STUDIA MATHEMATICA FOREST FIRE MODEL ON CONFIGURATION GRAPHS WITH RANDOM NODE DEGREE DISTRIBUTION

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FOREST FIRE MODEL ON CONFIGURATION GRAPHS WITH RANDOM NODE DEGREE DISTRIBUTION

Marina M. Leri

We consider two types of configuration graphs with node degrees being i.i.d. random variables following either the power-law or the Poisson distribution. The distribution parameter is a random variable following the uniform distribution on a predefined interval.

We consider a destructive process which can be interpreted as a fire spreading over the graph links, and could be used for modeling forest fires as well as banking system defaults. This process is often referred to as a "forest fire model". The probability of fire transfer over a graph link either possesses a predefined value and is fixed for all the graph links or is a random variable following the standard uniform distribution. By computer simulation we study the robustness of such graphs from a viewpoint of node survival in the two cases of starting a fire propagation process: the "random ignition" and the "targeted lightning-up". The results on finding the optimal interval of the node degree distribution parameter that would ensure maximum survival of trees in case of a fire are presented. A comparative analysis of various graph models in terms of their robustness to various fire propagation processes was performed.

1. Power-law random graph

We consider random graphs the number of nodes in which is equal to N. Node degrees $\xi_1, \xi_2, \ldots, \xi_N$ are independent identically distributed random variables

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drawn from either one of the following distributions. Power-law distribution:

$$\mathbf{P}\{\xi \geq k\} = k^{-\tau}, \qquad k = 1, 2, \dots, \ \tau > 1, \tag{1.1}$$

or the Poisson distribution with a single shift:

$$\mathbf{P}\{\xi = k + 1\} = \frac{\lambda^k}{k!} e^{-\lambda}, \qquad k = 0, 1, 2, \dots, \ \lambda > 0. \tag{1.2}$$

Node degrees form stubs, numbered in an arbitrary order. Then all stubs are joined one to another equiprobably to form links. If the sum of node degrees appears to be odd, one stub is added to a randomly chosen node to form a lacking connection. In our work we consider power-law graphs with the parameter τ of distribution (1.1) following the uniform distribution on a predefined interval $[a, b]$ and Poisson graphs with the parameter λ of distribution (1.2) uniformly distributed on a predefined interval $[c, d]$. The studies of configuration graphs have been attracting consistent interest because of the wide use of these models for representing massive data networks (see e.g. [3, 4, 6]).

2. Forest fire model on configuration graphs

An interesting trend in the studies of random graphs is the aspect connected with the transmission of various destructive influences through graph links. In particular, we consider the process which can be interpreted as spreading of a fire over the graph links [2], and could have some other applications, for example, modeling of banking system defaults [1].

To study a forest fire model we view graph nodes as trees growing in a certain area of a real forest. Since we consider this area to be limited, the number of trees in it also has to be limited. For this purpose we proposed to use auxiliary square lattice graphs sized 100×100 (see [5]). If a real fire can be transferred from one tree to another, corresponding graph nodes are connected by a link. Thus, in case of full packing every inner auxiliary graph node has 8 adjacent neighbours. Having considered different link allocations on our lattice graphs for about 14 values of an inner node degree m (see [5]) and knowing that $m = \zeta(\tau)$ (where $\zeta(x)$ is the Riemann zeta function), we derive a regression relation between graph size $N \leq 10000$ and the parameter τ of node degree distribution (1.1):

$$N = [9256\tau^{-1.05}], \qquad R^2 = 0.97. \tag{2.1}$$

The expectation of the distribution (1.2) is equal to $\lambda + 1$, therefore $\lambda = m - 1$ and using our auxiliary square lattice graphs we can derive the relation between

N and the parameter λ by analogy with (2.1):

$$N = [907.5\lambda + 2509.4], \qquad R^2 = 0.98. \tag{2.2}$$

Figures 1 and 2 show regression relations between the number of graph nodes N and the parameters of node degree distributions τ for power-law graphs and λ for Poisson graphs.

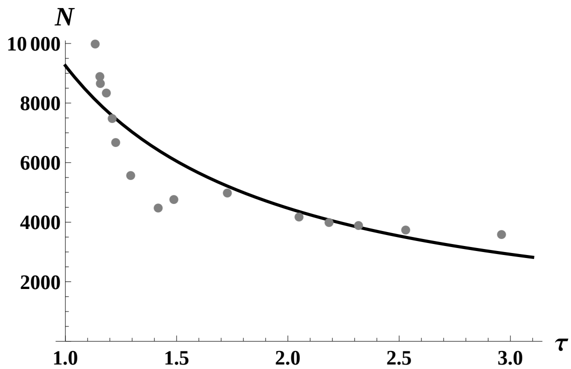

Figure 1: Regression relation between N and τ

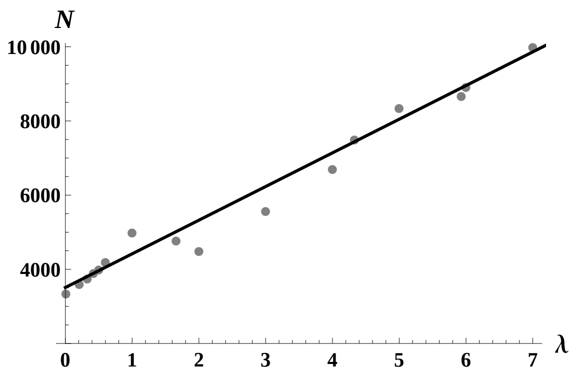

Figure 2: Regression relation between N and λ

These relations (2.1) and (2.2), where $\tau = \overline{\tau} = \dfrac{a+b}{2}$ and $\lambda = \overline{\lambda} = \dfrac{c+d}{2}$, confine the number of nodes N in corresponding configuration graphs. Thus, we study a fire propagation process on configuration graphs with N nodes, where N is calculated from (2.1) for the power-law distribution and from (2.2) for the

Poisson distribution with the parameters τ and λ uniformly distributed on various intervals $[a, b]$ and $[c, d]$, correspondingly. The probability of fire transition p over the graph links is either a predefined value and is fixed for all the graph links or is a random variable following the standard uniform distribution. The purpose is to find for each graph topology the optimal interval for the node degree distribution parameter (τ or λ) that would ensure maximum survival of graph nodes in case of a fire.

We consider the three intervals $[a, b]$: $(1, 2)$, $[2, 3]$, $(1, 3]$ with the N values obtained from (2.1) being equal to 6046, 3536 and 4470, respectively, and the three intervals $[c, d]$: $(0, 1]$, $(0, 2]$, $(0, 3]$ with the following N obtained from (2.2): 2963, 3416 and 3870, respectively. The two considered cases of starting a fire propagation process are the "targeted lightning-up", when the fire starts from the node with the highest degree, and the "random ignition" starting from an equiprobably chosen node. For both cases we found the relations between the average number of nodes surviving in a fire n and the probability of fire transition p. Figures 3 and 4 show how the number of remaining nodes n depends on the probability p for the two cases of starting a fire for the power-law graph topology.

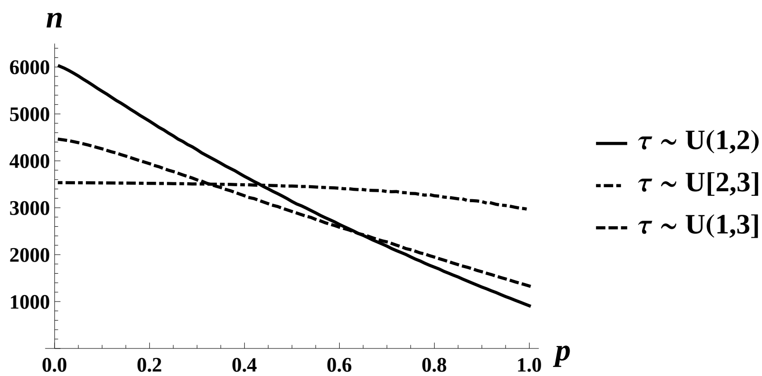

Figure 3: Relation between n and p for power-law graphs ("targeted lightning-up")

Obviously, the number of remaining nodes n is decreasing with the increase of p. The results show that graphs with the distribution parameter $\tau \sim \mathbf{U}(1, 2)$ are more resilient to fire at lower values of the probability p in both cases of fire ignition. But as the value of p increases the topology with the parameter $\tau \sim \mathbf{U}[2, 3]$ will ensure a better survival of graphs nodes. As for power-law graphs with $\tau \sim \mathbf{U}(1, 3]$, they are the most vulnerable to fire in both fire start cases.

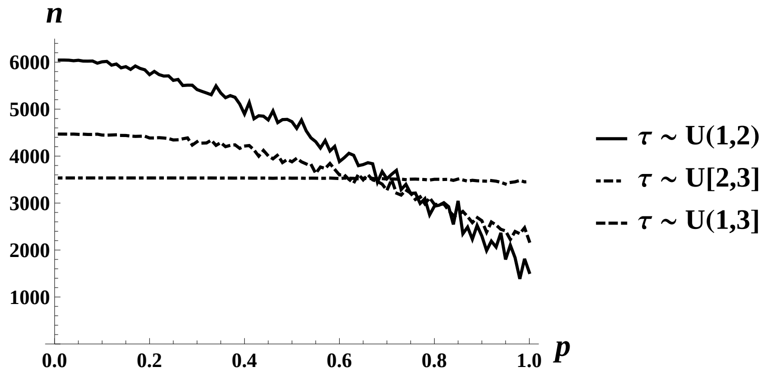

Figure 4: Relation between n and p for power-law graphs ("random ignition")

Similar results were obtained for configuration graphs of the Poisson topology. Here for both fire start cases, which one of the three topologies would ensure a better survival of nodes in case of a fire depends on the value of the probability of fire transition p (see Figures 5 and 6).

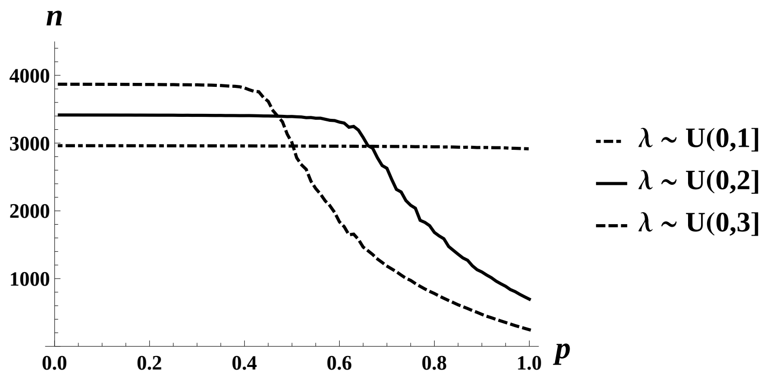

Figure 5: Relation between n and p for Poisson graphs ("targeted lightning-up")

Furthermore, we study a case when the probability of fire transition p is a random variable drawn from the standard uniform distribution. Table 1 shows the results obtained for power-law configuration graphs.

Thus, in the case of targeted lightning-up more nodes will survive in a fire if the graph topology has the parameter $\tau \sim \mathbf{U}[2,3]$. But in the case when a fire starts from an equiprobably chosen node, graphs with $\tau \sim \mathbf{U}(1,2)$ will be more

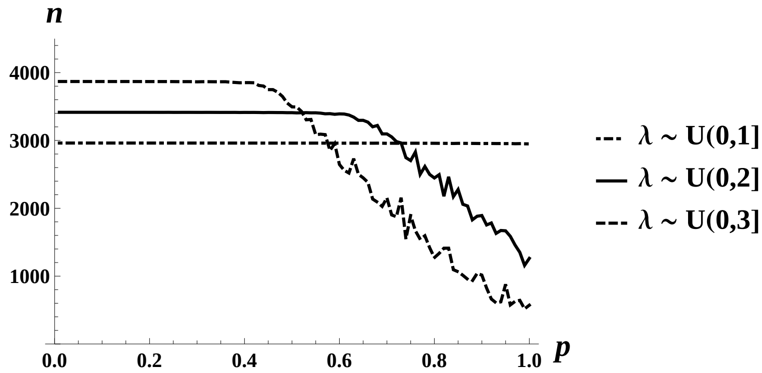

Figure 6: Relation between n and p for Poisson graphs (“random ignition”)

Table 1: An average number of nodes surviving in a fire $\overline{n}$ for power-law graphs

$\tau \sim \mathbf{U}[a,b]$	“targeted lightning-up”	“random ignition”
$(1,2)$	3146	4933
$[2,3]$	3464	3533
$(1,3]$	2917	3927

resilient. As for the Poisson node degree distribution (see Table 2), graphs with the parameter $\lambda \sim \mathbf{U}(0,2]$ are more resilient to fire in the case of a targeted lightning-up, whereas in the case of random ignition graphs with $\lambda \sim \mathbf{U}(0,3]$ will ensure a better nodes survival.

Table 2: An average number of nodes surviving in a fire $\overline{n}$ for Poisson graphs

$\tau \sim \mathbf{U}[c,d]$	“targeted lightning-up”	“random ignition”
$(0,1]$	2957	2960
$(0,2]$	3392	3409
$(0,3]$	2975	3498

Furthermore we have the aim to compare the two considered configuration graph models: with the power-law node degree distribution (1.1) and the Poisson distribution (1.2). Here we have to fix the initial number of nodes N. Then using auxiliary lattice graphs and the correspondence between the inner node degree

m and the number of nodes N (see above) we derive the relations between the parameters τ and λ and the number of nodes N:

$$\tau = 16813N^{-1.067}, \qquad R^2 = 0.87, \tag{2.3}$$

$$\lambda = 0.0011N - 3.61, \qquad R^2 = 0.98, \ N \geq 3350. \tag{2.4}$$

We consider configuration graphs of the same sizes $4000 \leq N \leq 7000$ calculating the corresponding values of τ and λ from (2.3) and (2.4). Here we consider a random probability of fire transition $p \sim \mathbf{U}(0,1)$. For each N were carried out 100 simulations and calculated the values $\mathrm{V} = \mathrm{V}(N)$ that are equal to the difference between the number of nodes remaining after a fire in the model (1.1) and in the model (1.2). Figures 7 and 8 show how values of V depend on N in two cases of fire ignition.

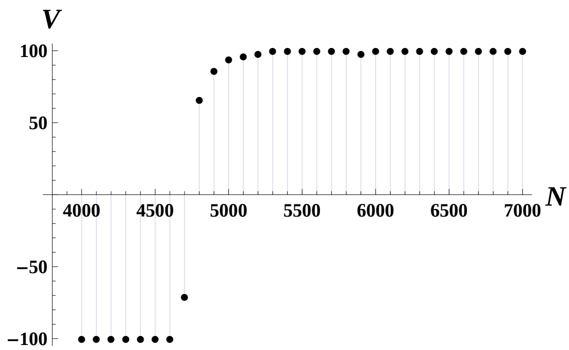

Figure 7: Comparing power-law and Poisson graphs ("targeted lightning-up")

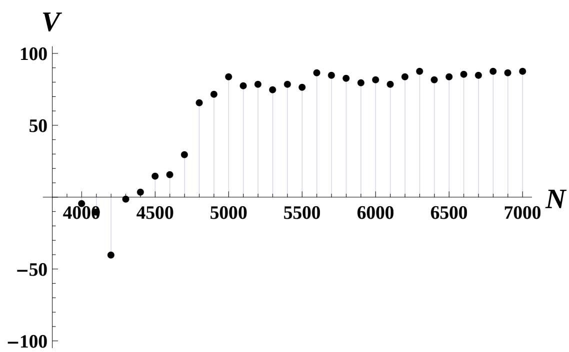

Figure 8: Comparing power-law and Poisson graphs ("random ignition")

Thus in the case of “targeted lightning-up” if $N < 4800$ graphs with the Poisson node degree distribution will ensure a better survival of nodes but, as the number of graph nodes increases, the power-law topology will be more resilient to the fire. In the case of random ignition this threshold for the number of nodes will be 4400.

Applications of these results to the real data may include forest planting procedures, as well as ways to stabilization of banking systems [1].

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