Optimization

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Contents

Mathematical Preliminaries

1 Vectors

Theorem 1.1: Vector Space $\mathbb{R}^n$

The vector space $\mathbb{R}^n$ is the set of all n-dimensional column vectors with real components. The space $\mathbb{R}^n$ is equipped with the following operations:

- component-wise addition:

$$x + y = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{bmatrix}$$

- scalar multiplication:

$$\alpha x = \begin{bmatrix} \alpha x_1 \\ \alpha x_2 \\ \vdots \\ \alpha x_n \end{bmatrix}$$

Theorem 1.2: Dot product

The dot product of two vectors $x, y \in \mathbb{R}^n$ is defined as the scalar

$$x \cdot y = x^T y = \sum_{i=1}^{n} x_i y_i.$$

Theorem 1.3: Norm

A norm $\|\cdot\|$ on $\mathbb{R}^n$ is a function $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}$ satisfying the following:

- non-negativity: $\|x\| \geq 0$ for any $x \in \mathbb{R}^n$; $\|x\| = 0$ if and only if $x = 0$,
- positive homogeneity: $\|\alpha x\| = |\alpha|\|x\|$ for any $x \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$,
- triangle inequality: $\|x + y\| \leq \|x\| + \|y\|$ for any $x, y \in \mathbb{R}^n$.

Theorem 1.4: ℓ_p-norms

The class of ℓ_p-norms is defined by

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}}$$

which includes the following special cases:

- ℓ_1-norm: $\|x\|_1 = \sum_{i=1}^{n} |x_i|$,
- ℓ_2-norm: $\|x\| = \|x\|_2 = \left(\sum_{i=1}^{n} x_i^2\right)^{\frac{1}{2}}$,
- ℓ_∞-norm: $\|x\|_\infty = \max_{i=1,\ldots,n} |x_i|$.

Theorem 1.5: Angle between two vectors

The angle $\angle(x, y)$ between two vectors $x, y \in \mathbb{R}^n$ is given by

$$\angle(x, y) = \arccos\left(\frac{x \cdot y}{\|x\|\|y\|}\right).$$

We say that two vectors $x, y \in \mathbb{R}^n$ are

- orthogonal if $\angle(x, y) = \frac{\pi}{2}$, i.e. $x \cdot y = 0$,
- aligned if $\angle(x, y) = 0$,
- anti-aligned if $\angle(x, y) = \pi$,
- parallel if $\angle(x, y) = 0$ or $\angle(x, y) = \pi$,
- at an acute angle if $\angle(x, y) < \frac{\pi}{2}$,
- at an obtuse angle if $\angle(x, y) > \frac{\pi}{2}$.

Theorem 1.6: Cauchy-Schwarz inequality

For any two vectors $x, y \in \mathbb{R}^n$, the following inequality holds:

$$|x \cdot y| \leq \|x\|\|y\|.$$

Theorem 1.7: Hölder inequality

Let $p \geq 1$. For any $x, y \in \mathbb{R}^n$, the following inequality holds:

$$|x \cdot y| \leq \|x\|_p \|y\|_q,$$

where $q = \frac{p}{p-1}$ is the Hölder conjugate of p.

2 Independence, Subspaces, Basis and Dimension

Theorem 2.1: Linear Independence

A set of vectors $a_1, a_2, \ldots, a_n$ in $\mathbb{R}^n$ is said to be linearly independent if no vector in the collection can be expressed as a combination of the others. In other words

$$\sum_{i=1}^{m} \lambda_i a_i = 0 \;\Rightarrow\; \forall i \in [1, m] : \; \lambda_i = 0$$

Theorem 2.2: Subspace

A nonempty subset S of $\mathbb{R}^n$ is called a subspace if for any real numbers λ_1, λ_2

$$a_1, a_2 \in S \;\Rightarrow\; \lambda_1 a_1 + \lambda_2 a_2 \in S$$

A subspace always contains the zero element.

Theorem 2.3: Span

Given a set of vectors $a_1, a_2, \ldots, a_n$ in $\mathbb{R}^n$, the set of linear combinations of these vectors is called the span, i.e.

$$\text{span}\{a_1, a_2, \ldots, a_n\} = \left\{ y \in \mathbb{R}^n \mid y = \sum_{i=1}^{m} \lambda_i a_i \right\}$$

Theorem 2.4: Basis

The subset $B = \{a_1, a_2, \ldots, a_n\}$ of $\mathbb{R}^n$ is called a basis if it is linearly independent and spans $\mathbb{R}^n$, i.e. it is a naximally independent subset of $\mathbb{R}^n$.

3 Orthogonality and Orthogonal Complements

Theorem 3.1: Orthogonality

A set of nonzero vectors $a_1, a_2, \ldots, a_n$ in $\mathbb{R}^n$ is said to be orthogonal if the dot product of any two distinct vectors is zero, i.e.

$$\forall i, j \in [1, n] : \; i \neq j \;\Rightarrow\; a_i \cdot a_j = 0$$

Consequently, two subspaces S_1 and S_2 of $\mathbb{R}^n$ are orthogonal if every vector in S_1 is orthogonal to every vector in S_2. In that case, S_2 is called the orthogonal complement of S_1 and denoted by $S_1^\perp$. The following now holds true for a subspace S of $\mathbb{R}^n$:

$$\mathbb{R}^n = S \oplus S^\perp$$

Theorem 3.2: Orthonormality

A set of nonzero vectors $a_1, a_2, \ldots, a_n$ in $\mathbb{R}^n$ is said to be orthonormal if it is orthogonal and each vector has a unit length, i.e.

$$\forall i, j \in [1, n] : \; i \neq j \;\Rightarrow\; a_i \cdot a_j = 0 \;\wedge\; \|a_i\| = 1$$

4 Matrices

Example 4.1: Types of matrices

A matrix $A \in \mathbb{R}^{m \times n}$ is said to be

- the zero matrix, denoted by 0, if all its entries are zero,
- a square matrix if $m = n$,
- the identity matrix if it is square and all its diagonal entries are one,
- a diagonal matrix if all its off-diagonal entries are zero,
- an upper triangular matrix if all its entries below the diagonal are zero,
- a lower triangular matrix if all its entries above the diagonal are zero,
- a symmetric matrix if it is square and $A = A^T$, the set of these matrices is denoted by $\mathbb{S}^n$,
- an orthogonal matrix if it is square and $AA^T = A^T A = I$,
- a non-singular matrix if it is square and there exists another square matrix $B \in \mathbb{R}^{n \times n}$, the inverse of A, such that
$$AB = BA = I$$
- a dyadic matrix if it is of the form $A = uv^T$ for some vectors $u, v \in \mathbb{R}^n$.

Theorem 4.1: Range

Given a matrix $A \in \mathbb{R}^{m \times n}$, the range of A is the set of m-dimensional vectors that can be expressed as Ax for some n-dimensional vector x, and we denote it by

$$\mathcal{R}(A) = \{Ax \mid x \in \mathbb{R}^n\}$$

In other words, it is the set of vectors that can be expressed as linear combinations of the columns of A.

Theorem 4.2: Kernel

Given a matrix $A \in \mathbb{R}^{m \times n}$, the kernel of A is the set of n-dimensional vectors that are mapped to the zero vector by A, and we denote it by

$$\mathcal{N}(A) = \{x \in \mathbb{R}^n \mid Ax = 0\}$$

In other words, it is the set of vectors that are orthogonal to the columns of A.

Theorem 4.3: Rank

Given a matrix $A \in \mathbb{R}^{m \times n}$, the rank of A is the dimension of its range, i.e.

$$\operatorname{rank}(A) = \dim(\mathcal{R}(A))$$

Note: The sum of the dimensions of the range of A and the null space (kernel) of A is equal to the number of columns n:

$$\dim(\mathcal{R}(A)) + \dim(\mathcal{N}(A)) = n$$

Theorem 4.4: Fundamental theorem of Linear Algebra

For any matrix $A \in \mathbb{R}^{m \times n}$, it holds that $\mathcal{N}(A) \perp \mathcal{R}(A^T)$ and $\mathcal{N}(A^T) \perp \mathcal{R}(A)$, therefore we have

$$\begin{aligned} \mathbb{R}^n &= \mathcal{N}(A) \oplus \mathcal{R}(A^T) \\ \mathbb{R}^m &= \mathcal{N}(A^T) \oplus \mathcal{R}(A) \end{aligned}$$

Theorem 4.5: Singular Value Decomposition

Every matrix $A \in \mathbb{R}^{m \times n}$ of rank r can be written as

$$A = U \Sigma V^T = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1^\top \\ V_2^\top \end{bmatrix}$$

where $U \in \mathbb{R}^{m \times m}$ and $V \in \mathbb{R}^{n \times n}$ are orthogonal matrices, $U_1 \in \mathbb{R}^{m \times r}$, $V_1 \in \mathbb{R}^{n \times r}$ and

$$\Sigma_1 = \operatorname{diag}(\sigma_1, \sigma_2, \ldots, \sigma_r) \in \mathbb{R}^{r \times r}$$

is a diagonal matrix, where $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_r > 0$ are the singular values of A. The columns of U and V are called the left and right singular vectors of A, respectively.

Property 4.1: Singular Value Decomposition

The singular value decomposition of a matrix $A \in \mathbb{R}^{m \times n}$ gives orthogonal bases for the four fundamental subspaces related to A:

$$\begin{aligned} \mathcal{R}(A) = \mathcal{R}(U_1), &\quad \mathcal{N}(A^T) = \mathcal{R}(U_2) \\ \mathcal{R}(A^T) = \mathcal{R}(V_1), &\quad \mathcal{N}(A) = \mathcal{R}(V_2) \end{aligned}$$

Theorem 4.6: Moore Penrose Pseudo Inverse

Assume $J \in \mathbb{R}^{m\times n}$ and that the singular value decomposition (SVD) of J is given by $J = U\Sigma V^T$. Then, the Moore-Penrose pseudo-inverse J^+ is given by

$$J^+ = VS^+U^T,$$

where for

$$S = \left[\begin{array}{ccccccc} \sigma_1 & & & & & & \\ & \sigma_2 & & & & & \\ & & \ddots & & & & \\ & & & \sigma_r & & & \\ & & & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 0 \\ \hline 0 & \cdots & \cdots & 0 & \cdots & \cdots & 0 \end{array}\right] \quad \text{holds} \quad S^+ = \left[\begin{array}{ccccccc|c} \sigma_1^{-1} & & & & & & & 0 \\ & \sigma_2^{-1} & & & & & & \vdots \\ & & \ddots & & & & & \vdots \\ & & & \sigma_r^{-1} & & & & 0 \\ & & & & 0 & & & \vdots \\ & & & & & \ddots & & \vdots \\ & & & & & & 0 & 0 \end{array}\right]$$

Property 4.2: Moore Penrose Pseudo Inverse

If matrix $A \in \mathbb{R}^{m\times n}$

- is non-singular then
$$A^+ = A^{-1},$$
- has full column rank, that is $r = n$, then
$$A^+A = VV^T = I,$$
i.e. A^+ is a left inverse of A,
- has full row rank, that is $r = m$, then
$$AA^+ = UU^T = I,$$
i.e. A^+ is a right inverse of A.

Theorem 4.7: Orthogonal-triangular decomposition (QR)

If $A \in \mathbb{R}^{m\times n}$, then there exists an orthogonal matrix $Q \in \mathbb{R}^{m\times m}$ and an upper triangular matrix $R \in \mathbb{R}^{m\times n}$ such that
$$A = QR.$$

Theorem 4.8: Eigenvalue decomposition

Any real symmetric matrix $A \in \mathbb{R}^{n\times n}$ can be decomposed as

$$A = Q\Lambda Q^T,$$

where $Q \in \mathbb{R}^{n\times n}$ is orthogonal, i.e. $Q^TQ = I$, and $\Lambda \in \mathbb{R}^{n\times n}$ is diagonal with the eigenvalues of A on the diagonal. The columns of Q form an orthonormal set of eigenvectors.

Theorem 4.9: Symmetric positive semi-definite matrices

We write for a symmetric matrix $B = B^T$, $B \in \mathbb{R}^{n\times n}$ that "$B \succeq 0$" if and only if B is positive semi-definite, i.e.,

$$\forall z \in \mathbb{R}^n : \; z^T B z \geq 0,$$

or, equivalently, if all (real) eigenvalues of B are non-negative.The set of all symmetric positive semi-definite matrices is denoted by $\mathbb{S}^n_+$.

Property 4.3: Symmetric positive semi-definite matrices

Let $Q \in \mathbb{S}^n$ be a symmetric matrix. Then the following statements are equivalent:

1. Q is positive semi-definite, i.e. $Q \succeq 0$,
2. all eigenvalues of Q are non-negative, i.e. $\lambda_i(Q) \geq 0$ for all $i \in [1, n]$,
3. all principal minors of Q, i.e. the determinant of a submatrix obtained from Q when the same set of rows and columns are stricken out, are non-negative,
4. Q can be written as $Q = AA^T$ for some matrix $A \in \mathbb{R}^{n\times r}$ and r is the rank of Q.

Theorem 4.10: Symmetric positive definite matrices

A symmetric is positive definite if $B \succ 0$, i.e.,

$$\forall z \in \mathbb{R}^n \backslash \{0\} : \; z^T B z > 0,$$

and the set of symmetric positive definite matrices is denoted by $\mathbb{S}^n_{++}$.

Theorem 4.11: Cholesky factorization

If $A \in \mathbb{R}^{n\times n}$ is symmetric positive definite, then there exists a unique lower triangular matrix $L \in \mathbb{R}^{n\times n}$ with positive diagonal entries such that

$$A = LL^T.$$

Theorem 4.12: Matrix norms

For a matrix $A \in \mathbb{R}^{m\times n}$ and two vector norms $\|\cdot\|_p$ and $\|\cdot\|_q$, the induced matrix norm is defined as

$$\|A\|_{p,q} = \max_x\{\|Ax\|_q \|x\|_p \leq 1\}.$$

When $p = q$, we simply write $\|A\|_p$.

Example 4.2: Spectral norm

The ℓ_2-induced norm or spectral norm of a matrix $A \in \mathbb{R}^{m\times n}$ is defined as

$$\|A\|_2 = \sqrt{\lambda_{\max}(A^T A)} = \sigma_{\max}(A),$$

where $\lambda_{\max}(A^T A)$ denotes the largest eigenvalue of $A^T A$ and $\sigma_{\max}(A)$ is the largest singular value of A.

Example 4.3: Frobenius norm

The Frobenius norm of a matrix $A \in \mathbb{R}^{m\times n}$ is defined as

$$\|A\|_F = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}^2} = \sqrt{\operatorname{tr}(A^T A)}.$$

5 Sequences

Lemma 5.1: Convergence

If $\{x_k\}$ is a non-increasing and bounded below or non-decreasing and bounded above sequence, it converges to a finite real number.

Theorem 5.1: $O(\cdot)$

For a function $f : \mathbb{R}^n \to \mathbb{R}^m$, we write

$$f(x) = O(g(x))$$

if and only if there exists a constant $C > 0$ and a neighborhood $\mathcal{N}$ of 0 such that

$$\forall x \in \mathcal{N} : \ \|f(x)\| \leq Cg(x),$$

i.e. "f shrinks as fast as g".

Theorem 5.2: $o(\cdot)$

For a function $f : \mathbb{R}^n \to \mathbb{R}^m$, we write

$$f(x) = o(g(x))$$

if and only if there exists a neighborhood $\mathcal{N}$ of 0 and a function $c : \mathcal{N} \to \mathbb{R}$ with $\lim_{x\to 0} c(x) = 0$ such that

$$\forall x \in \mathcal{N} : \ \|f(x)\| \leq c(x)g(x),$$

i.e. "f shrinks faster than g".

6 Differential Calculus

Theorem 6.1: Lipschitz continuity

A mapping $F : \mathbb{R}^n \to \mathbb{R}^m$ is Lipschitz continuous with Lipschitz constant L if

$$\|F(x) - F(y)\| \leq L\|x - y\| \quad \forall x, y \in \mathbb{R}^n.$$

Theorem 6.2: Linear mapping

A mapping $F : \mathbb{R}^n \to \mathbb{R}^m$ is linear if

$$\forall x, y \in \mathbb{R}^n, \ \forall \lambda_1, \lambda_2 \in \mathbb{R} : \ F(\lambda_1 x + \lambda_2 y) = \lambda_1 F(x) + \lambda_2 F(y).$$

Theorem 6.3: Affine mapping

A mapping $F : \mathbb{R}^n \to \mathbb{R}^m$ is affine if it is the sum of a linear mapping and a constant vector, i.e.

$$\exists A \in \mathbb{R}^{m\times n}, \ \exists b \in \mathbb{R}^m : \ F(x) = Ax + b.$$

Theorem 6.4: Quadratic function

A function $f : \mathbb{R}^n \to \mathbb{R}$ is quadratic if it can be written as

$$f(x) = \frac{1}{2}x^T Qx + q^T x + c$$

for some matrix $Q \in \mathbb{R}^{n\times n}$, vector $q \in \mathbb{R}^n$, and scalar $c \in \mathbb{R}$.

Theorem 6.5: First-order Taylor expansion

Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable at $x \in \mathbb{R}^n$. Then for all $y \in \mathbb{R}^n$, it holds that

$$f(y) = f(x) + \nabla f(x)^T (y - x) + o(\|y - x\|).$$

Lemma 6.1: Mean-value theorem

Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable. Then for every $x, y \in \mathbb{R}^n$ there exists a $\tau \in (0, 1)$ such that

$$f(y) = f(x) + \nabla f(x + \tau(y - x))^T (y - x),$$

Moreover, if f is continuously differentiable, then

$$f(y) = f(x) + \int_0^1 \nabla f(x + t(y - x))^T (y - x) dt$$

Theorem 6.6: Hessian

The Hessian of a function $f : \mathbb{R}^n \to \mathbb{R}$ is the matrix of second partial derivatives, i.e.

$$\nabla^2 f(x) = \left[\frac{\partial^2 f(x)}{\partial x_i \partial x_j} \right]_{i,j=1}^n$$

Theorem 6.7: Second-order Taylor expansion

Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is twice continuously differentiable at $x \in \mathbb{R}^n$. Then for all $y \in \mathbb{R}^n$, it holds that

$$f(y) = f(x) + \nabla f(x)^T (y - x) + \frac{1}{2} (y - x)^T \nabla^2 f(x)(y - x) + o(\|y - x\|^2).$$

Lemma 6.2: Taylor Rest Term theorem

Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable. Then for every $x, y \in \mathbb{R}^n$ there exists a $\theta \in [0, 1]$ such that

$$f(y) = f(x) + \nabla f(x)^T (y - x) + \frac{1}{2} (y - x)^T \nabla^2 f(x + \theta(y - x))(y - x).$$

Theorem 6.8: Jacobian

The Jacobian of a mapping $F : \mathbb{R}^n \to \mathbb{R}^m$ is the matrix of transposed gradients, i.e.

$$J_F(x) = \begin{bmatrix} \nabla f_1(x)^\top \\ \nabla f_2(x)^\top \\ \vdots \\ \nabla f_m(x)^\top \end{bmatrix} \in \mathbb{R}^{m\times n}$$

Application 6.1: Mean-value theorem for vector-valued functions

Suppose that $F : \mathbb{R}^n \to \mathbb{R}^m$ is continuously differentiable on $\mathbb{R}^n$. Then for every $x, y \in \mathbb{R}^n$ the following holds:

$$F(y) = F(x) + \int_0^1 J_F(x + t(y-x))(y-x)dt.$$

Theorem 6.9: Implicit function theorem

Let $F : \mathbb{R}^{n+m} \to \mathbb{R}^n$ be a continuously differentiable mapping of $x \in \mathbb{R}^n$ and $p \in \mathbb{R}^m$. If $x^* \in \mathbb{R}^n, p^* \in \mathbb{R}^m$ are such that

1. $F(x^*, p^*) = 0$,
2. the partial Jacobian $J_{F_x}(x^*, p^*)$ is non-singular,

then there exist open sets $S_{x^*} \subset \mathbb{R}^n, S_{p^*} \subset \mathbb{R}^m$ and a continuously differentiable function $g : S_{p^*} \to S_{x^*}$ such that

$$x^* = g(p^*) \quad \text{and} \quad F(g(p), p) = 0 \quad \forall p \in S_{p^*},$$

and

$$J_g(p^*) = -(J_{F_x}(g(x^*), p^*))^{-1} J_{F_p}(g(x^*), p^*).$$

Fundamental Concepts

7 Fundamental Concepts of Optimization

Theorem 7.1: Optimization problem in standard form

$$\begin{aligned}\underset{x \in \mathbb{R}^n}{\text{minimize}} \quad & f(x) \\ \text{subject to} \quad & g(x) = 0, \\ & h(x) \geq 0.\end{aligned}$$

Theorem 7.2: Level Set

The set

$$\{x \in \mathbb{R}^n \mid f(x) = c\}$$

is the level set of f for the value c, i.e. the set of all points that map to the same value c.

Theorem 7.3: Feasible set

The set

$$\{x \in \mathbb{R}^n \mid g(x) = 0 \;\wedge\; h(x) \geq 0\}$$

is the feasible set Ω, i.e. the set of all points that satisfy the constraints.

Theorem 7.4: Global minimizer

The point x^* is a global minimizer if and only if

$$x^* \in \Omega, \; \forall x \in \Omega : \; f(x^*) \leq f(x).$$

Theorem 7.5: Strict global minimizer

The point x^* is a strict global minimzer if and only if

$$x^* \in \Omega, \; \forall x \in \Omega \setminus \{x^*\} : \; f(x^*) < f(x).$$

Theorem 7.6: Strict local minimizer

The point x^* is a strict local minimizer if and only if $x^* \in \Omega$ and there exists a neighborhood $\mathcal{N}$ of x^* such that

$$\forall x \in (\Omega \cap \mathcal{N}) \setminus \{x^*\} : \; f(x^*) < f(x).$$

Theorem 7.7: Weierstrass

If $\Omega \in \mathbb{R}^n$ is compact, i.e. limited and closed, and $f : \Omega \to \mathbb{R}$ is continuous, then there exists a global minimizer (a solution) of the optimization problem

$$\begin{aligned} \underset{x \in \mathbb{R}^n}{\text{minimize}} \quad & f(x) \\ \text{subject to} \quad & x \in \Omega. \end{aligned}$$

Proof 7.1: Weierstrass

Regard the graph of f, $G = \{(x, f(x)) \in \mathbb{R}^n \times \mathbb{R} \mid x \in \Omega\}$. G is a compact set, and so is the projection of G onto its last coordinate, the set $G = \{s \in \mathbb{R} \mid \exists x \text{ such that } (x, s) \in G\}$, which is a compact interval $[f_{\min}, f_{\max}] \subset \mathbb{R}$. By construction, there must be at least one x^* such that $(x^*, f(x^*)) \in G$.

$\square$

8 Types of Optimization Problems

Theorem 8.1: Convex set

A set $\Omega \subset \mathbb{R}^n$ is convex if

$$\forall x, y \in \Omega,\ \forall \lambda \in [0, 1] :\ x + \lambda(y - x) \in \Omega.$$

or if "all connecting lines lie inside the set".

Theorem 8.2: Convex function

A function $f : \Omega \to \mathbb{R}$ is convex if

$$\forall x, y \in \Omega,\ \forall \lambda \in [0, 1] :\ f(x + \lambda(y - x)) \leq f(x) + \lambda(f(y) - f(x)).$$

or if "all secants (i.e. a line segment between two points on the graph) are above graph". This definition is equivalent to saying that the Epigraph of f, i.e. the set $\{(x, s) \in \mathbb{R}^n \times \mathbb{R} \mid x \in \Omega,\ s \geq f(x)\}$, is a convex set.

Note: a concave function is the same but then with $\geq$ instead of $\leq$.

Property 8.1: Convex function

If $f :\ D \to \mathbb{R}$ and $\Omega_f = \{(x, y) \mid x \in D, y \geq f(x)\}$ then the following holds:

$$f \text{ is convex } \Leftrightarrow\ \Omega_f \text{ is convex.}$$

Theorem 8.3: Convex optimization problem

An optimization problem with convex feasible set Ω and convex objective function $f : \Omega \to \mathbb{R}$ is called a convex optimization problem.

Property 8.2: Globality of local minima of convex optimization problem

For a convex optimization problem, a local minimum is also a global one.

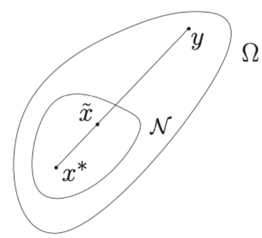
Proof 8.1: Globality of local minima of convex optimization problem

Regard a local minimum x^* of the convex optimization problem

$$\begin{aligned} \underset{x \in \mathbb{R}^n}{\text{minimize}} \quad & f(x) \\ \text{subject to} \quad & x \in \Omega. \end{aligned}$$

We will show that for any point $y \in \Omega$ holds $f(y) \geq f(x^*)$, i.e. x^* is a global minimum.

First we choose, using local optimality, a neighborhood $\mathcal{N}$ of x^* such that for all $\tilde{x} \in \Omega \cap \mathcal{N}$ holds $f(\tilde{x}) \geq f(x^*)$. Second, we regard the connecting line between x^* and y. This line is completely contained in Ω due to the convexity of Ω. Now we choose a point $\tilde{x}$ on this line such that it is in the neighborhood, but not equal to x^*, i.e. $\tilde{x} = x^* + \lambda(y - x^*)$ for some $\lambda \in (0, 1)$, and $\tilde{x} \in \Omega \cap \mathcal{N}$. Due to local optimality, we have $f(\tilde{x}) \geq f(x^*)$, and due to convexity we have



$$f(\tilde{x}) = f(x^* + \lambda(y - x^*)) \leq f(x^*) + \lambda(f(y) - f(x^*)).$$

It follows that $\lambda(f(y) - f(x^*)) \geq 0$, and since $\lambda \in (0, 1)$, we have $f(y) \geq f(x^*)$, as desired.

$\square$

Example 8.1: Nonlinear Programming (NLP)

Nonlinear Programming Problems are poblems of the following form:

$$\begin{aligned} \underset{x \in \mathbb{R}^n}{\text{minimize}} \quad & f(x) \\ \text{subject to} \quad & g(x) = 0, \\ & h(x) \geq 0, \end{aligned}$$

where $f : \mathbb{R}^n \to \mathbb{R}$, $g : \mathbb{R}^n \to \mathbb{R}^p$, and $h : \mathbb{R}^n \to \mathbb{R}^q$, are assumed to be continuously differentiable at least once.

Example 8.2: Linear Programming (LP)

When the functions f, g, h are affine (i.e., they can be expressed in the form $f(x) = a^T x + b$ for some vector a and scalar b) in the general formulation (see Theorem 8.1), the general NLP gets something easier to solve, namely a Linear Program (LP), which can be written as follows:

$$\begin{aligned} \underset{x \in \mathbb{R}^n}{\text{minimize}} \quad & c^T x \\ \text{subject to} \quad & Ax - b = 0, \\ & Cx - d \geq 0, \end{aligned}$$

where $c \in \mathbb{R}^n$, $A \in \mathbb{R}^{p \times n}$, $b \in \mathbb{R}^p$, $C \in \mathbb{R}^{q \times n}$, and $d \in \mathbb{R}^q$.

Example 8.3: Quadratic Programming (QP)

When the functions g, h are affine (as for an LP in Theorem 8.2), and the objective function f is a linear-quadratic function, the NLP becomes a Quadratic Program (QP), which can be written as follows:

$$\begin{aligned} \underset{x \in \mathbb{R}^n}{\text{minimize}} \quad & c^T x + \frac{1}{2} x^T B x \\ \text{subject to} \quad & Ax - b = 0, \\ & Cx - d \geq 0, \end{aligned}$$

where, in addition to the LP parameters, $B \in \mathbb{R}^{n \times n}$ is the Hessian matrix. Specifically, for the objective function $f(x) = c^T x + \frac{1}{2} x^T B x$, the Hessian is given by:

$$\nabla^2 f(x) = B.$$

Theorem 8.4: Convex QP

If the Hessian matrix B is positive semi-definite (i.e. if $\forall z \in \mathbb{R}^n \setminus \{0\} : z^T B z \geq 0$) we call the QP a convex QP. Convex QPs are tremendously easier to solve globally than “non-convex QPs” (i.e., where the Hessian B is not positive semi-definite), which might have different local minima (i.e. have a non-convex solution set).

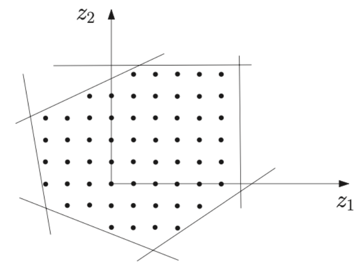
Theorem 8.5: Strictly convex QP

If the Hessian matrix B is positive definite (i.e. if $\forall z \in \mathbb{R}^n : z^T B z > 0$) we call the QP a strictly convex QP. Strictly convex QPs are a subset of convex QPs (see Theorem 8.4).

Example 8.4: Mixed-Integer Programming (MIP)

A mixed-integer programming problem or mixed-integer program (MIP) is a problem with both real and integer decision variables. A MIP can be formulated as follows:

$$\begin{aligned} \underset{\substack{x\in\mathbb{R}^n \\ z\in\mathbb{Z}^m}}{\text{minimize}} \quad & f(x,z) \\ \text{subject to} \quad & g(x,z)=0, \\ & h(x,z)\geq 0, \end{aligned}$$



where $x \in \mathbb{R}^n$ are the continuous variables, and $z \in \mathbb{Z}^m$ are the integer variables.

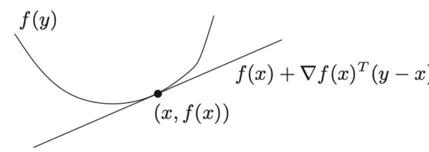
9 Convex Optimization

Theorem 9.1: Convexity for C^1 functions

Assume that $f: \Omega \to \mathbb{R}$ is continuously differentiable and Ω is convex. Then holds that f is convex if and only if

$$\forall x, y \in \Omega: \; f(y) \geq f(x) + \nabla f(x)^T (y-x)$$

i.e. tangents lie below the graph.



Proof 9.1: Convexity for C^1 functions

"$\Rightarrow$": Due to convexity of f holds for given $x, y \in \Omega$ and for any $\lambda \in [0,1]$ that

$$f(x+\lambda(y-x)) - f(x) \leq \lambda(f(y)-f(x)).$$

and therefore that

$$\nabla f(x)^T(y-x) = \lim_{\lambda\to 0} \frac{f(x+\lambda(y-x))-f(x)}{\lambda} \leq f(y)-f(x).$$

"$\Leftarrow$": To prove that $z = x+\lambda(y-x) = (1-\lambda)x + \lambda y$ holds that $f(z) \leq (1-\lambda)f(x) + \lambda f(y)$, we can use the equation from Theorem 9.1 twice to get

$$f(x) \geq f(z) + \nabla f(z)^T(x-z) \;\text{ and }\; f(y) \geq f(z) + \nabla f(z)^T(y-z),$$

which yield, when weighted with $(1-\lambda)$ and λ respectively, that

$$(1-\lambda)f(x) + \lambda f(y) \geq f(z) + \nabla f(z)^T \underbrace{[(1-\lambda)(x-z)+\lambda(y-z)]}_{=0}$$

□

Theorem 9.2: Convexity for C^2 functions

Assume that $f:\ \Omega \to \mathbb{R}$ is twice continuously differentiable and Ω is convex and open. Then holds that f is convex if and only if

$$\forall x \in \Omega:\ \nabla^2 f(x) \succeq 0,$$

i.e. the Hessian of f is positive semi-definite.

Proof 9.2: Convexity for C^2 functions

"$\Rightarrow$": Recall that a second order Taylor expansion of y at x in an arbitrary direction p is given by the following:

$$f(x+tp) = f(x) + t\nabla f(x)^T p + \frac{t^2}{2} p^T \nabla^2 f(x) p + o(t^2\|p\|).$$

From this we obtain that

$$p^T \nabla^2 f(x) p = \lim_{t\to 0} \frac{2}{t^2}\left(\underbrace{f(x+tp) - f(x) - t\nabla f(x)^T p}_{(9.1):\ \geq 0}\right) \geq 0.$$

"$\Leftarrow$": Conversely, to prove the other direction, we use Theorem 6.2 with some arbitrary $\theta \in [0,1]$:

$$f(y) = f(x) + \nabla f(x)^T (y-x) + \underbrace{\frac{t^2}{2}(y-x)^T \nabla^2 f(x+\theta(y-x))(y-x)}_{(9.2):\ \geq 0},$$

and thus f is convex, since Theorem 9.1 trivially holds.

$\square$

Property 9.1: Convexity perserving operations on convex functions

The following operations preserve the convexity of a function:

1. Non-negative weighted sum: Suppose that $\forall i \in [i,m]:\ \ f_i : \mathbb{R}^n \to \mathbb{R}$ are convex functions and $\forall i \in [1,m]:\ \lambda_i \geq 0$. Then the function

$$f(x) = \sum_{i=1}^{m} \lambda_i f_i(x)$$

is convex.

2. Affine input transformation: If $f:\ \Omega \to \mathbb{R}$ is convex, then also

$$A \in \mathbb{R}^{n\times m}:\ \tilde{f}(x) = f(Ax+b)$$

is convex on the domain $\tilde{\Omega} = \{x \mid Ax+b \in \Omega\}$.

3. Concatenation with monotone convex function: If $f:\ \Omega \to \mathbb{R}$ is convex and $g:\ \mathbb{R} \to \mathbb{R}$ is convex and monotonely increasing, then the composition $g \circ f$ is convex.

4. Pointwise supremum: The supremum over a set of convex functions $f_i(x)$, $i \in I$, where I can be an infinite set, i.e.,
$$f(x) = \sup_{i \in I} f_i(x)$$
is convex.
5. Composition: Let $h : \mathbb{R}^m \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}^m$ with $g = (g_1, \dots, g_m)$. Then $f(x) = (h \circ g)(x) = h(g(x))$ is convex if any of the followings holds:
   (a) h is convex and non-decreasing in each argument and $\forall i \in [1, m]:\ g_i$ is convex.
   (b) h is convex and non-decreasing in each argument and $\forall i \in [1, m]:\ g_i$ is concave, i.e. $-g_i$ is convex.

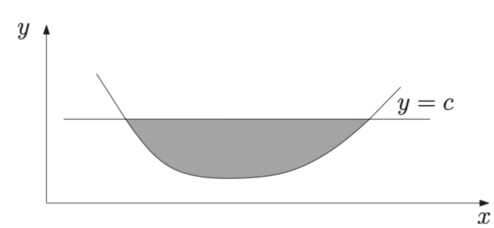
Proof 9.3: Convexity perserving operations on convex functions

1. TODO (use the definition of convexity).
2. TODO (use the definition of convexity).
3. Recall that g is a convex and monotonely increasing function, then:
$$\nabla^2 (g \circ f) = \underbrace{g''(f(x))}_{\geq 0} \underbrace{\nabla f(x) \nabla f(x)^T}_{\succeq 0} + \underbrace{g'(f(x))}_{\geq 0} \underbrace{\nabla^2 f(x)}_{\succeq 0} \succeq 0,$$
i.e. $g \circ f$ is convex, since the Hessian is positive semi-definite.
4. Epigraph of f is the intersection of the epigraphs of f_i, which are convex.
5. Recall that g_i is convex and that h is convex and non-decreasing in each argument. Then:
$$\begin{aligned} f(\lambda x + (1-\lambda) y) &= h(g(\lambda x + (1-\lambda) y)) \\ &\leq h(\lambda g(x) + (1-\lambda) g(y)) \qquad (1) \\ &\leq \lambda h(g(x)) + (1-\lambda) h(g(y)) \qquad (2) \\ &= \lambda f(x) + (1-\lambda) f(y), \end{aligned}$$
where we used the convexity of g_i and the fact that h is non-decreasing to obtain inequality (1), and the convexity of h to obtain inequality (2).

□

Theorem 9.3: Convexity of Sublevel subsets

If $f:\ \Omega \to \mathbb{R}$ is a convex function, then all its level sets
$$\text{lev}_{\leq \gamma} f = \{x \in \Omega \mid f(x) \leq \gamma\}$$
are convex.



Proof 9.4: Convexity of Sublevel subsets

If $f(x) \leq c$ and $f(y) \leq c$, then for any $\lambda \in [0,1]$ holds also

$$f((1-\lambda)x + \lambda y) \leq (1-\lambda)f(x) + \lambda f(y) \leq \underbrace{(1-\lambda)c + \lambda c}_{=c}.$$

$\square$

Property 9.2: Convexity perserving operations on convex sets

The following operations preserve the convexity of a set:

1. The intersection of finitely or infinitely many convex sets is convex.
2. Affine image: if Ω is convex, then for $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$ also the set $A\Omega + b = \{y \in \mathbb{R}^m \mid \exists x \in \Omega : y = Ax + b\}$ is convex.
3. Affine pre-image: if Ω is convex, then for $A \in \mathbb{R}^{n \times m}$, $b \in \mathbb{R}^n$ also the set $\{z \in \mathbb{R}^m \mid Az + b \in \Omega\}$ is convex.

Example 9.1: Convex Feasible set

If $\forall i \in [1,m] : f_i : \ \mathbb{R}^n \to \mathbb{R}$ are convex functions, then the set

$$\Omega = \{x \in \mathbb{R}^n \mid f_i(x) \leq 0, \ i \in [1,m]\}$$

is a convex set, because it is the intersection of sublevel sets Ω_i of convex functions f_i, i.e.

$$\begin{aligned}\Omega &= \bigcap_{i=1}^{m} \Omega_i \\ &= \bigcap_{i=1}^{m} \{x \in \mathbb{R}^n \mid f_i(x) \leq 0\}.\end{aligned}$$

Theorem 9.4: Optimality condition for convex problems

Regard a convex optimization problem with continuously differentiable objective function f. A point $x^* \in \Omega$ is a global optimizer if and only if

$$\forall y \in \Omega : \ \nabla f(x^*)^T (y - x^*) \geq 0.$$

Proof 9.5: Optimality condition for convex problems

"$\Rightarrow$": Assume for the sake of contradiction that $\exists y \in \Omega : \nabla f(x^*)^T (y - x^*) < 0$, then we could regard a Taylor expansion

$$f(x^* + \lambda(y - x^*)) = f(x^*) + \lambda \underbrace{\nabla f(x^*)^T (y - x^*)}_{<0} + \underbrace{o(\lambda)}_{\to 0}.$$

This implies that for sufficiently small positive λ, we have

$$f(x^* + \lambda(x - x^*)) < f(x^*),$$

which contradicts the optimality of x^*.

"$\Leftarrow$": Due to the C^1 characterization of convexity of f in Theorem 9.1, we have for any feasible $y \in \Omega$:

$$f(y) \geq f(x^*) + \underbrace{\nabla f(x^*)^T (y - x^*)}_{\geq 0} \geq f(x^*),$$

which implies that x^* is a global optimizer.

$\square$

Theorem 9.5: Sufficient Condition for Convex NLP

For a nonlinear optimization problem (NLP) in standard form

$$\begin{aligned} \underset{x \in \mathbb{R}^n}{\text{minimize}} \quad & f(x) \\ \text{subject to} \quad & g_i(x) = 0,\ i \in [1, m], \\ & h_i(x) \geq 0,\ j \in [1, p], \end{aligned}$$

the following conditions are necessary for convexity:

- the objective function $f : \mathbb{R}^n \to \mathbb{R}$ must be convex,
- the constraint set

  $$X = \{x \in \mathbb{R}^n \mid g(x) = 0,\ h(x) \geq 0\}$$

  must be convex. Since we know that the intersection of convex sets is convex, we can write X as the intersection of the sets G and H:

  $$\begin{aligned} X &= \{x \in \mathbb{R}^n \mid g(x) = 0,\ h(x) \geq 0\} \\ &= \{x \in \mathbb{R}^n \mid g(x) = 0\} \cap \{x \in \mathbb{R}^n \mid h(x) \geq 0\} \\ &= \left(\bigcap_{i=1}^{m} \{x \in \mathbb{R}^n \mid g_i(x) = 0\} \right) \cap \left(\bigcap_{i=1}^{m} \{x \in \mathbb{R}^n \mid h_i(x) \geq 0\} \right) \\ &= G \cap H \end{aligned}$$

  Now we must consider the requirements for the sets G and H to be convex:

  - Suppose $-H_i = \{x \in \mathbb{R}^n \mid h_i(x) \leq 0\}$, the zero sublevel set of the function h_i. If h_i is convex, the set $-H_i$ is convex, as seen in Theorem 9.3. Now, consider the case where h_i is concave.

Since h_i being concave means that $-h_i$ is convex, the set $H_i = \{x \in \mathbb{R}^n \mid h_i(x) \geq 0\}$ is a sublevel set of the convex function $-h_i$, and is therefore convex. Thus, the set H_i is convex when h_i is concave.

– On the other hand, G is the level set of of g_i. Therefore it is certainly a convex set whenever g_i is a affine function

$$\forall i \in [i, m] : \; g_i(x) = a_i^T x + b_i.$$

Example 9.2: Halfspace

A halfspace

$$H_{\leq} = \{x \in \mathbb{R}^n \mid a^T x \leq b\}$$

is a convex set, as the zero sublevel set of the affine (and therefore convex) function $f(x) = a^T x - b$, cf. Theorem 9.3.

Note: The opposite is not true; a function that has all its level sets convex is not necessarily convex.

Example 9.3: Polyhedral Set

A polyhedral set $C \subset \mathbb{R}^n$ is defined as the intersection of a finite number of halfspaces, i.e.,

$$C = \bigcap_{i=1,\ldots,m} \{x \in \mathbb{R}^n \mid a_i^T x \leq b_i\}.$$

Since the intersection of convex sets is convex, C is a convex set if all the halfspaces are convex.

Note: A polyhedral set might contain equalities, i.e.,

$$C = \{x \in \mathbb{R}^n \mid Ax \leq b, \; Cx = d\}.$$

which can be written with just inequalities as such:

$$C = \{x \in \mathbb{R}^n \mid Ax \leq b, \; Cx \leq d, \; -Cx \leq -d\}.$$

Example 9.4: Ellipsoid

If $P \in \mathbb{R}^{n \times n}$ is symmetric positive definite, then the ellipsoid

$$C = \{x \in \mathbb{R}^n \mid (x - x_c)^T P^{-1}(x - x_c) \leq 1\}$$

is convex, as the 1-sublevel set of the convex function $f(x) = (x - x_c)^T P^{-1}(x - x_c) - 1$, where x_c is the center of the ellipsoid and P is the shape matrix. The latter determines how far the ellipsoid extends in each direction from x_c; the lengths of the semi-axis are given by $\sqrt{\lambda_i}$, where λ_i are the eigenvalues of P, while the eigenvectors of P determine the orientation of the ellipsoid. When $P = r^2 I$, the ellipsoid is a ball with radius r around x_c, i.e.,

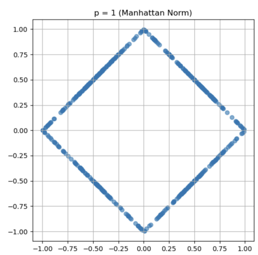
$$C = \{x \in \mathbb{R}^n \mid \|x - x_c\|_2 \leq r\},$$

or, in general, the p-norm ball

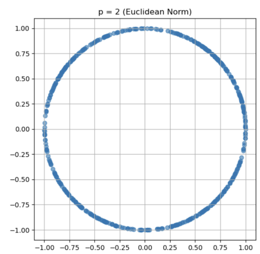
$$C = \{x \in \mathbb{R}^n \mid \|x - x_c\|_p \leq r\}.$$

where the value of p determines the shape of the ball, i.e.,

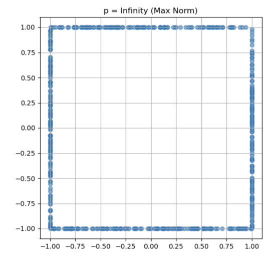
- $p = 1$:



- $p = 2$:



- $p = \infty$:



Note: All images above are for $r = 1$.

Example 9.5: Convex cones

A set C is said to be cone if

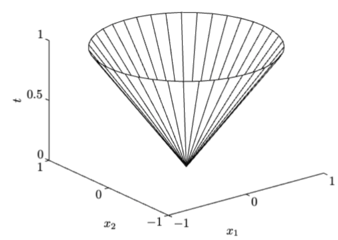
$$\forall x: \; x \in C \;\Rightarrow\; \forall \lambda \geq 0: \; \lambda x \in C.$$

Moreover, C is a convex cone if and only if it is closed under addition and multiplication with non-negative scalars, i.e.,

$$\forall x, y \in C, \; \forall \lambda, \mu \geq 0: \; \lambda x + \mu y \in C.$$

The following are examples of convex cones:

- Non-negative orthant: The set $C = \{x \in \mathbb{R}^n \mid x \geq 0\}$ is a convex cone.
- Norm cones: The set $C = \{(x, t) \in \mathbb{R}^n \times \mathbb{R} \mid \|x\|_p \leq t\}$ is a convex cone. For instance, the Lorenz cone or ice-cream cone, when $p = 2$, is shown on the right.
- Positive semi-definite cone: The set $\mathbb{S}^n_+$ is a convex cone.



10 The Lagrangian Function and Duality

Theorem 10.1: Primal Optimization Problem

We will denote the globally optimal value of the objective function subject to the constraints as the primal optimal value p^*, i.e.,

$$p^* = \left(\min_{x \in \mathbb{R}^n} f(x) \quad \text{s.t.} \quad g(x) = 0 \;\wedge\; h(x) \geq 0 \right).$$

and we will denote this optimization problem as the primal optimization problem.

Theorem 10.2: Lagrangian Function and Lagrange Multipliers

We define the Lagrangian function to be

$$\begin{aligned}\mathcal{L}(x, \lambda, \mu) &= f(x) + \sum_{i=1}^{l} \lambda_i g_i(x) + \sum_{i=1}^{m} \mu_i h_i(x) \\ &= f(x) + \lambda^T g(x) + \mu^T h(x).\end{aligned}$$

where $\lambda \in \mathbb{R}^l$ and $\mu \geq 0 \in \mathbb{R}^m$ are the Lagrange multipliers or dual variables.

Lemma 10.1: Lower Bound Property of Lagrangian

If $\tilde{x}$ is a feasible point of (10.1) and $\mu \geq 0$, then

$$\mathcal{L}(\tilde{x}, \lambda, \mu) \leq f(\tilde{(x)}).$$

Proof 10.1: Lower Bound Property of Lagrangian

Since $\tilde{x}$ is feasible, we have $g(\tilde{x}) = 0$ and $h(\tilde{x}) \geq 0$. Therefore, with $\mu \geq 0$, we have:

$$\mathcal{L}(\tilde{x}, \lambda, \mu) = f(\tilde{x}) + \lambda^T \underbrace{g(\tilde{x})}_{=0} + \underbrace{\mu^T}_{\geq 0} \underbrace{h(\tilde{x})}_{\geq 0} \leq f(\tilde{x}).$$

$\square$

Theorem 10.3: Lagrange Dual Function

The Lagrange dual function is defined as the unconstrained infimum of the Lagrangian function over x, for fixed multipliers λ and μ:

$$q(\lambda, \mu) = \inf_{x \in \mathbb{R}^n} \mathcal{L}(x, \lambda, \mu).$$

Lemma 10.2: Lower Bound property of Lagrange Dual

If $\mu \geq 0$, then the dual function $q(\lambda, \mu)$ is a lower bound on the primal optimal value p^*, i.e.,

$$q(\lambda, \mu) \leq p^*.$$

Proof 10.2: Lower Bound property of Lagrange Dual

Since the Lagrange function is bounded from below by Lemma 10.1, we have

$$q(\lambda, \mu) = \inf_{x \in \mathbb{R}^n} \mathcal{L}(x, \lambda, \mu) \leq f(\tilde{x}) \quad \text{for any feasible } \tilde{x}.$$

Naturally, this inequality holds in particular for the global minimizer x^*, which yields:

$$q(\lambda, \mu) \leq f(x^*) = p^*.$$

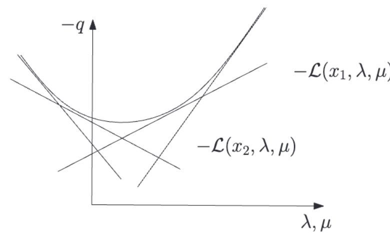
□

Theorem 10.4: Concavity of Lagrange Dual

The function $q : \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}$ is concave, even if the original NLP was not convex.

Proof 10.3: Concavity of Lagrange Dual

We will show that $-q$ is convex. The Lagrangian $\mathcal{L}$ is an affine function in the multipliers λ and μ, which in particular implies that $-\mathcal{L}$ is convex in λ and μ. Thus, the function $-q(\lambda, \mu) = \sup_x -\mathcal{L}(x, \lambda, \mu)$ is the supremum of convex functions in λ and μ that are indexed by x, and therefore convex.



□

Theorem 10.5: Dual Problem

The dual problem is defined as the convex maximization problem, i.e.,

$$d^* = \left(\max_{\lambda \in R^p, \mu \in R^q} q(\lambda, \mu) \quad \text{s.t.} \quad \mu \geq 0 \right).$$

Theorem 10.6: Weak Duality

Consider a primal-dual pair. Then, the following inequality holds:

$$d^* \leq p^*.$$

Theorem 10.7: Strong Duality

If the primal optimization problem is convex and the Slater condition (see Theorem 10.8) holds, then strong duality holds, i.e.,

$$d^* = p^*.$$

Theorem 10.8: Slater Condition

If there exist one feasible point $\tilde{x}$ such that all non-linear inequalities are strictly satisfied of a primal convex optimization problem hold, then the Slater condition is satisfied. More explicitly, for a convex problem we must have affine equality constraints, $g(x) = Ax+b$, and the inequality constraint functions can be either affine or concave functions, thus without loss of generality assume that the first $q_1 \leq q$ inequalities are affine and the remaining ones concave. Then the Slater condition holds if and only if there exists an $\tilde{x}$ such that

$$\begin{aligned} A\tilde{x} + b &= 0, \\ h_i(\tilde{x}) &\geq 0, \quad \text{for } i = 1, \dots, q_1, \\ h_i(\tilde{x}) &> 0, \quad \text{for } i = q_1 + 1, \dots, q. \end{aligned}$$

Note: This is trivially satisfied for LP and QP problems.

Theorem 10.9: KKT

Consider the convex optimization problem with equality and inequality constraints, i.e.,

$$\begin{aligned} \min_{x} \quad & f(x) \\ \text{s.t.} \quad & g(x) = 0, \\ & h(x) \geq 0, \end{aligned}$$

and assume that f is convex, g is affine, and h is concave. Also assume that Slater's condition holds and all functions are differentiable. Then, the following statements are equivalent:

1. x^* is a primal optimal and (λ^*, μ^*) are dual optimal.
2. (x^*, λ^*, μ^*) satisfy the Karush-Kuhn-Tucker (KKT) conditions:

1. Stationarity:
$$\nabla f(x) - \nabla g(x)\lambda - \nabla h(x)\mu = 0$$
2. Primal feasibility:
$$g(x) = 0$$
3. Dual feasibility:
$$h(x) \geq 0$$
4. Non-negativity of Lagrange multipliers:
$$\mu \geq 0$$
5. Complementary slackness:
$$\mu^T h(x) = 0$$

These KKT conditions, under the given assumptions, are necessary and sufficient for optimality.

Proof 10.4: KKT

"⇒": Because of the assumptions, there is no duality gap. Thus, there is strong duality and we have:

$$\begin{aligned} p^* = d^* &= q(\lambda^*, \mu^*) \\ &= \inf_x \mathcal{L}(x, \lambda^*, \mu^*) \\ &= \inf_x f(x) + {\lambda^*}^T g(x) + {\mu^*}^T h(x) \\ &\leq f(x^*) + {\lambda^*}^T g(x^*) + {\mu^*}^T h(x^*) \\ &= p^* - \underbrace{{\mu^*}^T h(x^*)}_{\geq 0}. \end{aligned}$$

This implies that ${\mu^*}^T h(x^*) = 0$ and $\nabla f(x) - \nabla g(x)\lambda^* - \nabla h(x)\mu^* = 0$. The remaining KKT conditions follow trivially.

"⇐": Since the assumptions still hold and the KKT condition of stationarity is satisfied for (x^*, λ^*, μ^*), we conclude that x^* is a global minimizer of the convex function $\mathcal{L}(x, \lambda^*, \mu^*)$. Therefore we have:

$$\begin{aligned} q(\lambda^*, \mu^*) &= \mathcal{L}(x^*, \lambda^*, \mu^*) \\ &= f(x^*) - \underbrace{\lambda^* g(x^*)}_{=0} - \underbrace{\mu^* h(x^*)}_{=0} \\ &= f(x^*), \end{aligned}$$

with the last line following from the other KKT conditions. We conclude:

$$p^* \leq f(x^*) = q(\lambda^*, \mu^*) \leq d^*.$$

The assumptions imply strong duality, and thus that $p^* = d^*$. Since we always have $d^* \leq p^*$, we conclude that the inequalities are in fact equalities. Consequently, x^* is primal optimal and (λ^*, μ^*) are dual optimal.

□

Example 10.1: Dual Decomposition

Unconstrained Optimization Algorithms

11 Optimality Conditions

Theorem 11.1: Unconstrained optimization Problems

We define the unconstrained optimization problem as

$$\min_{x \in D} f(x),$$

where we regard objective function $f : D \to \mathbb{R}$ that are defined on some open domain $D \subseteq \mathbb{R}^n$.

Theorem 11.2: Stationary Point

A point $\tilde{x}$ is called a stationary point of f if and only if

$$\nabla f(\tilde{x}) = 0.$$

Theorem 11.3: Descent Direction

A vector $p \in \mathbb{R}^n$ is called a descent direction at x if

$$\nabla f(x)^T p < 0.$$

Theorem 11.4: First Order Necessary Conditions (FONC)

If $x^* \in D$ is a local minimizer of $f : D \to \mathbb{R}$ and $f \in C^1$ then

$$\nabla f(x^*) = 0.$$

Proof 11.1: First Order Necessary Conditions (FONC)

Let us assume for contradiction that $\nabla f(x^*) \neq 0$. Then $p = -\nabla f(x^*)$ would be a descent direction in which the objective could be improved, as follows:

As D is open and $f \in C^1$, we could find a $t > 0$ that is small enough so that for all $\tau \in [0, t]$ holds $x^* + \tau p \in D$ and $\nabla f(x^* + \tau p)^T p < 0$. By Taylor's theorem, we would have for some $\theta \in (0, 1)$ that

$$f(x^* + tp) = f(x^*) + \underbrace{t \nabla f(x^* + \theta t p)^T p}_{<0} < f(x^*).$$

$\square$

Theorem 11.5: Second Order Necessary Conditions (SONC)

If $x^* \in D$ is a local minimizer of $f : D \to \mathbb{R}$ and $f \in C^2$ then

$$\nabla^2 f(x^*) \succeq 0.$$

Proof 11.2: Second Order Necessary Conditions (SONC)

Assume, for the sake of contradiction, that $\nabla^2 f(x^*) \prec 0$. This implies the existence of a vector $p \in \mathbb{R}^n$ such that $p^T \nabla^2 f(x^*) p < 0$. In this case, the objective function could be improved in the direction of p. By choosing a sufficiently small $t > 0$, we can ensure that for all $\tau \in [0, t]$, the following holds:

$$p^T \nabla^2 f(x^* + \tau p) p < 0.$$

Applying Taylor's theorem, we would have for some $\theta \in (0, 1)$ that

$$f(x^* + tp) = f(x^*) + \underbrace{t \nabla f(x^*)^T p}_{=0} + \frac{t^2}{2} \underbrace{p^T \nabla^2 f(x^* + \theta p)}_{<0} p < f(x^*),$$

which leads to a contradiction.

□

Theorem 11.6: Convex First Order Sufficient Conditions (cFOSC)

Assume that $f : D \to \mathbb{R}$ is convex and $f \in C^1$. If $x^* \in D$ is a stationary point of f, then x^* is a global minimizer of f.

Theorem 11.7: Second Order Sufficient Conditions (SOSC)

Assume that $f : D \to \mathbb{R}$ and $f \in C^2$. If $x^* \in D$ is a stationary point of f and $\nabla^2 f(x^*) \succ 0$, then x^* is a strict local minimizer of f.

Proof 11.3: Second Order Sufficient Conditions (SOSC)

We can choose a sufficiently small closed ball B around x^* so that for all $x \in B$ holds $\nabla^2 f(x) \succ 0$. Restricted to this ball, we have a convex problem, so that Theorem 11.6 together with stationarity of x^* implies that x^* is a global minimizer of f. To prove that it is strict, we look for any $x \in B \backslash x^*$ at the Taylor expansion, which yields with some $\theta \in (0, 1)$:

$$f(x) = f(x^*) + \underbrace{\nabla f(x^*)^T (x - x^*)}_{=0} + \frac{1}{2} \underbrace{(x - x^*)^T \nabla^2 f(x^* + \theta(x - x^*))(x - x^*)}_{>0} > f(x^*).$$

□

Theorem 11.8: Stability of Parametric Solutions

Assume that $f : D \times \mathbb{R}^m \to \mathbb{R}$, and regard the minimization of $f(\cdot, \tilde{a})$ for a given fixed value of $\tilde{a} \in \mathbb{R}^m$. If $\tilde{x} \in D$ satisfies the SOSC (see Theorem 11.7), then there is a neighborhood $\mathcal{N} \subset \mathbb{R}^m$ around $\tilde{a}$ such that the parametric minimizer function $x^*(a)$ is well-defined for all $a \in \mathcal{N}$ (i.e. there is a unique minimizer $x^*(a)$ for each $a \in \mathcal{N}$), is differentiable in $\mathcal{N}$, and $x^*(\tilde{a}) = \tilde{x}$. Its derivative at $\tilde{a}$ is given by

$$\frac{\partial(x^*(\tilde{a}))}{\partial a} = -\left(\nabla_x^2 f(\tilde{x}, \tilde{a})\right)^{-1} \frac{\partial\left(\nabla_x f(\tilde{x}, \tilde{a})\right)}{\partial a}.$$

Moreover, each such $x^*(a)$ with $a \in \mathcal{N}$ satisfies again the SOSC and is thus a strict local minimizer.

Proof 11.4: Stability of Parametric Solutions

The existence of the differentiable map $x^* : \mathcal{N} \to D$ follows from the implicit function theorem applied to the stationarity condition $\nabla_x f(x^*(a), a) = 0$. The derivative of $x^*(a)$ at $\tilde{a}$ is given by

$$\frac{d(\nabla_x f(x^*(a), a))}{da} = \underbrace{\frac{\partial\left(\nabla_x f(x^*(a), a)\right)}{\partial x}}_{=\nabla_x^2 f} \frac{\partial x^*(a)}{\partial a} + \frac{\partial\left(\nabla_x f(x^*(a), a)\right)}{\partial a} = 0$$

The fact that all points $x^*(a)$ satisfy the SOSC follows from the continuity of the second derivative.

$\square$

12 Estimation and Fitting Problems

Theorem 12.1: Estimation and Fitting Problems

Estimation and fitting problems are optimization problems with a special objective, namely a least squares objective. We define the estimation problem as

$$\min_{x \in \mathbb{R}^n} \frac{1}{2} \|\eta - M(x)\|_2^2,$$

where $\eta \in \mathbb{R}^m$ is the measurement vector, $M : \mathbb{R}^n \to \mathbb{R}^m$ is the model function, and $x \in \mathbb{R}^n$ is the parameter vector. Many models in estimation and fitting problems are linear functions of x. If M is linear, $M(x) = Jx$, then $f(x) = \frac{1}{2}\|\eta - Jx\|_2^2$ which is a convex function, as $\nabla^2 f(x) = J^T J \succeq 0$. Therefore local minimizers are found by:

$$\begin{aligned} \nabla f(x) = 0 &\Leftrightarrow J^T J x^* - J^T \eta = 0 \\ &\Leftrightarrow x^* = \underbrace{\left(J^T J\right)^{-1} J^T}_{J^+} \eta \end{aligned}$$

Theorem 12.2: Pseudo-inverse

J^+ is called the pseudo-inverse and is a generalization of the inverse matrix. If $J^T J \succ 0$, J^+ is given by

$$J^+ = {(J^T J)}^{-1} J^T.$$

So far, ${(J^T J)}^{-1}$ is only defined if $J^T J \succ 0$. THis holds if and only if $rank(J) = n$, i.e. if the collumds of J are linearly independent.

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