_{i \text{ OR } j = k} a_i b_j$.

e4e659, 21 lines

```
void convOr(VI& a, int k, bool inverse)
{
  FOR(i, 0, k)
    FOR(j, 0, 1 << k)
      if((j & (1 << i)) == 0)
      {
        if(inverse)
          updSub(a[j + (1 << i)], a[j]);
        else
          updAdd(a[j + (1 << i)], a[j]);
      }
}
VI multOr(VI a, VI b, int k)
{
  convOr(a, k, false);
  convOr(b, k, false);
  FOR(i, 0, 1 << k)
    a[i] = mult(a[i], b[i]);
  convOr(a, k, true);
  return a;
}
```

subset-convolution.hpp

Description: $c[S] = \sum_{T \subseteq S} a[T] \cdot b[S \setminus T]$.

Time: $\mathcal{O}\left(n^2 \cdot 2^n\right)$, 1.5s for $n = 20$.

5f8849, 27 lines

```
vector<VI> rankedMobius(VI a, int n)
{
  vector<VI> res(n + 1, VI(1 << n));
  FOR(mask, 0, 1 << n)
    res[__builtin_popcount(mask)][mask] = a[mask];
  FOR(sz, 0, n + 1)
```

```
    convOr(res[sz], n, false);
  return res;
}
VI subsetConvolution(VI a, VI b, int n)
{
  auto f = rankedMobius(a, n);
  auto g = rankedMobius(b, n);

  vector<VI> conv(n + 1, VI(1 << n));
  FOR(sz, 0, n + 1)
  {
    FOR(i, 0, sz + 1)
      FOR(mask, 0, 1 << n)
        updAdd(conv[sz][mask], mult(f[i][mask], g[sz - i][mask
             ]));
    convOr(conv[sz], n, true);
  }
  VI res(1 << n);
  FOR(mask, 0, 1 << n)
    res[mask] = conv[__builtin_popcount(mask)][mask];
  return res;
}
```

Polynomials and FFT

fft.hpp

Description: Number-theoretic transform. If you need complex-valued FFT, use the commented out code.
Time: $\mathcal{O}(n \log n)$

b018b9, 73 lines

```
const int LEN = 1 << 23;
const int GEN = 31;

/*typedef complex<db> com;
com pw[LEN];
void init()
{
  db phi = (db)2 * PI / LEN;
  FOR(i, 0, LEN)
    pw[i] = com(cos(phi * i), sin(phi * i));
}*/

void fft(VI& a, bool inverse)
{
  const int IGEN = binpow(GEN, mod - 2);
  int lg = __builtin_ctz(SZ(a));
  FOR(i, 0, SZ(a))
  {
    int k = 0;
    FOR(j, 0, lg)
      k |= ((i >> j) & 1) << (lg - j - 1);
    if(i < k)
      swap(a[i], a[k]);
  }
  for(int len = 2; len <= SZ(a); len *= 2)
  {
    // int diff = inv ? LEN - LEN / len : LEN / len;
    int ml = binpow(inverse ? IGEN : GEN, LEN / len);
    for(int i = 0; i < SZ(a); i += len)
    {
      // int pos = 0;
      int pw = 1;
      FOR(j, 0, len / 2)
      {
        int u = a[i + j];
        int v = mult(a[i + j + len / 2], pw); // * pw[pos]
        a[i + j] = add(u, v);
        a[i + j + len / 2] = sub(u, v);
        // pos = (pos + diff) % LEN;
        pw = mult(pw, ml);
      }
    }
  }
  if (inverse)
  {
    int m = binpow(SZ(a), mod - 2);
    FOR(i, 0, SZ(a))
      // a[i] /= SZ(a);
      a[i] = mult(a[i], m);
  }
}

VI mult(VI a, VI b)
{
  int n = SZ(a), m = SZ(b);
  if (n == 0 || m == 0)
    return {};
  int sz = 1, szRes = n + m - 1;
  while(sz < szRes)
    sz *= 2;
  a.resize(sz);
  b.resize(sz);

  fft(a, false);
  fft(b, false);

  FOR(i, 0, sz)
    a[i] = mult(a[i], b[i]);

  fft(a, true);
  a.resize(szRes);
  return a;
}
```

mult-arbitrary-mod.hpp

Description: Multiplies polynomials modulo arbitrary `mod` (or without modulo). Add the modulo parameter to the modular arithmetics functions (`int add(int a, int b, int m = mod)`). `LEN` must be 2^{24}. Change signature of the fft function into `void fft(VI& a, bool inverse, int nttMod, int GEN)`. `GEN` will not be a constant anymore. You must add `nttMod` inside the `fft` function 10 times in 8 lines of code. Change signature of the original `mult` function into `VI mult(VI a, VI b, int nttMod, int GEN)`. You must add `nttMod` inside the original `mult` function 4 times in 4 lines of code.

de7a5b, 32 lines

```
VI mult(const VI& a, const VI& b)
{
  int n = SZ(a), m = SZ(b);
  if (n == 0 || m == 0)
    return {};
  const int mods[3] = {754974721, 167772161, 469762049};
  const int invs[3] = {190329765, 58587104, 187290749};
  const int gens[3] = {362, 2, 40};
  vector<VI> fa(3, VI(n)), fb(3, VI(m));
  vector<VI> c(3);
  FOR(i, 0, 3)
  {
    FOR(j, 0, n)
      fa[i][j] = a[j] % mods[i];
    FOR(j, 0, m)
      fb[i][j] = b[j] % mods[i];
    c[i] = mult(fa[i], fb[i], mods[i], gens[i]);
  }
  __int128 modsProd = (__int128)mods[0] * mods[1] * mods[2];
  VI res(n + m - 1);
  FOR(i, 0, n + m - 1)
  {
    __int128 cur = 0;
    FOR(j, 0, 3)
    {
      cur += (__int128)mods[(j + 1) % 3] * mods[(j + 2) % 3]
        * mult(invs[j], c[j][i], mods[j]);
    }
    res[i] = cur % modsProd % mod;
  }
  return res;
}
```

inverse.hpp

Description: $\frac{1}{A(x)}$ modulo x^n.
Time: $\mathcal{O}(n \log n)$

ccfd8a, 32 lines

```
VI inverse(const VI& a, int n)
{
  assert(SZ(a) == n && a[0] != 0);
  if(n == 1)
    return {binpow(a[0], mod - 2)};

  VI ra = a;
  FOR(i, 0, SZ(ra))
    if(i & 1)
      ra[i] = sub(0, ra[i]);

  int nn = (n + 1) / 2;
  VI t = mult(a, ra);
  t.resize(n);

  FOR(i, 0, nn)
    t[i] = t[2 * i];

  t.resize(nn);
  t = inverse(t, nn);
  t.resize(n);

  RFOR(i, nn, 1)
  {
    t[2 * i] = t[i];
    t[i] = 0;
  }

  VI res = mult(ra, t);
  res.resize(n);
  return res;
}
```

log.hpp

Description: $\log(A(x))$ modulo x^n.
Time: $\mathcal{O}(n \log n)$

93688a, 26 lines

```
VI deriv(const VI& a)
{
  int n = SZ(a);
  VI res(max(0, n - 1));
  FOR(i, 0, n - 1)
    res[i] = mult(a[i + 1], i + 1);
  return res;
}

VI integr(const VI& a)
{
  int n = SZ(a);
  VI res(n + 1);
  RFOR(i, n, 1)
    res[i] = mult(a[i - 1], inv[i]);
  res[0] = 0;
  return res;
}

VI log(const VI& a, int n)
{
  assert(SZ(a) == n && a[0] == 1);
```

```
  VI res = integr(mult(deriv(a), inverse(a, n)));
  res.resize(n);
  return res;
}
```

exp.hpp

Description: $\exp(A(x))$ modulo x^n.
Time: $\mathcal{O}(n \log n)$ 7dff42, 21 lines

```
VI exp(const VI& a, int n)
{
  assert(SZ(a) == n && a[0] == 0);
  VI q = {1};
  for (int k = 2; k <= 2 * n; k *= 2)
  {
    q.resize(k);
    VI lnQ = log(q, k);
    FOR(i, 0, k)
    {
      if(i < n)
        lnQ[i] = sub(a[i], lnQ[i]);
      else
        lnQ[i] = sub(0, lnQ[i]);
    }
    lnQ[0] = add(lnQ[0], 1);
    q = mult(q, lnQ);
  }
  q.resize(n);
  return q;
}
```

divide.hpp

Description: Finds $Q(x)$ and $R(x)$ such that $A(x) = Q(x)B(x) + R(x)$ and $\deg R < \deg B$.
Time: $\mathcal{O}(n \log n)$ 2471a2, 28 lines

```
void removeLeadingZeros(VI& a)
{
  while(SZ(a) > 0 && a.back() == 0)
    a.pop_back();
}
pair<VI, VI> divide(VI a, VI b)
{
  removeLeadingZeros(a);
  removeLeadingZeros(b);
  int n = SZ(a), m = SZ(b);
  assert(m > 0);
  if(m > n)
    return {{}, a};
  reverse(ALL(a));
  reverse(ALL(b));
  VI q = b;
  q.resize(n - m + 1);
  q = mult(a, inverse(q, n - m + 1));
  q.resize(n - m + 1);
  reverse(ALL(a));
  reverse(ALL(b));
  reverse(ALL(q));
  VI r = mult(b, q);
  FOR(i, 0, n)
    r[i] = sub(a[i], r[i]);
  removeLeadingZeros(r);
  return {q, r};
}
```

multipoint-eval.hpp

Description: Evaluates the polynomial $P(x)$ of degree m at points $x_0, \ldots, x_{n-1}$.
Time: $\mathcal{O}\left(n \log^2 n + m \log m\right)$ 62eab9, 44 lines

```
VI multipointEval(const VI& p, const VI& x)
{
  int n = SZ(x);
  vector<VI> t;
  int _n = 1;
  while (_n < 2 * n)
    _n *= 2;
  t.resize(_n);

  function<void(int, int, int)> build = [&](int v, int tl, int
      tr)
  {
    if(tl + 1 == tr)
    {
      t[v] = {sub(0, x[tl]), 1};
      return;
    }
    int tm = (tl + tr) / 2;
    build(2 * v + 1, tl, tm);
    build(2 * v + 2, tm, tr);
    t[v] = mult(t[2 * v + 1], t[2 * v + 2]);
  };

  build(0, 0, n);
  VI ans(n);

  function<void(int, int, int, VI)> solve
    = [&](int v, int tl, int tr, VI q)
  {
    q = divide(q, t[v]).S;
    if (q.empty())
      return;
    if(tl + 1 == tr)
    {
      ans[tl] = q[0];
      return;
    }
    int tm = (tl + tr) / 2;
    solve(2 * v + 1, tl, tm, q);
    solve(2 * v + 2, tm, tr, q);
  };

  solve(0, 0, n, p);
  return ans;
}
```

shift-eval-values.hpp

Description: Let $P(x)$ be the polynomial of degree at most $n-1$. Given $P(0), P(1), \ldots, P(n-1)$. Computes $P(c), P(c+1), \ldots, P(c+m-1)$.
Time: $\mathcal{O}((n+m)\log(n+m))$ 5471a6, 35 lines

```
VI shiftEvalValues(VI a, int c, int m)
{
  int n = SZ(a);
  VI q(n);
  FOR(i, 0, n)
  {
    q[i] = mult(a[i], mult(ifact[i], ifact[n - i - 1]));
    if ((n - i) % 2 == 0)
      q[i] = sub(0, q[i]);
  }
  VI s(n + m - 1);
  FOR(i, 0, SZ(s))
    s[i] = binpow(sub(add(c, i), n - 1), mod - 2);
  VI res = mult(q, s);
  res = {res.begin() + n - 1, res.begin() + n + m - 1};
  int prod = 1;
  FOR(i, 0, n)
  {
    int cur = sub(c, i);
    if (cur != 0)
      prod = mult(prod, cur);
  }
  FOR(i, 0, m)
  {
    int j = add(c, i);
    res[i] = j < n ? a[j] : mult(res[i], prod);
    int r = add(c, i + 1);
    if (r != 0)
      prod = mult(prod, r);
    int l = sub(add(c, i), n - 1);
    if (l != 0)
      prod = mult(prod, binpow(l, mod - 2));
  }
  return res;
}
```

Newton's method

Usable to find the solution of equation $F(Q) = 0$.

For example $F(Q) = x \cdot Q^2 + A - Q = 0$.

Newton's method approximates the solution of the equation using the formula:

$$Q_{k+1} = Q_k - \frac{F(Q_k)}{F'(Q_k)}, \text{where } F' = \frac{dF}{dQ}$$

Example of the derivative: $F'(Q) = 2 \cdot x \cdot Q - 1$.

Keep in mind that $|Q_k| = 2^k$.

FFT tricks

Two-dimensional FFT

The complexity is $O(nm(\log n + \log m))$. The main problem is to resize the matrix. You must add non-empty vectors.

Divide-and-conquer FFT

Suppose we have the following DP relation:
$f(t) = g(t) - \sum_{0 \le u < t} f(u)h(t-u)$, where $g(t)$ and $h(t)$ are known and we want to compute $f(t)$. We can apply divide-and-conquer FFT.

Let $m = \lfloor \frac{l+r}{2} \rfloor$. We guarantee the following invariant conditions.

By the time we compute the values for the segment $[l, r)$, the following conditions are already met:

- The values for $[0, l)$ on the DP is already determined.
- The sum of contributions from $[0, l)$ through $[l, r)$ is already applied to the DP in $[l, r)$.

When calculate the values for the segment $[l, r)$ do:

- Calculate the values for the segment $[l, m)$ recursively.
- Calculate the contributions from $[l, m)$ to $[m, r)$.
- Calculate the values for the segment $[m, r)$ recursively.

Properties of the discrete Fourier transform

$$DFT(x)_k = \sum_{n=0}^{N-1} x_n \cdot e^{-i2\pi \frac{k}{N} n}$$

Let $x_n^R = x_{N-n \mod N}$.

$DFT(x^R) = \overline{DFT(x)}$.

For real x, $DFT(x)^R = \overline{DFT(x)}$.

Interpolation

When $x_0, x_1, \ldots, x_d$ and $y_0, y_1, \ldots, y_d$ are given (where x_i are pairwise distinct), a polynomial $f(x)$ of degree no more than d such that $f(x_i) = y_i (i = 0, \ldots, d)$ is uniquely determined.

Lagrange polynomial

Lagrange basis polynomial: $L_i(x) = \prod_{j \neq i} \frac{x - x_j}{x_i - x_j}$.

$f(x) = y_0 L_0(x) + y_1 L_1(x) + \cdots + y_d L_d(x)$.

Newton polynomial

Divided differences:

$[y_i] = y_i$

$[y_i, y_{i+1}] = \frac{y_{i+1} - y_i}{x_{i+1} - x_i}$

$[y_i, \ldots, y_j] = \frac{[y_{i+1}, \ldots, y_j] - [y_i, \ldots, y_{j-1}]}{x_j - x_i}$

Newton basis polynomial: $N_i(x) = \prod_{j=0}^{i-1}(x - x_j)$.

$f(x) = [y_0]N_0(x) + \cdots + [y_0, y_1, \ldots, y_d]N_d(x)$.

Linear recurrence

berlekamp-massey.hpp

Description: Finds a sequence of d integers $c_1, \ldots, c_d$ of the minimum length d such that $a_i = \sum_{j=1}^{d} c_j a_{i-j}$.

0f35d6, 36 lines

```
VI berlekampMassey(const VI& a)
{
  VI c = {1}, bp = {1};
  int l = 0, b = 1, x = 1;
  FOR(j, 0, SZ(a))
  {
    assert(SZ(c) == l + 1);
    int d = a[j];
    FOR(i, 1, l + 1)
      updAdd(d, mult(c[i], a[j - i]));
    if (d == 0)
    {
      x++;
      continue;
    }
    VI t = c;
    int coef = mult(d, binpow(b, mod - 2));
    if (SZ(bp) + x > SZ(c))
      c.resize(SZ(bp) + x);
    FOR(i, 0, SZ(bp))
      updSub(c[i + x], mult(coef, bp[i]));
    if (2 * l > j)
    {
      x++;
      continue;
    }
    l = j + 1 - l;
    bp = t;
    b = d;
    x = 1;
  }
  c.erase(c.begin());
  for (int& ci : c)
    ci = mult(ci, mod - 1);
  return c;
}
```

bostan-mori.hpp

Description: Computes the n-th term of a given linearly recurrent sequence $a_i = \sum_{j=1}^{d} c_j a_{i-j}$. The first d terms $a_0, a_1, \ldots, a_{d-1}$ are given.
The problem reduces to determining $[x^n]P(x)/Q(x)$.
$\frac{P(x)}{Q(x)} = \frac{P(x)Q(-x)}{Q(x)Q(-x)} = \frac{U_e(x^2)}{V(x^2)} + x \cdot \frac{U_o(x^2)}{V(x^2)}$.
Time: $\mathcal{O}(d \log d \log n)$.

7b5eea, 25 lines

```
int bostanMori(const VI& c, VI a, LL n)
{
  int k = SZ(c);
  assert(SZ(a) == k);
  VI q(k + 1);
  q[0] = 1;
  FOR(i, 0, k)
    q[i + 1] = sub(0, c[i]);
  VI p = mult(a, q);
  p.resize(k);
  while (n)
  {
    VI qMinus = q;
    for (int i = 1; i <= k; i += 2)
      qMinus[i] = sub(0, qMinus[i]);
    VI newP = mult(p, qMinus);
    VI newQ = mult(q, qMinus);
    FOR(i, 0, k)
      p[i] = newP[2 * i + (n & 1)];
    FOR(i, 0, k + 1)
      q[i] = newQ[2 * i];
    n >>= 1;
  }
  return mult(p[0], binpow(q[0], mod - 2));
}
```

P-recursive sequences

find-coefs-of-p-recursive.hpp

Description: Finds the polynomials P_j such that $\sum_{j=0}^{d} P_j(i) \cdot a_{i+d-j} = 0$. Returns an empty vector if unable to find such polynomials. The first k terms $a_0, a_1, \ldots, a_{k-1}$ are given.
Time: $\mathcal{O}\left(k^3\right)$

3c4580, 30 lines

```
vector<VI> findCoefsOfPRecursive(const VI& a, int d)
{
  int m = (SZ(a) - d) / (d + 1) - 1;
  if (m < 0)
    return {};
  int n = (m + 1) * (d + 1);
  vector<VI> matr(SZ(a) - d, VI(n));
  FOR(i, 0, SZ(a) - d)
  {
    FOR(j, 0, d + 1)
    {
      int pw = 1;
      FOR(k, 0, m + 1)
      {
        matr[i][(m + 1) * j + k] = mult(pw, a[i + d - j]);
        pw = mult(pw, i);
      }
    }
  }
  auto [v, w] = solveLinearSystem(matr, VI(SZ(a) - d));
  if(w.empty())
    return {};
  vector<VI> p(d + 1);
  FOR(j, 0, d + 1)
  {
    p[j] = {w[0].begin() + (m + 1) * j, w[0].begin() + (m + 1)
        * (j + 1)};
    removeLeadingZeros(p[j]);
  }
  return p;
}
```

find-nth-of-p-recursive.hpp

Description: Computes the n-th term of a given linearly recurrent sequence with polynomial coefficients $\sum_{j=0}^{d} P_j(i) \cdot a_{i+d-j} = 0$. The first d terms $a_0, a_1, \ldots, a_{d-1}$ are given. Let m be the maximum degree of P_j.
Time: $\mathcal{O}\left(d^2\sqrt{nm}\log nm + d^3\sqrt{nm}\right)$

9e174d, 134 lines

```
VI add(const VI& a, const VI& b)
{
  int n = SZ(a), m = SZ(b);
  VI c(max(n, m));
  FOR(i, 0, n)
    updAdd(c[i], a[i]);
  FOR(i, 0, m)
    updAdd(c[i], b[i]);
  return c;
}

int evalPoly(const VI& p, int x)
{
  int res = 0;
  RFOR(i, SZ(p), 0)
    res = add(mult(res, x), p[i]);
  return res;
}

VI mult(const vector<VI>& a, const VI& b)
{
  int n = SZ(a);
  VI c(n);
  FOR(i, 0, n)
    FOR(j, 0, n)
      updAdd(c[i], mult(a[i][j], b[j]));
  return c;
}

vector<VI> mult(const vector<VI>& a, const vector<VI>& b)
{
  int n = SZ(a);
  vector<VI> c(n, VI(n));
  FOR(i, 0, n)
    FOR(k, 0, n)
      FOR(j, 0, n)
        updAdd(c[i][j], mult(a[i][k], b[k][j]));
  return c;
}

typedef vector<vector<VI>> PolyMatr;

PolyMatr mult(const PolyMatr& a, const PolyMatr& b)
{
  int n = SZ(a);
  PolyMatr c(n, vector<VI>(n));
  FOR(i, 0, n)
    FOR(k, 0, n)
      FOR(j, 0, n)
        c[i][j] = add(c[i][j], mult(a[i][k], b[k][j]));
  return c;
}

int findNthOfPRecursive(const vector<VI>& p, VI a, int n)
{
  int d = SZ(p) - 1;
```

```
  assert(SZ(a) == d);
  if (n < d)
    return a[n];
  auto polyMatrProd = [](const PolyMatr& polyMatr, int k, VI u)
  {
    int h = SZ(polyMatr);

    auto shiftEvalMatrs =
      [&](const vector<vector<VI>>& matrices, int c, int m)
    {
      int cnt = SZ(matrices);
      vector<vector<VI>> res(m, vector<VI>(h, VI(h)));
      FOR(i, 0, h)
      {
        FOR(j, 0, h)
        {
          VI b(cnt);
          FOR(l, 0, cnt)
            b[l] = matrices[l][i][j];
          b = shiftEvalValues(b, c, m);
          FOR(l, 0, m)
            res[l][i][j] = b[l];
        }
      }
      return res;
    };

    int m = 0;
    FOR(i, 0, h)
      FOR(j, 0, h)
        m = max(m, SZ(polyMatr[i][j]) - 1);
    int s = 1;
    while ((LL)m * s * s < k)
      s *= 2;
    int invS = binpow(s, mod - 2);
    vector<vector<VI>> matrices(m + 1, vector<VI>(h, VI(h)));
    FOR(l, 0, m + 1)
    {
      FOR(i, 0, h)
        FOR(j, 0, h)
          matrices[l][i][j] = evalPoly(polyMatr[i][j], l * s);
    }
    for (int r = 1; r < s; r *= 2)
    {
      auto sh = shiftEvalMatrs(matrices, r * m + 1, SZ(matrices)
        ) - 1);
      matrices.insert(matrices.end(), ALL(sh));
      sh = shiftEvalMatrs(matrices, mult(r, invS), SZ(matrices)
        );
      FOR(l, 0, SZ(matrices))
        matrices[l] = mult(sh[l], matrices[l]);
    }
    int l = 0;
    for (; l + s <= k; l += s)
      u = mult(matrices[l / s], u);
    vector<VI> matr(h, VI(h));
    for (; l < k; l++)
    {
      FOR(i, 0, h)
        FOR(j, 0, h)
          matr[i][j] = evalPoly(polyMatr[i][j], l);
      u = mult(matr, u);
    }
    return u;
  };

  PolyMatr polyMatr(d, vector<VI>(d));
  FOR(i, 0, d - 1)
    polyMatr[i][i + 1] = p[0];
  FOR(i, 0, d)
  {
    polyMatr[d - 1][i] = p[d - i];
    for (int& coef : polyMatr[d - 1][i])
      coef = sub(0, coef);
  }
  PolyMatr denom = {{p[0]}};
  a = polyMatrProd(polyMatr, n - d + 1, a);
  const VI& x = polyMatrProd(denom, n - d + 1, {1});
  return mult(binpow(x[0], mod - 2), a.back());
}
```

Mathematical analysis and numerical methods

Taylor series

$$f(x)=\sum_{k=0}^{n}\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k+o((x-x_0)^n)$$

$$e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}\qquad \ln(1+x)=\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n}$$

$$\cos x=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}\qquad \sin x=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}$$

Green's theorem

$$\oint_C(L\mathrm{d}x+M\mathrm{d}y)=\iint_D\left(\frac{\partial M}{\partial x}-\frac{\partial L}{\partial y}\right)\mathrm{d}x\mathrm{d}y$$

Runge-Kutta 4th Order

$$\frac{\mathrm{d}y}{\mathrm{d}x}=f(x,y),y(0)=y_0,x_{i+1}-x_i=h$$

$$y_{i+1}=y_i+\frac{1}{6}(k_1+2k_2+2k_3+k_4)h$$

$$k_1=f(x_i,y_i)\qquad k_2=f(x_i+\frac{1}{2}h,y_i+\frac{1}{2}k_1h)$$

$$k_3=f(x_i+\frac{1}{2}h,y_i+\frac{1}{2}k_2h)\qquad k_4=f(x_i+h,y_i+k_3h)$$

List of integrals

$$\int\frac{\mathrm{d}x}{a^2+x^2}=\frac{1}{a}\operatorname{arctg}\frac{x}{a}+C$$

$$\int\frac{\mathrm{d}x}{a^2-x^2}=\frac{1}{2a}\ln\left|\frac{x+a}{x-a}\right|+C$$

$$\int\frac{\mathrm{d}x}{\sqrt{a^2-x^2}}=\arcsin\frac{x}{a}+C$$

$$\int\frac{\mathrm{d}x}{\sqrt{x^2+a}}=\ln\left|x+\sqrt{x^2+a}\right|+C$$

$$\int\frac{\mathrm{d}x}{\cos^2 x}=\operatorname{tg}x+C$$

$$\int\frac{\mathrm{d}x}{\sin^2 x}=-\operatorname{ctg}x+C$$

Simpson's rule

n – even number, $h=\frac{b-a}{n}$, $x_i=a+ih$

$$\int_a^b f(x)\mathrm{d}x\approx\frac{h}{3}\left[f(x_0)+4\sum_{i=1}^{\frac{n}{2}}f(x_{2i-1})+\right.$$
$$\left.+2\sum_{i=1}^{\frac{n}{2}-1}f(x_{2i})+f(x_n)\right]$$

Vandermonde matrix

$$V=V(x_0,x_1,\cdots,x_m)=\begin{bmatrix}1&x_0&x_0^2&\dots&x_0^n\\1&x_1&x_1^2&\dots&x_1^n\\1&x_2&x_2^2&\dots&x_2^n\\\vdots&\vdots&\vdots&\ddots&\vdots\\1&x_m&x_m^2&\dots&x_m^n\end{bmatrix}$$

$$V_{i,j}=x_i^j,\qquad\det(V)=\prod_{0\le i<j\le n}(x_j-x_i).$$

Hadamard matrix

$H_1=\begin{bmatrix}1\end{bmatrix},\qquad H_{2^k}=\begin{bmatrix}H_{2^{k-1}}&H_{2^{k-1}}\\H_{2^{k-1}}&-H_{2^{k-1}}\end{bmatrix}$

$\det(H_n)=\pm n^{\frac{n}{2}}$

For a matrix M such that $|M_{ij}|\le 1$, holds $|\det(M)|\le n^{n/2}$.

Number theory

Calculation of $a^b \mod m$

if $b\ge\phi(m)$, then value $a^b\equiv a^{[b \mod \phi(m)]+\phi(m)} \pmod m$.

Generators

A generator exists only for $n=1,2,4,p^k,2p^k$ for odd primes p and positive integers k.

g is a generator modulo n if any number coprime with n can be represented as $\left[g^i \mod n\right], 0\le i<\phi(n)$.

To find a generator:

- find $\phi(n)$ and $p_1,...,p_m$ — the prime factors of $\phi(n)$
- g is generator only if $g^{\frac{\phi(n)}{p_j}}\not\equiv 1 \pmod n$ for each j
- check $g=2,3,4,...,p-1$

Wilson's theorem

p is prime if and only if $(p-1)!\equiv(p-1) \pmod p$.

Quadratic residues

q is a quadratic residue modulo p if there exists an integer x such that $x^2\equiv q \pmod p$. If p is odd prime then there exist $\frac{p+1}{2}$ residues (including 0).

Number theory functions

$$n = p_1^{\alpha_1} \cdot \dots \cdot p_k^{\alpha_k}$$

$$\phi(n) = \prod p_i^{\alpha_i - 1}(p_i - 1) \text{ -- the number of coprimes}$$

$$F(n) = \frac{n \cdot \phi(n)}{2} \text{ -- the sum of coprimes for } n > 1$$

$$\mu(n) = (-1)^k \text{ if } \max(\alpha_i) = 1, \text{ else } 0$$

$$\sigma_k(n) = \sum_{d|n} d^k$$

$$\sigma_0(n) = \prod(\alpha_i + 1)$$

$$\sigma_{k>0}(n) = \prod \frac{p_i^{(\alpha_i+1)\cdot k} - 1}{p_i^k - 1}$$

Möbius

$$g(n) = \sum_{d|n} f(d) \iff f(n) = \sum_{d|n} \mu(d) g\left(\frac{n}{d}\right)$$

$$M(n) = \sum_{k=1}^{n} \mu(k), \quad \sum_{d=1}^{n} M\left(\left\lfloor \frac{n}{d} \right\rfloor\right) = 1$$

$$\sum_{d|n} \phi(d) = n, \quad \sum_{d|n} \mu(d) = [n = 1]$$

Combinatorics

Binomials

$$\sum_{k=0}^{n} C_n^k = 2^n \qquad \sum_{k=0}^{m} C_{n+k}^k = C_{n+m+1}^m$$

$$\sum_{m=0}^{n} C_m^k = C_{n+1}^{k+1} \qquad \sum_{k=0}^{n} (C_n^k)^2 = C_{2n}^n$$

$$\sum_{j=0}^{k} C_m^j C_{n-m}^{k-j} = C_n^k \qquad \sum_{j=0}^{m} C_m^j C_{n-m}^{k-j} = C_{n+1}^{k+1}$$

$$\sum_{k=0}^{n} C_{n-k}^k = F_{n+1}$$

Catalan numbers

$$C_n = \sum_{k=0}^{n-1} C_k C_{n-1-k} = \frac{1}{n+1} C_{2n}^n = C_{2n}^n - C_{2n}^{n-1}$$

1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786

Fibonacci numbers

$$F_1 = F_2 = 1 \qquad \gcd(F_m, F_n) = F_{\gcd(n,m)}$$

$$F_n = F_{n-1} + F_{n-2} \qquad F_{n+1}F_{n-1} - F_n^2 = (-1)^n$$

$$F_{n+k} = F_k F_{n+1} + F_{k-1} F_n \qquad F_{47} \approx 2.9 \cdot 10^9$$

$$F_n = \frac{(\frac{1+\sqrt{5}}{2})^n - (\frac{1-\sqrt{5}}{2})^n}{\sqrt{5}} \qquad F_{88} \approx 1.1 \cdot 10^{18}$$

Stirling numbers of the second kind

$S(n,k)$ – the number of ways to divide n element into k non-empty groups.

$S(n,n) = 1,\ n \geq 0$

$S(n,0) = 0,\ n > 0$

$S(n,k) = S(n-1,k-1) + S(n-1,k) \cdot k.$

$B_n = \sum_{k=0}^{n} S(n,k)$ from $n = 0$:

1, 1, 2, 5, 15, 52, 203, 877, 4140, 21147, 115975, 678570, 4213597, 27644437, 190899322, 1382958545, 10480142147, 82864869804,...

Generating functions

$$[x^i](1+x)^n = C_n^i \quad [x^i](1-x)^{-n} = C_{n+i-1}^i$$

$$C_\alpha^n = \frac{\alpha(\alpha-1)\dots(\alpha-n+1)}{n!}$$

$$\prod_{n=1}^{\infty}(1 - x^n) = \sum_{k=-\infty}^{\infty} (-1)^k x^{\frac{k(3k-1)}{2}} \text{(pentagonal number theorem)}$$

Hook length formula

A standard Young tableau is a filling of the n cells of the Young diagram with a permutation, such that each row and each column form increasing sequences. The **hook** $h_\lambda(i,j)$ is number of cells (a,b) in diagram such that $a = i$ and $b \geq j$ or $a \geq i$ and $b = j$.

7	4	3	1
5	2	1	
2			
1			

A tableau listing the hook length of each cell in the Young diagram $(4,3,1,1)$

The number of standard Young tableaux of shape λ:

$$f^\lambda = \frac{n!}{\prod h_\lambda(i,j)}$$

Burnside's lemma

Let G be a finite group that acts on a set X.

The *orbit* of an element x in X is the set of elements in X to which x can be moved by the elements of G. The orbit of x is denoted by $G \cdot x$:

$$G \cdot x = \{g \cdot x \mid g \in G\}.$$

For each g in G, let X^g denote the set of elements in X that are fixed by g (also said to be left invariant by g), that is, $X^g = \{x \in X \mid g \cdot x = x\}$. Burnside's lemma asserts the following formula for the number of orbits, denoted $|X/G|$:

$$|X/G| = \frac{1}{|G|} \sum_{g \in G} |X^g|.$$

Graphs

Prüfer sequence

At step i, remove the leaf with the smallest label and set the i-th element of the Prüfer sequence to be the label of this leaf's neighbour. The Prüfer sequence of a labeled tree is unique and has length $n-2$.

The number of spanning trees of K_n is n^{n-2}.
The number of spanning trees of $K_{L,R}$ number is $L^{R-1} \cdot R^{L-1}$.

Let $T_{n,k}$ be the number of labelled forests on n vertices with k connected components, such that vertices $1, \dots, k$ all belong to different components. $T_{n,k} = k \cdot n^{n-k-1}$.

The number of spanning trees in a complete graph K_n with the fixed degrees d_i is equal to: $\frac{(n-2)!}{\prod(d_i-1)}$

For a forest graph with connected components of sizes $s_0, \dots, s_{k-1}$, the number of ways to add edges to make a spanning tree is equal to: $n^{k-2} \cdot \prod s_i$

Chromatic polynomial

For a graph G, $\chi(G,\lambda) = \chi(\lambda)$ counts the number of its vertex λ-colorings. There is a unique polynomial $\chi(\lambda)$.
Deletion-contraction:

- The graph G/uv is obtained by merging u and v.
- The graph $G - uv$ is obtained by deleting the edge uv.
- $\chi(G,\lambda) = \chi(G - uv, \lambda) - \chi(G/uv, \lambda)$.

G is tree	$\chi(\lambda) = \lambda(\lambda-1)^{n-1}$
G is cycle C_n	$\chi(\lambda) = (\lambda-1)^n + (-1)^n(\lambda-1)$

Proposition. *$\chi(\lambda)$ is equal to the number of pairs (σ, O), where σ is any map $\sigma : V \to \{1, \dots, \lambda\}$ and O is an orientation of G, subject to the two conditions:*

- *The orientation O is acyclic.*
- *If $u \to v$ in O, then $\sigma(u) > \sigma(v)$.*

Define $\overline{\chi}(\lambda)$ to be the number of pairs (σ, O), where σ is any map $\sigma : V \to \{1, \dots, \lambda\}$ and O is an orientation of G, subject to the two conditions:

- The orientation O is acyclic.
- If $u \to v$ in O, then $\sigma(u) \geq \sigma(v)$.

Theorem. *Suppose that $|V| = n$. Then for all non-negative integers λ holds:*

$$\overline{\chi}(\lambda) = (-1)^n \chi(-\lambda)$$

Corollary. *$(-1)^n \chi(G,-1)$ is equal to the number of acyclic orientations of G.*

Kirchhoff's theorem

Let G be a finite graph, allowing multiple edges but not loops.

The laplacian matrix L of G is the $n \times n$ matrix whose (i,j)-entry L_{ij} is given by

$$L_{ij} = \begin{cases} -m_{ij}, & \text{if } i \neq j,\ m_{ij} \text{ edges between } v_i \text{ and } v_j, \\ \deg(v_i), & \text{if } i = j. \end{cases}$$

Let L_0 denote L with the i-th row and column removed for any i. Then for a connected graph, $\det(L_0)$ equals the number of spanning trees of G.

Karp's minimum mean-weight cycle algorithm

Let $G = (V, E)$ be a directed graph with weight function $w : E \to \mathbb{R}$, and let $n = |V|$. We define the ***mean weight*** of a cycle $c = \langle e_1, e_2, \dots, e_k \rangle$ of edges in E to be

$$\mu(c) = \frac{1}{k}\sum_{i=1}^{k} w(e_i).$$

Let $\mu^* = \min_c \mu(c)$, where c ranges over all directed cycles in G. We call a cycle c for which $\mu(c) = \mu^*$ a ***minimum mean-weight cycle***.

Assume without loss of generality that every vertex $v \in V$ is reachable from a source vertex $s \in V$. Let $\delta_k(s, v)$ be the weight of a shortest path from s to v consisting of *exactly* k edges. If there is no path from s to v with exactly k edges, then $\delta_k(s, v) = \infty$.

$$\mu^* = \min_{v \in V} \max_{0 \le k \le n-1} \frac{\delta_n(s,v) - \delta_k(s,v)}{n-k}.$$

This can be computed in time $O(VE)$.

Erdős–Gallai theorem

A sequence of non-negative integers $d_1 \geq \dots \geq d_n$ can be represented as the degree sequence of a finite simple graph on n vertices if and only if $d_1 + \dots + d_n$ is even and $\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{i=k+1}^{n} \min(d_i, k)$ holds for every k in $1 \leq k \leq n$.

Planar graph properties

For a simple, **connected**, planar graph with v vertices, e edges and f faces, the following simple conditions hold for $v \geq 3$:

- Theorem 1. $e \leq 3 \cdot v - 6$.
- Theorem 2. If there are no cycles of length 3, then $e \leq 2 \cdot v - 4$.
- Theorem 3. $f \leq 2 \cdot v - 4$.
- Euler's formula. $v - e + f = 2$.
- Theorem 4. $3 \cdot f \leq 2 \cdot e$.
- Theorem 5. The dual graph is also planar.
- Theorem 6. There exists a vertex v with $\deg(v) \leq 5$.

Dilworth's theorem

A partially ordered set is a set S with a relation $\leq$ on S satisfying:

1. $a \leq a$ for all $a \in S$ (reflexivity);
2. if $a \leq b$ and $b \leq a$, then $a = b$ (antisymmetry);
3. if $a \leq b$ and $b \leq c$, then $a \leq c$ (transitivity).

A chain is a subset of a set where each pair of distinct elements is comparable. An antichain is a subset of a set where every pair of elements is incomparable.

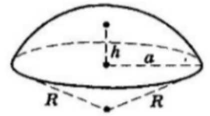
Dilworth's theorem states that, in any finite partially ordered set, the **largest antichain** has the same size as the **smallest chain decomposition**. Here, the size of the antichain is its number of elements, and the size of the chain decomposition is its number of chains.

Geometry

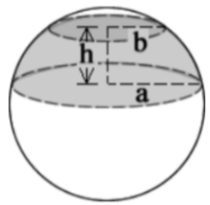
Trigonometry formulas

$$\sin(v + w) = \sin v \cos w + \cos v \sin w$$
$$\sin(v - w) = \sin v \cos w - \cos v \sin w$$
$$\tan(v + w) = \frac{\tan v + \tan w}{1 - \tan v \tan w}$$
$$\sin v + \sin w = 2 \sin \tfrac{v+w}{2} \cos \tfrac{v-w}{2}$$
$$\cos v + \cos w = 2 \cos \tfrac{v+w}{2} \cos \tfrac{v-w}{2}$$

Ball formulas



$$a = \sqrt{h \cdot (2R - h)}$$
$$V = \pi \cdot h^2 (R - \tfrac{h}{3})$$



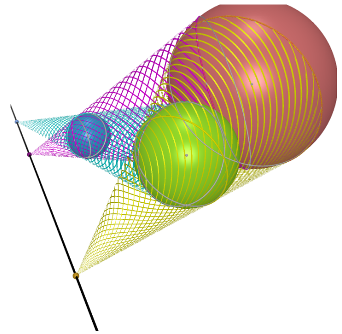
$$V = \tfrac{1}{6}\pi h(3a^2 + 3b^2 + h^2)$$
$$R = \sqrt{\tfrac{((a-b)^2 + h^2)((a+b)^2 + h^2)}{4h^2}}$$

Triangle formulas

$$S = \sqrt{p(p-a)(p-b)(p-c)} = \frac{abc}{4R}$$
$$m_a^2 = \frac{2b^2 + 2c^2 - a^2}{4} \text{(median)}$$
$$w_a^2 = \frac{bc((b+c)^2 - a^2)}{(b+c)^2} \text{(bisector)}$$
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$
$$a^2 = b^2 + c^2 - 2bc \cos A$$

Monge's theorem

There are three circles(balls) of different radii, for each pair of circles find the point of intersection of the external tangents. All three obtained points **lie on a line**. The point from the pair of the largest and the smallest **lies between** the other two.



Pick's theorem

Suppose that a polygon has integer coordinates for all of its vertices. Let i be the number of integer points inside, and let b be the number of integer points on boundary. Then the area $S = i + \frac{b}{2} - 1$.

Ptolemy's theorem

For a general quadrilateral $ABCD$ holds: $AB \cdot CD + AD \cdot BC \geq AC \cdot BD$.

Equality holds if and only if the quadrilateral is cyclic.

Euler line

For a general triangle, the orthocenter H, the centroid G, and the circumcenter O, in this order, lie on the same line (Euler line) and $\frac{|HG|}{|GO|} = \frac{2}{1}$.

Fermat point

In a given triangle $\triangle ABC$ the Fermat point is the point X, which minimizes the sum of distances from A, B, and C, $|AX| + |BX| + |CX|$.

If all angles of the triangle are less than $120°$, the the Fermat point is the interior point X from which each side subtends an angle of $120°$, i.e., $\angle BXC = \angle CXA = \angle AXB = 120°$.

If any angle of the triangle formed by those points is $120°$ or more, then the Fermat point is the vertex of that angle.

Various (7)

Find n bits using $O(\frac{n}{\log n})$ sums

To find n bits $x_0, \dots, x_{n-1}$ using $\approx 2.67 \cdot \frac{n}{\log_2(n)}$ sums of values in asked subsequences.

Define $Y + d$ as $\{y + d, \text{ for } y \in Y\}$.

Define f_k as maximum number of bits we can find using 2^k queries. We can make $f_{k+1} = 2 \cdot f_k + 2^k - 1$ using such queries:

- Ask sum of elements $[f_k, 2 \cdot f_k)$.
- Iterate through the first $2^k - 1$ queries Q_i, ask $Q_i \cup (Q_i + f_k)$ and $Q_i \cup [f_k, 2 \cdot f_k) \setminus (Q_i + f_k) \cup x_{2 \cdot f_k + i}$.
- Ask sum of elements $[0, f_{k+1})$.

If we have sums of queries $[f_k, 2 \cdot f_k)$, $Q \cup (Q + f_k)$ and $Q \cup [f_k, 2 \cdot f_k) \setminus (Q + f_k) \cup x$ as a, b and c respectively.

$x = (c + b - a)\%2, S(Q) = \frac{b+c-a-x}{2}, S(Q + f_k) = \frac{b+a-c+x}{2}$.

k	0	1	2	3	4	5	10	13
2^k	1	2	4	8	16	32	$\approx 10^3$	$\approx 8 \cdot 10^3$
f_k	1	2	5	13	33	81	$\approx 5 \cdot 10^3$	$\approx 5 \cdot 10^4$

Matroid

In terms of independence, a finite matroid M is a pair (E, I), where E is a finite set (called the ground set) and I is a family of subsets of E (called the independent sets) with the following properties:

1. The empty set is independent, i.e., $\emptyset \in I$;
2. Every subset of an independent set is independent, i.e., for each $A' \subseteq A \subseteq E$, if $A \in I$ then $A' \in I$;
3. If $A, B \in I$, and $|A| > |B|$, then there exists $x \in A \setminus B$ such that $B \cup \{x\} \in I$.

Matroid intersection

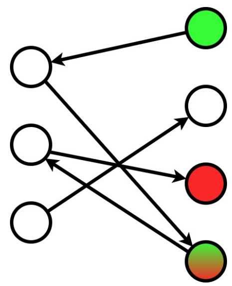
Matroids intersection of several matroids $M_1 = (X, I_1)$, $M_2 = (X, I_2)$, $\dots$, $M_k = (X, I_k)$ defined on the same ground set X represents "good" subsets as an intersection $I_1 \cap I_2 \cap \dots \cap I_k$. The task is to find a set of objects $S \subseteq X$ with maximum size such that $S \in (I_1 \cap I_2 \cap \dots \cap I_k)$.

Intersection of **three** or more matroids in **NP-complete** task. However, the intersection of two matroids can be done in polynomial time $\approx O(|X|^2 + |X| \cdot F(|X|))$, where $F(|X|)$ is the time to build the graph below.

Starting from $S = \emptyset$ we can increase it's size by one with following algorithm until maximum size. To do this, let's build the following directed bipartite graph. The left part of this graph will contain all objects from S, while the right part will contain all **other** objects from X.

Consider objects from the right part of the graph such that adding them into S keeps independence in I_1. Let's paint these objects in green. Similarly, paint in a red color objects that keeps independence in I_2.

Let's add an edge from the left vertex u to the right vertex v when $S \setminus \{u\} \cup \{v\} \in I_1$. Symmetrically let's add an edge from the right vertex v to the left vertex u when $S \setminus \{u\} \cup \{v\} \in I_2$.



Let's find the path from some green vertex to some red vertex. Objects on the path are added or removed from the set S, respectively, objects from the right part are added and the left part is removed.

three-sat.hpp

Description: solve can be called again if needed, but it searches for the solution from scratch.
Time: solve() works in less than $\mathcal{O}\left(1.84^n \cdot n^2\right)$.
a7ef7f, 93 lines

```
struct ThreeSAT
{
  int n;
  VI ans;
  vector<vector<VI>> hasRule;

  void init(int _n)
  {
    n = _n;
    hasRule.assign(2 * n, vector<VI>(2 * n, VI(2 * n, 0)));
  }
  //(x_a == valA) || (x_b == valB) || (x_c == valC)
  void addRule(int a, int valA, int b, int valB, int c, int
     valC)
  {
    a = 2 * a + valA;
    b = 2 * b + valB;
    c = 2 * c + valC;

    hasRule[a][b][c] = hasRule[a][c][b] = true;
    hasRule[b][a][c] = hasRule[b][c][a] = true;
    hasRule[c][a][b] = hasRule[c][b][a] = true;
  }
  VI findRule(int k)
  {
    FOR(i, 0, 2 * n)
      FOR(j, 0, 2 * n)
        FOR(t, 0, 2)
          if(hasRule[2 * k + t][i][j])
            return {2 * k + t, i, j};
    return {};
  }

  VI fixed;
  bool add(int k, int t)
  {
    if(ans[k] != -1)
      return ans[k] == t;

    ans[k] = t;
    fixed.PB(k);
    FOR(i, 0, n)
    {
      if(ans[i] == -1)
        continue;
      int xi = 2 * i + ans[i];
      int xk = 2 * k + t;
      FOR(j, 0, n)
        FOR(tj, 0, 2)
          if(hasRule[xi ^ 1][xk ^ 1][2 * j + tj] && !add(j, tj)
            )
            return false;
    }
    return true;
  }

  bool check()
  {
    int var = 0;
    while(var < n && ans[var] != -1)
      var++;
    if(var == n)
      return true;

    VI rule = findRule(var);
    FOR(i, 0, 3)
    {
      fixed.clear();
      bool ok = add(rule[i] / 2, rule[i] % 2);
      FOR(j, 0, i)
        ok &= add(rule[j] / 2, (rule[j] % 2) ^ 1);

      VI curFixed = fixed;
      if(ok && check())
        return true;

      for(int v : curFixed)
        ans[v] = -1;
    }
    return false;
  }

  VI solve()
  {
    ans.assign(n, -1);
    FOR(var, 0, n)
    {
      if(findRule(var).empty())
        add(var, 0);
    }
    if(!check())
      return {};
    return ans;
  }
};
```

gaussian-integer.hpp

Description: $n = am + b$, $\frac{n}{m} = a, n\%m = b$. use __gcd instead of gcd.
Facts: Primes of the form $4n + 3$ are Gaussian primes. Uniqueness of prime factorization.
cb938e, 41 lines

```
LL closest(LL u, LL d)
{
  if(d < 0)
    return closest(-u, -d);
  if(u < 0)
    return -closest(-u, d);
  return (2 * u + d) / (2 * d);
}
struct num : complex<LL>
{
```

```
  num(LL a, LL b = 0) : complex(a, b) {}
  num(complex a) : complex(a) {}
  num operator/ (num x)
  {
    num prod = *this * conj(x);
    LL D = (x * conj(x)).real();

    LL m = closest(prod.real(), D);
    LL n = closest(prod.imag(), D);

    return num(m, n);
  }
  num operator% (num x)
  {
    return *this - x * (*this / x);
  }
  bool operator == (num b)
  {
    FOR(it, 0, 4)
    {
      if(real() == b.real() && imag() == b.imag())
        return true;
      b = b * num(0, 1);
    }
    return false;
  }
  bool operator != (num b)
  {
    return !(*this == b);
  }
};
```

golden-section-search.hpp

4c0990, 27 lines

```
db goldenSectionSearch(db l, db r)
{
  const db c = (-1 + sqrt(5)) / 2;
  const int M = 474;
  db m1 = r - c * (r - l), fm1 = f(m1),
    m2 = l + c * (r - l), fm2 = f(m2);
  FOR(i, 0, M)
  {
    if (fm1 < fm2)
    {
      r = m2;
      m2 = m1;
      fm2 = fm1;
      m1 = r - c * (r - l);
      fm1 = f(m1);
    }
    else
    {
      l = m1;
      m1 = m2;
      fm1 = fm2;
      m2 = l + c * (r - l);
      fm2 = f(m2);
    }
  }
  return (l + r) / 2;
}
```

nim-product.hpp

Description: The Nim sum $\oplus$: $a \oplus b := \text{mex}(\{a' \oplus b | a' < a\} \cup \{a \oplus b' | b' < b\})$. The Nim product $\otimes$: $a \otimes b := \text{mex}\{(a' \otimes b) \oplus (a \otimes b') \oplus (a' \otimes b') | a' < a, b' < b\}$. Let A be the set consisting of integers between 0 (inclusive) and 2^{2^n} (exclusive) (where n is an integer). Then the algebraic structure whose addition is $\oplus$ and multiplication is $\otimes$ forms a field. Such a field is called **Nimber**.
Time: About 64 references to the precalculation table.

9d1a59, 30 lines

```
typedef unsigned long long ULL;

const int S = 8;

int small[1 << S][1 << S];

void init()
{
  FOR(i, 0, 1 << S)
    FOR(j, 0, 1 << S)
      small[i][j] = -1;
}

ULL nimProduct(ULL a, ULL b, int p = 64)
{
  if (min(a, b) <= 1)
    return a * b;
  if (p <= S && small[a][b] != -1)
    return small[a][b];
  p >>= 1;
  ULL a1 = a >> p, a2 = a & ((1ULL << p) - 1);
  ULL b1 = b >> p, b2 = b & ((1ULL << p) - 1);
  ULL c = nimProduct(a1, b1, p);
  ULL d = nimProduct(a2, b2, p);
  ULL e = nimProduct(a1 ^ a2, b1 ^ b2, p);
  ULL res = nimProduct(c, 1ULL << (p - 1), p) ^ d ^ ((d ^ e) <<
       p);
  if (p <= S / 2)
    small[a][b] = res;
  return res;
}
```