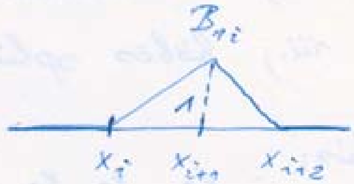
B-Splinek:

$h = x_i - x_{i+1}$, ekvidisztans felosztás esetén

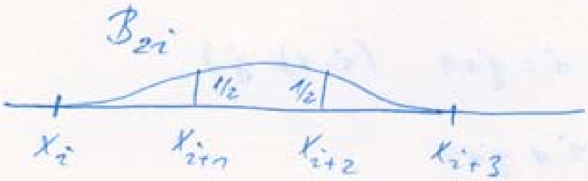
$l = 1$:

$$B_{1i}(x) = \frac{1}{h}\begin{cases} x - x_i & x \in [x_i, x_{i+1}] \\ x_{i+2} - x & x \in [x_{i+1}, x_{i+2}] \\ 0 & \text{különben} \end{cases}$$



$l = 2$:

$$B_{2i}(x) = \frac{1}{2h^2}\begin{cases} (x - x_i)^2 & x \in [x_i, x_{i+1}] \\ h^2 + 2h(x - x_{i+1}) - 2(x - x_{i+1})^2 & x \in [x_{i+1}, x_{i+2}] \\ (x_{i+3} - x)^2 & x \in [x_{i+2}, x_{i+3}] \\ 0 & \text{különben} \end{cases}$$



$l = 3$:

$$B_{3i}(x) = \frac{1}{6h^2}\begin{cases} (x - x_i)^3 \\ h^3 + 3h^2(x - x_{i+1}) + 3h(x - x_{i+1})^2 - 3(x - x_{i+1})^3 \\ h^3 + 3h^2(x_{i+3} - x) + 3h(x_{i+3} - x)^2 - 3(x_{i+3} - x)^3 \\ (x_{i+4} - x)^3 \\ 0 \end{cases}$$

Feladat: Legyen $f(x) = \cos x$, $x \in [-\pi, \pi]$

Írjuk fel az f-et a $-\pi, 0, \pi$-n interpoláló

i.) lineáris spline-t

ii.) kvadratikus splinet, peremf: $f'(-\pi) = 0$

iii.) köbös splinet, Hermite peremf: $f'(-\pi) = 0 = f'(\pi)$

Megoldás:

	x_0	x_1	x_2
x_i	$-\pi$	0	π
y_i	-1	1	-1

$h = \pi$

i.) $S_1 = \sum_{j=-1}^{1} c_j B_{1j}$ (3db B-Spline kell hozzá)

$$S_1(x_i) = \sum_{j=-1}^{1} c_j B_j(x_i) = c_{i-1} = f(x_i)$$

$$B_j(x_i) = \begin{cases} 1 & i = j+1 \quad (i-1 = j) \\ 0 & i \neq j+1 \end{cases}$$

$$\Rightarrow \left.\begin{aligned} c_{-1} &= -1 \\ c_0 &= 1 \\ c_1 &= -1 \end{aligned}\right\} \Rightarrow S_1(x) = (-1) B_{1,-1} + B_{1,0}(x) + (-1) B_{1,1}(x)$$

Intervallumokra felbontva:

$$x \in [-\pi, 0]: \quad S_1(x) = -\frac{1}{\pi}(0 - x) + \frac{1}{\pi}(x - (-\pi)) = \frac{1}{\pi}(2x + \pi) = \frac{2}{\pi}x + 1$$

$$x \in [0, \pi]: \quad S_1(x) = \frac{1}{\pi}(\pi - x) - \frac{1}{\pi}(x - 0) = -\frac{2}{\pi}x + 1$$

ii.) $$S_2 = \sum_{j=-2}^{1} c_j B_{2j} \quad \text{(4 db B-Spline kell hozzá)}$$

$$\left.\begin{aligned} &B_{2j}(x_j) = 0 \\ &B_{2j}(x_{j+1}) = B_{2j}(x_{j+2}) = \frac{1}{2} \\ &B_{2j}(x_{j+7}) = 0 \end{aligned}\right\} \Longrightarrow \dots$$

$$\dots \Rightarrow S_2(x_i) = \sum_{j=-2}^{1} c_j B_{2j}(x_i) = f(x_i) = \dots \qquad (i=0,1,2)$$

$$\dots = c_{i-2} B_{2,i-2}(x_i) + c_{i-1} B_{2,i-1}(x_i) = \frac{1}{2}(c_{i-2} + c_{i-1})$$

$$\Rightarrow \quad -1 = f(x_0) = \frac{1}{2}(c_{-2} + c_{-1})$$

$$1 = f(x_1) = \frac{1}{2}(c_{-1} + c_0)$$

$$-1 = f(x_2) = \frac{1}{2}(c_0 + c_1)$$

Peremfeltétel:

$$f'(-\pi) = 0 = \sum_{j=-2}^{1} c_j B_{2j}'(-\pi) = -\frac{1}{\pi} c_{-2} + \frac{1}{\pi} c_{-1}$$

$$\left(B_{-2}'(-\pi) = -\frac{1}{h} = -\frac{1}{\pi},\ B_{-1}'(-\pi) = \frac{1}{h} = \frac{1}{\pi},\ B_0'(-\pi) = 0,\ B_1'(\pi) = 0\right)$$

LER:

$$\begin{bmatrix} 0 \\ -1 \\ 1 \\ -1 \end{bmatrix} = \frac{1}{2} \cdot \begin{bmatrix} -2/\pi & 2/\pi & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} c_{-2} \\ c_{-1} \\ c_0 \\ c_1 \end{bmatrix} \Rightarrow \begin{aligned} c_{-2} &= c_{-1} = (-1) \\ c_0 &= 3 \\ c_1 &= -5 \end{aligned}$$

Tehát:

$$S_2(x) = (-1) B_{2,-2}(x) + (-1) B_{2,-1}(x) + 3 B_{2,0}(x) - 5 B_{2,1}(x)$$

iii.)

$$S_3 = \sum_{j=-3}^{1} c_j \cdot B_{3j} \quad , \quad f(x_i) = S_3(x_i) = \sum_{j=-3}^{1} c_j \cdot B_{3j}(x_i)$$

ahol:

$$B_i(x_{i+1}) = \frac{1}{6} = B_i(x_{i+3})$$

$$B_i(x_{i+2}) = \frac{4}{6}$$

$$B_j(x_i) = 0 \qquad \text{ha } j \neq i-1,\ i-2,\ i-3$$

$$f(x_i) = \frac{1}{6} c_{i-3} + \frac{4}{6} c_{i-2} + \frac{1}{6} c_{i-2} \qquad (i = 0, 1, 2)$$

Megj: Könnyítés:

	x_k	x_{k+1}	x_{k+2}	x_{k+3}	x_{k+4}
$B_{3k}(x)$	0	$1/6$	$4/6$	$1/6$	0
$B'_{3k}(x)$	0	$1/2h$	0	$-1/2h$	0
$B''_{3k}(x)$	0	$1/h^2$	$-\frac{2}{h^2}$	$\frac{1}{h^2}$	0

Hermite perem feltétel:

$$B'_{-3}(-\pi) = -\frac{1}{2\pi} = \frac{1}{6}\left(-\frac{3}{\pi}\right)$$

$$B'_{-2}(-\pi) = 0$$

$$B'_{-1}(-\pi) = \frac{1}{2\pi} = \frac{1}{6}\left(\frac{3}{\pi}\right)$$

$$B'_{-1}(\pi) = -\frac{1}{2\pi} = \frac{1}{6}\left(-\frac{3}{\pi}\right)$$

$$B'_0(\pi) = 0$$

$$B'_1(\pi) = \frac{1}{2\pi} = \frac{1}{6}\left(\frac{3}{\pi}\right)$$

$$\Rightarrow f'(-\pi) = 0 = \frac{1}{6}\left[\left(-\frac{3}{\pi}\right)c_3 + \left(\frac{3}{\pi}\right)c_{-1}\right]$$

$$f'(\pi) = 0 = \frac{1}{6}\left[\left(-\frac{3}{\pi}\right)c_1 + \left(\frac{3}{\pi}\right)c_1\right]$$

LER:

$$\frac{1}{6}\begin{bmatrix} -3/\pi & 0 & 3/\pi & 0 & 0 \\ 1 & 4 & 1 & 0 & 0 \\ 0 & 1 & 4 & 1 & 0 \\ 0 & 0 & 1 & 4 & 1 \\ 0 & 0 & -3/\pi & 0 & 3/\pi \end{bmatrix} \begin{bmatrix} c_{-3} \\ c_{-2} \\ c_{-1} \\ c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \\ -1 \\ 0 \end{bmatrix}$$

$$\Rightarrow \underline{c} = \begin{bmatrix} 3 \\ -3 \\ 3 \\ -3 \\ 3 \end{bmatrix}$$

Tehát: $S_3(x) = 3 \cdot \left(B_{-3} - B_{-2} + B_{-1} - B_0 + B_1\right)$