Modeling ½ spin down of electrons

Kinematic Analysis for the relationship of Gears

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ABSTRACT

Context. The purpose of this report is to analyze the kinematics and kinetics of a system of rigid bodies in a real-world system application. We chose to analyze an existing model that demonstrates the motion of an electron.
Methods. The force applied and angular momentum of the half down spin model was determined by 3-D printing and using the model with tracker software to record data. This data was analyzed with kinetic and kinematic equations of motions to graph angular velocity of the larger gear over the time period of the revolution, as well as the applied push force against time. A python script was used to outline and calculate angular momentum by using the odeint (ordinary differential solver) function.

1. Introduction

The inspiration for the focus on the motion of the electron lead to the analysis of its angular momentum and minute magnetic field. Although the electron should not be considered a spinning subatomic particle, as seen in in Figure 1, it does have the mathematical properties of angular momentum. This allows for the interpretation of its abstract moment as a spin. The spin is the total angular momentum of the body. For example, the spin of a planet is composed of the orbital angular momenta of each of the elementary particles that makes up the planet. In this case, the half spin of an electron indicates that it takes a rotation of 360 degrees, two full rotations, for the electron to reach the same state that it was in initially.

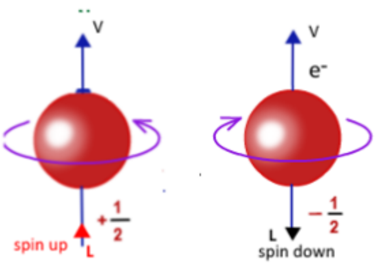

Fig. 1: Model of the electron spin theory which displays the positive and negative ½ spin of an electron.

To demonstrate this, the ½ spin down model in Figure 2 was utilized to replicate this motion . The model replicates the mathematical properties of the electron spin, using two interlocking gears with a gear ratio of 2:1. This displays the half spin of an electron, as it takes a two rotations for the electron to reach its initial state. Shown with a marking, the user can spin the larger gear around and visualize the initial state with the markings on the platform and the large gear, once the two lines are in line with one another.

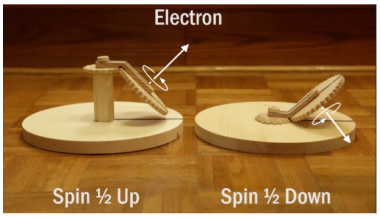

Fig. 2: A wooden demonstration for the ½ spin of an electron.

2. Process timeline & Manufacturing

At first, the prototype in mind was to be a Minimal Viable Product (MVP) that would encapsulate the principles and the objective concept. In consequence, the interlocking of the gear teeth was not of the highest priority, and so the three dimensional model made using Onshape had triangular, pointy-tipped teeth for both gears. This stage is shown in Figure 3 below.

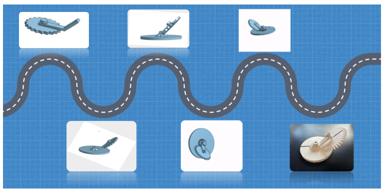

Fig. 3: The manufacturing timeline of the ½ spin down electron model

Even in stage 3, where the parts were optimized to bring the two gears together, tt was challenging to see whether they would actually interlock as the software could not render the motion. However, due to the expansive resources provided at the Cooper Union, as well as the time at hand, it was possible to manufacture multiple prototypes and iterations to have a working prototype by the end of it. Apart from the self-satisfaction of having it function properly, it meant that data could be recorded from the physical model itself, as opposed to a digital animation through

Onshape. This provided for a first-hand experience of what steps in engineering a product entailed, as well as how to operate three dimensional printers.

As a result, the gears were entirely refurbished to more sophisticated ones, with semicircular teeth that would be known to interlock. Once the mathematics indicated that the ratios were right and that in simulation the gears would operate as expected, the prototype was manufactured and stage 6 was reached. The assembly and fastening of all the indiidual parts is shown in Figure 4.

Fig. 4: Assembly of the prototype demonstrating the negative ½ spin of an electron

3. Free Body Diagram Analysis

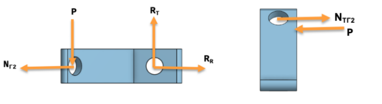

Fig. 5: Isolated free body diagram of holder for the prototype demonstrating the negative ½ spin of an electron

The free body diagrams of the individual components of the electron spin model are displayed in Figures 5 and 6 . They are labeled using an xy-coordinate system.As for the gears in specific, Figure 6 displays the normal force on both the big and small gear from their interlocking, NB1 and NB2.For ease, a legend is included with the nomenclature of variables expressed. The normal force of the peg from gear one is displayed on the holder opening.

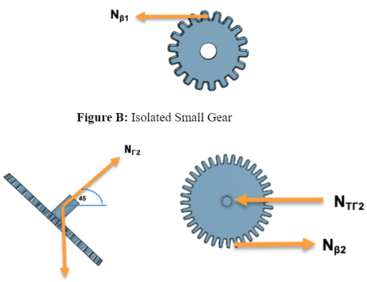

Fig. 6: Isolated free body diagram of the gears for the prototype demonstrating the negative ½ spin of an electron

Legend

$N\Gamma_2$	normal force of gear 2
P	applied force
R_T	tangential reaction force
$NT\Gamma_2$	normal tangential force gear 2
$N\beta_2$	normal force of big gear onto small gear
N_1	normal force from small gear onto big gear
M_2G	weight

The kinetic diagram shown in Figure 7 displays the angular acceleration of the large gear itself which rotates about the holder, as well as of the holder that rotates about the platform. The normal and tangential acceleration is shown as well, with it being aligned with the direction of rotation of the gear and handle. The normal acceleration is at the same angle as the handle and the large gear while the tangential remains parallel with the platform.

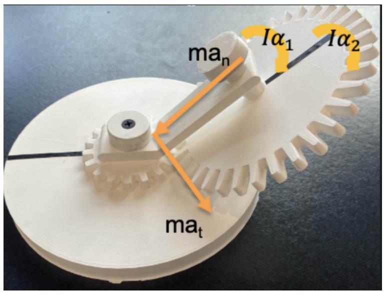

Fig. 7: Kinetic diagram of the system for the prototype demonstrating the negative ½ spin of an electron

4. Calculations

4.1. Dynamics for the system

Angular velocity of the Big Gear:

The first objective was to find a relationship between the angular velocity of the handle, ω_H, with the angular velocity of the larger gear in our system, ω_2. By using the vector form to compare the relative velocities of both the handle and gear, we were able to determine this relationship.

$$\hat{V}_B = \hat{V}_A + \hat{\omega}_{AB} \times \hat{r}_{AB}$$
$$\hat{V}_B = 0 + \hat{\omega}_2 k \times -\hat{r}_{AB} j$$
$$\hat{V}_B = \hat{r}_2 \hat{\omega}_2 i \quad (1)$$

As shown above, the velocity of the handle,V_2, was shown to be equal to the cross product between the angular velocity of the second gear and the radius of the second gear. Because this system was tilted, this relationship was found at an angle of 45 degrees.In order to compensate for this, the frame of reference was tilted to make the j axis tangent to the orientation of the second gear.Also, it's important to note that the velocity of, $\hat{V}_A$, was equal to zero, as that value represents the velocity between the second gear and the first gear- which is always a 'static' point. From there the relationship between the angular velocity of the handle, $\hat{\omega}_H$, and the velocity of the handle was determined (at the same point of its connection to the second gear). An important note is that for simplicity,the point at the center of the gear was considered to be in the same place as where the gear connects to the handle. In reality, there is a small difference but it was negligible. From this decision, the relationship was found to be:

$$\hat{V}_H = 0 + \hat{\omega}_H k \times -\hat{r}_H j$$
$$\hat{V}_H = \hat{r}_H \hat{\omega}_H i \quad (2)$$

The two velocities can then be equated, $V_2 = V_H$ as they're set at the same point. From there, the wanted relationship is found:

$$\omega_2 = \frac{\omega_H r_H}{r_2} \quad (3)$$

The value of r_H and r_2 were predetermined values that were dimensioned values on Onshape. An important note is that the value of r_H is the distance between the two connection points (or holes) of the handle. Later on, the experimental value of ω_H will be found using tracker.

4.2. Determination of applied force P

In order to simplify the results, the variables in they key below were used:

a	0.109
b	0.05m
r_2	0.078m
d_2	0.01m

The figure below outlines the dimensions for the handle. The value of d_2 represents the thickness of the big gear.

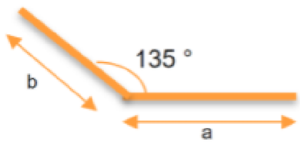

Fig. 8: Simple graphic outlining dimensions of prototypes handle that connects the two gears

The first step in finding the applied push force was to determine the moment of inertia of the entire system. This includes the handle and the big gea. First, the moment of inertia of the bottom part of the handle was shown to be:

$$\frac{1}{12}a^2 + m(\frac{a}{2})^2$$

This originates from the standard moment of inertia for a rod, $\frac{1}{12}a^2$, and as the center of mass for the rod was shifted by a certain distance,the parallel axis theorem was needed to compensate.

For the top part of the handle, the same moment of inertia for a rod can be used, but the distance between the center of mass of the rod and the central axis was different and more complex to find. This radius was found by using the law of cosines:

$$c = \sqrt{a^2 + b^2 - 2ab\cos(c)}$$

Utilizing this, the moment of inertia for the top part of the gear can be found:

$$m[(\frac{a}{2})^2 + (\frac{b}{2})^2 - 2(\frac{a}{2})(\frac{b}{2})\cos(135)]^2$$

The final calculation for the total moment of inertia was the moment of inertia of the gear from the central axis. This was more challenging when compared to the other moments as the gear was at an angle with relation to the central axis. This complex moment of inertia was found by multiplying a factor of $(1 + \sin(\theta)^2)$ to compensate for this.Along with the standard moment of inertia of a solid cylinder about its central diameter, the following equation is derived:

$$(\frac{mr^2}{4} + \frac{md^2}{12}) \cdot (1 + \sin(\theta)^2)$$

Agglomerating all of these terms give a total moment of inertia to be:

$$I_{total} = \frac{1}{12}a^2 + m(\frac{a}{2})^2 + \frac{1}{12}mb^2 + m[(\frac{a}{2})^2 + (\frac{b}{2})^2 - 2(\frac{a}{2})(\frac{b}{2})\cos(135)] + (m\frac{r_2^2}{4} + \frac{md^2}{12}) \cdot (1 + \sin(45)^2) + m[(\frac{a}{2})^2 + b^2 - 2(\frac{a}{2}b\cos(135)]^2$$

The sum of moments were then able to be set as the product of the total moment of inertia and the angular acceleration of the handle:

$$\alpha_H I_{total} = -P(b\cos(45) + \frac{a}{2}) + N_{\beta 1}(r_1)$$

In order to find the value of N_1, the normal force of the second gear acting on the first gear, the value of $N_{\beta 2}$ has to first be found. By finding the moment of inertia of the second gear, and the moments acting along that axis on the second gear,$N_{\beta 2}$ was isolated.

The moment of inertia for the second gear:

$$I_2 = \frac{1}{2}m(r_2)^2$$
$$\alpha_2 I_2 = N_{\beta 2} r_2$$
$$\frac{\alpha_2 I_2}{r_2} = N_{\beta 2} = N_{\beta 1}$$

Moreover, in using the same process as before, the relationship between the two angular accelerations was found:

$$\alpha_2 I_2 = \frac{\alpha_H r_H}{r_2}$$

The value of α_H being the angular acceleration of the holder was found using the tracker software. By substituting all the the variables above, and with some rearrangement,the force applied on the holder can be found.

$$P(b\cos(45) + \frac{a}{2}) = \frac{\alpha_2 I_2}{r_2}(r_1) - I_{total}\alpha_{total}$$
$$P = \frac{\frac{\alpha_2 I_2}{r_2}(r_1) - I_{total}\alpha_{total}}{(b\cos(45) + \frac{a}{2})} \quad (4)$$

Alternative method for finding applied push force: Angular Momentum

The angular momentum of this system is relatively simple to find, since the angular velocities acting on the handle, and its moment of inertia.

$$L = I\omega \Rightarrow L_{total} = I_{total}\omega_H$$

With this information, the derivative of the angular momentum, L, can be taken to give the net moment of the system.Combining all these principles provides an alternative form of the push force.

$$P = \frac{\frac{\alpha_2 I_2}{r_2}(r_1) - \frac{dL}{dt}}{(b\cos(45) + \frac{a}{2})}$$

These two forms will be used to compare the values of the experimentally and mathematically calculated values for the force applied, P.

5. Python

Using the tracker software shown above, a recording of the 3-D printed gear system spinning was used to record data. Using these data points, both the angular velocity and the angular acceleration of the handle, ω_H and α_H were calculated. This returned the angular velocities and accelerations at several different points of time during video which allowed for plotting against time. Utilizing Tracker limited the measurement of the angular velocity and angular acceleration of the system, as it could only measure the movement of the holder and not the large gear. This was due to the angle of the gear. The depth of gear could not be accounted and it does not stay facing a consistent direction throughout video-making, making it inaccurate to extract data from its motion. We also had to track the center of the model to set the coordinate axes to a fixed location. This way, throughout the shakiness of the video, the coordinate axes moved to stay centered with the model.

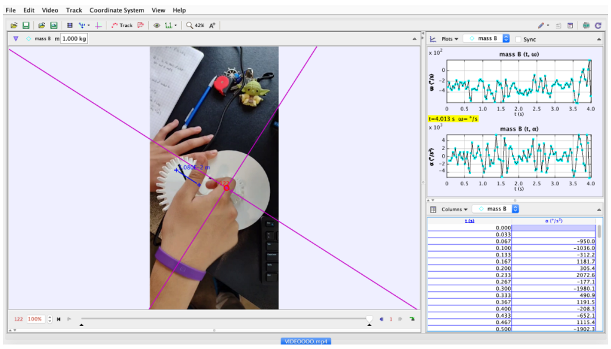

Fig. 9: Tracker software used to record data off of the electron demonstration model

With this gathered information, and the calculations which we previously determined, we were able to find several different variables related to our video and our structure. These produced values for the angular velocity of the second gear, and the force P in two different ways (one by using the data found from the angular acceleration, the other through using the angular velocity).

The force P was found to be very sporadic and at large magnitudes due to the inconsistent spinning of the handle. With the small jolts present from this inconsistent force being applied to the system, the angular acceleration was made to spike. This altered the overall value of the force of P as it needed to compensate for this quick accelerations.

Using python and its odeint function, an initial value ordinary differential equation was intended to be fed for it to calculate the necessary integration and output the desired outputs.

6. Results & Analysis

Figures 10 and 11 are the results that were generated with the use of python and the calculations listed above.

Figure 10 describes the value of ω_2 with respect to the change of time. This was found by using the ratio between the radii and the otherω_H. The average value found for $\omega_2$was equal to -88.4 rotations per second. This value was much higher than expected and must have been due to the tracker data not being set in a proper unit, or because of the reasons suggested in the Python section. Similarly the values for the applied push force, indicated in Figure 11, was also significantly higher than expected, with its average force being equal to -71.4 newtons.

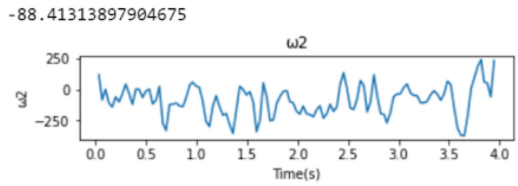

Fig. 10: The angular velocity of the second gear plotted against time

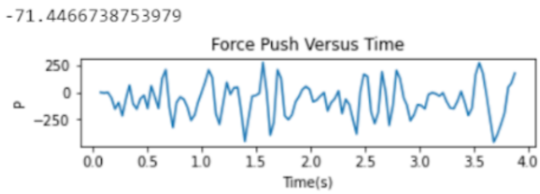

Fig. 11: Applied force onto model's handle plotted against time

There were many issues with finding the angular momentum using the ODE, which is why this was not included

7. Conlusion

Overall the calculation methods were reliable, but the numerical results were inaccurate considering the inconsistent turning of the gears. When recording the rotation of the large gear and extracting numerical data from it, due to friction, the turning was considerably shaky. This resulted in spikes in our data giving an imprecise and inaccurate result. Overall, we succeeded in creating the replication of the electron spin model and analyzing the kinetics and kinematics of the real-world rigid body system.

References

[1]Watts, L., 2021. Demonstration of Spin 1/2. Available at: <https://www.youtube.com/watch?v=JFSU9X11wyY> [Accessed 17 December 2021].

[2]F. Beer, E. Johnston, D. Mazurek, P. Cornwell and B. Self, Vector mechanics for engineers, 12th ed. McGraw HIll Education.