Optimization

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Contents

Fundamental Concepts

1 Fundamental Concepts of Optimization

Theorem 1.1: Optimization problem in standard form

$$\begin{aligned} \underset{x \in \mathbb{R}^n}{\text{minimize}} \quad & f(x) \\ \text{subject to} \quad & g(x) = 0, \\ & h(x) \geq 0. \end{aligned}$$

Theorem 1.2: Level Set

The set

$$\{x \in \mathbb{R}^n \mid f(x) = c\}$$

is the level set of f for the value c, i.e. the set of all points that map to the same value c.

Theorem 1.3: Feasible set

The set

$$\{x \in \mathbb{R}^n \mid g(x) = 0 \ \wedge \ h(x) \geq 0\}$$

is the feasible set Ω, i.e. the set of all points that satisfy the constraints.

Theorem 1.4: Global minimizer

The point x^* is a global minimizer if and only if

$$x^* \in \Omega, \ \forall x \in \Omega : \ f(x^*) \leq f(x).$$

Theorem 1.5: Strict global minimizer

The point x^* is a strict global minimzer if and only if

$$x^* \in \Omega, \ \forall x \in \Omega \setminus \{x^*\} : \ f(x^*) < f(x).$$

Theorem 1.6: Strict local minimizer

The point x^* is a strict local minimizer if and only if $x^* \in \Omega$ and there exists a neighborhood $\mathcal{N}$ of x^* such that

$$\forall x \in (\Omega \cap \mathcal{N}) \setminus \{x^*\} : \ f(x^*) < f(x).$$

Theorem 1.7: Weierstrass

If $\Omega \in \mathbb{R}^n$ is compact, i.e. limited and closed, and $f : \Omega \to \mathbb{R}$ is continuous, then there exists a global minimizer (a solution) of the optimization problem

$$\begin{aligned} \underset{x \in \mathbb{R}^n}{\text{minimize}} \quad & f(x) \\ \text{subject to} \quad & x \in \Omega. \end{aligned}$$

Proof 1.1: Weierstrass

Regard the graph of f, $G = \{(x, f(x)) \in \mathbb{R}^n \times \mathbb{R} \mid x \in \Omega\}$. G is a compact set, and so is the projection of G onto its last coordinate, the set $G = \{s \in \mathbb{R} \mid \exists x \text{ such that } (x, s) \in G\}$, which is a compact interval $[f_{\min}, f_{\max}] \subset \mathbb{R}$. By construction, there must be at least one x^* such that $(x^*, f(x^*)) \in G$.

$\square$

2 Types of Optimization Problems

Theorem 2.1: Convex set

A set $\Omega \subset \mathbb{R}^n$ is convex if

$$\forall x, y \in \Omega,\ \forall \lambda \in [0,1]:\ x + \lambda(y - x) \in \Omega.$$

or if "all connecting lines lie inside the set".

Theorem 2.2: Convex function

A function $f : \Omega \to \mathbb{R}$ is convex if

$$\forall x, y \in \Omega,\ \forall \lambda \in [0,1]:\ f(x + \lambda(y - x)) \le f(x) + \lambda(f(y) - f(x)).$$

or if "all secants (i.e. a line segment between two points on the graph) are above graph". This definition is equivalent to saying that the Epigraph of f, i.e. the set $\{(x, s) \in \mathbb{R}^n \times \mathbb{R} \mid x \in \Omega,\ s \ge f(x)\}$, is a convex set.

Note: a concave function is the same but then with $\ge$ instead of $\le$.

Property 2.1: Convex function

If $f:\ D \to \mathbb{R}$ and $\Omega_f = \{(x, y) \mid x \in D, y \ge f(x)\}$ then the following holds:

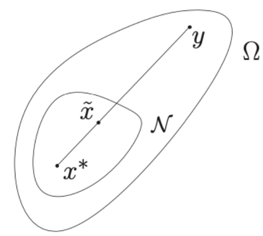
$$f \text{ is convex } \Leftrightarrow\ \Omega_f \text{ is convex.}$$

Property 2.2: Globality of local minima of convex function

For a convex function, a local minimum is also a global one.

Proof 2.1: Globality of local minima of convex function

First we choose, using local optimality, a neighborhood $\mathcal{N}$ of x^* such that for all $\tilde{x} \in \Omega \cap \mathcal{N}$ holds $f(\tilde{x}) \geq f(x^*)$. Second, we regard the connecting line between x^* and y. This line is completely contained in Ω due to the convexity of Ω. Now we choose a point $\tilde{x}$ on this line such that it is in the neighborhood, but not equal to x^*, i.e. $\tilde{x} = x^* + \lambda(y - x^*)$ for some $\lambda \in (0, 1)$, and $\tilde{x} \in \Omega \cap \mathcal{N}$. Due to local optimality, we have $f(\tilde{x}) \geq f(x^*)$, and due to convexity we have



$$f(\tilde{x}) = f(x^* + \lambda(y - x^*)) \leq f(x^*) + \lambda(f(y) - f(x^*)).$$

It follows that $\lambda(f(y) - f(x^*)) \geq 0$, and since $\lambda \in (0, 1)$, we have $f(y) \geq f(x^*)$.

$\square$

Example 2.1: Nonlinear Programming (NLP)

Nonlinear Programming Problems are poblems of the following form:

$$\begin{aligned} \underset{x \in \mathbb{R}^n}{\text{minimize}} \quad & f(x) \\ \text{subject to} \quad & g(x) = 0, \\ & h(x) \geq 0, \end{aligned}$$

where $f : \mathbb{R}^n \to \mathbb{R}$, $g : \mathbb{R}^n \to \mathbb{R}^p$, and $h : \mathbb{R}^n \to \mathbb{R}^q$, are assumed to be continuously differentiable at least once.

Example 2.2: Linear Programming (LP)

When the functions f, g, h are affine (i.e., they can be expressed in the form $f(x) = a^T x + b$ for some vector a and scalar b) in the general formulation (see Theorem 2.1), the general NLP gets something easier to solve, namely a Linear Program (LP), which can be written as follows:

$$\begin{aligned} \underset{x \in \mathbb{R}^n}{\text{minimize}} \quad & c^T x \\ \text{subject to} \quad & Ax - b = 0, \\ & Cx - d \geq 0, \end{aligned}$$

where $c \in \mathbb{R}^n$, $A \in \mathbb{R}^{p \times n}$, $b \in \mathbb{R}^p$, $C \in \mathbb{R}^{q \times n}$, and $d \in \mathbb{R}^q$.

Example 2.3: Quadratic Programming (QP)

When the functions g, h are affine (as for an LP in Theorem 2.2), and the objective function f is a linear-quadratic function, the NLP becomes a Quadratic Program (QP), which can be written as follows:

$$\begin{aligned} \underset{x \in \mathbb{R}^n}{\text{minimize}} \quad & c^T x + \frac{1}{2} x^T B x \\ \text{subject to} \quad & Ax - b = 0, \\ & Cx - d \geq 0, \end{aligned}$$

where, in addition to the LP parameters, $B \in \mathbb{R}^{n \times n}$ is the Hessian matrix. Specifically, for the objective function $f(x) = c^T x + \frac{1}{2} x^T B x$, the Hessian is given by:

$$\nabla^2 f(x) = B.$$

Theorem 2.3: Convex QP

If the Hessian matrix B is positive semi-definite (i.e. if $\forall z \in \mathbb{R}^n \setminus \{0\} : z^T B z \geq 0$) we call the QP a convex QP. Convex QPs are tremendously easier to solve globally than "non-convex QPs" (i.e., where the Hessian B is not positive semi-definite), which might have different local minima (i.e. have a non-convex solution set).

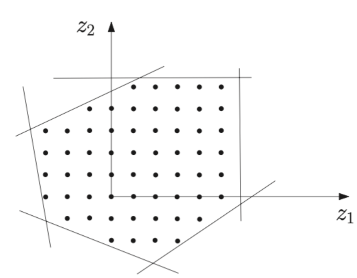
Theorem 2.4: Strictly convex QP

If the Hessian matrix B is positive definite (i.e. if $\forall z \in \mathbb{R}^n : z^T B z > 0$) we call the QP a strictly convex QP. Strictly convex QPs are a subset of convex QPs (see Theorem 2.3).

Example 2.4: Mixed-Integer Programming (MIP)

A mixed-integer programming problem or mixed-integer program (MIP) is a problem with both real and integer decision variables. A MIP can be formulated as follows:

$$\begin{aligned} \underset{\substack{x \in \mathbb{R}^n \\ z \in \mathbb{Z}^m}}{\text{minimize}} \quad & f(x, z) \\ \text{subject to} \quad & g(x, z) = 0, \\ & h(x, z) \geq 0, \end{aligned}$$



where $x \in \mathbb{R}^n$ are the continuous variables, and $z \in \mathbb{Z}^m$ are the integer variables.

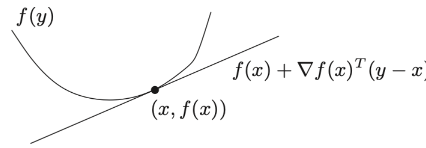
3 Convex Optimization

Theorem 3.1: Convexity for C^1 functions

Assume that $f : \Omega \to \mathbb{R}$ is continuously differentiable and Ω is convex. Then holds that f is convex if and only if

$$\forall x, y \in \Omega : \; f(y) \geq f(x) + \nabla f(x)^T (y - x)$$

i.e. tangents lie below the graph.



Proof 3.1: Convexity for C^1 functions

"$\Rightarrow$": Due to convexity of f holds for given $x, y \in \Omega$ and for any $\lambda \in [0, 1]$ that

$$f(x + \lambda(y - x)) - f(x) \leq \lambda(f(y) - f(x)),$$

and therefore that

$$\frac{f(x + \lambda(y - x)) - f(x)}{\lambda} \leq f(y) - f(x).$$

Furthermore, we can deduce that

$$\lim_{\lambda \to 0} \frac{f(x + \lambda(y - x)) - f(x)}{\lambda} = \nabla f(x)^T (y - x) \leq f(y) - f(x),$$

which proves the statement.

"$\Leftarrow$": To prove that for $z = x + \lambda(y - x) = (1 - \lambda)x + \lambda y$ holds that $f(z) \leq (1 - \lambda)f(x) + \lambda f(y)$, we can use the equation from Theorem 3.1 twice to get

$$f(x) \geq f(z) + \nabla f(z)^T (x - z) \quad \text{and} \quad f(y) \geq f(z) + \nabla f(z)^T (y - z),$$

which yield, when weighted with $(1 - \lambda)$ and λ respectively, that

$$(1 - \lambda)f(x) + \lambda f(y) \geq f(z) + \nabla f(z)^T \underbrace{[(1 - \lambda)(x - z) + \lambda(y - z)]}_{=0}$$

□

Theorem 3.2: Convexity for C^2 functions

Assume that $f : \Omega \to \mathbb{R}$ is twice continuously differentiable and Ω is convex and open. Then holds that f is convex if and only if

$$\forall x \in \Omega : \; \nabla^2 f(x) \succeq 0,$$

i.e. the Hessian of f is positive semi-definite.

Proof 3.2: Convexity for C^2 functions

"⇒": Recall that a second order Taylor expansion of y at x in an arbitrary direction p is given by the following:

$$f(x+tp) = f(x) + t\nabla f(x)^T p + \frac{t^2}{2} p^T \nabla^2 f(x) p + o(t^2\|p\|).$$

From this we obtain that

$$p^T \nabla^2 f(x) p = \lim_{t\to 0} \frac{2}{t^2}\left(\underbrace{f(x+tp) - f(x) - t\nabla f(x)^T p}_{(3.1):\ \geq 0}\right) \geq 0.$$

"⇐": Conversely, to prove the other direction, we use Theorem 19.2 with some arbitrary $\theta \in [0,1]$:

$$f(y) = f(x) + \nabla f(x)^T (y-x) + \underbrace{\frac{t^2}{2}(y-x)^T \nabla^2 f(x+\theta(y-x))(y-x)}_{(3.2):\ \geq 0}.$$

□

Property 3.1: Convexity perserving operations on convex functions

1. Affine input transformation: If $f:\ \Omega \to \mathbb{R}$ is convex, then

$$A \in \mathbb{R}^{n\times m}:\ \tilde{f}(x) = f(Ax+b)$$

   is convex on the domain $\tilde{\Omega} = \{x \mid Ax + b \in \Omega\}$.

2. Concatenation with monotone convex function: If $f:\ \Omega \to \mathbb{R}$ is convex and $g:\ \mathbb{R} \to \mathbb{R}$ is convex and monotonely increasing, then the composition $g \circ f$ is convex.

3. Pointwise supremum: The supremum over a set of convex functions $f_i(x)$, $i \in I$, where I can be an infinite set, i.e.,

$$f(x) = \sup_{i\in I} f_i(x)$$

   is convex.

Proof 3.3: Convexity perserving operations on convex functions

1. ...

2. Recall that g is a convex and monotonely increasing function, then:

$$\nabla^2(g\circ f) = \underbrace{g''(f(x))}_{\geq 0}\underbrace{\nabla f(x)\nabla f(x)^T}_{\succeq 0} + \underbrace{g'(f(x))}_{\geq 0}\underbrace{\nabla^2 f(x)}_{\succeq 0} \succeq 0,$$

   i.e. $g \circ f$ is convex, since the Hessian is positive semi-definite.

3. Epigraph of f is the intersection of the epigraphs of f_i, which are convex.

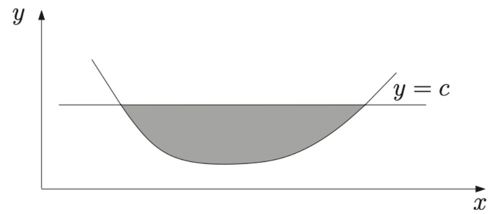
□

Theorem 3.3: Convexity of Sublevel subsets

If $f: \ \Omega \to \mathbb{R}$ is a convex function, then all its level sets

$$\text{lev}_{\leq\gamma} f = \{x \in \Omega \mid f(x) \leq \gamma\}$$

are convex.



Note: the opposite is not true; a function that has all its level sets convex is not necessarily convex.

Proof 3.4: Convexity of Sublevel subsets

If $f(x) \leq c$ and $f(y) \leq c$, then for any $\lambda \in [0,1]$ holds also

$$f((1-\lambda)x + \lambda y) \leq (1-\lambda)f(x) + \lambda f(y) \leq \underbrace{(1-\lambda)c + \lambda c}_{=c}.$$

$\square$

Property 3.2: Convexity perserving operations on convex sets

1. The intersection of finitely or infinitely many convex sets is convex.
2. Affine image: if Ω is convex, then for $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$ also the set $A\Omega + b = \{y \in \mathbb{R}^m \mid \exists x \in \Omega : y = Ax + b\}$ is convex.
3. Affine pre-image: if Ω is convex, then for $A \in \mathbb{R}^{n\times m}$, $b \in \mathbb{R}^n$ also the set $\{z \in \mathbb{R}^m \mid Az + b \in \Omega\}$ is convex.

Example 3.1: Convex Feasible set

If $\forall i \in [1,m] : f_i : \ \mathbb{R}^n \to \mathbb{R}$ are convex functions, then the set

$$\Omega = \{x \in \mathbb{R}^n \mid f_i(x) \leq 0, \ i \in [1,m]\}$$

is a convex set, because it is the intersection of sublevel sets Ω_i of convex functions f_i, i.e.

$$\Omega = \bigcap_{i=1}^{m} \{x \in \mathbb{R}^n \mid f_i(x) \leq 0\}.$$

Example 3.2: Halfspace

A halfspace

$$H_{\leq} = \{x \in \mathbb{R}^n \mid a^T x \leq b\}$$

is a convex set, as the zero sublevel set of the affine (and therefore convex) function $f(x) = a^T x - b$, cf. Theorem 3.3.

Example 3.3: Polyhedral Set

A polyhedral set $C \subset \mathbb{R}^n$ is defined as the intersection of a finite number of halfspaces, i.e.,

$$C = \bigcap_{i=1,\ldots,m} \{x \in \mathbb{R}^n \mid a_i^T x \leq b_i\}.$$

Since the intersection of convex sets is convex, C is a convex set if all the halfspaces are convex.

Note: A polyhedral set might contain equalities, i.e.,

$$C = \{x \in \mathbb{R}^n \mid Ax \leq b,\ Cx = d\}.$$

which can be written with just inequalities as such:

$$C = \{x \in \mathbb{R}^n \mid Ax \leq b,\ Cx \leq d,\ -Cx \leq -d\}.$$

Example 3.4: Ellipsoid

If $P \in \mathbb{R}^{n \times n}$ is symmetric positive definite, then the ellipsoid

$$C = \{x \in \mathbb{R}^n \mid (x - x_c)^T P^{-1}(x - x_c) \leq 1\}$$

is convex, as the 1-sublevel set of the convex function $f(x) = (x - x_c)^T P^{-1}(x - x_c) - 1$, where x_c is the center of the ellipsoid and P is the shape matrix. The latter determines how far the ellipsoid extends in each direction from x_c; the lengths of the semi-axis are given by $\sqrt{\lambda_i}$, where λ_i are the eigenvalues of P, while the eigenvectors of P determine the orientation of the ellipsoid. When $P = r^2 I$, the ellipsoid is a ball with radius r around x_c, i.e.,

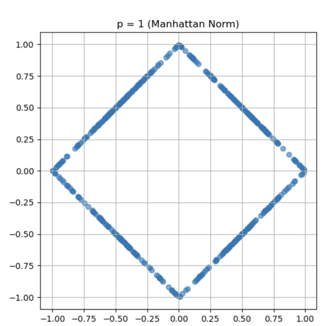
$$C = \{x \in \mathbb{R}^n \mid \|x - x_c\|_2 \leq r\},$$

or, in general, the p-norm ball

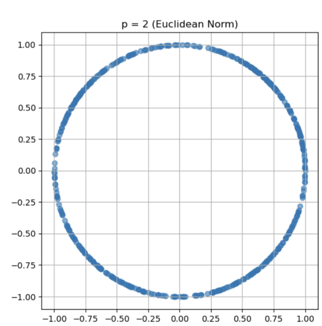
$$C = \{x \in \mathbb{R}^n \mid \|x - x_c\|_p \leq r\}.$$

where the value of p determines the shape of the ball, i.e.,

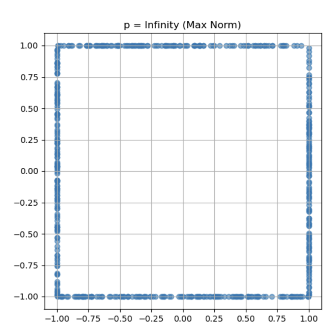
- $p = 1$:



- $p = 2$:



- $p = \infty$:



Note: All images above are for $r = 1$.

Example 3.5: Convex cones

A set C is said to be cone if

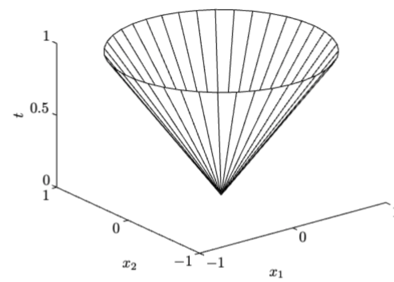
$$\forall x:\ x \in C \ \Rightarrow\ \forall \lambda \geq 0:\ \lambda x \in C.$$

Moreover, C is a convex cone if and only if it is closed under addition and multiplication with non-negative scalars, i.e.,

$$\forall x, y \in C,\ \forall \lambda, \mu \geq 0:\ \lambda x + \mu y \in C.$$

The following are examples of convex cones:

- Non-negative orthant: The set $C = \{x \in \mathbb{R}^n \mid x \geq 0\}$ is a convex cone.
- Norm cones: The set $C = \{(x,t) \in \mathbb{R}^n \times \mathbb{R} \mid \|x\|_p \leq t\}$ is a convex cone. For instance, the Lorenz cone or ice-cream cone, when $p = 2$, is shown on the right.
- Positive semi-definite cone: The set $\mathbb{S}^n_+$ is a convex cone.



Theorem 3.4: Sufficient Condition for Convex NLP

For a nonlinear optimization problem (NLP) in standard form

$$\begin{array}{ll} \underset{x \in \mathbb{R}^n}{\text{minimize}} & f(x) \\ \text{subject to} & g_i(x) = 0,\ i \in [1, m], \\ & h_i(x) \geq 0,\ j \in [1, p], \end{array}$$

the following conditions are necessary for convexity:

- the objective function $f : \mathbb{R}^n \to \mathbb{R}$ must be convex,
- the constraint set

  $$X = \{x \in \mathbb{R}^n \mid g(x) = 0,\ h(x) \geq 0\}$$

  must be convex. Since we know that the intersection of convex sets is convex, we can write X as the intersection of the sets G and H:

  $$\begin{aligned} X &= \{x \in \mathbb{R}^n \mid g(x) = 0,\ h(x) \geq 0\} \\ &= \{x \in \mathbb{R}^n \mid g(x) = 0\} \cap \{x \in \mathbb{R}^n \mid h(x) \geq 0\} \\ &= \left(\bigcap_{i=1}^{m} \{x \in \mathbb{R}^n \mid g_i(x) = 0\}\right) \cap \left(\bigcap_{i=1}^{m} \{x \in \mathbb{R}^n \mid h_i(x) \geq 0\}\right) \\ &= G \cap H \end{aligned}$$

  Now we must consider the requirements for the sets G and H to be convex:

  - Suppose $-H_i = \{x \in \mathbb{R}^n \mid h_i(x) \leq 0\}$, the zero sublevel set of the function h_i. If h_i is convex, the set $-H_i$ is convex, as seen in Theorem 3.3. Now, consider the case where h_i is concave. Since h_i being concave means that $-h_i$ is convex, the set $H_i = \{x \in \mathbb{R}^n \mid h_i(x) \geq 0\}$ is a sublevel set of the convex function $-h_i$, and is therefore convex. Thus, the set H_i is convex when h_i is concave.

- On the other hand, G is the level set of of g_i. Therefore it is certainly a convex set whenever g_i is a affine function

$$\forall i \in [i, m]: \; g_i(x) = a_i^T x + b_i.$$

Theorem 3.5: First Order Optimality condition for convex problems

Regard a convex optimization problem with continuously differentiable objective function f. A point $x^* \in \Omega$ is a global optimizer if and only if

$$\forall y \in \Omega: \; \nabla f(x^*)^T (y - x^*) \geq 0.$$

Proof 3.5: Optimality condition for convex problems

"$\Rightarrow$": Assume for the sake of contradiction that $\exists y \in \Omega: \; \nabla f(x^*)^T (y - x^*) < 0$, then we could regard a Taylor expansion

$$f(x^* + \lambda(y - x^*)) = f(x^*) + \lambda \underbrace{\nabla f(x^*)^T (y - x^*)}_{<0} + \underbrace{o(\lambda)}_{\to 0}.$$

This implies that for sufficiently small positive λ, we have

$$f(x^* + \lambda(x - x^*)) < f(x^*),$$

which contradicts the optimality of x^*.

"$\Leftarrow$": Due to the C^1 characterization of convexity of f in Theorem 3.1, we have for any feasible $y \in \Omega$:

$$f(y) \geq f(x^*) + \underbrace{\nabla f(x^*)^T (y - x^*)}_{\geq 0} \geq f(x^*),$$

which implies that x^* is a global optimizer.

$\square$

4 The Lagrangian Function and Duality

Theorem 4.1: Primal Optimization Problem

We will denote the globally optimal value of the objective function subject to the constraints as the primal optimal value p^*, i.e.,

$$p^* = \left(\min_{x \in \mathbb{R}^n} f(x) \quad \text{s.t.} \quad g(x) = 0 \;\wedge\; h(x) \geq 0 \right).$$

Theorem 4.2: Lagrangian Function and Lagrange Multipliers

We define the Lagrangian function to be

$$\begin{aligned}\mathcal{L}(x, \lambda, \mu) &= f(x) - \sum_{i=1}^{l} \lambda_i g_i(x) - \sum_{i=1}^{m} \mu_i h_i(x) \\ &= f(x) - \lambda^T g(x) - \mu^T h(x).\end{aligned}$$

where $\lambda \in \mathbb{R}^l$ and $\mu \geq 0 \in \mathbb{R}^m$ are the Lagrange multipliers or dual variables.

Lemma 4.1: Lower Bound Property of Lagrangian

If $\tilde{x}$ is a feasible point of (4.1) and $\mu \geq 0$, then

$$\mathcal{L}(\tilde{x}, \lambda, \mu) \leq f(\tilde{x}).$$

Proof 4.1: Lower Bound Property of Lagrangian

Since $\tilde{x}$ is feasible, we have $g(\tilde{x}) = 0$ and $h(\tilde{x}) \geq 0$. Therefore, with $\mu \geq 0$, we have:

$$\mathcal{L}(\tilde{x}, \lambda, \mu) = f(\tilde{x}) + \lambda^T \underbrace{g(\tilde{x})}_{=0} + \underbrace{\mu^T}_{\geq 0} \underbrace{h(\tilde{x})}_{\geq 0} \leq f(\tilde{x}).$$

$\square$

Theorem 4.3: Lagrange Dual Function

The Lagrange dual function is defined as the unconstrained infimum of the Lagrangian function over x, for fixed multipliers λ and μ:

$$q(\lambda, \mu) = \inf_{x \in \mathbb{R}^n} \mathcal{L}(x, \lambda, \mu).$$

Lemma 4.2: Lower Bound property of Lagrange Dual

If $\mu \geq 0$, then the dual function $q(\lambda, \mu)$ is a lower bound on the primal optimal value p^*, i.e.,

$$q(\lambda, \mu) \leq p^*.$$

Proof 4.2: Lower Bound property of Lagrange Dual

Since the Lagrange function is bounded from below by Lemma 4.1, we have

$$q(\lambda, \mu) = \inf_{x \in \mathbb{R}^n} \mathcal{L}(x, \lambda, \mu) \leq f(\tilde{x}) \quad \text{for any feasible } \tilde{x}.$$

Naturally, this inequality holds in particular for the global minimizer x^*, which yields:

$$q(\lambda, \mu) \leq f(x^*) = p^*.$$

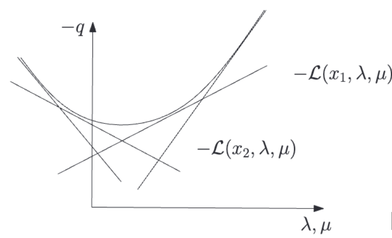
□

Theorem 4.4: Concavity of Lagrange Dual

The function $q : \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}$ is concave, even if the original NLP was not convex.

Proof 4.3: Concavity of Lagrange Dual

We will show that $-q$ is convex. The Lagrangian $\mathcal{L}$ is an affine function in the multipliers λ and μ, which in particular implies that $-\mathcal{L}$ is convex in λ and μ. Thus, the function $-q(\lambda, \mu) = \sup_x -\mathcal{L}(x, \lambda, \mu)$ is the supremum of convex functions in λ and μ that are indexed by x, and therefore convex.



□

Theorem 4.5: Dual Problem

The dual problem is defined as the convex maximization problem, i.e.,

$$d^* = \left(\max_{\lambda \in R^p, \mu \in R^q} q(\lambda, \mu) \quad \text{s.t.} \quad \mu \geq 0 \right).$$

Theorem 4.6: Weak Duality

Consider a primal-dual pair. Then, the following inequality holds:

$$d^* \leq p^*.$$

Theorem 4.7: Strong Duality

If the primal optimization problem is convex and the Slater condition (see Theorem 4.8) holds, then strong duality holds, i.e.,

$$d^* = p^*.$$

Theorem 4.8: Slater Condition

There exists at least one feasible point $\tilde{x} \in \Omega$ such that all non-linear (affine or concave) inequalities are strictly satisfied. For ℓ affine inequalities and $q - \ell$ concave inequalities, the Slater condition is satisfied if and only if there exists an $\tilde{x}$ such that

1. Affine equality:
$$A\tilde{x} + b = 0$$
2. Affine inequality:
$$\forall i \in [1, \ell]: \; h_i(\tilde{x}) \geq 0$$
3. Concave inequality:
$$\forall i \in (\ell, q]: \; h_i(\tilde{x}) > 0$$

Note: This is trivially satisfied for LP and QP problems.

Theorem 4.9: KKT

For a given optimization problem with convex objective and constraints, we have the following equivalence:

1. x^* is a primal optimal and (λ^*, μ^*) are dual optimal.
2. (x^*, λ^*, μ^*) satisfy the Karush-Kuhn-Tucker (KKT) conditions:
   (a) Lagrangian Stationarity (LS):
   $$\nabla f(x) - \nabla g(x)\lambda - \nabla h(x)\mu = 0$$
   (b) Primal Feasibility (PF):
   $$g(x) = 0 \;\wedge\; h(x) \geq 0$$
   (c) Dual Feasibility (DF):
   $$\mu \geq 0$$
   (d) Complementary slackness (CS):
   $$\forall i \in [1, q]: \; \mu_i h_i(x) = 0$$

Unconstrained Optimization Algorithms

5 Optimality Conditions

Theorem 5.1: Unconstrained optimization Problems

We define the unconstrained optimization problem as

$$\min_{x \in D} f(x),$$

where we regard objective function $f : D \to \mathbb{R}$ that are defined on some open domain $D \subseteq \mathbb{R}^n$.

Theorem 5.2: Stationary Point

A point $\tilde{x}$ is called a stationary point of f if and only if

$$\nabla f(\tilde{x}) = 0.$$

Theorem 5.3: Descent Direction

A vector $p \in \mathbb{R}^n$ is called a descent direction at x if

$$\nabla f(x)^T p < 0.$$

Theorem 5.4: First Order Necessary Conditions (FONC)

If $x^* \in D$ is a local minimizer of $f : D \to \mathbb{R}$ and $f \in C^1$ then

$$\nabla f(x^*) = 0.$$

Proof 5.1: First Order Necessary Conditions (FONC)

Let us assume for contradiction that $\nabla f(x^*) \neq 0$. Then $p = -\nabla f(x^*)$ would be a descent direction in which the objective could be improved, as follows:

As D is open and $f \in C^1$, we could find a $t > 0$ that is small enough so that for all $\tau \in [0, t]$ holds $x^* + \tau p \in D$ and $\nabla f(x^* + \tau p)^T p < 0$. By Taylor's theorem, we would have for some $\theta \in (0, t)$ that

$$f(x^* + tp) = f(x^*) + t \underbrace{\nabla f(x^* + \theta p)^T p}_{<0} < f(x^*).$$

$\square$

Theorem 5.5: Second Order Necessary Conditions (SONC)

If $x^* \in D$ is a local minimizer of $f : D \to \mathbb{R}$ and $f \in C^2$ then

$$\nabla^2 f(x^*) \succeq 0.$$

Proof 5.2: Second Order Necessary Conditions (SONC)

Assume, for the sake of contradiction, that $\nabla^2 f(x^*)$ is not positive semi-definite. This implies the existence of a vector $p \in \mathbb{R}^n$ such that $p^T \nabla^2 f(x^*) p < 0$. In this case, the objective function could be improved in the direction of p, as follows:

We could find a $t > 0$ that is small enough so that for all $\tau \in [0, t]$ holds $p^T \nabla^2 f(x^* + \tau p) p < 0$. By Taylor's theorem, we would have for some $\theta \in (0, t)$ that

$$f(x^* + tp) = f(x^*) + \underbrace{t \nabla f(x^*)^T p}_{=0} + \frac{t^2}{2} \underbrace{p^T \nabla^2 f(x^* + \theta p)}_{<0} p < f(x^*).$$

□

Theorem 5.6: Convex First Order Sufficient Conditions (cFOSC)

Assume that $f : D \to \mathbb{R}$ is convex and $f \in C^1$. If $x^* \in D$ is a stationary point of f, then x^* is a global minimizer of f.

Theorem 5.7: Second Order Sufficient Conditions (SOSC)

Assume that $f : D \to \mathbb{R}$ and $f \in C^2$. If $x^* \in D$ is a stationary point of f and $\nabla^2 f(x^*) \succ 0$, then x^* is a strict local minimizer of f.

Proof 5.3: Second Order Sufficient Conditions (SOSC)

We can choose a sufficiently small closed ball B around x^* so that for all $x \in B$ holds $\nabla^2 f(x) \succ 0$. Restricted to this ball, we have a convex problem, so that Theorem 5.6 together with stationarity of x^* implies that x^* is a global minimizer of f. To prove that it is strict, we look for any $x \in B \backslash x^*$ at the Taylor expansion, which yields with some $\theta \in (0, 1)$:

$$f(x) = f(x^*) + \underbrace{\nabla f(x^*)^T (x - x^*)}_{=0} + \frac{1}{2} \underbrace{(x - x^*)^T \nabla^2 f(x^* + \theta(x - x^*))(x - x^*)}_{>0} > f(x^*).$$

□

Theorem 5.8: Solution map

For a parametric optimization problem of the form

$$\min_{x \in D} f(x, a),$$

the dependency of x^* on a in the neighborhood of a fixed value $\overline{a}$, $x^*(a)$ is called the solution map.

Theorem 5.9: Stability of Parametric Solutions

Assume that $f : D \times \mathbb{R}^m \to \mathbb{R}$, and regard the minimization of $f(\cdot, \tilde{a})$ for a given fixed value of $\tilde{a} \in \mathbb{R}^m$. If $\tilde{x} \in D$ satisfies the SOSC (see Theorem 5.7), then there is a neighborhood $\mathcal{N} \subset \mathbb{R}^m$ around $\tilde{a}$ such that the parametric minimizer function $x^*(a)$ is well-defined for all $a \in \mathcal{N}$ (i.e. there is a unique minimizer $x^*(a)$ for each $a \in \mathcal{N}$), is differentiable in $\mathcal{N}$, and $x^*(\tilde{a}) = \tilde{x}$. Its derivative at $\tilde{a}$ is given by

$$\frac{\partial(x^*(\tilde{a}))}{\partial a} = -\left(\nabla_x^2 f(\tilde{x}, \tilde{a})\right)^{-1} \frac{\partial\left(\nabla_x f(\tilde{x}, \tilde{a})\right)}{\partial a}.$$

Moreover, each such $x^*(a)$ with $a \in \mathcal{N}$ satisfies again the SOSC and is thus a strict local minimizer.

Proof 5.4: Stability of Parametric Solutions

The existence of the differentiable map $x^* : \mathcal{N} \to D$ follows from the implicit function theorem applied to the stationarity condition $\nabla_x f(x^*(a), a) = 0$. The derivative of $x^*(a)$ at $\tilde{a}$ is given by

$$\frac{d(\nabla_x f(x^*(a), a))}{da} = \underbrace{\frac{\partial\left(\nabla_x f(x^*(a), a)\right)}{\partial x}}_{=\nabla_x^2 f} \frac{\partial x^*(a)}{\partial a} + \frac{\partial\left(\nabla_x f(x^*(a), a)\right)}{\partial a} = 0$$

The fact that all points $x^*(a)$ satisfy the SOSC follows from the continuity of the second derivative.

□

6 Estimation and Fitting Problems

7 Newton Type Optimization

8 Globalisation Strategies

9 Calculating Derivatives

Constrained Optimization Algorithms

10 Optimality Conditions for Constrained Optimization

Theorem 10.1: Unconstrained optimization Problems

We define the constrained optimization problem as

$$\begin{aligned} \underset{x \in \mathbb{R}^n}{\text{minimize}} \quad & f(x) \\ \text{subject to} \quad & g(x) = 0, \\ & h(x) \geq 0. \end{aligned}$$

in which $f : \mathbb{R}^n \to \mathbb{R}$, $g : \mathbb{R}^n \to \mathbb{R}^m$ and $h : \mathbb{R}^n \to \mathbb{R}^q$ are smooth.

Theorem 10.2: Tangent

A vector $p \in \mathbb{R}^n$ is called a tangent to Ω at $x^* \in \Omega$ if there exists a smooth curve $\overline{x}(t) : [0, \epsilon) \to \mathbb{R}^n$ with

$$\begin{aligned} & \overline{x}(0) = x^* \\ \forall t \in [0, \epsilon) : \; & x(t) \in \Omega \\ & \frac{d}{dp}\overline{x}(0) = p \end{aligned}$$

Theorem 10.3: Tangent Cone

The tangent cone $T_\Omega(x^*)$ of Ω is the set of all tangent vectors at x^*.

Theorem 10.4: First Order Necessary Conditions (FONC): variant 0

If x^* is a local minimizer, then

1. $x^* \in \Omega$
2. $\forall p \in T_\Omega(x^*) : \nabla f(x^*)^T p \geq 0$

Proof 10.1: First Order Necessary Conditions (FONC): variant 0

If $\exists p \in T_\Omega(x^*) : \nabla f(x^*)^T p < 0$, then there would be a feasible curve $\overline{x}(t)$ with $\frac{d}{dt} f(\overline{x}(t))|_{t=0} = \nabla f(x^*)^T p < 0$.

$\square$

Theorem 10.5: (In)active Constraints

An inequality constraint $h_i(x) \geq 0$ is called active at $x^* \in \Omega$ iff $h_i(x^*) = 0$ and otherwise inactive. Inactive constraints do not influence the Tangent Cone.

Theorem 10.6: Active Set

The (1-indexed) index set $\mathcal{A}(x^*) \subset \{1, \ldots, q\}$ of active constraints is called the active set.

Theorem 10.7: Linear Independence Constraint Qualification (LICQ)

The linear independence constraint qualification (LICQ) holds at $x^* \in \Omega$ iff all vectors $\nabla g_i(x^*)$ for $i \in \{1, \ldots, m\}$ and $\nabla h_i(x^*)$ with $i \in \mathcal{A}(x^*)$ are linearly independent.

Theorem 10.8: Linearized Feasible Cone

The linearized feasible cone at $x^* \in \Omega$ is defined as:

$$\mathcal{F}(x^*) = \{p \mid \forall i \in [1, m] : \nabla g_i(x^*)^T p = 0 \ \wedge \ \forall i \in \mathcal{A}(x^*) : \nabla h_i(x^*)^T p \geq 0\}$$

Now we have that at any $x^* \in \Omega$ holds

1. $T_\Omega(x^*) \subset \mathcal{F}(x^*)$
2. If LICQ holds at x^*, then $T_\Omega(x^*) = \mathcal{F}(x^*)$

Theorem 10.9: First Order Necessary Conditions (FONC): variant 1

If LICQ holds at x^* and x^* is a local minimizer, then

1. $x^* \in \Omega$
2. $\forall p \in \mathcal{F}(x^*) : \nabla f(x^*)^T p \geq 0$

Lemma 10.1: Farkas' Lemma

For any matrices $G \in \mathbb{R}^{m \times n}$, $H \in \mathbb{R}^{q \times n}$ and vector $c \in \mathbb{R}^n$ holds exactly one of the following mutually-exclusive statements:

1. $\exists \lambda \in \mathbb{R}^m, \ \mu \in \mathbb{R}^q : \ \mu \geq 0 \ \wedge \ c = G^T \lambda + H^T \mu$
2. $\exists p \in \mathbb{R}^n : Gp = 0 \ \wedge \ Hp \geq 0 \ \wedge \ c^T p < 0$

Proof 10.2: Farkas' Lemma

In the proof we use the separating hyperplane theorem with respect to the point $c \in \mathbb{R}^n$ and the set $S = \{G^T\lambda + H^T\mu \mid \lambda \in \mathbb{R}^m, \mu \in \mathbb{R}^q, \mu \geq 0\}$. Notice that S is a convex cone. The separating hyperplane theorem states that two convex sets - in our case the set S and the point c - can always be separated by a hyperplane. In our case, the hyperplane touches the set S at the origin, and is described by a normal vector p. Separation of S and c means that for all $y \in S$ holds that $y^T p \geq 0$ and on the other hand, $c^T p < 0$. Now we find that either $c \in S$, which would imply the first exclusive statement of the lemma, or $c \notin S$. In the latter case, we can find the following equalities:

$$\begin{aligned}
c \notin S &\Leftrightarrow \exists p \in \mathbb{R}^n, \forall y \in S: \; p^T y \geq 0 \;\wedge\; p^T c < 0 \\
&\Leftrightarrow \exists p \in \mathbb{R}^n, \forall \lambda \in \mathbb{R}^m, \forall \mu \in \mathbb{R}^q, \mu \geq 0: \; p^T(G^T\lambda + H^T\mu) \geq 0 \;\wedge\; p^T c < 0 \\
&\Leftrightarrow \exists p \in \mathbb{R}^n: \; \big(\forall \lambda: \; \lambda^T G p \geq 0 \;\wedge\; \forall \mu \geq 0: \; \mu^T H p \geq 0\big) \;\wedge\; p^T c < 0 \\
&\Leftrightarrow \exists p \in \mathbb{R}^n: Gp = 0 \;\wedge\; Hp \geq 0 \;\wedge\; p^T c < 0
\end{aligned}$$

The last line is equivalent to the second exclusive statement of the lemma.

□

Theorem 10.10: First Order Necessary Conditions (FONC): variant 2

If x^* is a local minimizer and LICQ holds at x^*, then there exists a $\lambda^* \in \mathbb{R}^m$ and $\mu^* \in \mathbb{R}^q$ such that

$$\begin{aligned}
\nabla f(x^*) - \nabla g(x^*)\lambda^* - \nabla h(x^*)\mu^* &= 0 \\
g(x^*) &= 0 \\
h(x^*) &\geq 0 \\
\mu^* &\geq 0 \\
\forall i \in [1, q]: \; \mu_i^* h_i(x^*) &= 0
\end{aligned}$$

Note: The KKT conditions are the First Order Necessary Conditions for Optimality for constrained optimization, and thus are the equivalent to $\nabla f(x^*) = 0$ for unconstrained optimization.

Proof 10.3: First Order Necessary Conditions (FONC): variant 2

Due to the feasability of x^*, we have that $g(x^*) = 0$ and $h(x^*) \geq 0$. Using Farkas' Lemma, we have:

$$\begin{aligned}
\forall p \in \mathcal{F}(x^*): \; p^T \nabla f(x^*) \geq 0 \;&\Leftrightarrow\; \nexists p \in \mathcal{F}(x^*): \; p^T \nabla f(x^*) < 0 \\
&\Leftrightarrow\; \forall \lambda^*, \mu_i^* \geq 0: \; \nabla f(x^*) = \sum_{i=1}^{m} \nabla g_i(x^*)\lambda_i^* + \sum_{i \in \mathcal{A}(x^*)} \nabla h_i(x^*)\mu_i^*
\end{aligned}$$

Now we set all component of μ that are not element of $\mathcal{A}(x^*)$ to zero, i.e. $\mu_i^* = 0$ if $h_i(x^*) > 0$. Now the KKT conditions are fulfilled; the last two trivially, the first one is satisfied due to

$$\forall i \notin \mathcal{A}(x^*): \; \mu_i^* = 0 \;\Rightarrow\; \sum_{i \in \mathcal{A}(x^*)} \nabla h_i(x^*)\mu_i^* = \sum_{i \in \{1,\dots,q\}} \nabla h_i(x^*)\mu_i^*.$$

□

Theorem 10.11: Equivalance of optimality and KKT conditions

Regard a convex NLP and a point $x^* \in \Omega$ at which LICQ holds. Then:

$$x^* \text{ is a global minimizer } \Leftrightarrow \exists \lambda, \mu \text{ so that KKT conditions hold.}$$

Theorem 10.12: Complementarity

Regard a KKT point (x^*, λ, μ). For $i \in \mathcal{A}(x^*)$ we say h_i is weakly active if $\mu_i = 0$, otherwise if $\mu_i > 0$ we say h_i is strictly active. We say that strict complementarity holds at this KKT point iff all active constraints are strictly active. We define the set of weakly active constraints to be $\mathcal{A}_0(x^*, \mu)$ and the set of strictly active constraints to be $\mathcal{A}_+(x^*, \mu)$. The sets are disjoint and their union is the active set, i.e.:

$$\mathcal{A}(x^*) = \mathcal{A}_0(x^*, \mu) \cup \mathcal{A}_+(x^*, \mu).$$

Theorem 10.13: Critical Cone

Regard the KKT point (x^*, λ, μ). The critical cone $C(x^*, \mu)$ is the following set:

$$C(x^*, \mu) = \{p \mid \nabla g_i(x^*)^T p = 0 \ \wedge \ \forall i \in \mathcal{A}_+(x^*, \mu): \ \nabla h_i(x^*)^T p = 0 \ \wedge \ \mathcal{A}_+(x^*, \mu): \ \nabla h_i(x^*)^T p \geq 0\}.$$

Note: $C(x^*, \mu) \subset \mathcal{F}(x^*)$. In case that LICQ holds, even $C(x^*, \mu) = T_\Omega(x^*)$. Thus, the critical cone is a subset of all feasible directions. In fact: it contains all feasible directions which are from first order information neither uphill or downhill directions.

11 Equality Constrained Optimization Algorithms

12 Inequality Constrained Optimization Algorithms

13 Optimal Control Problems

Appendix: Mathematical Preliminaries

14 Vectors

Theorem 14.1: Vector Space $\mathbb{R}^n$

The vector space $\mathbb{R}^n$ is the set of all n-dimensional column vectors with real components. The space $\mathbb{R}^n$ is equipped with the following operations:

- component-wise addition:

$$x + y = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{bmatrix}$$

- scalar multiplication:

$$\alpha x = \begin{bmatrix} \alpha x_1 \\ \alpha x_2 \\ \vdots \\ \alpha x_n \end{bmatrix}$$

Theorem 14.2: Dot product

The dot product of two vectors $x, y \in \mathbb{R}^n$ is defined as the scalar

$$x \cdot y = x^T y = \sum_{i=1}^{n} x_i y_i.$$

Theorem 14.3: Norm

A norm $\|\cdot\|$ on $\mathbb{R}^n$ is a function $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}$ satisfying the following:

- non-negativity: $\|x\| \geq 0$ for any $x \in \mathbb{R}^n$; $\|x\| = 0$ if and only if $x = 0$,
- positive homogeneity: $\|\alpha x\| = |\alpha|\|x\|$ for any $x \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$,
- triangle inequality: $\|x + y\| \leq \|x\| + \|y\|$ for any $x, y \in \mathbb{R}^n$.

Theorem 14.4: ℓ_p-norms

The class of ℓ_p-norms is defined by

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}}$$

which includes the following special cases:

- ℓ_1-norm: $\|x\|_1 = \sum_{i=1}^{n} |x_i|$,
- ℓ_2-norm: $\|x\| = \|x\|_2 = \left(\sum_{i=1}^{n} x_i^2\right)^{\frac{1}{2}}$,
- ℓ_∞-norm: $\|x\|_\infty = \max_{i=1,\dots,n} |x_i|$.

Theorem 14.5: Angle between two vectors

The angle $\angle(x, y)$ between two vectors $x, y \in \mathbb{R}^n$ is given by

$$\angle(x, y) = \arccos\left(\frac{x \cdot y}{\|x\|\|y\|}\right).$$

We say that two vectors $x, y \in \mathbb{R}^n$ are

- orthogonal if $\angle(x, y) = \frac{\pi}{2}$, i.e. $x \cdot y = 0$,
- aligned if $\angle(x, y) = 0$,
- anti-aligned if $\angle(x, y) = \pi$,
- parallel if $\angle(x, y) = 0$ or $\angle(x, y) = \pi$,
- at an acute angle if $\angle(x, y) < \frac{\pi}{2}$,
- at an obtuse angle if $\angle(x, y) > \frac{\pi}{2}$.

Theorem 14.6: Cauchy-Schwarz inequality

For any two vectors $x, y \in \mathbb{R}^n$, the following inequality holds:

$$|x \cdot y| \leq \|x\|\|y\|.$$

Theorem 14.7: Hölder inequality

Let $p \geq 1$. For any $x, y \in \mathbb{R}^n$, the following inequality holds:

$$|x \cdot y| \leq \|x\|_p \|y\|_q,$$

where $q = \frac{p}{p-1}$ is the Hölder conjugate of p.

15 Independence, Subspaces, Basis and Dimension

Theorem 15.1: Linear Independence

A set of vectors $a_1, a_2, \ldots, a_n$ in $\mathbb{R}^n$ is said to be linearly independent if no vector in the collection can be expressed as a combination of the others. In other words

$$\sum_{i=1}^{m} \lambda_i a_i = 0 \;\Rightarrow\; \forall i \in [1, m] : \;\lambda_i = 0$$

Theorem 15.2: Subspace

A nonempty subset S of $\mathbb{R}^n$ is called a subspace if for any real numbers λ_1, λ_2

$$a_1, a_2 \in S \;\Rightarrow\; \lambda_1 a_1 + \lambda_2 a_2 \in S$$

A subspace always contains the zero element.

Theorem 15.3: Span

Given a set of vectors $a_1, a_2, \ldots, a_n$ in $\mathbb{R}^n$, the set of linear combinations of these vectors is called the span, i.e.

$$\text{span}\{a_1, a_2, \ldots, a_n\} = \left\{ y \in \mathbb{R}^n \mid y = \sum_{i=1}^{m} \lambda_i a_i \right\}$$

Theorem 15.4: Basis

The subset $B = \{a_1, a_2, \ldots, a_n\}$ of $\mathbb{R}^n$ is called a basis if it is linearly independent and spans $\mathbb{R}^n$, i.e. it is a naximally independent subset of $\mathbb{R}^n$.

16 Orthogonality and Orthogonal Complements

Theorem 16.1: Orthogonality

A set of nonzero vectors $a_1, a_2, \ldots, a_n$ in $\mathbb{R}^n$ is said to be orthogonal if the dot product of any two distinct vectors is zero, i.e.

$$\forall i, j \in [1, n] : \; i \neq j \;\Rightarrow\; a_i \cdot a_j = 0$$

Consequently, two subspaces S_1 and S_2 of $\mathbb{R}^n$ are orthogonal if every vector in S_1 is orthogonal to every vector in S_2. In that case, S_2 is called the orthogonal complement of S_1 and denoted by $S_1^\perp$. The following now holds true for a subspace S of $\mathbb{R}^n$:

$$\mathbb{R}^n = S \oplus S^\perp$$

Theorem 16.2: Orthonormality

A set of nonzero vectors $a_1, a_2, \ldots, a_n$ in $\mathbb{R}^n$ is said to be orthonormal if it is orthogonal and each vector has a unit length, i.e.

$$\forall i, j \in [1, n] : \; i \neq j \;\Rightarrow\; a_i \cdot a_j = 0 \;\wedge\; \|a_i\| = 1$$

17 Matrices

Example 17.1: Types of matrices

A matrix $A \in \mathbb{R}^{m \times n}$ is said to be

- the zero matrix, denoted by 0, if all its entries are zero,
- a square matrix if $m = n$,
- the identity matrix if it is square and all its diagonal entries are one,
- a diagonal matrix if all its off-diagonal entries are zero,
- an upper triangular matrix if all its entries below the diagonal are zero,
- a lower triangular matrix if all its entries above the diagonal are zero,
- a symmetric matrix if it is square and $A = A^T$, the set of these matrices is denoted by $\mathbb{S}^n$,
- an orthogonal matrix if it is square and $AA^T = A^T A = I$,
- a non-singular matrix if it is square and there exists another square matrix $B \in \mathbb{R}^{n \times n}$, the inverse of A, such that
$$AB = BA = I$$
- a dyadic matrix if it is of the form $A = uv^T$ for some vectors $u, v \in \mathbb{R}^n$.

Theorem 17.1: Range

Given a matrix $A \in \mathbb{R}^{m \times n}$, the range of A is the set of m-dimensional vectors that can be expressed as Ax for some n-dimensional vector x, and we denote it by

$$\mathcal{R}(A) = \{Ax \mid x \in \mathbb{R}^n\}$$

In other words, it is the set of vectors that can be expressed as linear combinations of the columns of A.

Theorem 17.2: Kernel

Given a matrix $A \in \mathbb{R}^{m \times n}$, the kernel of A is the set of n-dimensional vectors that are mapped to the zero vector by A, and we denote it by

$$\mathcal{N}(A) = \{x \in \mathbb{R}^n \mid Ax = 0\}$$

In other words, it is the set of vectors that are orthogonal to the columns of A.

Theorem 17.3: Rank

Given a matrix $A \in \mathbb{R}^{m\times n}$, the rank of A is the dimension of its range, i.e.

$$\operatorname{rank}(A) = \dim(\mathcal{R}(A))$$

Note: The sum of the dimensions of the range of A and the null space (kernel) of A is equal to the number of columns n:

$$\dim(\mathcal{R}(A)) + \dim(\mathcal{N}(A)) = n$$

Theorem 17.4: Fundamental theorem of Linear Algebra

For any matrix $A \in \mathbb{R}^{m\times n}$, it holds that $\mathcal{N}(A) \perp \mathcal{R}(A^T)$ and $\mathcal{N}(A^T) \perp \mathcal{R}(A)$, therefore we have

$$\begin{aligned}\mathbb{R}^n &= \mathcal{N}(A) \oplus \mathcal{R}(A^T)\\ \mathbb{R}^m &= \mathcal{N}(A^T) \oplus \mathcal{R}(A)\end{aligned}$$

Theorem 17.5: Singular Value Decomposition

Every matrix $A \in \mathbb{R}^{m\times n}$ of rank r can be written as

$$A = U\Sigma V^T = \begin{bmatrix} U_1 & U_2 \end{bmatrix}\begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} V_1^\top \\ V_2^\top \end{bmatrix}$$

where $U \in \mathbb{R}^{m\times m}$ and $V \in \mathbb{R}^{n\times n}$ are orthogonal matrices, $U_1 \in \mathbb{R}^{m\times r}$, $V_1 \in \mathbb{R}^{n\times r}$ and

$$\Sigma_1 = \operatorname{diag}(\sigma_1, \sigma_2, \dots, \sigma_r) \in \mathbb{R}^{r\times r}$$

is a diagonal matrix, where $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_r > 0$ are the singular values of A. The columns of U and V are called the left and right singular vectors of A, respectively.

Property 17.1: Singular Value Decomposition

The singular value decomposition of a matrix $A \in \mathbb{R}^{m\times n}$ gives orthogonal bases for the four fundamental subspaces related to A:

$$\begin{aligned}\mathcal{R}(A) &= \mathcal{R}(U_1), \quad \mathcal{N}(A^T) = \mathcal{R}(U_2)\\ \mathcal{R}(A^T) &= \mathcal{R}(V_1), \quad \mathcal{N}(A) = \mathcal{R}(V_2)\end{aligned}$$

Theorem 17.6: Orthogonal-triangular decomposition (QR)

If $A \in \mathbb{R}^{m\times n}$, then there exists an orthogonal matrix $Q \in \mathbb{R}^{m\times m}$ and an upper triangular matrix $R \in \mathbb{R}^{m\times n}$ such that

$$A = QR.$$

Theorem 17.7: Eigenvalue decomposition

Any real symmetric matrix $A \in \mathbb{R}^{n\times n}$ can be decomposed as

$$A = Q\Lambda Q^T,$$

where $Q \in \mathbb{R}^{n\times n}$ is orthogonal, i.e. $Q^T Q = I$, and $\Lambda \in \mathbb{R}^{n\times n}$ is diagonal with the eigenvalues of A on the diagonal. The columns of Q form an orthonormal set of eigenvectors.

Theorem 17.8: Symmetric positive semi-definite matrices

We write for a symmetric matrix $B = B^T$, $B \in \mathbb{R}^{n\times n}$ that "$B \succeq 0$" if and only if B is positive semi-definite, i.e.,

$$\forall z \in \mathbb{R}^n: \; z^T B z \geq 0,$$

or, equivalently, if all (real) eigenvalues of B are non-negative.The set of all symmetric positive semi-definite matrices is denoted by $\mathbb{S}^n_+$.

Property 17.2: Symmetric positive semi-definite matrices

Let $Q \in \mathbb{S}^n$ be a symmetric matrix. Then the following statements are equivalent:

1. Q is positive semi-definite, i.e. $Q \succeq 0$,
2. all eigenvalues of Q are non-negative, i.e. $\lambda_i(Q) \geq 0$ for all $i \in [1, n]$,
3. all principal minors of Q, i.e. the determinant of a submatrix obtained from Q when the same set of rows and columns are stricken out, are non-negative,
4. Q can be written as $Q = AA^T$ for some matrix $A \in \mathbb{R}^{n\times r}$ and r is the rank of Q.

Theorem 17.9: Symmetric positive definite matrices

A symmetric is positive definite if $B \succ 0$, i.e.,

$$\forall z \in \mathbb{R}^n \backslash \{0\}: \; z^T B z > 0,$$

and the set of symmetric positive definite matrices is denoted by $\mathbb{S}^n_{++}$.

Theorem 17.10: Cholesky factorization

If $A \in \mathbb{R}^{n\times n}$ is symmetric positive definite, then there exists a unique lower triangular matrix $L \in \mathbb{R}^{n\times n}$ with positive diagonal entries such that

$$A = LL^T.$$

Theorem 17.11: Matrix norms

For a matrix $A \in \mathbb{R}^{m \times n}$ and two vector norms $\|\cdot\|_p$ and $\|\cdot\|_q$, the induced matrix norm is defined as

$$\|A\|_{p,q} = \max_x \{\|Ax\|_q \|x\|_p \leq 1\}.$$

When $p = q$, we simply write $\|A\|_p$.

Example 17.2: Spectral norm

The ℓ_2-induced norm or spectral norm of a matrix $A \in \mathbb{R}^{m \times n}$ is defined as

$$\|A\|_2 = \sqrt{\lambda_{\max}(A^T A)} = \sigma_{\max}(A),$$

where $\lambda_{\max}(A^T A)$ denotes the largest eigenvalue of $A^T A$ and $\sigma_{\max}(A)$ is the largest singular value of A.

Example 17.3: Frobenius norm

The Frobenius norm of a matrix $A \in \mathbb{R}^{m \times n}$ is defined as

$$\|A\|_F = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}^2} = \sqrt{\operatorname{tr}(A^T A)}.$$

18 Sequences

Lemma 18.1: Convergence

If $\{x_k\}$ is a non-increasing and bounded below or non-decreasing and bounded above sequence, it converges to a finite real number.

Theorem 18.1: $O(\cdot)$

For a function $f : \mathbb{R}^n \to \mathbb{R}^m$, we write

$$f(x) = O(g(x))$$

if and only if there exists a constant $C > 0$ and a neighborhood $\mathcal{N}$ of 0 such that

$$\forall x \in \mathcal{N} : \ \|f(x)\| \leq C g(x),$$

i.e. "f shrinks as fast as g".

Theorem 18.2: $o(\cdot)$

For a function $f:\mathbb{R}^n \to \mathbb{R}^m$, we write
$$f(x) = o(g(x))$$
if and only if there exists a neighborhood $\mathcal{N}$ of 0 and a function $c:\mathcal{N}\to\mathbb{R}$ with $\lim_{x\to 0} c(x) = 0$ such that
$$\forall x \in \mathcal{N}: \ \|f(x)\| \le c(x)g(x),$$
i.e. "f shrinks faster than g".

19 Differential Calculus

Theorem 19.1: Lipschitz continuity

A mapping $F:\mathbb{R}^n \to \mathbb{R}^m$ is Lipschitz continuous with Lipschitz constant L if
$$\|F(x) - F(y)\| \le L\|x - y\| \quad \forall x, y \in \mathbb{R}^n.$$

Theorem 19.2: Linear mapping

A mapping $F:\mathbb{R}^n \to \mathbb{R}^m$ is linear if
$$\forall x, y \in \mathbb{R}^n,\ \forall \lambda_1, \lambda_2 \in \mathbb{R}: \ F(\lambda_1 x + \lambda_2 y) = \lambda_1 F(x) + \lambda_2 F(y).$$

Theorem 19.3: Affine mapping

A mapping $F:\mathbb{R}^n \to \mathbb{R}^m$ is affine if it is the sum of a linear mapping and a constant vector, i.e.
$$\exists A \in \mathbb{R}^{m\times n},\ \exists b \in \mathbb{R}^m: \ F(x) = Ax + b.$$

Theorem 19.4: Quadratic function

A function $f:\mathbb{R}^n \to \mathbb{R}$ is quadratic if it can be written as
$$f(x) = \frac{1}{2}x^T Q x + q^T x + c$$
for some matrix $Q \in \mathbb{R}^{n\times n}$, vector $q \in \mathbb{R}^n$, and scalar $c \in \mathbb{R}$.

Theorem 19.5: First-order Taylor expansion

Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable at $x \in \mathbb{R}^n$. Then for all $y \in \mathbb{R}^n$, it holds that

$$f(y) = f(x) + \nabla f(x)^T (y - x) + o(\|y - x\|).$$

Lemma 19.1: Mean-value theorem

Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable. Then for every $x, y \in \mathbb{R}^n$ there exists a $\tau \in (0, 1)$ such that

$$f(y) = f(x) + \nabla f(x + \tau(y - x))^T (y - x),$$

Moreover, if f is continuously differentiable, then

$$f(y) = f(x) + \int_0^1 \nabla f(x + t(y - x))^T (y - x) dt$$

Theorem 19.6: Hessian

The Hessian of a function $f : \mathbb{R}^n \to \mathbb{R}$ is the matrix of second partial derivatives, i.e.

$$\nabla^2 f(x) = \left[\frac{\partial^2 f(x)}{\partial x_i \partial x_j} \right]_{i,j=1}^n$$

Theorem 19.7: Second-order Taylor expansion

Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is twice continuously differentiable at $x \in \mathbb{R}^n$. Then for all $y \in \mathbb{R}^n$, it holds that

$$f(y) = f(x) + \nabla f(x)^T (y - x) + \frac{1}{2}(y - x)^T \nabla^2 f(x)(y - x) + o(\|y - x\|^2).$$

Lemma 19.2: Taylor Rest Term theorem

Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable. Then for every $x, y \in \mathbb{R}^n$ there exists a $\theta \in [0, 1]$ such that

$$f(y) = f(x) + \nabla f(x)^T (y - x) + \frac{1}{2}(y - x)^T \nabla^2 f(x + \theta(y - x))(y - x).$$

Theorem 19.8: Jacobian

The Jacobian of a mapping $F : \mathbb{R}^n \to \mathbb{R}^m$ is the matrix of transposed gradients, i.e.

$$J_F(x) = \begin{bmatrix} \nabla f_1(x)^\top \\ \nabla f_2(x)^\top \\ \vdots \\ \nabla f_m(x)^\top \end{bmatrix} \in \mathbb{R}^{m \times n}$$

Application 19.1: Mean-value theorem for vector-valued functions

Suppose that $F : \mathbb{R}^n \to \mathbb{R}^m$ is continuously differentiable on $\mathbb{R}^n$. Then for every $x, y \in \mathbb{R}^n$ the following holds:

$$F(y) = F(x) + \int_0^1 J_F(x + t(y-x))(y-x)dt.$$

Theorem 19.9: Implicit function theorem

Let $F : \mathbb{R}^{n+m} \to \mathbb{R}^n$ be a continuously differentiable mapping of $x \in \mathbb{R}^n$ and $p \in \mathbb{R}^m$. If $x^* \in \mathbb{R}^n, p^* \in \mathbb{R}^m$ are such that

1. $F(x^*, p^*) = 0$,
2. the partial Jacobian $J_{F_x}(x^*, p^*)$ is non-singular,

then there exist open sets $S_{x^*} \subset \mathbb{R}^n, S_{p^*} \subset \mathbb{R}^m$ and a continuously differentiable function $g : S_{p^*} \to S_{x^*}$ such that

$$x^* = g(p^*) \quad \text{and} \quad F(g(p), p) = 0 \quad \forall p \in S_{p^*},$$

and

$$J_g(p^*) = -(J_{F_x}(g(x^*), p^*))^{-1} J_{F_p}(g(x^*), p^*).$$