

Optimization

Pieter Vanderschueren

Year 2024-2025

Contents

Fundamental Concepts	2
1 Fundamental Concepts of Optimization	3
2 Types of Optimization Problems	4
3 Convex Optimization	7
4 The Lagrangian Function and Duality	7
Unconstrained Optimization Algorithms	8
5 Optimality Conditions	9
6 Estimation and Fitting Problems	10
7 Newton Type Optimization	11
8 Globalisation Strategies	12
9 Calculating Derivatives	13
Constrained Optimization Algorithms	14
10 Optimality Conditions for Constrained Optimization	15
11 Equality Constrained Optimization Algorithms	16
12 Inequality Constrained Optimization Algorithms	17
13 Optimal Control Problems	18

Fundamental Concepts

1 Fundamental Concepts of Optimization

Theorem 1.1: Optimization problem in standard form

$$\begin{array}{ll}\text{minimize} & f(x) \\ x \in \mathbb{R}^n & \\ \text{subject to} & g(x) = 0, \\ & h(x) \geq 0.\end{array}$$

Theorem 1.2: Level Set

The set

$$\{ x \in \mathbb{R}^n \mid f(x) = c \}$$

is the level set of f for the value c .

Theorem 1.3: Feasible set

The set

$$\{ x \in \mathbb{R}^n \mid g(x) = 0, h(x) \geq 0 \}$$

is the feasible set Ω .

Theorem 1.4: Global minimizer

The point x^* is a global minimizer if and only if

$$x^* \in \Omega, \forall x \in \Omega : f(x^*) \leq f(x).$$

Theorem 1.5: Strict global minimizer

The point x^* is a strict global minimizer if and only if

$$x^* \in \Omega, \forall x \in \Omega \setminus \{x^*\} : f(x^*) < f(x).$$

Theorem 1.6: Strict local minimizer

The point x^* is a strict local minimizer if and only if $x^* \in \Omega$ and there exists a neighborhood \mathcal{N} of x^* such that

$$\forall x \in (\Omega \cap \mathcal{N}) \setminus \{x^*\} : f(x^*) < f(x).$$

Theorem 1.7: Weierstrass

if $\Omega \subset \mathbb{R}^n$ is compact and $f : \Omega \rightarrow \mathbb{R}$ is continuous, then there exists a global minimizer of the optimization problem

$$\begin{aligned} & \underset{x \in \mathbb{R}^n}{\text{minimize}} && f(x) \\ & \text{subject to} && x \in \Omega. \end{aligned}$$

Proof 1.1: Weierstrass

Regard the graph of f , $G = \{(x, f(x)) \in \mathbb{R}^n \times \mathbb{R} \mid x \in \Omega\}$. G is a compact set, and so is the projection of G onto its last coordinate, the set $G = \{s \in \mathbb{R} \mid \exists x \text{ such that } (x, s) \in G\}$, which is a compact interval $[f_{\min}, f_{\max}] \subset \mathbb{R}$. By construction, there must be at least one x^* such that $(x^*, f(x^*)) \in G$. □

2 Types of Optimization Problems

Theorem 1.8: Nonlinear Programming (NLP)

Nonlinear Programming Problems are problems of the following form:

$$\begin{aligned} & \underset{x \in \mathbb{R}^n}{\text{minimize}} && f(x) \\ & \text{subject to} && g(x) = 0, \\ & && h(x) \geq 0, \end{aligned}$$

where $f : \mathbb{R}^n \rightarrow \mathbb{R}$, $g : \mathbb{R}^n \rightarrow \mathbb{R}^p$, and $h : \mathbb{R}^n \rightarrow \mathbb{R}^q$, are assumed to be continuously differentiable at least once.

Theorem 1.9: Linear Programming (LP)

When the functions f , g , h are affine (i.e., they can be expressed in the form $f(x) = a^T x + b$ for some vector a and scalar b) in the general formulation (see Theorem 1.8), the general NLP gets something easier to solve, namely a Linear Program (LP), which can be written as follows:

$$\begin{aligned} & \underset{x \in \mathbb{R}^n}{\text{minimize}} && c^T x \\ & \text{subject to} && Ax - b = 0, \\ & && Cx - d \geq 0, \end{aligned}$$

where $c \in \mathbb{R}^n$, $A \in \mathbb{R}^{p \times n}$, $b \in \mathbb{R}^p$, $C \in \mathbb{R}^{q \times n}$, and $d \in \mathbb{R}^q$.

Theorem 1.10: Quadratic Programming (QP)

When the functions g, h are affine (as for an LP in Theorem 1.9), and the objective function f is a linear-quadratic function, the NLP becomes a Quadratic Program (QP), which can be written as follows:

$$\begin{aligned} & \underset{x \in \mathbb{R}^n}{\text{minimize}} && c^T x + \frac{1}{2} x^T B x \\ & \text{subject to} && Ax - b = 0, \\ & && Cx - d \geq 0, \end{aligned}$$

where, in addition to the LP parameters, $B \in \mathbb{R}^{n \times n}$ is the Hessian matrix. Specifically, for the objective function $f(x) = c^T x + \frac{1}{2} x^T B x$, the Hessian is given by:

$$\nabla^2 f(x) = B.$$

Theorem 1.11: Convex QP

If the Hessian matrix B is positive semi-definite (i.e. if $\forall z \in \mathbb{R}^n \setminus \{0\} : z^T B z \geq 0$) we call the QP a convex QP. Convex QPs are tremendously easier to solve globally than “non-convex QPs” (i.e., where the Hessian B is not positive semi-definite), which might have different local minima (i.e. have a non-convex solution set).

Theorem 1.12: Strictly convex QP

If the Hessian matrix B is positive definite (i.e. if $\forall z \in \mathbb{R}^n \setminus \{0\} : z^T B z > 0$) we call the QP a strictly convex QP. Strictly convex QPs are a subset of convex QPs (see Theorem 1.11).

Theorem 1.13: Convex set

A set $\Omega \subset \mathbb{R}^n$ is convex if

$$\forall x, y \in \Omega, \forall \lambda \in [0, 1] : x + \lambda(y - x) \in \Omega.$$

or if “all connecting lines lie inside the set”.

Theorem 1.14: Convex function

A function $f : \Omega \rightarrow \mathbb{R}$ is convex if

$$\forall x, y \in \Omega, \forall \lambda \in [0, 1] : f(x + \lambda(y - x)) \leq f(x) + \lambda(f(y) - f(x)).$$

or if “all secants (i.e. a line segment between two points on the graph) are above graph”. This definition is equivalent to saying that the Epigraph of f , i.e. the set $\{(x, s) \in \mathbb{R}^n \times \mathbb{R} \mid x \in \Omega, s \geq f(x)\}$, is a convex set.

Theorem 1.15: Convex optimization problem

An optimization problem with convex feasible set Ω and convex objective function $f : \Omega \rightarrow \mathbb{R}$ is called a convex optimization problem.

Theorem 1.16: Globality of local minima of convex optimization problem

For a convex optimization problem, a local minimum is also a global one.

Proof 1.2: Globality of local minima of convex optimization problem

Regard a local minimum x^* of the convex optimization problem

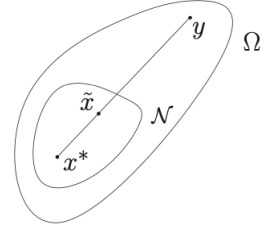
$$\begin{aligned} & \underset{x \in \mathbb{R}^n}{\text{minimize}} && f(x) \\ & \text{subject to} && x \in \Omega. \end{aligned}$$

We will show that for any point $y \in \Omega$ holds $f(y) \geq f(x^*)$, i.e. x^* is a global minimum.

First we choose, using local optimality, a neighborhood \mathcal{N} of x^* such that for all $\tilde{x} \in \Omega \cap \mathcal{N}$ holds $f(\tilde{x}) \geq f(x^*)$. Second, we regard the connecting line between x^* and y . This line is completely contained in Ω due to the convexity of Ω . Now we choose a point \tilde{x} on this line such that it is in the neighborhood, but not equal to x^* , i.e. $\tilde{x} = x^* + \lambda(y - x^*)$ for some $\lambda \in (0, 1)$, and $\tilde{x} \in \Omega \cap \mathcal{N}$. Due to local optimality, we have $f(\tilde{x}) \geq f(x^*)$, and due to convexity we have

$$f(\tilde{x}) = f(x^* + \lambda(y - x^*)) \leq f(x^*) + \lambda(f(y) - f(x^*)).$$

It follows that $\lambda(f(y) - f(x^*)) \geq 0$, and since $\lambda \in (0, 1)$, we have $f(y) \geq f(x^*)$, as desired. □



Theorem 1.17: Unconstrained optimization problem

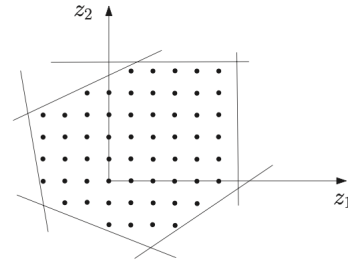
An optimization problem with no constraints, i.e. $g(x) = 0$ and $h(x) \geq 0$ are empty, is called an unconstrained optimization problem.

Theorem 1.18: Mixed-Integer Programming (MIP)

A mixed-integer programming problem or mixed-integer program (MIP) is a problem with both real and integer decision variables. A MIP can be formulated as follows:

$$\begin{array}{ll}\text{minimize} & f(x, z) \\ & x \in \mathbb{R}^n \\ & z \in \mathbb{Z}^m \\ \text{subject to} & g(x, z) = 0, \\ & h(x, z) \geq 0,\end{array}$$

where $x \in \mathbb{R}^n$ are the continuous variables, and $z \in \mathbb{Z}^m$ are the integer variables.



3 Convex Optimization

4 The Lagrangian Function and Duality

Unconstrained Optimization Algorithms

5 Optimality Conditions

6 Estimation and Fitting Problems

7 Newton Type Optimization

8 Globalisation Strategies

9 Calculating Derivatives

Constrained Optimization Algorithms

10 Optimality Conditions for Constrained Optimization

11 Equality Constrained Optimization Algorithms

12 Inequality Constrained Optimization Algorithms

13 Optimal Control Problems