Optimization

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Mathematical Preliminaries

Fundamental Concepts

7 Fundamental Concepts of Optimization

Theorem 7.1: Optimization problem in standard form

$$\begin{aligned} \underset{x \in \mathbb{R}^n}{\text{minimize}} \quad & f(x) \\ \text{subject to} \quad & g(x) = 0, \\ & h(x) \geq 0. \end{aligned}$$

Theorem 7.2: Level Set

The set

$$\{\ x \in \mathbb{R}^n \mid f(x) = c\ \}$$

is the level set of f for the value c, i.e. the set of all points that map to the same value c.

Theorem 7.3: Feasible set

The set

$$\{\ x \in \mathbb{R}^n \mid g(x) = 0\ \wedge\ h(x) \geq 0\ \}$$

is the feasible set Ω, i.e. the set of all points that satisfy the constraints.

Theorem 7.4: Global minimizer

The point x^* is a global minimizer if and only if

$$x^* \in \Omega,\ \forall x \in \Omega:\ f(x^*) \leq f(x).$$

Theorem 7.5: Strict global minimizer

The point x^* is a strict global minimzer if and only if

$$x^* \in \Omega,\ \forall x \in \Omega \setminus \{x^*\}:\ f(x^*) < f(x).$$

Theorem 7.6: Strict local minimizer

The point x^* is a strict local minimizer if and only if $x^* \in \Omega$ and there exists a neighborhood $\mathcal{N}$ of x^* such that

$$\forall x \in (\Omega \cap \mathcal{N}) \setminus \{x^*\}:\ f(x^*) < f(x).$$

Theorem 7.7: Weierstrass

If $\Omega \in \mathbb{R}^n$ is compact, i.e. limited and closed, and $f : \Omega \to \mathbb{R}$ is continuous, then there exists a global minimizer (a solution) of the optimization problem

$$\begin{aligned} \underset{x \in \mathbb{R}^n}{\text{minimize}} \quad & f(x) \\ \text{subject to} \quad & x \in \Omega. \end{aligned}$$

Proof 7.1: Weierstrass

Regard the graph of f, $G = \{(x, f(x)) \in \mathbb{R}^n \times \mathbb{R} \mid x \in \Omega\}$. G is a compact set, and so is the projection of G onto its last coordinate, the set $G = \{s \in \mathbb{R} \mid \exists x \text{ such that } (x, s) \in G\}$, which is a compact interval $[f_{\min}, f_{\max}] \subset \mathbb{R}$. By construction, there must be at least one x^* such that $(x^*, f(x^*)) \in G$.

$\square$

8 Types of Optimization Problems

Theorem 8.1: Convex set

A set $\Omega \subset \mathbb{R}^n$ is convex if

$$\forall x, y \in \Omega,\ \forall \lambda \in [0, 1]:\ x + \lambda(y - x) \in \Omega.$$

or if "all connecting lines lie inside the set".

Theorem 8.2: Convex function

A function $f : \Omega \to \mathbb{R}$ is convex if

$$\forall x, y \in \Omega,\ \forall \lambda \in [0, 1]:\ f(x + \lambda(y - x)) \leq f(x) + \lambda(f(y) - f(x)).$$

or if "all secants (i.e. a line segment between two points on the graph) are above graph". This definition is equivalent to saying that the Epigraph of f, i.e. the set $\{(x, s) \in \mathbb{R}^n \times \mathbb{R} \mid x \in \Omega,\ s \geq f(x)\}$, is a convex set.

Note: a concave function is the same but then with $\geq$ instead of $\leq$.

Property 8.1: Convex function

If $f:\ D \to \mathbb{R}$ and $\Omega_f = \{(x, y) \mid x \in D, y \geq f(x)\}$ then the following holds:

$$f \text{ is convex} \ \Leftrightarrow\ \Omega_f \text{ is convex.}$$

Theorem 8.3: Convex optimization problem

An optimization problem with convex feasible set Ω and convex objective function $f : \Omega \to \mathbb{R}$ is called a convex optimization problem.

Property 8.2: Globality of local minima of convex optimization problem

For a convex optimization problem, a local minimum is also a global one.

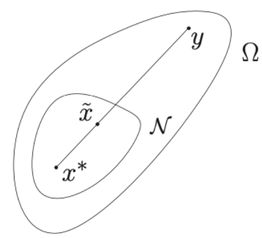
Proof 8.1: Globality of local minima of convex optimization problem

Regard a local minimum x^* of the convex optimization problem

$$\begin{aligned} \underset{x \in \mathbb{R}^n}{\text{minimize}} \quad & f(x) \\ \text{subject to} \quad & x \in \Omega. \end{aligned}$$

We will show that for any point $y \in \Omega$ holds $f(y) \geq f(x^*)$, i.e. x^* is a global minimum.

First we choose, using local optimality, a neighborhood $\mathcal{N}$ of x^* such that for all $\tilde{x} \in \Omega \cap \mathcal{N}$ holds $f(\tilde{x}) \geq f(x^*)$. Second, we regard the connecting line between x^* and y. This line is completely contained in Ω due to the convexity of Ω. Now we choose a point $\tilde{x}$ on this line such that it is in the neighborhood, but not equal to x^*, i.e. $\tilde{x} = x^* + \lambda(y - x^*)$ for some $\lambda \in (0, 1)$, and $\tilde{x} \in \Omega \cap \mathcal{N}$. Due to local optimality, we have $f(\tilde{x}) \geq f(x^*)$, and due to convexity we have



$$f(\tilde{x}) = f(x^* + \lambda(y - x^*)) \leq f(x^*) + \lambda(f(y) - f(x^*)).$$

It follows that $\lambda(f(y) - f(x^*)) \geq 0$, and since $\lambda \in (0, 1)$, we have $f(y) \geq f(x^*)$, as desired.

$\square$

Theorem 8.4: Unconstrained optimization problem

An optimization problem with no constraints, i.e. $g(x) = 0$ and $h(x) \geq 0$ are empty, is called an unconstrained optimization problem.

Example 8.1: Nonlinear Programming (NLP)

Nonlinear Programming Problems are poblems of the following form:

$$\begin{aligned} \underset{x \in \mathbb{R}^n}{\text{minimize}} \quad & f(x) \\ \text{subject to} \quad & g(x) = 0, \\ & h(x) \geq 0, \end{aligned}$$

where $f : \mathbb{R}^n \to \mathbb{R}$, $g : \mathbb{R}^n \to \mathbb{R}^p$, and $h : \mathbb{R}^n \to \mathbb{R}^q$, are assumed to be continuously differentiable at least once.

Example 8.2: Linear Programming (LP)

When the functions f, g, h are affine (i.e., they can be expressed in the form $f(x) = a^T x + b$ for some vector a and scalar b) in the general formulation (see Theorem 8.1), the general NLP gets something easier to solve, namely a Linear Program (LP), which can be written as follows:

$$\begin{aligned} \underset{x \in \mathbb{R}^n}{\text{minimize}} \quad & c^T x \\ \text{subject to} \quad & Ax - b = 0, \\ & Cx - d \geq 0, \end{aligned}$$

where $c \in \mathbb{R}^n$, $A \in \mathbb{R}^{p \times n}$, $b \in \mathbb{R}^p$, $C \in \mathbb{R}^{q \times n}$, and $d \in \mathbb{R}^q$.

Example 8.3: Quadratic Programming (QP)

When the functions g, h are affine (as for an LP in Theorem 8.2), and the objective function f is a linear-quadratic function, the NLP becomes a Quadratic Program (QP), which can be written as follows:

$$\begin{aligned} \underset{x \in \mathbb{R}^n}{\text{minimize}} \quad & c^T x + \frac{1}{2} x^T B x \\ \text{subject to} \quad & Ax - b = 0, \\ & Cx - d \geq 0, \end{aligned}$$

where, in addition to the LP parameters, $B \in \mathbb{R}^{n \times n}$ is the Hessian matrix. Specifically, for the objective function $f(x) = c^T x + \frac{1}{2} x^T B x$, the Hessian is given by:

$$\nabla^2 f(x) = B.$$

Theorem 8.5: Convex QP

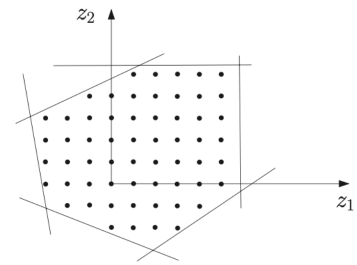
If the Hessian matrix B is positive semi-definite (i.e. if $\forall z \in \mathbb{R}^n \setminus \{0\} : z^T B z \geq 0$) we call the QP a convex QP. Convex QPs are tremendously easier to solve globally than “non-convex QPs” (i.e., where the Hessian B is not positive semi-definite), which might have different local minima (i.e. have a non-convex solution set).

Theorem 8.6: Strictly convex QP

If the Hessian matrix B is positive definite (i.e. if $\forall z \in \mathbb{R}^n : z^T B z > 0$) we call the QP a strictly convex QP. Strictly convex QPs are a subset of convex QPs (see Theorem 8.5).

Example 8.4: Mixed-Integer Programming (MIP)

A mixed-integer programming problem or mixed-integer program (MIP) is a problem with both real and integer decision variables. A MIP can be formulated as follows:



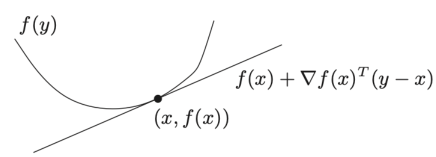
$$\begin{aligned} \underset{\substack{x \in \mathbb{R}^n \\ z \in \mathbb{Z}^m}}{\text{minimize}} \quad & f(x, z) \\ \text{subject to} \quad & g(x, z) = 0, \\ & h(x, z) \geq 0, \end{aligned}$$

where $x \in \mathbb{R}^n$ are the continuous variables, and $z \in \mathbb{Z}^m$ are the integer variables.

9 Convex Optimization

Theorem 9.1: Convexity for C^1 functions

Assume that $f : \Omega \to \mathbb{R}$ is continuously differentiable and Ω is convex. Then holds that f is convex if and only if



$$\forall x, y \in \Omega : \ f(y) \geq f(x) + \nabla f(x)^T(y - x)$$

i.e. tangents lie below the graph.

Proof 9.1: Convexity for C^1 functions

"$\Rightarrow$": Due to convexity of f holds for given $x, y \in \Omega$ and for any $\lambda \in [0, 1]$ that

$$f(x + \lambda(y - x)) - f(x) \leq \lambda(f(y) - f(x)).$$

and therefore that

$$\nabla f(x)^T(y - x) = \lim_{\lambda \to 0} \frac{f(x + \lambda(y - x)) - f(x)}{\lambda} \leq \frac{f(y) - f(x)}{y - x}.$$

"$\Leftarrow$": To prove that $z = x + \lambda(y - x) = (1 - \lambda)x + \lambda y$ holds that $f(z) \leq (1 - \lambda)f(x) + \lambda f(y)$, we can use the equation from Theorem 9.1 twice to get

$$f(x) \geq f(z) + \nabla f(z)^T(x - z) \quad \text{and} \quad f(y) \geq f(z) + \nabla f(z)^T(y - z),$$

which yield, when weighted with $(1 - \lambda)$ and λ respectively, that

$$(1 - \lambda)f(x) + \lambda f(y) \geq f(z) + \nabla f(z)^T \underbrace{[(1 - \lambda)(x - z) + \lambda(y - z)]}_{=0}$$

□

Theorem 9.2: Generalized Inequality for Symmetric Matrices

We write for a symmetric matrix $B = B^T$, $B \in \mathbb{R}^{n \times n}$ that "$B \succeq 0$" if and only if B is positive semi-definite, i.e.,

$$\forall z \in \mathbb{R}^n : \; z^T B z \geq 0,$$

or, equivalently, if all (real) eigenvalues of B are non-negative.

Theorem 9.3: $O(\cdot)$

For a function $f : \mathbb{R}^n \to \mathbb{R}^m$, we write

$$f(x) = O(g(x))$$

if and only if there exists a constant $C > 0$ and a neighborhood $\mathcal{N}$ of 0 such that

$$\forall x \in \mathcal{N} : \; \|f(x)\| \leq C g(x),$$

i.e. "f shrinks as fast as g".

Theorem 9.4: $o(\cdot)$

For a function $f : \mathbb{R}^n \to \mathbb{R}^m$, we write

$$f(x) = o(g(x))$$

if and only if there exists a neighborhood $\mathcal{N}$ of 0 and a function $c : \mathcal{N} \to \mathbb{R}$ with $\lim_{x \to 0} c(x) = 0$ such that

$$\forall x \in \mathcal{N} : \; \|f(x)\| \leq c(x) g(x),$$

i.e. "f shrinks faster than g".

Theorem 9.5: Convexity for C^2 functions

Assume that $f : \; \Omega \to \mathbb{R}$ is twice continuously differentiable and Ω is convex and open. Then holds that f is convex if and only if

$$\forall x \in \Omega : \; \nabla^2 f(x) \succeq 0,$$

i.e. the Hessian of f is positive semi-definite.

Proof 9.2: Convexity for C^2 functions

To prove that the Theorem 9.1 implies the Theorem 9.5, we can use a second order Taylor expansion of y at x in an arbitrary direction p:

$$f(x + tp) = f(x) + t \nabla f(x)^T p + \frac{t^2}{2} p^T \nabla^2 f(x) p + o(t^2 \|p\|).$$

From this we obtain that

$$p^T \nabla^2 f(x) p = \lim_{t \to 0} \frac{2}{t^2} \left(\underbrace{f(x+tp) - f(x) - t\nabla f(x)^T p}_{(9.1):\ \geq 0} \right) \geq 0.$$

Conversely, to prove the other direction, we use the Taylor rest term formula with some arbitrary $\theta \in (0,1)$:

$$f(y) = f(x) + \nabla f(x)^T (y-x) + \underbrace{\frac{1}{2} t^2 (y-x)^T \nabla^2 f(x + \theta(y-x))(y-x)}_{(9.5):\ \geq 0}.$$

□

Property 9.1: Convexity perserving operations on convex functions

The following operations preserve the convexity of a function:

1. Non-negative weighted sum: Suppose that $\forall i \in [i, m]: \ f_i : \mathbb{R}^n \to \mathbb{R}$ are convex functions and $\forall i \in [1, m]: \ \lambda_i \geq 0$. Then the function

$$f(x) = \sum_{i=1}^{m} \lambda_i f_i(x)$$

   is convex.

2. Affine input transformation: If $f: \ \Omega \to \mathbb{R}$ is convex, then also

$$A \in \mathbb{R}^{n \times m}: \ \tilde{f}(x) = f(Ax + b)$$

   is convex on the domain $\tilde{\Omega} = \{x \mid Ax + b \in \Omega\}$.

3. Concatenation with monotone convex function: If $f: \ \Omega \to \mathbb{R}$ is convex and $g: \ \mathbb{R} \to \mathbb{R}$ is convex and monotonely increasing, then the composition $g \circ f$ is convex.

4. Pointwise supremum: The supremum over a set of convex functions $f_i(x)$, $i \in I$, where I can be an infinite set, i.e.,

$$f(x) = \sup_{i \in I} f_i(x)$$

   is convex.

5. Composition: Let $h: \mathbb{R}^m \to \mathbb{R}$ and $g: \mathbb{R}^n \to \mathbb{R}^m$ with $g = (g_1, \ldots, g_m)$. Then $f(x) = (h \circ g)(x) = h(g(x))$ is convex if any of the followings holds:

   (a) h is convex and non-decreasing in each argument and $\forall i \in [1, m]: \ g_i$ is convex.

   (b) h is convex and non-decreasing in each argument and $\forall i \in [1, m]: \ g_i$ is concave, i.e. $-g_i$ is convex.

Proof 9.3: Convexity perserving operations on convex functions

1. TODO (use the definition of convexity).

2. TODO (use the definition of convexity).

3. Recall that g is a convex and monotonely increasing function, then:

$$\nabla^2(g \circ f) = \underbrace{g''(f(x))}_{\geq 0} \underbrace{\nabla f(x) \nabla f(x)^T}_{\succeq 0} + \underbrace{g'(f(x))}_{\geq 0} \underbrace{\nabla^2 f(x)}_{\succeq 0} \succeq 0,$$

i.e. $g \circ f$ is convex, since the Hessian is positive semi-definite.

4. Epigraph of f is the intersection of the epigraphs of f_i, which are convex.

5. Recall that g_i is convex and that h is convex and non-decreasing in each argument. Then:

$$\begin{aligned} f(\lambda x + (1-\lambda)y) &= h(g(\lambda x + (1-\lambda)y)) \\ &\leq h(\lambda g(x) + (1-\lambda)g(y)) \qquad (1) \\ &\leq \lambda h(g(x)) + (1-\lambda)h(g(y)) \qquad (2) \\ &= \lambda f(x) + (1-\lambda)f(y), \end{aligned}$$

where we used the convexity of g_i and the fact that h is non-decreasing to obtain inequality (1), and the convexity of h to obtain inequality (2).

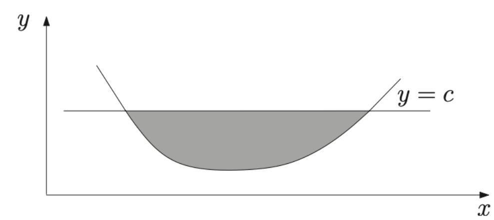
$\square$

Theorem 9.6: Convexity of Sublevel subsets

If $f: \ \Omega \to \mathbb{R}$ is a convex function, then all its level sets

$$\mathrm{lev}_{\leq \gamma} f = \{ \ x \in \Omega \mid f(x) \leq \gamma \ \}$$

are convex.



Proof 9.4: Convexity of Sublevel subsets

If $f(x) \leq c$ and $f(y) \leq c$, then for any $\lambda \in [0,1]$ holds also

$$f((1-\lambda)x + \lambda y) \leq (1-\lambda)f(x) + \lambda f(y) \leq \underbrace{(1-\lambda)c + \lambda c}_{=c}.$$

$\square$

Property 9.2: Convexity perserving operations on convex sets

The following operations preserve the convexity of a set:

1. The intersection of finitely or infinitely many convex sets is convex.
2. Affine image: if Ω is convex, then for $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$ also the set $A\Omega + b = \{y \in \mathbb{R}^m \mid \exists x \in \Omega : y = Ax + b\}$ is convex.
3. Affine pre-image: if Ω is convex, then for $A \in \mathbb{R}^{n \times m}$, $b \in \mathbb{R}^n$ also the set $\{z \in \mathbb{R}^m \mid Az + b \in \Omega\}$ is convex.

Example 9.1: Convex Feasible set

If $\forall i \in [1, m] : f_i : \mathbb{R}^n \to \mathbb{R}$ are convex functions, then the set

$$\Omega = \{x \in \mathbb{R}^n \mid f_i(x) \leq 0, \; i \in [1, m]\}$$

is a convex set, because it is the intersection of sublevel sets Ω_i of convex functions f_i, i.e.

$$\begin{aligned} \Omega &= \bigcap_{i=1}^{m} \Omega_i \\ &= \bigcap_{i=1}^{m} \{x \in \mathbb{R}^n \mid f_i(x) \leq 0\}. \end{aligned}$$

Theorem 9.7: Optimality condition for convex problems

Regard a convex optimization problem with continuously differentiable objective function f. A point $x^* \in \Omega$ is a global optimizer if and only if

$$\forall y \in \Omega : \; \nabla f(x^*)^T (y - x^*) \geq 0.$$

Proof 9.5: Optimality condition for convex problems

"$\Rightarrow$": Assume for the sake of contradiction that $\exists y \in \Omega : \; \nabla f(x^*)^T (y - x^*) < 0$, then we could regard a Taylor expansion

$$f(x^* + \lambda(y - x^*)) = f(x^*) + \lambda \underbrace{\nabla f(x^*)^T (y - x^*)}_{<0} + \underbrace{o(\lambda)}_{\to 0}.$$

This implies that for sufficiently small positive λ, we have

$$f(x^* + \lambda(x - x^*)) < f(x^*),$$

which contradicts the optimality of x^*.

"$\Leftarrow$": Due to the C^1 characterization of convexity of f in Theorem 9.1, we have for any feasible $y \in \Omega$:

$$f(y) \geq f(x^*) + \underbrace{\nabla f(x^*)^T (y - x^*)}_{\geq 0} \geq f(x^*),$$

which implies that x^* is a global optimizer.

$\square$

Theorem 9.8: Sufficient Condition for Convex NLP

For a nonlinear optimization problem (NLP) in standard form

$$\begin{aligned} \underset{x \in \mathbb{R}^n}{\text{minimize}} \quad & f(x) \\ \text{subject to} \quad & g_i(x) = 0,\ i \in [1, m], \\ & h_i(x) \geq 0,\ j \in [1, p], \end{aligned}$$

the following conditions are necessary for convexity:

- the objective function $f : \mathbb{R}^n \to \mathbb{R}$ must be convex,
- the constraint set
  $$X = \{\ x \in \mathbb{R}^n \mid g(x) = 0,\ h(x) \geq 0\ \}$$
  must be convex. Since we know that the intersection of convex sets is convex, we can write X as the intersection of the sets G and H:
  $$\begin{aligned} X &= \{x \in \mathbb{R}^n \mid g(x) = 0,\ h(x) \geq 0\} \\ &= \{x \in \mathbb{R}^n \mid g(x) = 0\} \cap \{x \in \mathbb{R}^n \mid h(x) \geq 0\} \\ &= \left(\bigcap_{i=1}^{m} \{x \in \mathbb{R}^n \mid g_i(x) = 0\} \right) \cap \left(\bigcap_{i=1}^{m} \{x \in \mathbb{R}^n \mid h_i(x) \geq 0\} \right) \\ &= G \cap H \end{aligned}$$
  Now we must consider the requirements for the sets G and H to be convex:
  - Suppose $-H_i = \{x \in \mathbb{R}^n \mid h_i(x) \leq 0\}$, the zero sublevel set of the function h_i. If h_i is convex, the set $-H_i$ is convex, as seen in Theorem 9.6. Now, consider the case where h_i is concave. Since h_i being concave means that $-h_i$ is convex, the set $H_i = \{x \in \mathbb{R}^n \mid h_i(x) \geq 0\}$ is a sublevel set of the convex function $-h_i$, and is therefore convex. Thus, the set H_i is convex when h_i is concave.
  - On the other hand, G is the level set of of g_i. Therefore it is certainly a convex set whenever g_i is a affine function
    $$\forall i \in [i, m] :\ \ g_i(x) = a_i^T x + b_i.$$

TODO: examples

10 The Lagrangian Function and Duality

Theorem 10.1: Primal Optimization Problem

We will denote the globally optimal value of the objective function subject to the constraints as the primal optimal value p^*, i.e.,

$$p^* = \left(\min_{x \in \mathbb{R}^n} f(x) \quad \text{s.t.} \quad g(x) = 0\ \wedge\ h(x) \geq 0 \right).$$

and we will denote this optimization problem as the primal optimization problem.

Theorem 10.2: Lagrangian Function and Lagrange Multipliers

We define the Lagrangian function to be

$$\begin{aligned}\mathcal{L}(x, \lambda, \mu) &= f(x) + \sum_{i=1}^{l} \lambda_i g_i(x) + \sum_{i=1}^{m} \mu_i h_i(x) \\ &= f(x) + \lambda^T g(x) + \mu^T h(x).\end{aligned}$$

Here, we have introduced Lagrange multipliers or dual variables $\lambda \in \mathbb{R}^l$ and strictly positive $\mu \in \mathbb{R}^m$.

Lemma 10.1: Lower Bound Property of Lagrangian

If $\tilde{x}$ is a feasible point of (10.1) and $\mu \geq 0$, then

$$\mathcal{L}(\tilde{x}, \lambda, \mu) \leq f(\tilde{(x)}).$$

Proof 10.1: Lower Bound Property of Lagrangian

Since $\tilde{x}$ is feasible, we have $g(\tilde{x}) = 0$ and $h(\tilde{x}) \geq 0$. Therefore, with $\mu \geq 0$, we have:

$$\mathcal{L}(\tilde{x}, \lambda, \mu) = f(\tilde{x}) + \lambda^T \underbrace{g(\tilde{x})}_{=0} + \underbrace{\mu^T}_{\geq 0} \underbrace{h(\tilde{x})}_{\geq 0} \leq f(\tilde{x}).$$

$\square$

Theorem 10.3: Lagrange Dual Function

The Lagrange dual function is defined as the unconstrained infimum of the Lagrangian function over x, for fixed multipliers λ and μ:

$$q(\lambda, \mu) = \inf_{x \in \mathbb{R}^n} \mathcal{L}(x, \lambda, \mu).$$

Lemma 10.2: Lower Bound property of Lagrange Dual

If $\mu \geq 0$, then the dual function $q(\lambda, \mu)$ is a lower bound on the primal optimal value p^*, i.e.,

$$q(\lambda, \mu) \leq p^*.$$

Proof 10.2: Lower Bound property of Lagrange Dual

Since the Lagrange function is bounded from below by Lemma 10.1, we have

$$q(\lambda, \mu) = \inf_{x \in \mathbb{R}^n} \mathcal{L}(x, \lambda, \mu) \leq f(\tilde{x}) \quad \text{for any feasible } \tilde{x}.$$

Naturally, this inequality holds in particular for the global minimizer x^*, which yields:

$$q(\lambda, \mu) \leq f(x^*) = p^*.$$

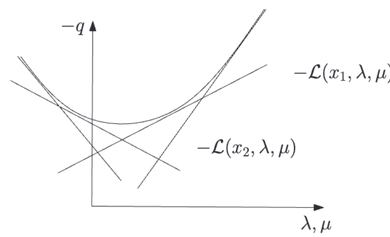
$\square$

Theorem 10.4: Concavity of Lagrange Dual

The function $q: \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}$ is concave, even if the original NLP was not convex.

Proof 10.3: Concavity of Lagrange Dual

We will show that $-q$ is convex. The Lagrangian $\mathcal{L}$ is an affine function in the multipliers λ and μ, which in particular implies that $-\mathcal{L}$ is convex in λ and μ. Thus, the function $-q(\lambda, \mu) = \sup_x -\mathcal{L}(x, \lambda, \mu)$ is the supremum of convex functions in λ and μ that are indexed by x, and therefore convex.



$\square$

Theorem 10.5: Dual Problem

The dual problem is defined as the convex maximization problem, i.e.,

$$d^* = \left(\max_{\lambda \in R^p, \mu \in R^q} q(\lambda, \mu) \quad \text{s.t.} \quad \mu \geq 0 \right).$$

Theorem 10.6: Weak Duality

Consider a primal-dual pair. Then, the following inequality holds:

$$d^* \leq p^*.$$

Theorem 10.7: Strong Duality

If the primal optimization problem is convex and the Slater condition (see Theorem 10.8) holds, then strong duality holds, i.e.,

$$d^* = p^*.$$

Theorem 10.8: Slater Condition

If there exist one feasible point $\tilde{x}$ such that all non-linear inequalities are strictly satisfied of a primal convex optimization problem hold, then the Slater condition is satisfied. More explicitly, for a convex problem we must have affine equality constraints, $g(x) = Ax+b$, and the inequality constraint functions can be either affine or concave functions, thus without loss of generality assume that the first $q_1 \leq q$ inequalities are affine and the remaining ones concave. Then the Slater condition holds if and only if there exists an $\tilde{x}$ such that

$$\begin{aligned} A\tilde{x} + b &= 0, \\ h_i(\tilde{x}) &\geq 0, \quad \text{for } i = 1, \dots, q_1, \\ g_i(\tilde{x}) &> 0, \quad \text{for } i = q_1 + 1, \dots, q. \end{aligned}$$

Unconstrained Optimization Algorithms

11 Optimality Conditions

Theorem 11.1: Unconstrained optimization Problems

We define the unconstrained optimization problem as

$$\min_{x \in D} f(x),$$

where we regard objective function $f : D \to \mathbb{R}$ that are defined on some open domain $D \subseteq \mathbb{R}^n$.

Theorem 11.2: Stationary Point

A point $\tilde{x}$ is called a stationary point of f if and only if

$$\nabla f(\tilde{x}) = 0.$$

Theorem 11.3: Descent Direction

A vector $p \in \mathbb{R}^n$ is called a descent direction at x if

$$\nabla f(x)^T p < 0.$$

Theorem 11.4: First Order Necessary Conditions (FONC)

If $x^* \in D$ is a local minimizer of $f : D \to \mathbb{R}$ and $f \in C^1$ then

$$\nabla f(x^*) = 0.$$

Proof 11.1: First Order Necessary Conditions (FONC)

Let us assume for contradiction that $\nabla f(x^*) \neq 0$. Then $p = -\nabla f(x^*)$ would be a descent direction in which the objective could be improved, as follows:

As D is open and $f \in C^1$, we could find a $t > 0$ that is small enough so that for all $\tau \in [0, t]$ holds $x^* + \tau p \in D$ and $\nabla f(x^* + \tau p)^T p < 0$. By Taylor's theorem, we would have for some $\theta \in (0, 1)$ that

$$f(x^* + tp) = f(x^*) + \underbrace{t \nabla f(x^* + \theta tp)^T p}_{<0} < f(x^*).$$

$\square$

Theorem 11.5: Second Order Necessary Conditions (SONC)

If $x^* \in D$ is a local minimizer of $f : D \to \mathbb{R}$ and $f \in C^2$ then

$$\nabla^2 f(x^*) \succeq 0.$$

Proof 11.2: Second Order Necessary Conditions (SONC)

Assume, for the sake of contradiction, that $\nabla^2 f(x^*) \prec 0$. This implies the existence of a vector $p \in \mathbb{R}^n$ such that $p^T \nabla^2 f(x^*) p < 0$. In this case, the objective function could be improved in the direction of p. By choosing a sufficiently small $t > 0$, we can ensure that for all $\tau \in [0, t]$, the following holds:

$$p^T \nabla^2 f(x^* + \tau p) p < 0.$$

Applying Taylor's theorem, we would have for some $\theta \in (0, 1)$ that

$$f(x^* + tp) = f(x^*) + \underbrace{t \nabla f(x^*)^T p}_{=0} + \frac{t^2}{2} \underbrace{p^T \nabla^2 f(x^* + \theta p)}_{<0} p < f(x^*),$$

which leads to a contradiction.

□

Theorem 11.6: Convex First Order Sufficient Conditions (cFOSC)

Assume that $f : D \to \mathbb{R}$ is convex and $f \in C^1$. If $x^* \in D$ is a stationary point of f, then x^* is a global minimizer of f.

Theorem 11.7: Second Order Sufficient Conditions (SOSC)

Assume that $f : D \to \mathbb{R}$ and $f \in C^2$. If $x^* \in D$ is a stationary point of f and $\nabla^2 f(x^*) \succ 0$, then x^* is a strict local minimizer of f.

Proof 11.3: Second Order Sufficient Conditions (SOSC)

We can choose a sufficiently small closed ball B around x^* so that for all $x \in B$ holds $\nabla^2 f(x) \succ 0$. Restricted to this ball, we have a convex problem, so that Theorem 11.6 together with stationarity of x^* implies that x^* is a global minimizer of f. To prove that it is strict, we look for any $x \in B \backslash x^*$ at the Taylor expansion, which yields with some $\theta \in (0, 1)$:

$$f(x) = f(x^*) + \underbrace{\nabla f(x^*)^T (x - x^*)}_{=0} + \frac{1}{2} \underbrace{(x - x^*)^T \nabla^2 f(x^* + \theta(x - x^*))(x - x^*)}_{>0} > f(x^*).$$

□

Theorem 11.8: Stability of Parametric Solutions

Assume that $f : D \times \mathbb{R}^m \to \mathbb{R}$, and regard the minimization of $f(\cdot, \tilde{a})$ for a given fixed value of $\tilde{a} \in \mathbb{R}^m$. If $\tilde{x} \in D$ satisfies the SOSC (see Theorem 11.7), then there is a neighborhood $\mathcal{N} \subset \mathbb{R}^m$ around $\tilde{a}$ such that the parametric minimizer function $x^*(a)$ is well-defined for all $a \in \mathcal{N}$ (i.e. there is a unique minimizer $x^*(a)$ for each $a \in \mathcal{N}$), is differentiable in $\mathcal{N}$, and $x^*(\tilde{a}) = \tilde{x}$. Its derivative at $\tilde{a}$ is given by

$$\frac{\partial(x^*(\tilde{a}))}{\partial a} = -\left(\nabla_x^2 f(\tilde{x}, \tilde{a})\right)^{-1} \frac{\partial\left(\nabla_x f(\tilde{x}, \tilde{a})\right)}{\partial a}.$$

Moreover, each such $x^*(a)$ with $a \in \mathcal{N}$ satisfies again the SOSC and is thus a strict local minimizer.

Proof 11.4: Stability of Parametric Solutions

The existence of the differentiable map $x^* : \mathcal{N} \to D$ follows from the implicit function theorem applied to the stationarity condition $\nabla_x f(x^*(a), a) = 0$. The derivative of $x^*(a)$ at $\tilde{a}$ is given by

$$\frac{d(\nabla_x f(x^*(a), a))}{da} = \underbrace{\frac{\partial\left(\nabla_x f(x^*(a), a)\right)}{\partial x}}_{=\nabla_x^2 f} \frac{\partial x^*(a)}{\partial a} + \frac{\partial\left(\nabla_x f(x^*(a), a)\right)}{\partial a} = 0$$

The fact that all points $x^*(a)$ satisfy the SOSC follows from the continuity of the second derivative.

□

12 Estimation and Fitting Problems

Theorem 12.1: Estimation and Fitting Problems

Estimation and fitting problems are optimization problems with a special objective, namely a least squares objective. We define the estimation problem as

$$\min_{x \in \mathbb{R}^n} \frac{1}{2} \|\eta - M(x)\|_2^2,$$

where $\eta \in \mathbb{R}^m$ is the measurement vector, $M : \mathbb{R}^n \to \mathbb{R}^m$ is the model function, and $x \in \mathbb{R}^n$ is the parameter vector. Many models in estimation and fitting problems are linear functions of x. If M is linear, $M(x) = Jx$, then $f(x) = \frac{1}{2}\|\eta - Jx\|_2^2$ which is a convex function, as $\nabla^2 f(x) = J^T J \succeq 0$. Therefore local minimizers are found by:

$$\begin{aligned} \nabla f(x) = 0 &\Leftrightarrow J^T J x^* - J^T \eta = 0 \\ &\Leftrightarrow x^* = \underbrace{\left(J^T J\right)^{-1} J^T}_{J^+} \eta \end{aligned}$$

Theorem 12.2: Pseudo-inverse

J^+ is called the pseudo-inverse and is a generalization of the inverse matrix. If $J^T J \succ 0$, J^+ is given by

$$J^+ = (J^T J)^{-1} J^T.$$

So far, $(J^T J)^{-1}$ is only defined if $J^T J \succ 0$. THis holds if and only if $rank(J) = n$, i.e. if the collumds of J are linearly independent.

Theorem 12.3: Moore Penrose Pseudo Inverse

Assume $J \in \mathbb{R}^{m \times n}$ and that the singular value decomposition (SVD) of J is given by $J = U \Sigma V^T$. Then, the Moore-Penrose pseudo-inverse J^+ is given by

$$J^+ = V S^+ U^T,$$

where for

$$
S = \left[\begin{array}{ccccccc}
\sigma_1 &&&&&& \\
& \sigma_2 &&&&& \\
&& \ddots &&&& \\
&&& \sigma_r &&& \\
&&&& 0 && \\
&&&&& \ddots & \\
&&&&&& 0 \\
\hline
0 & \cdots & \cdots & 0 & \cdots & \cdots & 0
\end{array}\right]
\quad \text{holds} \quad
S^+ = \left[\begin{array}{cccccc|c}
\sigma_1^{-1} &&&&&& 0 \\
& \sigma_2^{-1} &&&&& \vdots \\
&& \ddots &&&& \vdots \\
&&& \sigma_r^{-1} &&& 0 \\
&&&& 0 && \vdots \\
&&&&& \ddots & \vdots \\
&&&&& 0 & 0
\end{array}\right]
$$

Lemma 12.1

or $\epsilon \to 0$

13 Newton Type Optimization

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